Topics in Quantum Field Theory in Curved Spacetime: Unruh Effect, Hawking Radiation, and AdS/CFT

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Abstract: We study the framework of quantum field theory in curved spaces. The Unruh effect and Hawking radiation are described for both Minkowski and anti-de Sitter spaces, and greybody factors are approximated for asymptotically flat black holes. We also explore the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, which conjectures an exact equivalence for $\mathcal{N} = 4$ super Yang-Mills and string theory in $\mathrm{AdS}_5 \times S^5$. Specifically, we demonstrate how the Euclidean and Minkowski formulations can be employed to compute field theory correlation functions. We will see that the proposed correspondence is a powerful tool that can be used to address otherwise inaccessible questions in quantum gravity using field theoretic methods. The AdS/CFT correspondence has lead to significant developments in the study of magnetically charged anti-de Sitter black holes, which is our ultimate research goal.

Contents

1 Introduction

The general theory of relativity and quantum field theory are two pillars of modern physics which have been extremely successful in their respective regimes of validity. Two recent examples of their triumphs are the measurement of gravitational waves which precisely agrees with the theory, and the discovery of the Higgs particle as predicted by the Standard Model. Unlike the other fundamental forces which can be understood through quantum field theory (QFT), general relativity (GR) is supported solely by classical physics and hence is as of yet incomplete. In QFT, interactions are described through the exchange of quanta, unlike the continuum of spacetime in GR. As with particle accelerators, studying physics on the scale of quanta in QFT involves high-energy scattering, where higher energy collisions allow one to probe smaller distances. In GR, an analogous high-energy collision of gravitational particles would eventually create a singularity, whose horizon would continue to grow with the incident energy. From this it is apparent that GR cannot be a QFT.

The field of quantum gravity is the search for a complete theory that quantizes the gravitational field and can make meaningful physical predictions for interactions where quantum mechanical effects are not negligible — perhaps even unifying all of the fundamental forces within one framework. Quantum effects should become significant at high energies and when the frequency is comparable with a characteristic (inverse) time scale of the background spacetime. Therefore, such a theory is not just of academic interest but is necessary to understand the strong gravity of black holes or the big bang.

However, directly treating the gravitational quanta, or gravitons, as excitations of the classical gravitational field leads to problems. The equivalence principle on which gravity is founded states that all forms of matter and energy — including gravitational energy — couple equally strongly to gravity. Whenever a background gravitational field produces important effects and creates matter or energy, then this means it must also create gravitons. These gravitons carry energy, and in turn excite the gravitational field producing more gravitons. This nonlinear response would continue and diverge; consequently, it is quite difficult to ignore quantum effects. At high energies and short distance scales, this introduces unphysical infinities that are not renormalizabile.

However, it would be reasonable to suspect that for interactions on sufficiently large scales, quantum effects could be treated as a perturbation. If the entire spacetime metric is split up as the sum of the background metric plus the graviton field, it seems reasonable to linearize the system by moving the graviton contribution to the other side of the equality in Einstein's equations and treating it as part of the source. In this way, treating quantum fields in curved spacetime on scales larger than the Planck dimensions may be a semiclassical approximation to the theory of quantum gravity. If the gravitational field is treated with perturbation theory, the coupling constant which appears is the square of the Planck length $(G\hbar/c^3)^{1/2} = 1.62 \times 10^{-33}\,\mathrm{cm}$. Both this scale — along with the corresponding Planck time $(G\hbar/c^5)^{1/2} = 5.39 \times 10^{-44}\,\mathrm{s}$ — are small even compared to the scale of nucleons, and so this treatment of QFT in curved space seems justified.

It seems that the current situation of quantum gravity is analogous to the period of semiclassical approximations in old quantum theory that preceded the development of modern quantum mechanics. These techniques led to advancements such as the Bohr atomic model, which was able to predict the energy levels, orbital angular momenta, and the orbits' space quantization for the hydrogen atom. Hopefully, a semiclassical approximation to quantum gravity will likewise be fruitful in its predictions and lead us to a more complete theory, and perhaps even bear predictions that are observable to detectors such as LIGO and VIRGO.

Just as laboratory equipment was then incapable of measuring on the atomic scale, present-day detectors are incapable of measuring on the scale of the Planck length — twenty orders of magnitude smaller than the atomic nucleus. In the mid-nineteenth century, however, atomic physics was able to produce experimentally verifiable hypotheses for gas dynamics, long before the atomic hypothesis could be directly verified. We also hope that as we gain a better understanding of quantum gravity, we will see its intersection with a more tractable field of physics.

Despite the current lack of verifiable predictions, quantum field theory in curved spacetime has already proven itself to be rewarding. One of the key discoveries in this field is the thermal emission by quantum black holes by Hawking [1]. This phenomenon has been widely accepted, as it has been derived from multiple perspectives, and because it relates black holes and thermodynamics by describing the thermal interactions of a black hole with its universe. It is believed that the Hawking effect is the first step towards a fundamental juncture of gravity, QFT, and thermodynamics, the study of which is expected to yield key developments in physics. This alliance is immediately visible from the Bekenstein-Hawking formula for black hole entropy:

$$S_{\text{BH}} = \frac{k_B c^3}{4G\hbar} \cdot (\text{event horizon area})\,.$$

Within the right-hand side proportionality coefficient, we see four fields of physics being represented at once: the Boltzmann constant k_B from thermodynamics, the speed of light c from relativity, Newton's constant G from gravity, and the Planck constant $\hbar$ from quantum mechanics. The result illuminates only a small corner of what is hopefully a much more grand unification.

Although a fully satisfactory theory of quantum gravity remains elusive, the last few decades has seen significant progress towards this objective. One of the best contenders for a complete theory is string theory, and the proposal of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence two decades ago has brought with it a new approach to quantum gravity. The conjecture states that there is an exact equivalence between string theories in anti-de Sitter (AdS) space and gauge theories. The benefits of this are twofold: difficult computations in strongly coupled field theory dynamics such as Wilson loops can be tackled using weakly coupled gravity, while the problematic questions of the black hole paradox and the quantum entropy of black holes can be addressed through field theory. The correspondence provides us with the powerful ability to rephrase difficult questions regarding strong interactions in terms of the counterpart's more tractable weak interactions.

This paper is intended to provide an introduction to the framework of QFT in curved spaces. It is assumed the reader is comfortable with the notation and fundamental concepts of GR and QFT, but otherwise the presentation intends to be as self-contained as possible in its cited results. The first part of this paper reviews the necessary preliminaries of these subjects in sections 2 and 3. Section 2 is based on the presentation in [2], and begins with the quantization of scalar fields in flat Minkowski spacetime, before discussing the generalization to curved space and the notion of correlation functions (also referred to as correlators or Green functions). In section 3, we compare and contrast the properties of Minkowski and anti-de Sitter space, based primarily on the discussions of [3, 4].

Sections 4 through 6 comprise the second part of this paper, which study Hawking radiation and the Unruh effect in flat Minkowski spacetime. The Unruh effect [5], presented in section 4, is in some ways the formulation of Hawking radiation in terms of acceleration — as is guaranteed by the equivalence principle of GR. Specifically, it states that a uniformly accelerating observer through the Minkowski vacuum will measure a Planck thermal emission distribution, and our analysis of this phenomenon will follow that of [6]. Section 5 then examines Hawking radiation for the arbitrary-dimensional Schwarzschild black hole using the same method as did Hawking [1], but more along the lines of the reviews in [2, 6]. As we will see in section 6, black holes are not perfectly thermal blackbodies, and so we will approximate the corrective greybody factor for both the low- and high-frequency radiation limits — these calculations are based on the work of [7, 8].

In the third and final part of this paper, sections 7 through 9 depart from flat spacetime and focus primarily on anti-de Sitter (AdS) space. Section 7 briefly diverges back to QFT to discuss the powerful path-integral formulation of the theory (as in [2]), so that we can then understand the language in which the AdS/CFT correspondence is stated. This discussion begins with a heuristic and mathematical description of the Euclidean correspondence along the lines of [3], and then a presentation of both the difficulties and a proposed solution by [9] for the Minkowski formulation. Next, section 8 examines acceleration and the Unruh effect in AdS through the global embedding in Minkowski spacetime (GEMS) approach, following the analysis of [10]. Lastly, we turn to Hawking radiation for the Schwarzschild AdS black hole in section 9. We will choose to use the technique analogous to that of section 5, as presented in [11], except in this case we will have to pay closer attention to the boundary conditions in AdS and how to define the past vacuum.

2 Review of quantum field theory

Before we delve into the thermal emission calculations, we will first briefly review the techniques of QFT as in [2]. Section 2.1 summarizes the theory in flat Minkowski spacetime, noting the necessary ingredients that need to be adapted for general spacetime. We then take the step up to curved spacetimes that are globally hyperbolic in section 2.2, and then specialize to stationary spacetimes in section 2.2 so that we may generalize the notion of positive and negative frequencies. The analysis of the Unruh effect and Hawking radiation in flat spacetime presented in sections 4 and 5 will be of this type, and so in section 2.3 we will consider the key features of these spacetimes in order to prepare some calculations based upon [6] for this later use. We will then conclude with a brief venture in section 2.4 to Green functions, which are a fundamental tool in QFT that describe particle propagation in spacetime, and will be the language used to formulate the AdS/CFT correspondence in section 7.

QFT can be introduced in multiple ways, but the approach we will take is the canonical quantization scheme. An alternative method, the path-integral quantization of Feynman, is presented in section 7.1.

2.1 Quantum field theory in flat spacetime

Let $\phi(x)$ be a scalar field defined for all points $(t, x) = (t, x_1, \ldots, x_d)$ in the $(d+1)$-dimensional Minkowski spacetime, whose metric is

$$\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu dx^\nu = -\mathrm{d}t^2 + \sum_{i=1}^{d} \mathrm{d}x_i^2. \tag{2.1}$$

The dynamics of the field may be obtained by defining the action

$$S = \int \mathrm{d}^{d+1}x\, \mathscr{L}, \qquad \mathscr{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2. \tag{2.2}$$

Here, m will represent the mass of the field quanta once we quantize. Insisting that the variation of the action δS vanish, we obtain

$$(\Box - m^2)\phi = 0, \tag{2.3}$$

for the operator

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{2.4}$$

Equation (2.3) is the *Klein-Gordon equation* of motion. The conjugate momentum is defined by

$$\pi = \frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} = \partial_t \phi \tag{2.5}$$

The Klein-Gordon equation (2.3) can be solved via separation of variables. Doing so yields the set of plane wave solutions

$$f = f_0 e^{ik_\mu x^\mu}, \tag{2.6}$$

with f_0 constant, $k^\mu = (\omega, k^i)$, $k_\mu = (-\omega, k_i)$, and ω is the frequency and k^i the wave vector. The frequency and wave vector are related by

$$\omega = \sqrt{k^2 + m^2}. \tag{2.7}$$

Since the wave vector k determines ω up to a sign, we write $f \equiv f_k$. We define modes $\{f_k\}$ that satisfy

$$\partial_t f_k = -i\omega f_k, \quad \omega > 0 \tag{2.8}$$

to be *positive frequency modes*, and modes $\{f_k^*\}$ that satisfy

$$\partial_t f_k^* = i\omega f_k^*, \quad \omega > 0 \tag{2.9}$$

to be *negative frequency modes*. When we progress to curved spacetime, defining positive and negative frequency becomes more difficult. In section 2.2.2, we will see that insisting the spacetime be stationary is a sufficient condition for this definition.

In order to discuss completeness and orthonormality, we define an inner product on the space of solutions to the Klein-Gordon equation (2.3):

$$(\phi_1, \phi_2) = -i \int \mathrm{d}^d x \, (\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1). \tag{2.10}$$

The integral is taken over a hyperplane of constant t. The fact that this choice of hyperplane makes the definition (2.10) an inner product is noteworthy, since for curved spacetime such a natural choice of hyperplane will not be so apparent. The reason why hyperplanes of constant t suffice is because they are Cauchy surfaces (which will be discussed section 3.1.2), and in section 2.2 we will require that the spacetime be globally hyperbolic so that we are ensured the existence of such a Cauchy surface.

Together, the set of modes (2.8) and (2.9) is an orthogonal family with respect to (2.10) (i.e. $(f_k, f_{k'}) = 0$ for $k \neq k'$). Consequently, we can define the delta-function normalized positive frequency modes

$$f_k = \frac{1}{(2\pi)^{d/2}\sqrt{2\omega}} e^{ik_\mu x^\mu}, \tag{2.11}$$

which satisfy

$$(f_k, f_{k'}) = \delta(k - k'), \qquad (f_k, f_{k'}^*) = 0, \qquad (f_k^*, f_{k'}^*) = -\delta(k - k'). \tag{2.12}$$

Together, the modes (2.11) along with their complex conjugates are also complete. This allows us to write any classical field configuration $\phi(x)$ that is a solution to the Klein-Gordon equation (2.3) as

$$\phi(x) = \int \mathrm{d}^d k \, (a_k f_k + a_k^* f_k^*), \tag{2.13}$$

for complex coefficients a_k ($*$ denotes complex conjugate).

Now we quantize the scalar field. We replace the classical fields by operators acting on Hilbert space, and by imposing the equal-time commutation relations

$$[\phi(t,x), \phi(t,x')] = 0, \qquad [\pi(t,x), \pi(t,x')] = 0, \qquad [\phi(t,x), \pi(t,x)] = i\delta(x - x'). \tag{2.14}$$

After quantization, the coefficients a_k are now upgraded to operators, and so the classical field becomes the operator

$$\phi = \int \mathrm{d}^d k \, (a_k f_k + a_k^\dagger f_k^*). \tag{2.15}$$

The operators $a_k, a_k^\dagger$ are called *annihilation* and *creation operators*, respectively. Their function is analogous to the familiar annihilation and creation operators of the harmonic oscillator. Inserting the expansion (2.15) into the commutation relations (2.14), one obtains commutation relations for the annihilation and creation operators:

$$[a_k, a_{k'}] = 0, \qquad [a_k^\dagger, a_{k'}^\dagger] = 0, \qquad [a_k, a_{k'}^\dagger] = \delta_{kk'}. \tag{2.16}$$

In the Heisenberg picture, the quantum states span a Hilbert space. Defining the *vacuum* to be the state $|0\rangle$ from which no particles can be annihilated:

$$a_k |0\rangle = 0 \quad \text{for all } k, \tag{2.17}$$

one can construct an orthonormal basis from the creation operators applied to the vacuum. For example, a j-particle state is given by

$$\left|1_{k_1}, \ldots, 1_{k_j}\right\rangle = a^\dagger_{k_1} \ldots a^\dagger_{k_j} |0\rangle \tag{2.18}$$

for distinct modes $k_1, \ldots, k_j$. For repeated momenta, a normalization factor must be included in accordance to the eigenvalues

$$a^\dagger_k |n_k\rangle = (n+1)^{1/2} |(n+1)_k\rangle, \qquad a_k |n_k\rangle = n^{1/2} |(n-1)_k\rangle. \tag{2.19}$$

The orthonormality can then be shown using the commutation relations (2.16).

The *number operator* for the mode k is defined to be

$$N_k = a^\dagger_k a_k. \tag{2.20}$$

This is appropriately named, since from (2.17) and (2.19) one obtains

$$\langle 0| N_k |0\rangle = 0 \quad \text{for all } k, \qquad \left\langle {}^1n_{k_1}, \ldots, {}^jn_{k_j}\right| N_{k_i} \left|{}^1n_{k_1}, \ldots, {}^jn_{k_j}\right\rangle = {}^in, \tag{2.21}$$

where ${}^in$ corresponds with the state ${}^in_{k_i}$. We also define the total number operator

$$N = \int \mathrm{d}^d k \, N_k, \tag{2.22}$$

which by linearity satisfies (2.21), but with $\sum_i {}^in$ in place of ${}^in$.

Next, we will generalize the techniques of QFT in flat spacetime presented thus far to a suitable class of curved spacetimes.

2.2 Quantum field theory in curved spacetime

2.2.1 Globally hyperbolic spacetime

Let (M, g) be a $(d+1)$-dimensional globally hyperbolic spacetime with a Cauchy surface Σ. For a review of the definition of global hyperbolicity, the reader can jump ahead and read section 3.1.2. Our action for a scalar field ϕ is given by

$$S = \int \mathrm{d}^{d+1}x \, \sqrt{g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right), \tag{2.23}$$

where we define

$$g = |\det g|. \tag{2.24}$$

The principle of least action ($\delta S = 0$) yields

$$(\Box - m^2)\phi = 0, \tag{2.25}$$

for the operator

$$\Box = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\right). \tag{2.26}$$

Equation (2.25) is the *Klein-Gordon equation* of motion.

