UCLA
Department of Mathematics

Math 132 Section 2 Spring 2021

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Section: ☐

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Problem Set # 1

Problem (1) Write the complex number

$$(1+i)^2$$

in the form $a + ib$ with a and b real.

Problem (2) Write the complex number

$$\frac{8i-1}{i}$$

in the form $a + ib$ with a and b real.

Problem (3) Let z be any complex number. Show that $\text{Re}(z) = \frac{1}{2}[z + \overline{z}]$.

Problem (4) Let z be any complex number. Show that $\text{Im}(z) = \frac{1}{2i}[z - \overline{z}]$.

Problem (5) If z_1 & z_2 are complex numbers, show that in general $\mathrm{Re}(z_1 z_2) \neq \mathrm{Re}(z_1)\mathrm{Re}(z_2)$. Under precisely what conditions do you get equality?

Problem (6) Let z_1 and z_2 denote arbitrary complex numbers. Show that the product of the conjugates is the conjugate of the product; that is $\overline{[z_1 z_2]} = \overline{z}_1\,\overline{z}_2$.

Problem (7) Let z_1 and z_2 denote arbitrary but non–zero complex numbers. Show that the product of the inverse is the inverse of the products; that is

$$\frac{1}{z_1 z_2} = \frac{1}{z_1}\frac{1}{z_2}.$$

Hint: Although the correct answer will emerge by grinding out the relevant formulas, it is suggested that you try an *algebraic* approach.

Problem (8) Let z_1 and z_2 denote arbitrary but non–zero complex numbers. Show that the product of the inverse of the moduli is the inverse of the product of the moduli which in turn is equal to the modulus of the product of the inverses. I.e.,

$$\left|\frac{1}{z_1 z_2}\right| = \frac{1}{|z_1 z_2|} = \frac{1}{|z_1|}\frac{1}{|z_2|}$$

You may use the results of previous problems especially if you tried the (recommended) algebric approach.

Problem (9) Prove that if $z_1 z_2 = 0$ then either $z_1 = 0$ or $z_2 = 0$ (or both).

Hint: Do not panic, this is not so hard. At worst, write the purported z_1 as $a + ib$ and similarly for z_2. This will give you two real equations and the rest is straightforward. But, there are other algebraic methods you can now employ.

Problem (10) Write the complex number

$$\frac{-1+5i}{2+3i}$$

in the form $a + ib$ with a and b real.

Problem (11) Write the complex number

$$\frac{2+3i}{1+2i} + \frac{8+i}{6-i}$$

in the form $a + ib$ with a and b real.

Problem (12) Write the complex number

$$(2+i)(-1-i)(3-2i)$$

in the form $a + ib$ with a and b real.

Problem (13) Show that the complex number $z = -1 + i$ satisfies the equation

$$z^2 + 2z + 2 = 0.$$

Problem (14) Find the complex numbers z_1 and z_2 that satisfy the system of equations

$$\begin{aligned} &(1-i)z_1 + 3z_2 = 2 - 3i \\ &iz_1 + (1+2i)z_2 = 1. \end{aligned}$$

Quarternians: Quarternians are the next generalization beyond complex numbers. They are defined by

$$\underline{q} = q_0 + q_1\mathbf{i} + q_1\mathbf{j} + q_3\mathbf{k}$$

where $q_0, \ldots q_3$ are real numbers. The quantities $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ each individually act like our own unit imaginary number:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

but amongst themselves have the (strange) rules: $(\mathbf{i})(\mathbf{j}) = \mathbf{k} = -(\mathbf{j})(\mathbf{i})$; etc. Note that this portion is anti–commutative – this aspect of the quarternian is just like the cross product for 3D vectors. All other properties: associativity multiplication by scalars etc. follow the expected rules. For connivence, one can write

$$\underline{q} = (q_0, \vec{q})$$

where $\vec{q}$ may be identified as a 3–component vector. In the following, there are a couple of problems which involve these delightful objects that, unfortunately, have seen little use over the years.

Problem (15) Let $\underline{q} = (q_0, \vec{q})$ and $\underline{p} = (p_0, \vec{p})$ denote two quarternians. Derive a formula in terms of the standard 3D vector products (e.g., $\vec{q} \times \vec{p}$ and $\vec{q} \cdot \vec{p}$) for the quarternian product $(\underline{q})(\underline{p})$.

Problem (16) For complex numbers, the only solutions to $z^2 = -1$ are $z = \pm i$. Not so for quarternians. Show that here is a continuous (two–parameter) family of quarternian solutions to $\underline{q}^2 = -1$.

Problem (17) For the purpose of this problem, all capital letters are to be considered integers. It is a known and useful fact in number theory that if M and N are integers both of which can be written as a sum of squares of two integers:

$$M = A^2 + B^2$$
$$N = C^2 + D^2$$

then so can their product. Prove this using complex numbers. Hint: Write $z_M := A + iB$ and acquire a "Complex Analysis"–style expression for M. Similarly, ...

Problem (18) Show that $|z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leq \sqrt{2}|z|$.

Problem (19) Show that for every complex number z,

$$\mathrm{Re}(iz) = -\mathrm{Im}(z).$$

Problem (20) If z is a complex number with $\mathrm{Re}(z) > 1$, show that $0 \leq \mathrm{Re}(1/z) \leq 1$.

Problem (21) Let $\mathbb{Q}$ denote the rational numbers and consider the set $\mathbb{Q}[\frac{\sqrt{5}}{2}]$ which is the set all numbers of the form

$$a + \frac{\sqrt{5}}{2}b; \quad a, b \in \mathbb{Q}.$$

(This is a subfield of $\mathbb{R}$ with certain similarities to $\mathbb{C}$.)

Part A   Find the law of multiplication for distinct elements of $\mathbb{Q}[\frac{\sqrt{5}}{2}]$.

Part B   For $w = a + \frac{\sqrt{5}}{2}b$ with a and b *not* both zero, find the inverse of w that is to say a number w^{-1} which satisfies $w^{-1}w = 1$. Express your answer in the form $\alpha + \frac{\sqrt{5}}{2}\beta$.