UCLA Department of Mathematics

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Math 132 Section 2 Spring 2021

Problem Set # 2

Problem (1) What is to follow may or may not be of use but is anyway true. Let $z_1 = a + ib$ and $z_2 = c + id$; a, b, c and d real. Consider the corresponding vectors which we will regard as 3–component vectors with the third component zero:

$$\vec{w}_1 = \langle a, b, 0\rangle$$

$$\vec{w}_2 = \langle c, d, 0\rangle$$

Show that both the real and imaginary parts of $\overline{z}_1 z_2$ can be expressed in terms of the usual vector (cross) and scalar (dot) products of $\vec{w}_1$ and $\vec{w}_2$.

Problem (2) Let $\vec{A} = \langle a, b\rangle$ denote a two–component vector and $\hat{\alpha} = \langle \cos\alpha, \sin\alpha\rangle$ a two–component *unit* vector. Recall the usual formula/definition of vector projection

$$\mathrm{Proj}_{\hat{\alpha}}(\vec{A}) = (\vec{A}\cdot\hat{\alpha})\hat{\alpha}.$$

Let us identify $\vec{A}$ with the complex number $z = a + ib$ and, similarly, $\hat{\alpha}$ will be represented by $\mathrm{e}^{i\alpha}$. Derive a formula for the corresponding complex number that represents $\mathrm{Proj}_{\hat{\alpha}}(\vec{A})$ in terms of z, $\overline{z}$ and $\mathrm{e}^{i\alpha}$.

Problem (3) Suppose that z and w are (purportedly) complex numbers such that both $z + w$ and zw are real and negative. Show that this implies that z and w are in fact both real.

Problem (4) Which of the three points i, $2 - i$, or -3 (regarded as points in the complex plane $\mathbb{C}$) is farthest from the origin?

Problem (5) A complex number z satisfies

$$|z|^2 - 2z = 3 + 4i.$$

Find this z.

Problem (6) Let $a_0, a_1, \ldots a_n$ denote real constants and suppose that z is a solution to the equation

$$a_0 + a_1 z + ... + a_n z^n = 0.$$

Show that $\bar{z}$ is also a solution to this equation.

Problem (7) Write the complex number

$$\frac{\mathrm{e}^{1+3i\pi}}{\mathrm{e}^{-1+i\frac{\pi}{2}}}$$

in the form $a + ib$ with a and b real.

Problem (8) Write the complex number

$$e^{e^i}$$

in the form $a + ib$ with a and b real.

Problem (9) Write the complex number

$$(1 + i)^6$$

in the polar form $re^{i\theta}$.

Problem (10) Write the complex number

$$\frac{2i}{3e^{4+i}}$$

in the polar form $re^{i\theta}$.

Problem (11) Using complex exponentials, write expressions for $\cos 5\theta$ and $\sin 5\theta$ which are polynomials in $\sin\theta$ and $\cos\theta$.

Problem (12) Using the equation expressing cosines in terms of complex exponentials (not the other way around) derive an expression for $\cos\frac{1}{2}\theta$ in terms of $\cos\theta$.

Problem (13) Let z be any complex number with modulus one *except* $z = 1$: $|z| = 1$, $z \neq 1$. Show that

$$\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}$$

Problem (14) Let α denote an m^{th} root of unity and β an n^{th} root of unity (where, of course, m and n are integers). Show that the product is a k^{th} root of unity for some integer k.

Problem (15) Find all solutions to the equation

$$z^4 = \frac{z}{5+3i}$$

(You may express your solution in terms of some particular angle, φ, whose cosine you know.)

Problem (16) Find all solutions to the equation $z^5 = (z+1)^5$.

Problem (17) The following shows that multiplication properties of complex numbers can be used to derive difficult trigonometric identities (beyond our standard $e^{i\theta} = \cos\theta + i\sin\theta$). Consider the complex number

$$z = 1 + iT; \qquad\qquad -\infty < T < +\infty.$$

(i) Interpret T geometrically. (Easy.)

(ii) With the above in mind, knowing that $z^2 \propto 1 + iS$ for some $S \in (-\infty, +\infty)$ derive a formula for $\tan 2\theta$ in terms of $\tan\theta$. Comment on the range of validity of the formula (and make sure this makes sense).

(iii) Similarly, derive a formula for $\tan 3\theta$ in terms of $\tan\theta$ and, also, comment on the range of validity of the formula.

Problem (18) Consider the most general linear complex function

$$f(z) = Px + Qy$$

where P and Q denote arbitrary complex constants. Find the conditions on P and Q for which f has a complex derivative.

Problem (19) Write the function $f(z)$ in the form $u(x, y) + iv(x, y)$ – u and v real – where

$$f(z) = 4z^2 + 6z + i + 1.$$

Problem (20) Write the function $f(z)$ in the form $u(x, y) + iv(x, y)$ – u and v real – where

$$f(z) = \frac{1}{z}$$