We may proceed as in section 2.1, until the definition of the inner product (2.10). For Minkowski space, integrating over a hyperplane of constant t made this an inner product. As discussed in section 3.1.2, this choice of hyperplane is a Cauchy surface, which is what facilitated definition (2.10). Consequently, the inner product on the space of solutions to the Klein-Gordon equation is defined by

$$(\phi_1, \phi_2) = -i\int_\Sigma \mathrm{d}^d x\, \sqrt{\gamma} n^\mu(\phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1), \tag{2.27}$$

where Σ is a Cauchy surface, n^μ is a normal vector, and γ is the induced metric on Σ. This definition is independent of the choice of Σ. This inner product allows us to continue on as in section 2.1 to define (non-unique) orthonormal bases satisfying

$$(f_k, f_{k'}) = \delta(k-k'), \qquad (f_k^*, f_{k'}^*) = -\delta(k-k'). \tag{2.28}$$

The lack of uniqueness is a consequence of the lack of a preferred time coordinate. Next, we will further impose that our spacetime be stationary, and show that this introduces a natural choice for a time coordinate.

2.2.2 Stationary spacetime

In flat space, the time translational-invariance of the metric guarantees that the Lagrangian (2.2) is also time-independent, and consequently that energy is conserved. The analogue we seek here is that we require (M, g) to also be stationary. This means there exists a timelike Killing vector field $K = K^\mu\partial_\mu$ for the metric g – that is, the Lie derivative $\mathcal{L}_K g = 0$ vanishes. Choosing a coordinate system $\{x^\mu\}$ for which $K^\mu = (1, 0, 0, 0)$, the vanishing of the Lie derivative requires

$$0 = \mathcal{L}_K g = K^\sigma\partial_\sigma g_{\mu\nu} + g_{\sigma\nu}\partial_\mu K^\sigma + g_{\mu\sigma}\partial_\nu K^\sigma = \partial_t g_{\mu\nu}.$$

In other words, we can always find a coordinate system so that the metric is time-independent.

The Killing vector field commutes with the Klein-Gordon operator $\Box - m^2$. It is also anti-Hermitian, since

$$\begin{aligned}(f, Kg) &= -i\int_\Sigma d^3x\, \sqrt{\gamma}n^\mu(f\partial_\mu(\partial_t g^*) - (\partial_t g^*)\partial_\mu f) \\ &= -i\int_\Sigma d^3x\, \sqrt{\gamma}n^\mu((-\partial_t f)\partial_\mu g^* - g^*\partial_\mu(-\partial_t f)) = (-Kf, g).\end{aligned}$$

Consequently, all eigenvalues of K must be purely imaginary, since $Kf = \lambda f$ for normalized f (i.e. $(f, f) = 1$) implies

$$\overline{\lambda} = (f, \lambda f) = (f, Kf) = (-Kf, f) = (-\lambda f, f) = -\lambda.$$

This allows us to define positive and negative frequency modes in curved spacetime. The modes $\{f_j\}$ are *positive frequency modes* if

$$\mathcal{L}_K f_j = \partial_t f_j = -i\omega f_j, \quad \omega > 0, \tag{2.29}$$

and similarly the modes $\{f_j^*\}$ are *negative frequency modes* if

$$\mathcal{L}_K f_j^* = \partial_t f_j^* = i\omega f_j^*, \quad \omega > 0. \tag{2.30}$$

The quantization process is exactly the same as in section 2.1. Any field configuration $\phi(x)$ that solves the Klein-Gordon equation (2.25) can be expanded as

$$\phi(x) = \int \mathrm{d}^d k\, (a_k f_k + a_k^\dagger f_k^*). \tag{2.31}$$

The commutation relations (2.16) still hold for the creation and annihilation operators, and we define the vacuum sate in the analogous way. Stationary symmetry facilitated the selection of a preferred time coordinate, giving the timelike Killing vector field and allowing us to define positive and negative frequencies. This might not be possible in a general cosmological set-up, but for the cases we are interested in it is sufficient.

2.3 Sandwich spacetime

Consider a spacetime (M, g) which in the past $(t < t_1)$ and future $(t > t_2)$ is stationary, but not necessarily stationary in the period $(t_1 < t < t_2)$. Let us also assume that the Klein-Gordon equation of motion holds throughout the whole space. As in section 2.2.2, there are Killing vector fields K^1 and K^2 in the past and future regions respectively, that allow us to define the orthonormal (with respect to (2.27)) modes $\{f_j, f_j^*\}$ and $\{g_j, g_j^*\}$. Although the modes $\{f_j\}$ were found in the past region $(t < t_1)$ and $\{g_j\}$ in the future region $(t > t_2)$, they can both be analytically continued throughout the entire spacetime. The question is then, how do the modes $\{g_j\}$ in the future region relate to the $\{f_j\}$ obtained for the past region?

By completeness, any scalar field ϕ that solves the Klein-Gordon equation (2.25) can be expanded in terms of the two bases as

$$\phi(x) = \int \mathrm{d}^d k\, (a_k f_k + a_k^\dagger f_k^*) = \int \mathrm{d}^d k\, (b_k g_k + b_k^\dagger g_k^*) \tag{2.32}$$

for two sets of annihilation operators a_k, b_k which both satisfy the commutation relations (2.16). Moreover, we may also expand

$$g_j = \int \mathrm{d}^d k\, (A_{jk} f_j + B_{jk} f_j^*) \tag{2.33}$$

for A_j, B_j complex-valued functions of k. This is called a *Bogoliubov transformation*, and A, B are called *Bogoliubov coefficients*. We can combine this equation and its complex conjugate in the form of the matrix equation

$$\begin{pmatrix} g_j \\ g_j^* \end{pmatrix} = \int \mathrm{d}^d k\, M_{jk} \begin{pmatrix} f_k \\ f_k^* \end{pmatrix}, \qquad M_{jk} = \begin{pmatrix} A_{jk} & B_{jk} \\ B_{jk}^* & A_{jk}^* \end{pmatrix}. \tag{2.34}$$

2.3.1 Bogoliubov coefficients

We would like to invert the transformation (2.34), since along with (2.32) would allow us to relate the two sets of annihilation and creation operators. From the orthogonality (2.28) of the modes, we find

$$\delta(j-k) = (g_j, g_k) = \int \mathrm{d}^d\ell \, (A_{j\ell}A^*_{k\ell} - B_{j\ell}B^*_{k\ell}). \tag{2.35}$$

We also have

$$0 = (g_j, g^*_k) = \int \mathrm{d}^d\ell \, (A_{j\ell}B_{k\ell} - B_{j\ell}A_{k\ell}). \tag{2.36}$$

Together, (2.35) and (2.36) allow us to invert M_{jk}. Indeed, symbolically we have

$$\left[\int \mathrm{d}^dk \, M_{jk}\right]^{-1} = \int \mathrm{d}^d\ell \begin{pmatrix} A^*_{k\ell} & -B_{k\ell} \\ -B^*_{k\ell} & A_{k\ell} \end{pmatrix}, \tag{2.37}$$

since

$$\int \mathrm{d}^dk \, M_{jk}M^{-1}_{jk} = \int \mathrm{d}^dk\mathrm{d}^d\ell \begin{pmatrix} A_{j\ell}A^*_{k\ell} - B_{j\ell}B^*_{k\ell} & -A_{jk}B_{\ell k} + B_{jk}A_{\ell k} \\ B^*_{jk}A^*_{k\ell} - A^*_{jk}B^*_{k\ell} & -B^*_{jk}B_{\ell k} + A^*_{jk}A_{\ell k} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

courtesy of (2.35) and (2.36).

Returning to the field expansion (2.32), we have

$$\int \mathrm{d}^dk \, \begin{pmatrix} b_k & b^\dagger_k \end{pmatrix} \begin{pmatrix} g_k \\ g^*_k \end{pmatrix} = \phi(x) = \int \mathrm{d}^dk \, \begin{pmatrix} a_k & a^\dagger_k \end{pmatrix} \begin{pmatrix} f_k \\ f^*_k \end{pmatrix}. \tag{2.38}$$

Inverting (2.34) using (2.37) and plugging this in the right-hand side of (2.38), we obtain

$$\int \mathrm{d}^dk \, \begin{pmatrix} b_k & b^\dagger_k \end{pmatrix} \begin{pmatrix} g_k \\ g^*_k \end{pmatrix} = \int \mathrm{d}^dk\mathrm{d}^d\ell \, \begin{pmatrix} a_k & a^\dagger_k \end{pmatrix} \begin{pmatrix} A^*_{k\ell} & -B_{k\ell} \\ -B^*_{k\ell} & A_{k\ell} \end{pmatrix} \begin{pmatrix} g_\ell \\ g^*_\ell \end{pmatrix}.$$

Therefore, we conclude

$$\begin{pmatrix} b_k & b^\dagger_k \end{pmatrix} = \int \mathrm{d}^d\ell \, \begin{pmatrix} a_\ell & a^\dagger_\ell \end{pmatrix} \begin{pmatrix} A^*_{k\ell} & -B_{k\ell} \\ -B^*_{k\ell} & A_{k\ell} \end{pmatrix},$$

or equivalently

$$\begin{pmatrix} b_k \\ b^\dagger_k \end{pmatrix} = \int \mathrm{d}^d\ell \begin{pmatrix} A^*_{k\ell} & -B^*_{k\ell} \\ -B_{k\ell} & A_{k\ell} \end{pmatrix} \begin{pmatrix} a_\ell \\ a^\dagger_\ell \end{pmatrix}. \tag{2.39}$$

This is the desired relation between the two different sets of annihilation and creation operators. In fact, this is a key relation that lies at the foundation of many of the main results of QFT in curved space. We will now explicitly see one such fundamental implication.

2.3.2 Particle number

At the end of section 2.1, we saw that a set of annihilation and creation operators introduces a notion of vacuum and a particle number operator. The question then arises: how does the past region vacuum relate to the future region vacuum? We will now calculate the number of particles for the mode k in the past vacuum. Define the past vacuum $|0\rangle$ as in (2.17), and the operator $N_k = b_k^\dagger b_k$ whose expectation value is the number of particles for the mode k as observed by a stationary observer in the future region. Using (2.39), we find

$$\begin{aligned}\langle 0| N_k |0\rangle = \langle 0| b_k^\dagger b_k |0\rangle &= \int \mathrm{d}^d\ell \mathrm{d}^d\ell' \, \langle 0| (-B_{k\ell} a_\ell + A_{k\ell} a_\ell^\dagger)(A^*_{k\ell'} a_{\ell'} - B^*_{k\ell'} a^\dagger_{\ell'}) |0\rangle \\ &= \int \mathrm{d}^d\ell \mathrm{d}^d\ell' \, B_{k\ell} B^*_{k\ell'} \langle 0| a_\ell a^\dagger_{\ell'} |0\rangle = \int \mathrm{d}^d\ell \, |B_{k\ell}|^2. \end{aligned} \tag{2.40}$$

That is, the number of particles created as viewed by a stationary observer in the future ($t > t_2$) from the vacuum in the past ($t < t_1$) is given by the Bogoliubov coefficient B. This result lies at the heart of our analysis in sections 4, 5, and (9) when we derive the Unruh effect and Hawking radiation.

We will now recall one final idea from QFT for later use: Green functions.

2.4 Green functions

Green functions are two-point correlation functions, or two-point correlators, which tell us the propagation amplitude from an excitation or particle at one point in the spacetime to another point. They are a crucial ingredient in QFT, and has its roots in classical physics. Green functions and more general correlators will be examined again in section 7. We will begin with the scalar fields in flat Minkowski spacetime, and will then move to the cases of Euclidean spacetime and nonzero temperature, and encounter the technique of Wick rotation.

2.4.1 Zero-temperature flat spacetime

Green functions are not uniquely defined. Indeed, their definition is dependent on the choice of boundary condition, and consequently each Green function corresponds to a specific physical situation. We will show that vacuum expectation values of various products of free field operators can be identified with various Green functions of the wave equation. Let ϕ denote a scalar field solution to the Klein-Gordon equation (2.3), and define the following two-point correlators:

$$G^+(x,x') = \langle 0| \phi(x)\phi(x') |0\rangle \,, \qquad G^-(x,x') = \langle 0| \phi(x')\phi(x) |0\rangle \,, \tag{2.41}$$

$$iG_F(x,x') = \langle 0| T(\phi(x)\phi(x')) |0\rangle = \theta(t-t')G^+(x,x') + \theta(t'-t)G^-(x,x'), \tag{2.42}$$

$$\begin{aligned} G_R(x,x') &= i\theta(t-t') \left[G^+(x,x') - G^-(x,x')\right] , \\ G_A(x,x') &= -i\theta(t'-t) \left[G^+(x,x') - G^-(x,x')\right] , \end{aligned} \tag{2.43}$$

where $\theta(t)$ denotes the unit step function, and T is the time-ordering operator. $G^\pm$ are called the *Wightman functions*, G_F is the *Feynman propagator*, and $G_{R,A}$ are the *retarded* and *advanced Green functions* respectively. From the field equation (2.3), it is clear that the Wightman functions $G^{\pm}$ both satisfy the homogeneous equation

$$\left(\Box_x - m^2\right) G^{\pm}(x,x') = 0, \tag{2.44}$$

and so they are not Green functions in the classical sense. Using $\partial_t \theta(t-t') = \delta(t-t')$ and the equal time commutators (2.14), we find that the Feynman propagator G_F and the retarded and advanced Green functions $G_{R,A}$ satisfy the relations

$$\left(\Box_x - m^2\right) G_F(x,x') = \delta(x-x'), \qquad \left(\Box_x - m^2\right) G_{R,A}(x,x') = -\delta(x-x'). \tag{2.45}$$

Consequently, the correlators $G_{F,R,A}$ are Green functions in the traditional sense, and they each describe the propagation of field disturbances subject to certain boundary conditions. For example, the retarded Green function is identified by the boundary condition that the system be causal.

We can obtain integral representations of these by substituting the mode decomposition (2.13) of ϕ into the definitions (2.41)–(2.43) and using some residue calculus. We will manipulate the Wightman function G^+ as an example, but the other cases follow the same process. Putting the mode decomposition (2.13) into the definition (2.41) yields

$$\begin{aligned} iG^+(x,x') &= i \langle 0| \phi(x)\phi(x') |0\rangle \\ &= i \langle 0| \int \mathrm{d}^d k \mathrm{d}^d k' \left[a_k f_k(x) + a_k^\dagger f_k^*(x)\right]\left[a_{k'} f_{k'}(x') + a_{k'}^\dagger f_{k'}^*(x')\right] |0\rangle \,. \end{aligned}$$

Using the definition of the vacuum (2.17) and its Hermitian conjugate, only one of these terms survives. Recalling the definition (2.11), we find

$$\begin{aligned} iG^+(x,x') &= i\int \mathrm{d}^d k \mathrm{d}^d k' \; \langle 0| a_k a_{k'}^\dagger |0\rangle \, f_k(x) f_{k'}^*(x') = i \int \mathrm{d}^k \mathrm{d}^d k' \; \langle 1_k | 1_{k'}\rangle \, f_k(x) f_{k'}^*(x') \\ &= i \int \mathrm{d}^d k \, f_k(x) f_k^*(x') = \frac{i}{2(2\pi)^d} \int \mathrm{d}^d k \, \frac{e^{i[k\cdot(x-x') - \omega(t-t')]}}{\omega} \\ &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, e^{ik\cdot(x-x')} \left[i \frac{e^{-ik^0(t-t')}}{k^0+\omega} \right]_{k^0=\omega} \\ &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \, e^{ik\cdot(x-x')} \, \mathrm{Res}\left[i \frac{e^{-ik^0(t-t')}}{k^0+\omega}, \omega \right] \\ &= \frac{1}{(2\pi)^{d+1}} \int \mathrm{d}^d k \mathrm{d}k^0 \, \frac{e^{ik\cdot(x-x') - ik^0(t-t')}}{(k^0)^2 - k^2 - m^2}, \end{aligned}$$

where we have inserted the k^0 integral to be taken around a counterclockwise contour surrounding $k^0 = \omega$ in the complex k^0-plane. Recall that ω was defined back in (2.7). Doing this for all of the two-point correlators, one finds

$$\mathscr{G}(x,x') = \frac{1}{(2\pi)^{d+1}} \int \mathrm{d}^d k \mathrm{d}k^0 \, \frac{\exp\left[ik\cdot(x-x') - ik^0(t-t')\right]}{(k^0)^2 - |k|^2 - m^2} \tag{2.46}$$

for $\mathscr{G} = iG^{\pm}, G_{F,R,A}$. The d-dimensional integrals are taken over the whole k-space, and the k^0 integrals are taken to be the following contours in the complex k^0 plane:

- G^+: counterclockwise circle around the pole ω;
- G^-: clockwise circle around the pole $-\omega$;
- G_R: negatively oriented real axis deformed above both $\pm\omega$;
- G_A: negatively oriented real axis deformed below both $\pm\omega$; and
- G_F: positively oriented real axis deformed below both $-\omega$ and above ω.

