UCLA Department of Mathematics

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Math 132 Section 2 Spring 2021

Problem Set # 4

Problem (1) Let ψ be a harmonic function (in some region $\mathbb{D}$). Show that the function $\psi_x - i\psi_y$ is analytic (in this region).

Problem (2) The Jacobian of a generic map from the (x, y)–plane to the (u, v)–plane: $u = u(x, y)$, $v = v(x, y)$ is defined to be the determinant:

$$J(x_0, y_0) = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

where the partial derivatives are to be evaluated at the point (x_0, y_0). [It may or may not be recalled that these objects are essential for the consideration of change of variables in the context of multi–variable integration.] Show that if u and v are the real and imaginary parts of an analytic function, $f(z) = u(x, y) + iv(x, y)$, then $J(x, y)$ is given by $|f'(z)|^2$.

Problem (3) Show that if $\phi(x,y)$ and $\psi(x,y)$ are harmonic then the functions

$$u(x,y) = \phi_x\phi_y + \psi_x\psi_y$$

and

$$v(x,y) = \frac{1}{2}[\phi_x^2 + \psi_x^2 - \phi_y^2 - \psi_y^2]$$

are harmonic conjugates (satisfy the Cauchy–Riemann equations).

Problem (4) Suppose that both $f(z)$ and $\overline{f}(z)$ are analytic (in some suitable region $\mathbb{D}$ e.g., the inside of a circle). Show that this necessarily implies that f is constant.

Problem (5) Show that if $f(z)$ is analytic (in some region $\mathbb{D}$ e.g., the inside of a circle) and $|f|$ is constant then f itself is constant. Hint: Writing $f = u + iv$ use the fact that $|f|^2$ is supposed to be constant. Use CR or harmonicity, etc. to establish that the partial derivatives of u and v are zero; by 132–standards, this is sufficient. Or, use exponential form and C.R. equations.

Problem (6) Using the principle value definition, namely $-\pi < \theta \leq +\pi$, find the value of $(1+i)^{1+i}$.

Problem (7) Consider the function

$$f(z) = z^{\frac{1}{2}}$$

with branch cut along the positive real axis. To be specific, when expressed in polar form the angle θ satisfies $0 \leq \theta < 2\pi$. Show e.g., by taking derivatives along radial lines (θ fixed) that whenever the function is analytic,

$$\frac{d}{dz} z^{\frac{1}{2}} = \frac{1}{2} \frac{1}{z^{\frac{1}{2}}}.$$

See if you can generalize this.

Problem (8) A function $f(z)$ called $z^{1/4}$ has the following four properties:
I) It agrees with the usual real $x^{1/4}$ on the positive real axis.
II) It is discontinuous on the positive imaginary axis (where the function is not defined).
III) Everywhere except for the origin and the positive imaginary axis, it is analytic.
IV) Everywhere except for the positive imaginary axis, it satisfies $f^4 = z$.

Find an explicit formula for such a function e.g., using polar coordinates, and evaluate $f(-4)$ expressing your answer in the form $a + ib$ with a and b real.

Problem (9) The quantity denoted by i^i is somewhat ambiguous since there are different possible interpretations of how this might be computed. Write down two distinct values for i^i and justify your perspective(s).

Problem (10) Slightly hard. Here we will show that it is possible to construct functions with finite branch lines. Consider

$$f(z) = (z^2-1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}.$$

Now write $(z-1) = \rho_1 e^{i\varphi_1}$, $(z+1) = \rho_2 e^{i\varphi_2}$; here $\rho_1 = |z-1|$, $\rho_2 = |z+1|$ and the φ's are angles with the x–axis but for rays emanating from $z = \pm 1$. Show that by proper choice(s) of the range of the φ's you have a version of $\sqrt{z^2-1}$ which agrees with the usual $\sqrt{x^2-1}$ for x real and satisfying $|x| > 1$ and that this version is analytic except for $z = \pm 1$ and the segment joining these points. A picture might be helpful (but not required).