UCLA Department of Mathematics

Math 132 Section 2 Spring 2021

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Problem Set # 3

Problem (1) In many situations, one encounters coupled differential equations; often times complex analysis can be quite useful. A famous example is the system

$$\frac{dA}{dt} = B(t); \quad \frac{dB}{dt} = -A(t).$$

Here $A(t)$ and $B(t)$ are unknown functions of t; all quantities are real. Now consider the complex valued function (of the single real variable t)

$$Q(t) = A(t) + iB(t)$$

Note that by adding $i\times$[left equation] to the right equation, you get a straightforward complex equation for $Q(t)$. Using this method, solve this above system subject to the initial condition $A(t = 0) = A_0$, $B(t = 0) = B_0$.

Problem (2) Write the function $f(z)$ in the form $u(x,y) + iv(x,y)$ – u and v real – where

$$f(z) = \frac{2z^2 + 3}{|z - 1|}$$

Problem (3) Show directly that for any $z \neq 1$,

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

Problem (4) Using complex exponentials and the identity for $1 + z + z^2 + ... z^n$, derive the formula

$$1 + \cos\theta + \cos 2\theta + ... + \cos n\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}.$$

(Here, of course, we assume that θ lies strictly between 0 and 2π.)

Problem (5) Let a denote one of the n^{th} roots of unity which is not 1. Show that

$$1 + 2a + 3a^2 + \cdots + na^{n-1} = \frac{n}{a-1}$$

Problem (6) Find all complex numbers z for which $\frac{2-z}{2+z}$ is pure real.

Problem (7) Find all complex numbers z for which $\frac{2-z}{2+z}$ is pure imaginary.

Problem (8) Suppose $f(z) = u(x, y) + iv(x, y)$ is expressed in polar coordinates:$f(z) = A(r, \theta) + iB(r, \theta)$. Derive the polar Cauchy–Riemann equations satisfied by A and B if u and v obey the usual (Cartesian) CR equations.

Problem (9) Let $g(z)$ be a complex function which is (defined and) differentiable everywhere with derivative $g'(z)$. Now define $f(z)$ via $f(z) = [\overline{g}(\overline{z})]$ – the complex conjugate of g evaluated at the conjugate of z. Show, using directly the definition of derivative, that f is differentiable and write $f'(z)$ in terms of $g'(z)$.

Problem (10) Let $g(z)$ be a complex function which is (defined and) differentiable everywhere with derivative $g'(z)$. Now define $f(z)$ via $f(z) = [\overline{g}(\overline{z})]$ – the complex conjugate of g evaluated at the conjugate of z. Show, by (carefully) using the Cauchy Riemann equations, that $f(z)$ is everywhere differentiable.

Problem (11) For any complex number $c = a + ib$ (a and b real), write down the real and imaginary parts of the function $f(z) = \mathrm{e}^{cz}$ and show that the derivative exists and that in fact, $f'(z) = c\mathrm{e}^{cz}$

[You need not prove the existence of the derivative from first principles, you may use the fact that the CR equations are necessary and sufficient.]

Problem (12) Let $f(z) = u(x,y) + iv(x,y)$ be a complex function which is everywhere differentiable – so, explicitly, the Cauchy–Riemann equations are satisfied. Now consider the two level curves

$$u(x,y) = c_1 \;\;\&\;\; v(x,y) = c_2$$

where c_1 and c_2 are constants (whose particular value plays no role). It is supposed that the two curves intersect at some point (x_0, y_0). Show that they do so orthogonally; that is to say at the point of intersection, the tangents to the curves are at right angles.

Problem (13) Let $f(z) = (e^{x^2-y^2})(\cos 2xy + i \sin 2xy)$. Show that f is analytic everywhere.

Problem (14) Find a non–trivial polynomial function in x and y where the combined degree of each term is four – that is to say a polynomial is of the form

$$\sum_{a=0}^{4} c_a x^{4-a} y^a$$

with not all of the c's zero – such that the polynomial is harmonic.

Problem (15) Derive, from first principles, expressions for the real and imaginary parts of the function

$$f(z) = \sin z$$

in terms of the usual circular and hyperbolic functions. Show your work.

Problem (16) Derive, from first principles, expressions for the real and imaginary parts of the function

$$f(z) = \sinh z$$

in terms of the usual circular and hyperbolic functions. Show your work.

Problem (17) If $\hat{a}$ denotes a unit vector in the plane, let $D_{\hat{a}}$ denote the usual directional derivative in the direction of $\hat{a}$. [Explicitly, if we write $\hat{a} = (a_1, a_2)$ and $K(x, y)$ is a two variable function then $D_{\hat{a}}K = a_1K_x + a_2K_y = \hat{a} \cdot \nabla K$] Now let $f(z) = u + iv$ be an analytic function at z. Show that $D_{\hat{a}}u = D_{\hat{c}}v$ where $\hat{c}$ is the unit vector rotated 90° from $\hat{a}$.