Note that the contours for $G_{F,R,A}$ do not fully enclose either pole $\pm\omega$. Mathematically, this is caused by the step function θ in the definitions (2.42) and (2.43) being nonzero for times t, t' which make the exponential in (2.46) decay in the k^0 region necessary to complete the contour.

As an example, the integral for the Feynman propagator G_F can also be further manipulated to put it in the more tractable form

$$G_F(x,x') = \lim_{\epsilon\to 0^+} -\frac{i\pi}{(4\pi i)^{n/2}} \left(\frac{2m^2}{-\sigma + i\epsilon}\right)^{(n-2)/4} H^{(2)}_{n/2-1}\left\{\left[2m^2(\sigma - i\epsilon)\right]^{1/2}\right\},$$

with $\sigma = \frac{1}{2}(x-x')^2 = \frac{1}{2}\eta_{\alpha\beta}(x^\alpha - x'^\alpha)(x^\beta - x'^\beta)$ and $H^{(2)}$ is a Hankel function of the second kind. The addition of $-i\epsilon$ indicates that G_F is the limit of functions that are analytic in the lower half-σ-plane.

2.4.2 Wick rotation and Euclidean spacetime

The Feynman propagator G_F possesses a nice property that the other two-point correlators do not. Note that in the integral representation (2.46), the k^0 contour for G_F relative to the poles $\pm\omega$ is unchanged when rotated 90 degrees counterclockwise about the origin. That is, if we take (2.46) and implement the substitutions $\kappa = -ik^0$, $\tau = -it$, $\tau' = -it'$, we find

$$\begin{aligned} G_F(t,x;t',x') &= \frac{i}{(2\pi)^{d+1}} \int \mathrm{d}^d k \mathrm{d}\kappa \, \frac{\exp[ik\cdot(x-x') + i\kappa(\tau-\tau')]}{-\kappa^2 - |k|^2 - m^2} \\ &= -iG_E(i\tau, x; i\tau', x') \end{aligned} \tag{2.47}$$

for

$$G_E(\tau,x;\tau',x') = \frac{1}{(2\pi)^n} \int \mathrm{d}^d k \mathrm{d}\kappa \, \frac{\exp[ik\cdot(x-x') + i\kappa(\tau-\tau')]}{\kappa^2 + |k|^2 + m^2}. \tag{2.48}$$

Now the κ integration is over the real axis, since after rotating the k^0 contour for G_F, there are no poles to obstruct us from deforming the contour to the axis. G_E is the *Euclidean Green function*, in that it satisfies

$$\left(\Box_x - m^2\right) G_E(x,x') = -\delta^n(x-x') \tag{2.49}$$

for the d'Alembertian operator $\Box$ on n-dimensional Euclidean space:

$$\Box = \delta^{\mu\nu}\partial_\mu\partial_\nu = \frac{\partial^2}{\partial\tau^2} + \sum_{j=1}^{d}\frac{\partial^2}{\partial x_j^2}. \tag{2.50}$$

The Euclidean Green function G_E can also be found by starting QFT in Euclidean spacetime, and then taking an expectation value of scalar fields. The advantage of working with Euclidean field theory is that the operator $\Box$ now has a well-defined inverse, since the poles of (2.48) lie off-axis. Consequently, it is often convenient to work in Euclidean space and then "rotate" to Minkowski space via (2.47), a process which is called *Wick rotation*.

2.4.3 Nonzero Temperature

Since all of these Green functions have been obtained as expectation values of field operators in the vacuum state $|0\rangle$, they only describe systems at zero temperature. However, we would also like to discuss systems at nonzero temperatures, and so we will now introduce the appropriate adjustments that have to be made. A system at nonzero temperature is described by a statistical distribution over all possible states, and so the corresponding Green functions will be given by the average over all pure states of the expectation values of the corresponding field operators.

Suppose that $|\psi_i\rangle$ is a pure state — that is, an eigenstate of the normal-ordered Hamiltonian

$$:H: = \int \mathrm{d}^d k\, a_k^\dagger a_k \omega \tag{2.51}$$

with energy eigenvalue E_i. We have not motivated why this definition is correct, but the reader should recognize that if $a, a^\dagger$ are taken to be the creation and annihilation operators of the harmonic oscillator, then this definition (2.51) should look familiar other than the inserted integral and the missing added term $\omega/2$. Adding such a term here would cause the integral to diverge, and so it has been removed during renormalization by normal ordering operation ": :".

Referring to the definitions (2.20) and (2.22), we see that $|\psi_i\rangle$ is also an eigenstate of the total number operator with some number eigenvalue n_i. Since both the number of particles and the energy are variable, an equilibrium system at temperature T is described by a grand canonical ensemble of states. Therefore, the probability that the system will be in the state $|\psi_i\rangle$ is given by

$$\rho_i = \frac{e^{-\beta(E_i-\mu n_i)}}{Z}, \tag{2.52}$$

where β is the inverse temperature

$$\beta = \frac{1}{k_B T}, \tag{2.53}$$

k_B is Boltzmann's constant, μ is the chemical potential,

$$Z = \sum_j e^{-\beta(E_j-\mu n_j)} = e^{-\beta\Omega} \tag{2.54}$$

is the grand partition function, and Ω is the thermodynamic potential. The ensemble average at a temperature $T = (k_B \beta)^{-1}$ of any operator A is thus

$$\langle A \rangle_\beta = \sum_i \rho_i \langle \psi_i | A | \psi_i \rangle . \tag{2.55}$$

Introducing the quantum *density operator*, defined by

$$\rho = \exp\left[\beta(\Omega + \mu N - H)\right], \tag{2.56}$$

allows us to write

$$\rho_i = \frac{e^{\beta(\mu n_i - E_i)}}{Z} \langle \psi_i | \psi_i \rangle = \langle \psi_i | e^{\beta\Omega} e^{\beta(\mu n_i - E_i)} | \psi_i \rangle = \langle \psi_i | \rho | \psi_i \rangle . \tag{2.57}$$

We can also express requirement of unit total probability as

$$\operatorname{tr} \rho \equiv \sum_i \langle \psi_i | \rho | \psi_i \rangle = 1, \tag{2.58}$$

and the definition (2.55) takes the simpler form

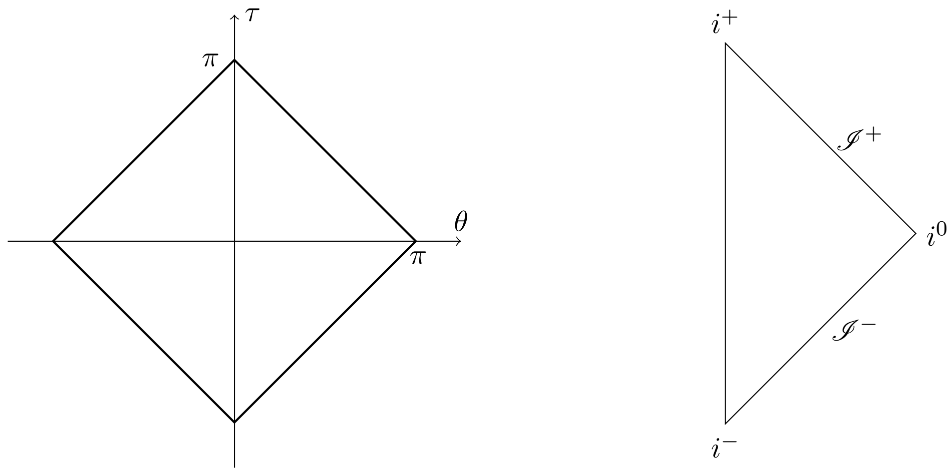
$$\langle A \rangle_\beta = \sum_i \rho_i \langle \psi_i | A | \psi_i \rangle = \sum_i \langle \psi_i | \rho | \psi_i \rangle \langle \psi_i | A | \psi_i \rangle = \sum_i \langle \psi_i | \rho A | \psi_i \rangle = \operatorname{tr} \rho A. \tag{2.59}$$

We are now prepared to define *thermal Green functions* by simply replacing the vacuum expectation values $\langle 0 | \cdot | 0 \rangle$ in the definitions (2.41)–(2.43) by the ensemble average $\langle \cdot \rangle_\beta$. One could continue from here to derive relations between thermal Green functions and their zero-temperature counterparts, but this is all we need at the moment. Thermal Green functions will return in section 7, when we discuss how the AdS/CFT correspondence allows one to generally compute correlators for gravity via field theory.

3 Causal structure of spacetime

In this section, we introduce the definitions of metrics and coordinates for Minkowski and anti-de Sitter (AdS) spacetimes of general dimension (as discussed in [3, 4]). We will also recall how to solve the geodesic equation using techniques from Lagrangian mechanics in section 3.1.1, and then apply these methods in order to contrast geodesics in each case. In particular, in section 3.2.1 we will show that free massive particles in AdS carry out bounded motion and that photons can travel to and from infinity in finite proper time. Consequently, we expect different heuristic behavior of the Unruh effect and Hawking radiation in AdS, and so these phenomena are worth studying.

Lastly, we will discuss the property of global hyperbolicity for a spacetime (based on [6]). As was seen in section 2.2, this concept played a key role in developing QFT in a general spacetime. After recalling the definition of global hyperbolicity in section 3.1.2, we argue that Minkowski spacetime possesses this property. In section 3.2.2, however, we will see that AdS is not globally hyperbolic, and how this can be seen from the geodesic behavior. The fact that AdS is not globally hyperbolic will significantly shape our discussion of the Unruh effect and Hawking radiation in how it differs from the case of flat spacetime.



(a) Two-dimensional spacetime.

(b) $(d+1)$-dimensional spacetime.

Figure 1: Penrose diagrams for flat Minkowski spacetime.

3.1 Flat spacetime

First, we consider the two-dimensional ($d = 1$) flat Minkowski space $\mathbb{R}^{1,1}$, with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2. \tag{3.1}$$

In terms of the Penrose coordinates τ, θ defined by

$$\tau \pm \theta = 2\arctan(t \pm x), \tag{3.2}$$

the metric becomes

$$\mathrm{d}s^2 = \frac{1}{4\cos^2[(\tau+\theta)/2]\cos^2[(\tau-\theta)/2]}(-\mathrm{d}\tau^2 + \mathrm{d}\theta^2). \tag{3.3}$$

This is conformal to (3.1), since they differ only by an overall factor. However, the coordinates τ, θ are now both bounded in magnitude by π, and so the spacetime is now contained within the finite rectangle $|\tau + \theta| < \pi$ in the (τ, θ)-plane. This is the compactification of two-dimensional Minkowski space, and allows one to visually display causal relations over the whole spacetime without distorting angles in a Penrose diagram. Figure 1a displays the Penrose diagram for this two-dimensional Minkowski spacetime. In these coordinates, the paths of photons trace rays that make an angle $\pi/4$ or $3\pi/4$ with the θ-axis.

Now we will generalize this to $(d+1)$-dimensional flat space $\mathbb{R}^{1,d}$, for $d \geq 2$. The metric is now

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i=1}^{d} \mathrm{d}x_i^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{d-1}^2 \tag{3.4}$$

in Cartesian and polar coordinates, respectively. Here, $\mathrm{d}\Omega_{d-1}^2$ denotes the metric on the $(d-1)$-dimensional sphere S^{d-1}. The appropriate Penrose coordinates are now

$$\tau \pm \theta = 2\arctan(t \pm r) \tag{3.5}$$

which transforms the metric to

$$\mathrm{d}s^2 = \frac{1}{4\cos^2[(\tau+\theta)/2]\cos^2[(\tau-\theta)/2]}\left(-\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_{d-1}^2\right) \tag{3.6}$$

This is conformal to (3.4), but the coordinates u, v are now bounded, and the spacetime occupies the triangle $|\tau| < \pi$, $0 \leq \theta < \pi$. Figure 1b displays the two-dimensional Penrose diagram for this $(d+1)$-dimensional Minkowski spacetime. This is the right half of the triangle in figure 1a, with the axes suppressed.

3.1.1 Geodesics

We will now review how to solve the geodesic equation using techniques of Lagrangian mechanics, so that we can observe how geodesics in AdS spacetime differ from those in flat spacetime.

In GR, particle trajectories are determined by the geodesic equation. That is, unaccelerated particles take the parameterized path $x(\tau)$ (with fixed endpoints) which minimize the path length ℓ with respect to the metric. The expression for the path length functional is

$$\ell[x] = -\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\,\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{3.7}$$

where "$\cdot$" denotes differentiation with respect to the affine parameter τ. From Lagrangian mechanics, we know that the minimization of this integral by a path $x(\tau)$ is equivalent to $x(\tau)$ solving the Euler-Lagrange equations with the integrand as the Lagrangian. Therefore, we may use the Euler-Lagrange equations in order to solve for geodesics.

It turns out that for affinely parameterized geodesics, rather than solving the Euler-Lagrange equations for the integrand of (3.7), is equivalent to solve them for the more convenient Lagrangian

$$\mathscr{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{3.8}$$

Unlike Lagrangian mechanics, however, we have the additional constraint of the line element. Since $\mathscr{L}$ is proportional to $\mathrm{d}s^2$, then we have two cases depending on the object whose trajectory is $x(\tau)$:

$$\begin{cases} \mathscr{L} < 0 & \text{timelike: the object is massive.} \\ \mathscr{L} = 0 & \text{null: the object is light.} \end{cases} \tag{3.9}$$

It is a simple exercise to show that for the case of Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu}$, we obtain the trajectories of inertial observers familiar from Newtonian mechanics.

3.1.2 Global hyperbolicity

As mentioned in section 2, Minkowski spacetime is globally hyperbolic. We will now define and explain the significance of this notion.

For a hypersurface Σ, the future domain of dependence $D^+(\Sigma)$ is the set of points p in the manifold for which every past inextendable causal curve through p intersects Σ. Likewise, the past domain of dependence $D^-(\Sigma)$ is the set of points p for which every future inextendable causal curve through p intersects Σ. A hypersurface Σ is said to be a *Cauchy surface* for the spacetime if the union of $D^+(\Sigma)$ and $D^-(\Sigma)$ is the entire manifold. A spacetime is called *globally hyperbolic* if there exists a Cauchy surface.

From the perspective of differential equations, complete knowledge of a function on a hypersurface Σ is enough to determine the solution on the whole future domain of dependence $D^+(\Sigma)$. Therefore, complete knowledge of data on a Cauchy surface determines the solution throughout the entire space and for all times. This is why requiring global hyperbolicity in section 2.2 was sufficient to develop QFT, and why (2.10) and (2.27) defined inner products. Formulating QFT in a non-globally hyperbolic spacetime is still possible, but is much more difficult.

The global hyperbolicity of Minkowski spacetime can be seen from figure 1b. Referring to the coordinates (3.5), we can see that hyperplanes of constant t in this diagram are curves connecting i^0 to the vertical axis. Since all inextendable causal curves connect the edge $\mathscr{I}^-$ to $\mathscr{I}^+$, then they all must intersect any given hyperplane of constant t.

3.2 Anti-de Sitter spacetime

Anti-de Sitter (AdS) space is defined to have constant negative curvature everywhere. Using this, the $(d+1)$-dimensional AdS spacetime (AdS_{d+1}) can be embedded in flat $(d+2)$-dimensional flat space as the hyperboloid

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2, \tag{3.10}$$

where R is the radius of curvature. This yields the induced metric

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 - \mathrm{d}X_{d+1}^2 + \sum_{i=1}^{d} \mathrm{d}X_i^2. \tag{3.11}$$

By construction, the space is homogeneous and isotropic, with the isometry group $SO(2, d-1)$.

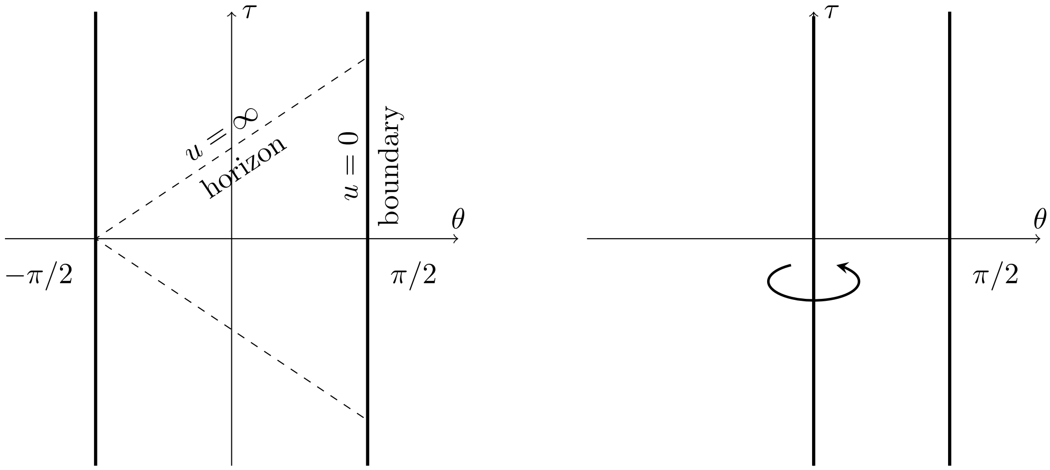
The Poincaré coordinates $(u, t, \vec{x})$ defined by

$$\begin{aligned} X_0 &= \frac{1}{2u}\left[1 + u^2(R^2 + x^2 - t^2)\right], \qquad & X_{d+1} &= Rut, \\ X_i &= Rux_i \quad (i = 1, \ldots, d-1), \qquad & X_d &= \frac{1}{2u}\left[1 - u^2(R^2 - x^2 + t^2)\right] \end{aligned} \tag{3.12}$$

for $0 < u$ and $\vec{x} \in \mathbb{R}^d$ are a solution to (3.10). Note that they only cover one half of the hyperboloid (3.10). Substituting these coordinates into the metric (3.11), we obtain the AdS_{d+1} metric:

$$\mathrm{d}s^2 = R^2\left[\frac{\mathrm{d}u^2}{u^2} + u^2\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right)\right] \tag{3.13}$$

Comparing this to the metric (3.4), we see that conformally AdS_{d+1} looks like d-dimensional Minkowski space in the coordinates $(t, \vec{x})$, with an additional warp factor in terms of the



(a) Two-dimensional spacetime.

(b) $(d+1)$-dimensional spacetime.

Figure 2: Penrose diagrams for anti-de Sitter spacetime.

coordinate u. Consequently, the Penrose diagram for the the Poincaré patch of AdS_{d+1} looks like the triangular diagram 1b. However, since the coordinates do not cover all of AdS_{d+1}, the resulting Penrose coordinates τ, θ can be analytically continued past the diagonal boundaries of figure 1b.

Equation (3.10) is also solved by

$$\begin{aligned} X_0 &= R\cosh\rho\cos\tau, \qquad X_{d+1} = R\cosh\rho\sin\tau, \\ X_i &= R\sinh\rho\,\Omega_i \qquad \sum_{i=1}^{d}\Omega_i^2 = 1. \end{aligned} \tag{3.14}$$

Unlike the Poincaré coordinates, these are global coordinates in that they cover the entire hyperboloid. Plugging these into the metric (3.11) yields another metric on AdS_{d+1}:

$$\mathrm{d}s^2 = R^2\left(-\cosh^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\Omega_{d-1}^2\right) \tag{3.15}$$

To study the causal structure, we make the coordinate change $\tan\theta = \sinh\rho$ for $0 \leq \theta < \pi/2$ for $d \geq 2$ and $|\theta| < \pi/2$ for $d = 1$, to obtain the metric

$$\mathrm{d}s^2 = \frac{R^2}{\cos^2\theta}\left(-\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_{d-1}^2\right) \tag{3.16}$$

This is conformal to the metric (3.6), except now θ is bounded by $\pi/2$ rather than π. The spacetime causal structure is unaffected by such a conformal rescaling, and so from here we can deduce the Penrose diagrams for AdS_2 and AdS_{d+1} for $d \geq 2$, which are depicted in figure 2. Since τ is unrestricted, the spacetimes occupy an infinite rectangular strip confined only in the θ-direction.

In the two-dimensional spacetime diagram 2a, the dashed line indicates the triangular Poincaré patch. In Poincaré coordinates, $u = 0$ corresponds to the spacetime boundary, while $u = \infty$ does not since τ and θ can be continued beyond this horizon. The circular arrow in the $(d+1)$-dimensional spacetime diagram 2b illustrates that each point in the rectangular region corresponds to a $(d-1)$-dimensional sphere. From this, it is apparent that the boundary of AdS_{d+1} is $\mathbb{R} \times S^{d-1}$.

There is another common form of the global coordinates that we will define here for later use. Letting $r = \sinh\rho$, the metric (3.15) becomes

$$\mathrm{d}s^2 = R^2 \left(-F(r)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{F(r)} + r^2\,\mathrm{d}\Omega_{d-1}^2 \right), \qquad F(r) = 1 + r^2. \tag{3.17}$$

Since R^2 is a constant, it is often absorbed, and the resulting metric is equivalent to the one above.

Before we move on, let us make a note on the time coordinate in the two expressions, (3.13) and (3.15), for the metric of AdS_{d+1}. In (3.15), the norm of the timelike Killing vector ∂_τ is everywhere nonzero. In particular, it has a constant norm in the conformally rescaled metric

$$\mathrm{d}s'^2 = -\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_{d-1}^2 \tag{3.18}$$

For this reason, τ is called the global time coordinate of AdS. Conversely, the timelike Killing vector ∂_t in (3.13) becomes null at the Killing horizon $u = 0$.

3.2.1 Geodesics

We will now examine radial timelike and null geodesics using the Lagrangian mechanics techniques of section 3.1.1 for the metric (3.17). Assuming that all angular coordinates are constant, the metric (3.17) reduces to that of AdS_2

$$\mathrm{d}s^2 = -F(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F(r)}, \qquad F(r) = 1 + r^2. \tag{3.19}$$

We have renamed the time coordinate "t" so that we may use τ to denote the proper time as in section 3.1.1. This yields the Lagrangian

$$\mathscr{L} = -\frac{1}{2}F(r)\dot{t}^2 + \frac{\dot{r}^2}{2F(r)}, \tag{3.20}$$

where "$\cdot$" denotes differentiation with respect to the proper time τ. Since this Lagrangian is not explicitly dependent on t, we have the conserved quantity

$$0 = \frac{\partial\mathscr{L}}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathscr{L}}{\partial\dot{t}} = -\frac{\mathrm{d}}{\mathrm{d}\tau}(F\dot{t}) \quad\Longrightarrow\quad F\dot{t} = E, \tag{3.21}$$

for E a constant. The Lagrangian (3.20) can be rewritten in terms of this as

$$\mathscr{L} = \frac{1}{2F(r)}\left(-E^2 + \dot{r}^2\right). \tag{3.22}$$

First, let us consider the radial motion $r(t)$ of a massive particle. Such a particle traces a timelike geodesic, and so in accordance with (3.9) we set the Lagrangian (3.22) equal to $-k^2/2$ and solve for $\dot{r}$:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + k^2 r^2 = E^2 - k^2.$$

The solution to this nonlinear differential equation is

$$r(\tau) = \sqrt{\frac{E^2}{k^2} - 1}\,\sin\left[k(\tau - \tau_0)\right],$$

where τ_0 is a constant of integration. That is, massive particles execute *bounded* harmonic motion when confined radially. In general, unaccelerated massive particles cannot escape to infinity in AdS.

Now let $r(t)$ denote the trajectory of a photon. Such a particle traces a null geodesic, and so in accordance with (3.9) we set the Lagrangian (3.22) equal to zero and solve for $\dot{r}$:

$$\dot{r} = E.$$

Diving through by the expression (3.21) for $\dot{t}$ yields

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\dot{r}}{\dot{t}} = F(r) = r^2 + 1$$

This is a separable differential equation, with solution

$$r(t) = \tan(t - t_0). \tag{3.23}$$

Therefore, we conclude that null geodesics reaches infinity in a *finite* amount of time. In the next section, we will discuss important consequences of this behavior.

3.2.2 Global hyperbolicity

Unlike Minkowski spacetime, AdS_{d+1} is not globally hyperbolic — as is illustrated in figure 3. Here, we see that the future domain of dependence of any spacelike hypersurface Σ has its boundary, called the future Cauchy horizon, within the spacetime. Likewise, the past domain of dependence also has its boundary within the spacetime, and so one cannot find a hypersurface whose future and past domains of dependence together cover the whole spacetime. Therefore, AdS is not globally hyperbolic.

This fact can also be seen from the radial null geodesics (3.23). Just as photons can reach infinity in finite time, they also can also reach any point in the spacetime from infinity in finite time. From a differential equations perspective, this means that knowledge of a function of a slice of the spacetime is not enough to evolve the function for all times — information at infinity will also be necessary.

Consequently, in order to apply our development of QFT in curved spacetime presented in section 2 to AdS, we will impose boundary conditions at infinity. Later, we will see that the appropriate choice for massless scalar fields is reflection at infinity. See [12] for more detail.

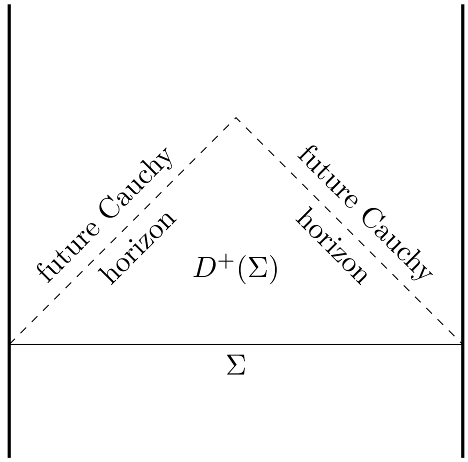


Figure 3: The future domain of dependence $D^+(\Sigma)$ and future Cauchy horizon for a spacelike hypersurface Σ in two-dimensional anti-de Sitter spacetime.

4 Unruh effect in flat spacetime

The equivalence principle of GR states that the gravitational forces observed locally can be formulated in terms of fictitious forces experienced in a non-inertial reference frame. Therefore, if a black hole releases a thermal emission of particles, then there should be an analogous phenomenon for accelerated observers — this is the Unruh effect [5]. Specifically, we will show that a uniformly accelerating observer in Minkowski vacuum measures a blackbody thermal emission of particles, based on the review [6]. Although the Unruh effect was discovered after Hawking radiation, it conceptually comes first, and so we shall explore this concept now before delving into Hawking radiation in section 5.

For simplicity, we will consider a massless scalar field ϕ (satisfying $\Box\phi = 0$) and a uniformly accelerating observer in two-dimensional Minkowski spacetime. Before we begin handling the QFT, we shall review the definition of uniform acceleration in section 4.1 (motivated by [13]), and in section 4.2 we define the hyperbolic coordinates and corresponding metric that is convenient for the observer. At this point we will have two metrics — the Minkowski metric for an inertial observer and the metric in terms of hyperbolic coordinates for the accelerating observer — similar to the scenario of the sandwich spacetime in section 2.3. As was done for the sandwich spacetime, section 4.3 defines an orthonormal set of modes in both frames, and then section 4.4 computes the resulting Bogoliubov coefficients for the transformation. Equation (2.40) then tells us how to use the coefficients to calculate the number of particles the accelerated observer views in a stationary observer's vacuum.

4.1 Uniformly accelerated observers

What does uniform acceleration mean? If the velocity of a particle in an inertial frame is $u(t)$, then the Newtonian definition of uniform acceleration — that du/dt is constant —

is unacceptable, since this implies that u will grow without bound and surpass the speed of light in vacuum. Intuitively, we would like constant acceleration to correspond with an observer attached to an engine exerting a constant thrust. For this reason, we define *uniform acceleration* to mean that at each instant, the acceleration in a co-moving frame has the same constant value. This requires that at each time t, an inertial primed frame traveling with the same velocity measures the particle's acceleration to be $du'/dt' = \alpha$, a constant. From special relativity, we can transform this acceleration to the first frame via

$$\frac{du}{dt} = \frac{\alpha}{\beta^3} = \left(1 - u^2\right)^{3/2} \alpha.$$

(Note that we are still using $c = 1$, and so u appears rather than u/c.) This is a separable differential equation with solution

$$u = \frac{\alpha(t - t_0)}{[1 + a^2(t - t_0)^2]^{1/2}}.$$

Setting $u = dx/dt$, we may now solve for $x(t)$ directly by integration to arrive at

$$[\alpha(x - x_0) + 1]^2 - \alpha^2(t - t_0)^2 = 1. \tag{4.1}$$

These describe hyperbolas in (x, t)-space centered at $(x_0 - 1/\alpha, t_0)$. The asymptotes $t = \pm x$ depict light rays, which illustrate an event horizon — an observer at rest in this frame at $x_0 - 1/\alpha$ can not pass any information to the particle after time t_0.

Now that we know the world line of a uniformly accelerated observer, we can parameterize their trajectory with conveniently selected coordinates and examine the induced spacetime metric.

4.2 Rindler spacetime in two-dimensional flat spacetime

Rindler spacetime is a sub-region of two-dimensional Minkowski spacetime (3.1) associated with a uniformly accelerating observer. The trajectory of such an observer is given by

$$x(\tau) = \frac{1}{\alpha}\cosh(\alpha\tau), \qquad t(\tau) = \frac{1}{\alpha}\sinh(\alpha\tau), \tag{4.2}$$

for α a constant. Note that this is a parameterization of the hyperbolic trajectories (4.1), where we have taken $t_0 = 0$ and $x_0 = 1/\alpha$. We can also calculate the acceleration directly via

$$a^\mu = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = (\alpha\sinh(\alpha\tau), \alpha\cosh(\alpha\tau)), \qquad a^2 = \eta_{\mu\nu}a^\mu a^\nu = \alpha^2,$$

and so the acceleration $\pm\alpha$ is indeed constant. The asymptotes are the lines $t = x$, and the region $x \leq t$ is inaccessible to the observer. If we were to divide the Penrose diagram 1a of Minkowski space with the lines $\tau = \pm\theta$, the accessible portions of the spacetime to the observer are the left and right sub-rectangles — these are called the *Rindler wedges*.

In the right Rindler wedge, define the coordinates (η, ξ) defined by

$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta), \qquad x = \frac{1}{a} e^{a\xi} \cosh(a\eta), \tag{4.3}$$

$$\eta = \frac{\alpha}{a}\tau, \qquad \xi = \frac{1}{a} \ln \frac{a}{\alpha}. \tag{4.4}$$

This way, τ is proportional to η and ξ is constant. An observer with constant acceleration $a = \alpha$ moves along $\eta = \tau$, $\xi = 0$. In these coordinates, the metric becomes

$$\mathrm{d}s^2 = e^{2a\xi}(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2). \tag{4.5}$$

Notice that this metric is independent of η, so the metric admits the timelike Killing vector field ∂_η and is stationary. Physically, this symmetry arises because the tangent vector ∂_η corresponds to a boost in the x direction:

$$\partial_\eta = \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x = a(x\partial_t + t\partial_x).$$

For the left Rindler wedge, we negate the definition of the coordinates:

$$t = -\frac{1}{a} e^{a\xi} \sinh(a\eta), \qquad x = -\frac{1}{a} e^{a\xi} \cosh(a\eta). \tag{4.6}$$

These coordinates now cover the left wedge, but the metric (4.5) is the same for both wedges. This makes the tangent vector ∂_η still future-pointing, but now in the opposite direction in comparison to ∂_η for the right wedge. It is important to note that we cannot use the coordinates (4.3) and (4.3) simultaneously, since they have the same ranges on both Rindler wedges.

Altogether, we have three different globally hyperbolic manifolds equipped with future-pointing Killing vectors for a uniformly accelerating observer: Minkowski space with ∂_t, the right Rindler wedge with ∂_η, and the left Rindler wedge with $-\partial_\eta$. All three spacetimes are both globally hyperbolic and stationary, and so the QFT developed in section 2.2 applies to all three. We will now quantize the scalar field in the first two spacetimes and apply the analysis of section 2.3.

4.3 Quantization

Specializing equation (2.11) to two-dimensional Minkowski spacetime ($d = 1$), we have the orthonormal plane wave solutions

$$f_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}, \qquad k^\mu = (\omega, k) \tag{4.7}$$

to the massless ($m = 0$) equation of motion (2.3). Referring to equation (2.8), we see that positive frequency modes in this flat spacetime are defined with respect to the Killing vector field ∂_t — that is, we require $\partial_t f_k = -i\omega f_k$ for $\omega > 0$. After canonical quantization, any field configuration ϕ that is a solution to the equation of motion (2.3) can be expanded in terms of the orthonormal basis $\{f_k, f_k^*\}$ as

$$\phi = \int \mathrm{d}k\, (a_k f_k + a_k^\dagger f_k^*). \tag{4.8}$$

As defined in (2.17), we also have the Minkowski vacuum $|0\rangle$ which satisfies $a_k |0\rangle = 0$ for all k.

In the right Rindler wedge, the massless equation of motion (2.25) reads

$$0 = \Box\phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi.$$

Dropping the exponential term, we see that this is the same as the equation of motion (2.3) for Minkowski space, and so we obtain the same orthonormal plane wave solutions in terms of the right Rindler wedge coordinates:

$$g_k^R = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}, \qquad x^\mu = (\eta, \xi). \tag{4.9}$$

As mentioned in section 4.2, our timelike Killing vector field for the right Rindler wedge is ∂_η. Referring to equation (2.29), we see that positive frequency modes are defined with respect to the Killing vector field ∂_η – that is, we require $\partial_\eta g_k^R = -i\omega g_k^R$ for $\omega > 0$. After canonical quantization, any field configuration ϕ solution to the equation of motion (2.25) can be expanded in terms of the orthonormal basis $\{g_k^R, g_k^{R*}\}$ as

$$\phi = \int \mathrm{d}k\, (b_k g_k^R + b_k^\dagger g_k^{R*}). \tag{4.10}$$

Analogous to the definition (2.17), we define the right Rindler wedge vacuum $\left|{}^R 0\right\rangle$ to be the state satisfying $b_k \left|{}^R 0\right\rangle = 0$ for all k.

Our aim is to answer the question: what does an observer in the right Rindler wedge measure for the Minkowski vacuum? In order to address this, we need to have a second field expansion to compare to (4.8) defined on a large enough section of the spacetime to be deterministic. Note that neither set of modes $\{g_k^R, g_k^{R*}\}$ nor $\{g_k^L, g_k^{L*}\}$ can possibly be bases for such a region since they are only defined in their respective regions. To solve this, we extend the Rindler modes $g_k^{R,L}$ so that they are defined on a Cauchy surface (e.g. $\eta = 0$) for the whole Minkowski space:

$$g_k^R = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{right Rindler wedge} \\ 0 & \text{left Rindler wedge} \end{cases} \tag{4.11}$$

$$g_k^L = \begin{cases} 0 & \text{right Rindler wedge} \\ \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{left Rindler wedge} \end{cases} \tag{4.12}$$

These latter modes have positive frequency with respect to the Killing vector field $-\partial_\eta$. By definition of Cauchy surfaces, this is a sufficient region of the spacetime to make our analysis well-defined.

Together, these $\{g_k^R, g_k^{R*}\}$ and $\{g_k^L, g_k^{L*}\}$ form a complete set of orthonormal modes on our Cauchy surface. As in section 2.3, this along with the complete orthonormal modes $\{f_k, f_k^*\}$ allows us to define the Bogoliubov transformation (2.33). The next subsection is devoted to calculating the resultant Bogoliubov coefficients.

4.4 Unruh effect

To answer our question in section 4.3, we want to compute the Bogoliubov coefficients of the transformation between the two sets of modes $\{f_k, f_k^*\}$ and $\{g_k^R, g_k^{R*}, g_k^L, g_k^{L*}\}$, and then use equation (2.40) to compute the expectation of the number operator. We have two possible expansions for any field configuration ϕ solving the equation of motion (2.3):

$$\int \mathrm{d}k\,(a_k f_k + a_k^\dagger f_k^*) = \phi = \int \mathrm{d}k\,(b_k g_k^R + c_k g_k^L + b_k^\dagger g_k^{R*} + c_k^\dagger g_k^{L*}). \tag{4.13}$$

The relation between the modes g_k^R and f_k are given by the Bogoliubov transformation

$$\begin{aligned} g_k^R(u) &= \int_{-\infty}^{\infty} \mathrm{d}\omega'\,(A_{\omega\omega'} f_{\omega'} + B_{\omega\omega'} f_{\omega'}^*) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega' \sqrt{\frac{\pi}{\omega'}}\left(A_{\omega\omega'} e^{-i\omega' u} + B_{\omega\omega'} e^{i\omega' u}\right) \end{aligned} \tag{4.14}$$

where $u = t - x$ (so that $ik_\mu x^\mu = -i\omega' t + ikx = -i\omega' u$). Note that for a massless scalar field, $d = 1$ and equation (2.7) tells us that k is a one-vector and $\omega = k$.

The Bogoliubov transformation (4.14) looks a lot like an inverse Fourier transform, and so to determine the coefficients we will compare this expression to the inverse Fourier transform of $g_k^R(u)$. Since g_k^R is a function of only u, let

$$\tilde{g}_\omega(\omega') = \int \mathrm{d}u\, e^{i\omega' u} g_k^R(u)$$

denote the Fourier transform of $g_k^R(u)$. Then we may write

$$\begin{aligned} g_k^R(u) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega'\, e^{-i\omega' u}\tilde{g}_\omega(\omega') \\ &= \frac{1}{2\pi}\int_{0}^{+\infty} \mathrm{d}\omega'\, e^{-i\omega' u}\tilde{g}_\omega(\omega') + \frac{1}{2\pi}\int_{0}^{+\infty} \mathrm{d}\omega'\, e^{i\omega' u}\tilde{g}_\omega(-\omega'). \end{aligned} \tag{4.15}$$

Comparing (4.14) with (4.15), we see that

$$A_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{g}_\omega(\omega'), \qquad B_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{g}_\omega(-\omega'). \tag{4.16}$$

If there is a relation between $\tilde{g}_\omega(-\omega')$ and $\tilde{g}_\omega(\omega')$, then from the relation (2.35), $|B|^2$ can be determined from $|A|^2$. The next section is devoted to demonstrating that this relation is

$$\tilde{g}_\omega(-\omega') = -e^{-\pi\omega/a}\tilde{g}_\omega(\omega'). \tag{4.17}$$

4.4.1 Integral manipulations

In this section, relation (4.17) is proved. The Fourier transform is defined in terms of $u = t - x$, which can be translated into (η, ξ) via (4.4) by

$$\eta - \xi = -\frac{1}{a}\ln(-au). \tag{4.18}$$

Plugging in g_k^R in the region $x \geq t$ ($u \leq 0$), we may now write the Fourier transform $\tilde{g}_\omega^R$ as

$$\begin{aligned}
\tilde{g}_\omega(\omega') &= \int_{-\infty}^{0} \mathrm{d}u\, e^{i\omega' u} \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u} = \frac{1}{\sqrt{4\pi\omega}} \int_{-\infty}^{0} \mathrm{d}u\, e^{i\omega' u} e^{i\omega \ln(-au)/a} \\
&= \frac{1}{\sqrt{4\pi\omega}} \int_{-\infty}^{0} \mathrm{d}u\, e^{i\omega' u} (-au)^{i\omega/a} = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} \int_{0}^{\infty} \mathrm{d}u\, e^{-i\omega' u} u^{i\omega/a} \\
&= \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} \frac{\omega}{a\omega'} \int_{0}^{\infty} \mathrm{d}u\, e^{-i\omega' u} u^{(i\omega/a)-1}.
\end{aligned} \tag{4.19}$$

Let us pause and manipulate just the integral term. Let b and s be parameters with positive real part. We have an integral of the form

$$\int_0^\infty dx\, e^{-bx} x^{s-1} = \int_0^\infty \frac{d(bx)}{b} e^{-bx} \frac{(bx)^{s-1}}{b^{s-1}} = b^{-s} \int_0^\infty dy\, e^{-y} y^{s-1} = e^{-s\ln b}\Gamma(s). \tag{4.20}$$

Writing $b = A + iB$, we pick the branch of the natural logarithm such that

$$\ln b = \ln\sqrt{A^2 + B^2} + i\arctan\left(\frac{B}{A}\right) = \ln\sqrt{A^2 + B^2} + i\arctan\left|\frac{B}{A}\right| \operatorname{sgn}\left(\frac{B}{A}\right). \tag{4.21}$$

For our integral, we take

$$b = i\omega' = i\omega' + \epsilon, \qquad s = i\omega/a = i\omega/a + \epsilon, \qquad \epsilon \to 0^+. \tag{4.22}$$

Then

$$\begin{aligned}
\ln b &= \ln\sqrt{\omega'^2 + \epsilon^2} + i\arctan\left|\frac{\omega'}{\epsilon}\right| \operatorname{sgn}\left(\frac{\omega'}{\epsilon}\right), \\
\lim_{\epsilon\to 0^+} \ln b &= \ln|\omega'| + \frac{i\pi}{2}\operatorname{sgn}(\omega').
\end{aligned} \tag{4.23}$$

Inserting this into our expression for $\tilde{g}_\omega(\omega')$ yields

$$\begin{aligned}
\tilde{g}_\omega(\omega') &= \lim_{\epsilon\to 0^+} \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} \frac{\omega}{a\omega'} e^{-(i\omega/a)\ln b}\Gamma\left(\frac{i\omega}{a}\right) \\
&= \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} \frac{\omega}{a\omega'} e^{-(i\omega/a)\ln|\omega'|} e^{\pi\omega/2a\operatorname{sgn}(\omega')}\Gamma\left(\frac{i\omega}{a}\right).
\end{aligned} \tag{4.24}$$

This is valid for all ω'. For the Bogoliubov coefficients we are interested in $\omega' > 0$ (cf. (4.15)), for which we have the relations

$$\tilde{g}_\omega(\omega') = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} \frac{\omega}{a|\omega'|} e^{-(i\omega/a)\ln|\omega'|} e^{\pi\omega/2a}\Gamma\left(\frac{i\omega}{a}\right) \tag{4.25}$$

$$\tilde{g}_\omega(-\omega') = -\frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} \frac{\omega}{a|\omega'|} e^{-(i\omega/a)\ln|\omega'|} e^{-\pi\omega/2a} \Gamma\left(\frac{i\omega}{a}\right), \tag{4.26}$$

from which we conclude

$$\tilde{g}_\omega(-\omega') = -e^{-\pi\omega/a} \tilde{g}_\omega(\omega'),$$

This is the relation (4.17) which we set out to prove.

4.4.2 Thermal bath

From the expression of the coefficients (4.16) and the relation (4.17), we can relate the Bogoliubov coefficients via

$$A_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}} \tilde{g}_\omega(\omega') = -\sqrt{\frac{\omega'}{\pi}} e^{\pi\omega/a} \tilde{g}_\omega(-\omega') = -e^{\pi\omega/a} B_{\omega\omega'}. \tag{4.27}$$

Using this and the identity (2.35),

$$|A|^2 - |B|^2 = 1,$$

we see that

$$|B|^2 = \frac{1}{e^{2\pi\omega/a} - 1}. \tag{4.28}$$

From equation (2.40), we know this expression is exactly the number of particles with frequency ω seen by an observer in the right Rindler wedge in the Minkowski vacuum. The function (4.28) is a Planck spectrum with temperature $T = a/2\pi$, which shows that a Rindler observer is immersed in a thermal bath of particles. This phenomenon is known as the *Unruh effect* [5].

5 Hawking radiation for asymptotically flat spacetimes

Despite their name, black holes are not in fact completely black, in that quantum effects cause them to emit radiation with a thermal spectrum. This result of Hawking [1] bears his name, and is one of the key triumphs of QFT in curved spacetime as a semiclassical approximation to a theory of quantum gravity. The thermal emission of a black hole leads to a decrease in the black hole's mass, and unless more mass is introduced into the system, the process continues until the black hole vanishes; consequently, this process is also referred to as black hole evaporation. Thermodynamically, black holes having finite temperature indicates that they also finite entropy, and so the discovery of Hawking radiation hinted at a new and promising intersection of the theories of gravity and thermodynamics.

As in Hawking's original paper [1], this section derives the phenomenon via the geometric optics approximation for a collapsing mass in asymptotically flat spacetime that remains spherically symmetric throughout its collapse (based on the reviews [2, 6]). Section 5.1 describes the structure of the resulting spacetime and discusses how we will construct orthonormal modes for both the infinite past and infinite future. We then task ourselves in section 5.2

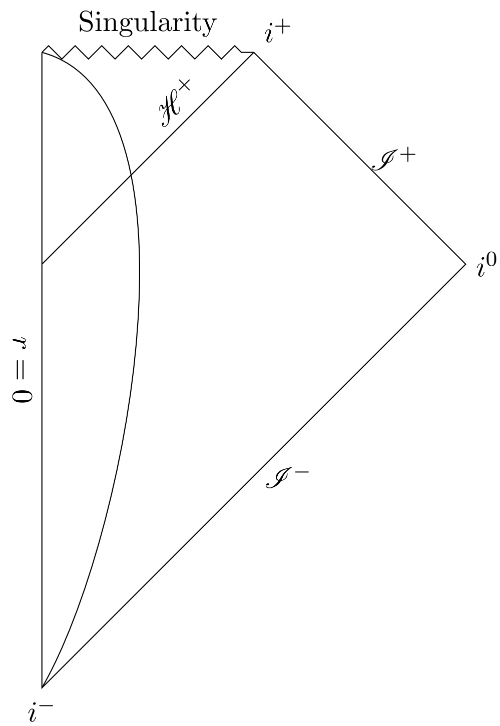


Figure 4: Penrose diagram for a symmetrically collapsing mass. The curved line represents the boundary of the mass.

with reducing the equation of motion for a massless scalar field to a Schrödinger-like wave equation, from which we can extract asymptotic solutions. This will give us two sets of orthonormal modes, which we will want to relate in section 5.4 via a physical argument so that we may apply the Bogoliubov transformation results of section 2.3. However, in order to make the physical relation realizable we will need to employ the geometric optics approximation, and so we will review this concept in section 5.3.

5.1 The set-up

Suppose a star were to collapse to a black hole, and the entire time remained completely spherically symmetric. Inside the star, there are complicated dynamics and the spacetime is not static. Outside, however, is the $(d+1)$-dimensional $(d \geq 3)$ Schwarzschild metric which describes the spacetime surrounding a spherical mass M, and consequently the spacetime is stationary at past null infinity ($\mathscr{I}^-$) and future null infinity ($\mathscr{I}^+$). The scenario is depicted in figure 4. This is an example of the sandwich spacetime considered in section 2.3. A massless scalar field ϕ propagating through the spacetime according to the Klein-Gordon equation (2.25) can be canonically quantized with respect to a set of modes defined at $\mathscr{I}^-$ and another at $\mathscr{I}^+$. We then would like to ask the question: what does an observer in the far future view in the past vacuum state?

Notice that $\mathscr{I}^-$ is a Cauchy surface, but $\mathscr{I}^+$ is not due to the region behind the horizon $\mathscr{H}^+$. However, $\mathscr{I}^+$ together with the future horizon $\mathscr{H}^+$ is a Cauchy surface. There is no timelike Killing vector on $\mathscr{H}^+$, so the notion of positive frequency modes cannot be described in the usual way. Our computation, however, will not rely on this since we are interested in the modes that reach $\mathscr{I}^+$. Altogether, this will allow us to define three sets of modes,

- $\{f_i\}$, positive frequency modes on $\mathscr{I}^-$;
- $\{g_i\}$, positive frequency modes on $\mathscr{I}^+$ and no Cauchy data on $\mathscr{H}^+$; and
- $\{h_i\}$, positive frequency modes on $\mathscr{H}^+$ and no Cauchy data on $\mathscr{I}^+$.

The modes $\{f_i, f_i^*\}$ form a complete set, and so do $\{g_i, g_i^*, h_i, h_i^*\}$. Any solution ϕ to the equation of motion can be written as

$$\int \mathrm{d}^d k\, (a_k f_k + a_k^\dagger f_k^*) = \phi = \int \mathrm{d}^d k\, (b_k g_k + c_k h_k + b_k^\dagger g_k^* + c_k^\dagger h_k^*). \tag{5.1}$$

The vacuum state with respect to the basis $\{f_k, f_k^*\}$ is defined to be the state $|\mathrm{in}\rangle$ such that $a_k\,|\mathrm{in}\rangle = 0$ for all k. The Bogoliubov transformations of section 2.3 provide an expansion for each mode g_k in terms of the modes $\{f_k, f_k^*\}$, and the number of particles seen by an observer in the far future in the vacuum $|\mathrm{in}\rangle$ will then given by (2.40) and the appropriate Bogoliubov coefficient. To compute this coefficient, we will find a solution to the equation of motion in the Schwarzschild metric.

5.2 Wave equation

For the $(d+1)$-dimensional Schwarzschild metric is

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -F(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F(r)} + r^2\,\mathrm{d}\Omega_{d-1}^2, \qquad F(r) = 1 - \frac{\mu}{r^{d-2}}. \tag{5.2}$$

The parameter μ is proportional to the mass M of the black hole

$$M = \frac{(d-1)\Omega_{d-1}}{16\pi G_{d+1}}\mu, \tag{5.3}$$

where G_{d+1} is Newtons constant in $d+1$ dimensions and $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a unit $(d-1)$-sphere. The horizon radius r_H is defined as the largest positive root of $F(r) = 0$. For $d = 3$, this has the familiar expression

$$r_H = 2MG_4. \tag{5.4}$$

The massless Klein-Gordon equation of motion (2.25) for the metric (5.2) reads

$$0 = \Box\phi = -\frac{1}{F(r)}\partial_t^2\phi + F(r)\partial_r^2\phi + \frac{[r^{d-1}F(r)]'}{r^{d-1}}\partial_r\phi + \frac{1}{r^{d-1}}\Delta_{S^{d-1}}\phi \tag{5.5}$$

where $\Delta_{S^{d-1}}$ is the operator on the $(d-1)$-dimensional sphere. For $d = 3$, this operator is

$$\Delta_{S^2} = \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right) + \frac{1}{\sin^2\theta}\partial_\varphi^2,$$

which we recognize as the spherical part of the three-dimensional Laplacian operator. Since the angular part of this equation is the same as that for the wave equation in Minkowski space, we have the ansatz for a spherical wave solution of the form

$$\phi = \frac{f(r,t)}{r^{(d-1)/2}} Y_{l,\{m\}}(\Omega), \tag{5.6}$$

where $Y_{l,\{m\}}(\Omega)$ are the spherical harmonics on the $(d-1)$-dimensional sphere. Their defining property is that they are eigenfunctions of the operator $\Delta_{S^{d-1}}$ with eigenvalue given by

$$r^{d-1}\Delta_{S^{d-1}} Y_{l,\{m\}}(\Omega) = -l(l+d-2)Y_{l,\{m\}}(\Omega). \tag{5.7}$$

Plugging the ansatz (5.6) into the wave equation (5.5) yields the bivariate differential equation

$$-\frac{1}{F(r)}\partial_t^2 f + F(r)\left[\partial_r^2 f - \frac{d-1}{r}\partial_r f + \frac{d^2-1}{4r^2}f\right] + \frac{[r^{d-1}F(r)]'}{r^{d-1}}\partial_r\left(\frac{f}{r^{(d-1)/2}}\right) - \frac{l(l+d-2)}{r^{d-1}}f = 0 \tag{5.8}$$

We would like to simplify this by hiding the cross derivatives ∂_r and by removing the factors of $F(r)$ in front of both second derivatives. This leads us to define the tortoise coordinate r_* by

$$r_* = \int_{r_H}^{r} \frac{\mathrm{d}r'}{F(r')}, \qquad \partial_{r^*} = F(r)\partial_r. \tag{5.9}$$

For later use, note the key property

$$\lim_{r\to r_H}\frac{dr}{dr_*} = \lim_{r\to r_H} F(r) = 0, \tag{5.10}$$

and so as a function $r(r_*)$ becomes more constant close to the horizon. For $d = 3$, the explicit formula for r_* is

$$r_* = r + r_H \ln\left(\frac{r}{r_H} - 1\right). \tag{5.11}$$

Changing variables $r \to r_*$, the wave equation (5.8) becomes

$$\left(-\partial_t^2 + \partial_{r_*}^2\right) f - V(r) f = 0, \tag{5.12}$$

for the potential-like term $V(r)$ given by

$$V(r) = \frac{(d-1)(d-3)}{4}\frac{F(r)^2}{r^2} + \frac{d-1}{2}\frac{F(r)\partial_r F(r)}{r} + l(l+d-2)\frac{F(r)}{r^2}. \tag{5.13}$$

For $d = 3$, the explicit formula for $V(r)$ is

$$V(r) = \left(1 - \frac{r_H}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{r_H}{r^3}\right). \tag{5.14}$$

In this section we will not be concerned with the specific form of $V(r)$, but only the key features

$$\lim_{r\to r_H^+} V(r) = \lim_{r_*\to -\infty} V(r) = 0, \qquad \lim_{r\to\infty} V(r) = \lim_{r_*\to\infty} V(r) = 0. \tag{5.15}$$

Between these two values, the potential V increases and forms a barrier — the exact form of which depends on l. From quantum mechanical scattering theory, we know that any solution f coming in from $r = \infty$ will be partially reflected and partially transmitted. At $\mathscr{I}^+$ (i.e. in the limit $r \to \infty$, $t \to \infty$, the potential vanishes and we are left with

$$(-\partial_t^2 + \partial_{r_*}^2)f = 0, \tag{5.16}$$

whose solutions are plane waves. In section 5.4, we will consider outgoing plane wave solutions to (5.16) given by

$$f = e^{ik_\mu x^\mu} = e^{-i\omega u}, \quad k_\mu = (-\omega, k), \quad x^\mu = (t, r_*), \quad u = t - r_*. \tag{5.17}$$

In the next section, however, we will depart on a brief interlude to recall the geometric optics approximation.

5.3 Geometric optics approximation

The aim of geometrical optics is to describe the propagation of a scalar wave f satisfying the massless Klein-Gordon equation (2.25). The approximation is to use the ansatz

$$f = Ae^{iS}, \tag{5.18}$$

where S is called the phase, and the amplitude A is assumed to be constant relative to S. This is analogous to the WKB approximation in quantum mechanics. The Klein-Gordon equation $\Box f = 0$ is reduced to

$$0 = \partial_\mu\partial^\mu f = \partial_\mu[i(\partial^\mu S)f] = i(\partial_\mu\partial^\mu S)f - (\partial_\mu S)(\partial^\mu S)f = iS\partial_\mu\partial^\mu f - (\partial_\mu S)(\partial^\mu S)f,$$

where the last equality is justified by integrating the equation over an arbitrary volume and applying integration by parts twice. The first term in the last expression vanishes according to the first equality, and so we conclude

$$0 = (\partial_\mu S)(\partial^\mu S) = g^{\mu\nu}(\partial_\mu S)(\partial_\nu S). \tag{5.19}$$

Let $S(x)$ denote the hypersurface of constant phase associated to the value x. The normal vector to $S(x)$ is $k_\mu = \partial_\mu S$ (note that this definition is consistent with our k_μ for the plane waves (5.17)), and the requirement (5.19) is equivalent to the condition that the normal vector is null:

$$k_\mu k^\mu = 0. \tag{5.20}$$

This makes sense, since f describes the propagation of light. Due to its lightlike nature, the vector k_μ is also normal to some tangent vector k^μ for a certain curve $x^\mu(\lambda)$ lying in the surface $S(x)$ — that is, $k^\mu = dx^\mu/d\lambda$.

We also claim that k^μ is the tangent vector field of null geodesic curves. Taking the covariant derivative of (5.20), we find

$$0 = 2k^\mu D_\sigma k_\mu = 2(k^\mu D_\sigma(\partial_\mu S)) = 2(k^\mu D_\mu(\partial_\sigma S)) = 2k^\mu D_\mu k_\sigma,$$
$$0 = k^\mu D_\mu k^\sigma = k^\mu \partial_\mu k^\sigma + k^\mu \Gamma^\sigma_{\mu\nu} k^\nu = \frac{d^2 x^\sigma}{d\lambda^2} + \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}. \tag{5.21}$$

This last equation is the geodesic equation. Therefore, we conclude the surface of constant phase of any wave $f = Ae^{iS}$ can be traced back in time by following null geodesics. This is the approximation used by Hawking to regress the asymptotic form of the waves (5.17) through time, and it is the argument we will present in the next section.

5.4 Hawking's computation

At $\mathscr{I}^+$, we found the outgoing asymptotic solution (5.17) to the wave equation. We will use the geometric optics approximation of section 5.3 to trace the outgoing solution back in time. Let γ_H denote the limiting null geodesic that passes through the mass at the last possible moment to not fall into the singularity, and instead stays at the horizon $\mathscr{H}^+$ (see figure 5). Let γ be the null geodesic associated with the outgoing solution (5.17), and consider the limit in which γ approaches γ_H. The affine distance at $\mathscr{I}^+$ is the Kruskal coordinate U, which is related to u by

$$U = -e^{-uk_H}, \tag{5.22}$$

and

$$k_H = \frac{F'(r_H)}{2} \tag{5.23}$$

is the surface gravity. For $d = 3$, the surface gravity is

$$k_H = \frac{1}{4\pi} \tag{5.24}$$

Outgoing null geodesics are given by constant U. At $\mathscr{I}^+$, if we translate U so that the affine distance of the limiting null geodesic γ_H to be $U = 0$, then one of the null geodesics γ of the outgoing solution is some $U = -\epsilon$ in the limit $\epsilon \to 0^+$. From the definition (5.22), this corresponds to

$$u = -\frac{1}{k_H} \ln \epsilon. \tag{5.25}$$

This coordinate u is the affine distance between γ_H and γ along an ingoing null geodesic. The outgoing plane wave (5.17) at $\mathscr{I}^+$ can now be written as

$$f = e^{-i\omega u} = e^{i\omega \ln(\epsilon)/k_H}. \tag{5.26}$$

As already mentioned, in the limit $\gamma \to \gamma_H$ we have $\epsilon \to 0^+$, and so the phase of (5.26) oscillates arbitrarily quickly in this limit. Therefore, the high-frequency approximation of geometric optics is justified.

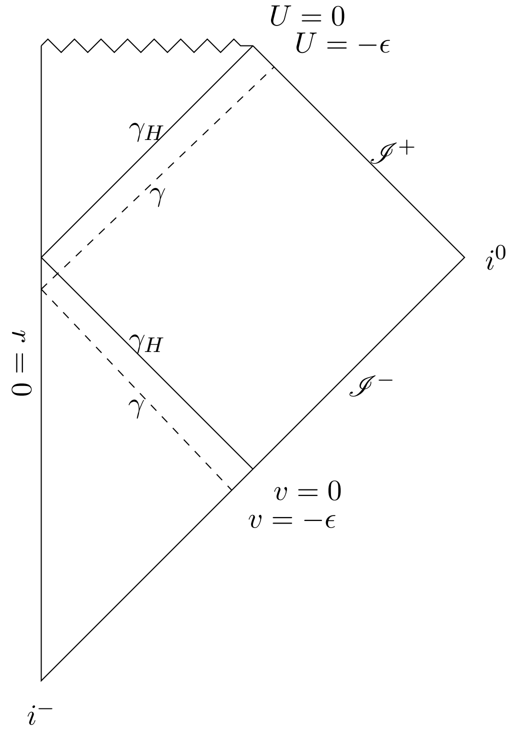


Figure 5: Tracing null geodesics a symmetrically collapsing mass. The line γ_H is the null geodesic that remains at the horizon, and the dashed line γ is the null geodesic an affine distance $U = -\epsilon \to 0$ away at $\mathscr{I}^+$.

According to section 5.3, surfaces of constant phase of the solution (5.26) can be traced back in time by following the null geodesic γ. This is depicted in figure 5. γ reaches $\mathscr{I}^-$ at an affine distance $\nu = -\epsilon$ with respect to the limiting null geodesic γ_H. Together, this tells us:

- at $\mathscr{I}^+$, the outgoing solution is $f = e^{i\omega \ln(\epsilon)/k_H}$; and
- at $\mathscr{I}^-$, the outgoing solution is $f = e^{i\omega \ln(-\nu)/k_H}$.

Recall from our analysis and notation of the Unruh effect in section 4 that we worked the outgoing solution given by (4.9) and (4.18):

$$g_k^R \propto e^{-i\omega(\eta-\xi)} = e^{i\omega \ln(-au)/a} = e^{i\omega \ln[a(r-t)]/a}.$$

Here, our outgoing solution is

$$f \propto e^{i\omega \ln(\epsilon)/k_H} = e^{i\omega \ln(-k_H u)/k_H} = e^{i\omega \ln[k_H(r_*-t)]/k_H}.$$

Recalling that $r_* \simeq r$ as $r \to \infty$, we see that these are identical and $\mathscr{I}^+$, except for k_H in place of a. Similarly, both the outgoing waves of section 4 and our current analysis agree at $\mathscr{I}^-$, since the former are plane waves in flat space and the latter are plane waves in asymptotically flat space. Therefore, repeating the calculation of section 4 would yield

$$|B|^2 = \frac{1}{e^{2\pi\omega/k_H} - 1}. \tag{5.27}$$

This is a Planck spectrum with the *Hawking temperature*

$$T_H = \frac{k_H}{2\pi}. \tag{5.28}$$

Therefore, an asymptotically black hole spacetime has a nonzero temperature T_H, and the corresponding thermal emission leads to a decrease in the mass. This phenomenon is called *Hawking radiation* [1]. Notice, however, that the outgoing radiation (5.27) is purely thermal and carries no information regarding the original state that collapsed into the black hole. In particular, a pure state of the system can become a thermal density matrix, violating the principle that the information of a wavefunction of a physical system at one point in time should in theory determine its value at any other time. This is known as the *information paradox*.

Our analysis here was entirely independent of the specific form of the potential $V(r)$. When we account for this in the next section, we will find that there is consequent back-scattering and the spectrum on the right-hand side of (5.27) gains a corrective greybody factor γ. This factor will be frequency dependent, and will in general also depend on the structure of the potential $V(r)$.

6 Greybody factors for asymptotically flat spacetimes

As is usually the case in physics, the physical object we are studying is not a perfect blackbody. To correct for this, we insert the greybody factor $\gamma(\omega)$ into the Planck spectrum to obtain the more realistic thermal spectrum

$$\frac{\gamma(\omega)}{e^{2\pi\omega/k_H} - 1}.$$

This section is dedicated to calculating $\gamma(\omega)$ for asymptotically flat black holes in two different regimes. In section 6.1, we will use the low frequency limit — that the radiation wavelength is large compared to the radius of the black hole horizon — to approximate the greybody factors for all static and spherically symmetric black holes (as in [7]). Here, the result will in fact be independent of the particular choice of black hole, and this is simply because the wavelength is too long to be affected by the black hole's structure. Section 6.2 will then examine the other extreme of high frequency radiation, for which the wavelength is small compared to the black hole radius. In this approximation, the particular computations depend significantly of the choice of black hole and on dimension, and so to observe how the technique is applied we will consider the simple case of the four-dimensional Schwarzschild black hole (based on [8]).

We also would like to make an important note regarding the stability of scalar perturbations to black hole spacetimes. For all asymptotically flat black holes with no electric charge, perturbations are stable for all dimensions, and so we are justified in working with generic dimension. For charged black holes, however, scalar perturbations have only been proved stable for four- and five-dimensional black holes. We will still work in generic dimension, but the results we find will be accurate exactly when the spacetime under consideration is stable.

6.1 Greybody factors at low frequency

This section aims to calculate greybody factors and the absorption cross-section at low frequency for all static and spherically symmetric black holes. We will maintain generality in our discussion for as long as possible so that much of our work applies to asymptotically curved spacetimes as well, since at low frequencies much of the analysis is the same. It is not until section 6.1.3 that we assume that the spacetime is asymptotically flat. Our definitions and manipulations introduced in section 5.2 carry over to this scenario, except we no longer assume that the metric is specifically the Schwarzschild black hole. We could not assume the same generality that we do here since physically our discussion was founded upon a spherically symmetric collapsing mass.

The greybody factor of a given black hole for low frequency scattering is identical to the absorption probability, since the scattering and absorption processes are reverse to each other. Therefore, we shall examine the absorption process of a scalar wave by a black hole. We consider a scalar wave propagating from infinity throughout spacetime and being partly reflected by the potential barrier of the black hole, so that near the horizon the transmitted radiation appears as purely incoming radiation into the black hole. Analogously, the specific physical process measured by the greybody factor is the emission of radiation from the black hole which gets partly reflected by the potential barrier just outside the horizon, and the transmitted radiation appears as purely outgoing radiation in the asymptotic region of spacetime.

We begin by considering a general, static and spherically symmetric, $(d+1)$-dimensional black hole metric of the form

$$\mathrm{d}s^2 = -F(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F(r)} + r^2\,\mathrm{d}\Omega_{d-1}^2 \tag{6.1}$$

for the general radial function

$$F(r) = F_a(r) + F_h(r). \tag{6.2}$$

Here, $F_a(r)$ is the asymptotic part of $F(r)$, meaning that

$$\lim_{r\to\infty} F_h(r) = 0. \tag{6.3}$$

The specific nature of the black hole is contained in $F_h(r)$, the precise form of which turns out not to be important. We will refer to the *asymptotic region* as the range for which $F_a(r) \gg F_h(r)$, and the *horizon region* for when $F_a(r)$ and $F_h(r)$ are of comparable magnitude.

We will also quickly upgrade some definitions from section 5 for our more general set-up. The horizon radius r_H is the largest real root r of $F(r)$. We also define

$$k_H = \frac{1}{2}F'(r_H), \tag{6.4}$$

$$T_H = \frac{k_H}{2\pi}, \tag{6.5}$$

$$A_H = \Omega_{d-1} r_H^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} r_H^{d-1} \tag{6.6}$$

to be the surface gravity, Hawking temperature, and the area of the event horizon, respectively.

The propagation of a massless scalar field ϕ in the background of the black hole spacetime (6.1) is dictated by the Klein-Gordon equation (2.25). From section 5.2, we know that an effective ansatz is the spherical wave

$$\phi(t, r, \Omega) = e^{i\omega t}\Phi_{\omega,l}(r) Y_{l,\{m\}}(\Omega). \tag{6.7}$$

This reduces the Klein-Gordon equation to the ordinary differential equation

$$\partial_r \left(r^{d-1} F(r) \partial_r \Phi_{\omega,l} \right) + \omega^2 \frac{r^{d-1}}{F(r)} \Phi_{\omega,l} - l(l+d-2) r^{d-3} \Phi_{\omega,l} = 0. \tag{6.8}$$

To clean this up, we again define the tortoise coordinate $r_* = r_*(r)$ as in (5.9), except now for our more general radial function (6.2). This puts (6.8) into the more tractable Schrödinger-like wave equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r_*^2} + \omega^2 - V(r) \right] \left(r^{(d-1)/2} \Phi_{\omega,l} \right) = 0, \tag{6.9}$$

for the potential

$$V(r) = \frac{(d-1)(d-3)}{4} \frac{F(r)^2}{r^2} + \frac{d-1}{2} \frac{F(r)F'(r)}{r} + l(l+d-2) \frac{F(r)}{r^2}. \tag{6.10}$$

Let us now consider the low frequency limit

$$\omega \ll T_H, \qquad \omega r_H \ll 1 \tag{6.11}$$

for the scalar wave (6.7). Note that the limit (6.11) requires the wavelength to be much larger than any characteristic scale of the black hole. In a method similar to the WKB approximation applied to quantum tunneling, this will enable us to match the behavior of the Schrödinger wavefunction across broad regions of spacetime with a very high degree of accuracy. In order to find the greybody factor, we split up the spacetime in three regions:

- Region I: The region near the event horizon, characterized by $r \simeq r_H$ and $V(r) \ll \omega^2$.
- Region II: The region between regions I and III, characterized by $V(r) \gg \omega^2$.
- Region III: The asymptotic region, characterized by $r \gg r_H$.

We shall then match the behavior of the wave function (6.7) between these regions. The leading contribution to the greybody factor in the low frequency limit (6.11) comes from the $l = 0$ mode, and so for the following we set $l = 0$:

$$\Phi_\omega(r) = \Phi_{\omega,0}(r). \tag{6.12}$$

6.1.1 Region I: the horizon region

Near the horizon, the metric radial function $F(r)$ is well-approximated by the linearization

$$F(r) \simeq 2k_H(r - r_H). \tag{6.13}$$

This yields the asymptotic form for the potential (6.10)

$$V(r) \simeq 2(d-1)\frac{k_H^2}{r_H}(r - r_H). \tag{6.14}$$

The horizon condition limit $V(r) \ll \omega^2$ now becomes

$$\frac{r - r_H}{r_H} \ll \frac{\omega^2}{k_H^2}, \tag{6.15}$$

and this, together with $\omega \ll T_H \sim k_H$, implies that we are working in the limit

$$r r_H \ll r_H. \tag{6.16}$$

Since in this region $V(r)$ is negligible compared to ω^2, the wave equation (6.9) simplifies to

$$\left[\frac{\mathrm{d}^2}{dr_*^2} + \omega^2\right]\left(r^{(d-1)/2}\Phi_\omega\right) = 0. \tag{6.17}$$

This of course has the incoming wave solution

$$\left(\frac{r}{r_H}\right)^{(d-1)/2}\Phi_\omega = A_I e^{i\omega r_*}. \tag{6.18}$$

Note that we do not need to utilize the other linearly independent solution, since we are assuming an incident wave from infinity. Moreover, since we are working in the limit (6.16), we can simply write

$$\Phi_\omega = A_I e^{i\omega x}. \tag{6.19}$$

To measure the associated flux near the horizon, we note that in terms of the tortoise coordinate r_*, we are effectively considering a one-dimensional Schrödinger-like equation with zero potential. Therefore, the flux per unit area is

$$j_{\text{hor}} = \frac{1}{2i}\left(\Phi_\omega^*\frac{\mathrm{d}\Phi_\omega}{\mathrm{d}x} - \Phi_\omega\frac{\mathrm{d}\Phi_\omega^*}{\mathrm{d}x}\right) = \omega|A_I|^2, \tag{6.20}$$

and total flux near the event horizon is

$$J_{\text{hor}} = A_H \omega |A_I|^2. \tag{6.21}$$

We now want to tailor the wavefunction (6.19) for a region that overlaps with region II but still lies within region I as well. From the linearization (6.13), the tortoise coordinate in region I has the form

$$r_* = \int_{r_H}^{r} \frac{\mathrm{d}r'}{F(r')} \simeq \frac{1}{2k_H} \ln\left(\frac{r - r_H}{r_H}\right). \tag{6.22}$$

We will move slightly away from the horizon, so that

$$\frac{r - r_H}{r_H} \gg e^{-2k_H/\omega}. \tag{6.23}$$

This is still within region I, since together with (6.15) this implies

$$e^{-2k_H/\omega} \ll \frac{\omega^2}{k_H^2}, \tag{6.24}$$

which follows from the low-frequency limit (6.11) alone. Using this, we may plug the tortoise coordinate asymptotic form (6.22) into the wavefuntion (6.19) to get

$$\Phi_\omega = A_I \exp\left[\frac{i\omega}{2k_H} \ln\left(\frac{r - r_H}{r_H}\right)\right] \simeq A_I \left[1 + \frac{i\omega}{2k_H} \ln\left(\frac{r - r_H}{r_H}\right)\right]. \tag{6.25}$$

We will later match (6.25) to a general solution of the scalar wave equation in region II.

6.1.2 Region II: the intermediate region

Since $\omega^2 \ll V(r)$ in region II, only the first term in differential equation (6.8) is dominant:

$$\partial_r \left(r^{d-1} F(r) \partial_r \Phi_\omega\right) = 0. \tag{6.26}$$

Integration immediately yields

$$\Phi_\omega(r) = A_{II} + B_{II} G(r), \tag{6.27}$$

for the function $G(r)$ defined by

$$G(r) = \int_\infty^r \frac{\mathrm{d}r'}{g(r')}, \qquad g(r) = r^{d-1} F(r). \tag{6.28}$$

In the part of region II that overlaps with region I, we have $r \approx r_H$, and so we can use the linearization (6.13) to obtain

$$G(r) \simeq \int_\infty^r \frac{\mathrm{d}r'}{r_H^{d-1} 2k_H (r' - r_H)} = \frac{1}{2k_H r_H^{d-1}} \ln\left(\frac{r - r_H}{r_H}\right). \tag{6.29}$$

Putting this into (6.27) and matching with the wavefunction (6.25) of region I immediately yields

$$A_{II} = A_I, \qquad B_{II} = i\omega r_H^{d-1} A_I. \tag{6.30}$$

Now for the overlap with region III we instead take $r \gg r_H$, which yields

$$G(r) \simeq \int_\infty^r \frac{\mathrm{d}r'}{r'^{d-1} F_a(r')}. \tag{6.31}$$

Putting this into (6.27) yields

$$\Phi_\omega(r) = A_I \left[1 + i\omega r_H^{d-1} \int_\infty^r \frac{\mathrm{d}r'}{r'^{d-1} F_a(r')} \right] \tag{6.32}$$

for $r \gg r_H$ within the region with $\omega^2 \ll V(r)$. We will match this expression (6.32) for the wavefunction to a general solution for the scalar wave equation in region III. Again, up to this stage, note that we have been completely generic on which type of black hole we are considering.

6.1.3 Region III: the asymptotic region

We will now make use of the assumption that our black hole resides within an asymptotically flat spacetime. This, along with the limit $r \gg r_H$ for region III, tells us that

$$F(r) \simeq F_a(r) = 1. \tag{6.33}$$

Since the metric (6.1) in this regime reduces to that of flat Minkowski space. The radial part of the general solution of the flat space wave equation (still with $l = 0$) can be expressed in terms of the linear combination

$$\Phi_\omega = \rho^{(2-d)/2} \left[C_1 H^{(1)}_{(d-2)/2}(\rho) + C_2 H^{(2)}_{(d-2)/2}(\rho) \right] \tag{6.34}$$

with $\rho = r\omega$, and H denoting the Hankel functions.

Note that we can take the limit $\rho \ll 1$ in order to overlap region III with region II, since along with the low-frequency limit (6.11) and the requirement $r \gg r_H$ of region III we get

$$r_H \ll r \ll \frac{1}{\omega}. \tag{6.35}$$

That is, r is large but is still bounded above. Taking the limit $\rho \ll 1$ of (6.34) yields

$$\Phi_\omega = \frac{C_1 + C_2}{\Gamma(d/2) 2^{(d-2)/2}} \left[1 + \mathcal{O}(\rho)\right] - i(C_1 - C_2) \frac{\Gamma[(d-2)/2] 2^{(d-2)/2}}{\pi \rho^{d-2}} \left[1 + \mathcal{O}(\rho)\right]. \tag{6.36}$$

Applying the assumption $F_a(r) = 1$ to the region II wavefunction (6.32) gives

$$\Phi_\omega = A_I \left[1 + i\omega r_H^{d-1} \int_\infty^r \frac{\mathrm{d}r'}{r'^{d-1}} \right] = A_I \left[1 - \frac{i\omega r_H^{d-1}}{(d-2) r^{d-2}} \right]. \tag{6.37}$$

Visually matching this to (6.36) tells us that

$$C_1 + C_2 = \Gamma\left(\frac{d}{2}\right) 2^{(d-2)/2} A_I, \qquad C_1 - C_2 = \frac{\pi\omega^{d-1} r_H^{d-1}}{(d-2)\Gamma[(d-2)/2]2^{(d-2)/2}} A_I. \tag{6.38}$$

As a sanity check, we note that from the low-frequency limit (6.11) we have $|C_1 - C_2| \ll |C_1 + C_2|$. Therefore, ignoring higher order terms $\rho = r\omega$ in the imaginary parts of the Hankel functions in (6.34) was a self-consistent decision.

We will now compare the flux in the asymptotic region to that of the horizon region (6.21). Splitting the total incoming flux into its inward and outward traveling components,

$$J_{\text{asy}} = r^{d-1}\Omega_{d-1}\frac{1}{2i}\left(\Phi_\omega^* \frac{\mathrm{d}\Phi_\omega}{\mathrm{d}r} - \Phi_\omega \frac{\mathrm{d}\Phi_\omega^*}{\mathrm{d}r}\right) = J_{\text{in}} - J_{\text{out}}, \tag{6.39}$$

we have for the general wave-function solution (6.34) that

$$J_{\text{in}} = \frac{2}{\pi}\Omega_{d-2}\omega^{3-d}|C_1|^2, \qquad J_{\text{out}} = \frac{2}{\pi}\Omega_{d-2}\omega^{3-d}|C_2|^2. \tag{6.40}$$

Using the coefficient relations (6.38), we find

$$J_{\text{asy}} = \frac{2}{\pi}\Omega_{d-1}\omega^{2-d}\left(|C_1|^2 - |C_2|^2\right) = \omega|A_I|^2\Omega_{d-1}r_H^{d-1}. \tag{6.41}$$

Comparing this with the horizon flux (6.21), we conclude that the flux is preserved from the horizon to the asymptotic region:

$$J_{\text{hor}} = J_{\text{asy}} = J_{\text{in}} - J_{\text{out}}. \tag{6.42}$$

Now, we are able to calculate the greybody factor and absorption cross-section in the low-frequency limit.

6.1.4 Greybody factor and absorption cross-section

As we previously discussed, the greybody factor for low frequency scattering is identical to the absorption probability, and this is because the scattering and absorption processes are reverse to each other. Recall that our calculations thus far were for a scalar wave propagating from infinity, being partly reflected by the potential barrier, and the transmitted radiation being the only wave present at the horizon. The greybody factor $\gamma(\omega)$ should therefore be defined as the ratio $J_{\text{hor}}/J_{\text{in}}$. Using the calculated fluxes (6.40) and the flux conservation (6.42), in addition to the coefficient relations (6.38), we find

$$\begin{aligned}\gamma(\omega) &= \frac{J_{\text{hor}}}{J_{\text{in}}} = 1 - \frac{J_{\text{out}}}{J_{\text{in}}} = 1 - \frac{|C_2|^2}{|C_1|^2} \\ &= 0 + (C_1 - C_2)\frac{4}{C_1 + C_2} + \mathcal{O}\left(\left|\frac{C_1 - C_2}{C_1 + C_2}\right|^2\right) = \frac{4\pi\omega^{d-1}r_H^{d-1}}{2^{d-1}[\Gamma(d/2)]^2}.\end{aligned} \tag{6.43}$$

This is the greybody factor in the low-frequency limit (6.11) for asymptotically flat black holes.

To find the absorption cross-section $\sigma(\omega)$ from this absorption probability, we need to project a plane-wave wavefunction $e^{i\omega z}$ onto an ingoing spherical s-wave (i.e. $l = 0$) of the form

$$\frac{e^{i\omega r}}{(r^{d-1}\Omega_{d-1})^{1/2}}\Psi.$$

For our purposes, this computation is not physically enlightening, and so we will state that the result is

$$|\Psi|^2 = \frac{(2\pi)^{d-1}}{\omega^{d-1}\Omega_{d-1}}.$$

This, together with the absorption probability (6.43), gives us the absorption cross-section

$$\sigma(\omega) = \gamma(\omega)|\Psi|^2 = A_H \tag{6.44}$$

low-frequency limit (6.11) for asymptotically flat black holes. Recall that A_H is the area of the horizon defined by (6.6). Notice that the cross-section (6.44) is entirely independent of the black hole, and hence is a purely geometrical and universal result.

6.2 Greybody factors at asymptotic frequency

In contrast with the low-frequency limit (6.11), the greybody factors at asymptotic frequency are much more dependent on the specific nature of the black hole. Physically this makes sense, since the asymptotic frequency limit would require that the wavelength is much smaller than the dimensions of the black hole, and so the wavefunctions must be significantly affected by the corresponding Schrödinger-like potential. The technique commonly used to calculate greybody factors is called monodromy matching, and mathematically its nature also is dependent on the choice of black hole and dimension.

To get a flavor of the monodromy-matching technique, we shall specialize to the simplest case of a four-dimensional ($d = 3$) Schwarzschild black hole in asymptotically flat spacetime. The idea of this method is to determine the reflection and transmission coefficients by tracking how they change as we follow closed contours in the complex r-plane. Specifically, we will study the behavior of the solution $r^{(d-1)/2}\Phi$ around the contour γ displayed in figure 6, with the large semi-circular arc taken to be arbitrarily close to infinity. Recall the four-dimensional tortoise coordinate given by (5.11). The $\mathrm{Re}\, r_* < 0$ region is shaded in the figure, and the points $r = 0$, $r = r_H$ and a contour γ for calculation of the transmission and reflection coefficients are marked. γ runs along the line $\mathrm{Re}\, r_* = 0$ to the limiting points $|\omega r_*| \gg 1$ in each direction. The symbol ★ indicates the point at which our boundary conditions at infinity will be imposed. We will calculate the effect of the trip around γ in two different ways, and the comparison of the two results will give enough information to calculate the transmission and reflection coefficients.

Let

$$\psi(r) = r^{(d-1)/2}\Phi(r) \tag{6.45}$$

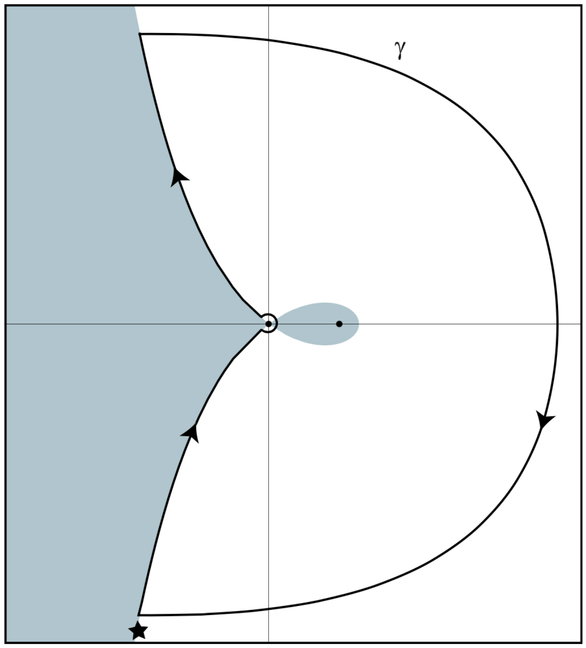


Figure 6: The monodromy contour γ in the complex r-plane for the four-dimensional asymptotically flat Schwarzschild black hole. The shaded region denotes $\mathrm{Re}\, r_* < 0$, and the two circular points indicate $r = 0$ and $r = r_H$. Figure borrowed from [8].

denote our wavefunction that solves the Schrödinger-like equation (6.9) for the specific potential (5.14) corresponding to our black hole. As in (5.17), since $V(r)$ vanishes as $r \to \infty$ and , then as $r \to \infty$ in any direction the asymptotic solution to (6.9) is of the form

$$\psi(r) \simeq C_+ e^{i\omega r_*} + C_- e^{-i\omega r_*}. \tag{6.46}$$

Notice that the logarithmic term in the tortoise coordinate (5.11) makes $r_*(r)$ multivalued, and so there is an ambiguity in defining both exponential terms and the coefficients $C_\pm$ in the above expression.

Take the transmission coefficient $T(\omega)$ and the reflection coefficient $R(\omega)$ to have a branch cut along the positive imaginary ω-axis. If we are on the right side of the branch cut (that is, we take $\mathrm{Re}\,\omega = \epsilon \to 0^+$ for ω on the positive imaginary axis), then $T(\omega)$ and $R(\omega)$ are defined through the asymptotic solutions

$$\psi \simeq \begin{cases} e^{i\omega r_*} + R(\omega) e^{-i\omega r_*} & r \to \bigstar \\ T(\omega) e^{i\omega r_*} & r \to r_H \end{cases} \tag{6.47}$$

since we are considering an incoming wave from infinity $\bigstar$ that is reflected and transmitted through the potential. Both of these regions ($r \to r_H, \bigstar$) are where the asymptotic solution (6.46) is valid.

We will also need an asymptotic form of ψ as $r \to 0$. In particular, we will need the leading order behaviors for both large and small ω of the solutions to the Schrödinger-like equation (6.9) for the specific potential (5.14). Obtaining this is a straightforward and physically unenlightening computation by expanding r_* in powers of r/r_H, and so we refer the reader to the appendix of REFERENCE and will take the following asymptotic forms as $r \to 0$ for granted:

$$\psi(r) \simeq \begin{cases} A(\omega r_*)^{1/2} & \omega \ll 1/r_* \\ B\cos(\omega r_* - \pi/4) & \omega \gg 1/r_* \end{cases} \qquad \text{as } r \to 0. \tag{6.48}$$

A, B denote arbitrary constants. Note that, to leading order, we are saying that the two linearly independent solutions to our second-order differential equaiton are equal (this is a special feature of the static Schwarzschild solution).

Now for our first analysis of the trip around γ. Macroscopically, notice that γ encloses a singularity $r = r_H$, around which both $r_*(r)$ and consequently $\psi(r)$ are multivalued. Consequently, the boundary condition (6.47) implies that $\psi(r)$ picks up a phase $e^{2\pi\omega r_H}$ on any path encircling r_H in the clockwise direction. Furthermore, $e^{-i\omega r_*}$ is multiplied by the phase $e^{-2\pi\omega r_H}$ on such a path. These two phases together imply that the coefficient of $e^{-i\omega r_*}$ must be multiplied by $e^{4\pi\omega r_H}$ after the trip around γ. Therefore, at the end of our trip around γ, the reflection coefficient R is multiplied by the monodromy $e^{4\pi\omega r_H}$ after a trip around γ — this is a fact of only the boundary condition at r_H and the form of $r_*(r)$. Note that in our context, $4\pi r_H$ is the inverse Hawking temperature denoted by $\beta = 1/T_H$, so the monodromy can be written $e^{\beta\omega}$.

Next, we will follow the contour γ beginning at ★ in more detail. From ★, we match ψ near the line $\operatorname{Re} r_* = 0$ traveling up towards a point slightly southwest of $r = 0$. Namely, if we write the solution ψ southwest of $r = 0$ as

$$\psi(r) = \alpha\left(e^{i(\omega r_* - \pi/4)} + e^{-i(\omega r_* - \pi/4)}\right)$$

in the limit $\omega \gg 1/r_*$ (cf. the asymptotics (6.48)), then visually matching this with the boundary conditions (6.47) we see that

$$\alpha(e^{-i\pi/4} + e^{i\pi/4}) \approx 1 + R(\omega). \tag{6.49}$$

Here $\approx$ means there are $\mathcal{O}(|\omega r_H|^{-1/2})$ corrections (this is negligible in the asymptotic frequency limit). Next we want to express $T(\omega)$ in terms of $A_\pm$. For this we first make a quarter-turn about $r = 0$ from southwest to southeast, then travel near the bubble $\operatorname{Re} r_* = 0$ until the $|\omega r_*| \gg 1$, and then run directly to $r = r_H$. Since on this trip $\operatorname{Re} r_* < 0$ and the ω^2 term dominates $V(r)$, the WKB (physical optics) approximation ensures that the coefficient of $e^{i\omega r_*}$ will not be modified as we travel to the horizon, to leading order in ω. (The main idea is that $e^{i\omega r_*}$ dominates the in the asymptotic form as $\omega \to i\infty$ when $\operatorname{Re} r_* < 0$, so any modification to its coefficient would cause those asymptotics to deviate from the physical optics approximation.) Therefore this coefficient at the horizon is determined simply by the asymptotics of ψ as we run southeast from $r = 0$, leading to

$$\alpha e^{i3\pi/4} \approx T(\omega). \tag{6.50}$$

Finally we want to complete our trip around the contour. So we could make a three-quarter turn from southwest to northwest, then matching coefficients again as we travel north until $|\omega x| \gg 1$. On the other hand, after doing so and following this coefficient around the big semicircle and back to $\bigstar$, we will have completed a full trip around γ. As discussed above, after the full round trip around γ the coefficient of $e^{-i\omega r_*}$ is $e^{\beta\omega}R(\omega)$, so we get

$$\alpha e^{i5\pi/4} \approx e^{\beta\omega}R(\omega) \tag{6.51}$$

The three relations (6.49)–(6.51) define an inhomogeneous linear system for the three unknowns: $\alpha, T(\omega), R(\omega)$. Solving it gives the final answer:

$$T(\omega) \approx \frac{e^{\beta\omega} - 1}{e^{\beta\omega} + 3}, \tag{6.52}$$

$$R(\omega) \approx \frac{2i}{e^{\beta\omega} + 3}. \tag{6.53}$$

Here the symbol $\approx$ means there can be $\mathcal{O}(|\omega r_H|^{-1/2})$ corrections both in the numerator and the denominator, which is justified in the asymptotic frequency limit. In particular, since the incident wave was normalized to unit amplitude, the transmission coefficient $T(\omega)$ given by (6.52) is also the greybody factor $\gamma(\omega)$.

This section ends our discussion of flat and asymptotically flat spacetimes. In the next section, we will begin examining anti-de Sitter spacetime.

7 Green functions and the AdS/CFT correspondence

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence is a powerful conjecture of the equivalence between string theories in AdS space and gauge theories. The benefits of this relationship run both ways: arduous computations in strongly coupled field theory dynamics can be tackled using weakly coupled gravity, and the problematic questions of quantum black holes and the black hole paradox can be addressed using field theory. In this section, we will examine the formulation of the correspondence in Euclidean spacetime, and how this adapts to Minkowski spacetime.

The language used to formulate the AdS/CFT correspondence is that of path-integral functionals. In section 7.1, we will see how to extract arbitrary correlation functions (or "correlators") — and hence the quantization of a field theory — from simply defining such a generating functional (based upon [2]). Section 7.2 then describes AdS/CFT both heuristically and formally for the well-understood case of Euclidean space (based upon [3]). The correspondence does not lift immediately to Minkowski space, and so section 7.3 examines both the issue and a proposed solution [9] for formulating AdS/CFT in Minkowski space. As was alluded to in section 3.2, a key concept here will be the selection of appropriate boundary conditions; this is a theme that we will see again in section 9.

7.1 Path-integral quantization

Our derivation of QFT in section 2 followed the path of canonical quantization, in the sense that we quantized the classical theory by enforcing the commutation relations (2.14). This, however, is not the only way to develop the theory; for example, we could have instead insisted the definitions of Green functions in terms of expectation values of the scalar field (see section 2.4), and from this obtained the rest of section 2. Here, we will explore the alternative method of Feynman's path-integral quantization, which will play a key role in the statement of the AdS/CFT correspondence.

Let ϕ denote a scalar field whose equation of motion is governed by an action $S = S[\phi]$ (e.g. the action (2.2)). The fundamental object here is the *generating functional* or *path-integral functional*, defined by

$$Z[J] = \langle \text{out}, 0|0, \text{in}\rangle = \int \mathscr{D}[\phi]\, \exp\left\{ iS[\phi] + i\int d^n x\, J(x)\phi(x)\right\}. \tag{7.1}$$

The quantity Z is the generating functional and gives the transition amplitude from the initial vacuum $|0, \text{in}\rangle$ to the final vacuum $|0, \text{out}\rangle$ in the presence of a source $J(x)$ which is producing particles. In (7.1), "$\mathscr{D}[\phi]$" denotes that the integral is taken over the space of solutions to the corresponding equation of motion. This is an infinite dimensional function space, and consequently the choice of measure for this integral is still not fully understood. However, in order to derive the QFT we only need to know how to manipulate the integral (7.1) as a formal expression, and the rules for this are well-defined. In application, there are many special cases (e.g. section (7.1.2)) for which the integral becomes finite dimensional and can be evaluated explicitly. As a side note, when $J(x) = 0$ in the definition (7.1), we have

$$Z[0] = \langle 0|0\rangle = \int \mathscr{D}[\phi]\, \exp\left\{iS[\phi]\right\}, \tag{7.2}$$

which is frequently normalized to unity.

7.1.1 Obtaining Green functions

To show how QFT can be derived from (7.1), we will demonstrate how to obtain arbitrary correlation functions through differentiation of the functional integral. Let $|\varphi, t\rangle$ denote a field configuration eigenstate at time t, so that $\phi(t, x)\,|\varphi, t\rangle = \varphi(x)\,|\varphi, t\rangle$. First, we will examine how to create a functional integral expression for the time-ordered expectation value from $|\varphi_a, t_a\rangle$ to $|\varphi_b, t_b\rangle$ of the product of a finite sequence of field operators. Recall the path integral from quantum mechanics, which has the property that

$$\langle \varphi_b, t_b|\varphi_a, t_a\rangle = \langle \varphi_b|\, e^{-iH(t_b - t_a)}\, |\varphi_a\rangle = \int \mathscr{D}[\phi]\, \exp\left(i\int_{t_a}^{t_b} \mathrm{d}t\, \mathscr{L}\right). \tag{7.3}$$

Here, H and $\mathscr{L}$ denote the Hamiltonian and Lagrangian associated with the equation of motion. Let T denote the time-ordering operator, θ the unit step function, and S_j the permutation group on j items. Then the expectation value of an arbitrary time-ordered product of field operators can be broken apart as

$$\begin{aligned}
&\langle\varphi_b, t_b|T(\phi(x_1)\dots\phi(x_j))\,|\varphi_a, t_a\rangle = \\
&\qquad = \sum_{\sigma\in S_j} \theta(t_{\sigma(1)} > \dots > t_{\sigma(j)})\,\langle\varphi_b, t_b|\,\phi(x_{\sigma(1)})\dots\phi(x_{\sigma(j)})\,|\varphi_a, t_a\rangle \\
&\qquad = \sum_{\sigma\in S_j} \theta(t_{\sigma(1)} > \dots > t_{\sigma(j)})\cdot \\
&\qquad\qquad \cdot\langle\varphi_b|\,e^{-iH(t_b-t_{\sigma(1)})}\phi(x_{\sigma(1)})e^{-iH(t_{\sigma(1)}-t_{\sigma(2)})}\dots\phi(x_{\sigma(j)})e^{-iH(t_{\sigma(j)}-t_a)}\,|\varphi_a\rangle\,.
\end{aligned}$$

We have removed the time dependence from the states and written them in terms of the evolution operators e^{-iHt}. Next, we will pairwise insert kets and bras so as to only multiply by the identity, and then pull out the consequent eigenvalues φ. Continuing with our calculation, we have

$$\begin{aligned}
&= \sum_{\sigma\in S_j} \theta(t_{\sigma(1)} > \dots > t_{\sigma(j)})\,\langle\varphi_b|\,e^{-iH(t_b-t_{\sigma(1)})}\phi(x_{\sigma(1)})\,\left|\varphi(x_{\sigma(1)})\right\rangle\cdot \\
&\qquad\cdot\left\langle\varphi(x_{\sigma(1)})\right|e^{-iH(t_{\sigma(1)}-t_{\sigma(2)})}\dots\phi(x_{\sigma(j)})\left|\varphi(x_{\sigma(j)})\right\rangle\left\langle\varphi(x_{\sigma(j)})\right|e^{-iH(t_{\sigma(j)}-t_a)}\,|\varphi_a\rangle \\
&= \sum_{\sigma\in S_j} \varphi(x_{\sigma(1)})\dots\varphi(x_{\sigma(j)})\,\theta(t_{\sigma(1)} > \dots > t_{\sigma(j)})\cdot \\
&\qquad\cdot\langle\varphi_b|\,e^{-iH(t_b-t_{\sigma(1)})}\left|\varphi(x_{\sigma(1)})\right\rangle\dots\left\langle\varphi(x_{\sigma(j)})\right|e^{-iH(t_{\sigma(j)}-t_a)}\,|\varphi_a\rangle\,.
\end{aligned}$$

Each of the j expectation values above can be expressed in terms of the path integral (7.3). The step function factor then ensures that the integration domains are in order and add constructively:

$$\begin{aligned}
&= \sum_{\sigma\in S_j} \varphi(x_{\sigma(1)})\dots\varphi(x_{\sigma(j)})\,\theta(t_{\sigma(1)} > \dots > t_{\sigma(j)}) \\
&\qquad\cdot\left[\int\mathscr{D}[\phi]\,\exp\left(i\int_{t_{\sigma(1)}}^{t_b} dt\,\mathscr{L}\right)\right]\cdot\dots\cdot\left[\int\mathscr{D}[\phi]\,\exp\left(i\int_{t_a}^{t_{\sigma(j)}} dt\,\mathscr{L}\right)\right] \\
&= \int_{\phi(t_a,x)=\varphi_a(x)}^{\phi(t_b,x)=\varphi_b(x)} \mathscr{D}[\phi]\,\phi(x_1)\dots\phi(x_j)e^{iS[\phi]}.
\end{aligned}$$

To recapitulate, we have just shown the relation

$$\langle\varphi_b, t_b|\,T(\phi(x_1)\dots\phi(x_j))\,|\varphi_a, t_a\rangle = \int_{\phi(t_a,x)=\varphi_a(x)}^{\phi(t_b,x)=\varphi_b(x)} \mathscr{D}[\phi]\,\phi(x_1)\dots\phi(x_j)e^{iS[\phi]}. \tag{7.4}$$

Although we carried this out for a finite time interval, we can take the length of the interval to infinity to get the corresponding vacuum expectation value:

$$\lim_{t\to(1-i\epsilon)\infty}\frac{\langle\varphi_b, t|\,T(\phi(x_1)\dots\phi(x_j))\,|\varphi_a, -t\rangle}{\langle\varphi_b, t|\varphi_a, -t\rangle} = \langle 0|\,T(\phi(x_1)\dots\phi(x_j))\,|0\rangle\,. \tag{7.5}$$

Together, (7.4) and (7.5) tells us how to express the vacuum expectation value necessary to define the Green functions of section 2.4 in terms of functional integrals. Now, we will demonstrate how differentiating the generating functional (7.1) yields the right-hand side of (7.4). For functional integrals like $Z[J]$, we define the functional derivative

$$\frac{\delta Z[J(x)]}{\delta J(y)} = \lim_{\epsilon \to 0} \frac{Z[J(x) + \epsilon\delta(x-y)] - Z[J(x)]}{\epsilon} \tag{7.6}$$

when the limit formally converges. Taking one derivative of the generating functional (7.1) yields

$$\begin{aligned}
\left(\frac{\delta Z}{\delta J(x_1)}\right)\bigg|_{J=0} &= \lim_{\epsilon\to 0} \frac{Z[J(x)+\epsilon\delta(x-x_1)] - Z[J(x)]}{\epsilon}\bigg|_{J=0} \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathscr{D}[\phi]\,\bigg\{\exp\left[iS[\phi(x)] + i\int d^n x\, J(x)\phi(x) + i\epsilon\phi(x_1)\right] \\
&\qquad\qquad - \exp\left[iS[\phi(x)] + i\int d^n x\, J(x)\phi(x)\right]\bigg\}\bigg|_{J=0} \\
&= \int \mathscr{D}[\phi]\,\exp\left\{iS[\phi(x)] + i\int d^n x\, J(x)\phi(x)\right\}\lim_{\epsilon\to 0}\left\{\frac{\exp\left[i\epsilon\phi(x_1)\right]-1}{\epsilon}\right\}\bigg|_{J=0} \\
&= \int \mathscr{D}[\phi]\,\exp\left\{iS[\phi(x)] + i\int d^n x\, J(x)\phi(x)\right\} i\phi(x_1)\bigg|_{J=0} \\
&= i\int \mathscr{D}[\phi]\,\exp\left[iS[\phi(x)]\right]\phi(x_1).
\end{aligned}$$

Using an inductive argument, we find

$$\left(\frac{\delta^j Z}{\delta J(x_1)\ldots\delta J(x_j)}\right)\bigg|_{J=0} = i^j \int \mathscr{D}[\phi]\,\phi(x_1)\ldots\phi(x_j)e^{iS[\phi]}. \tag{7.7}$$

Altogether, (7.4), (7.5), and (7.7) imply

$$\left(\frac{1}{Z[J]}\frac{\delta^j Z[J]}{\delta J(x_1)\ldots\delta J(x_j)}\right)\bigg|_{J=0} = i^j \langle 0|\, T(\phi(x_1)\ldots\phi(x_j))\,|0\rangle_c\,. \tag{7.8}$$

This allows us to define the time-ordered Green functions as in the beginning of section 2.4, and any other j-point correlation function we desire. The subscript c denotes that only connected Feynman diagrams are included in perturbation theory. Next, we will take the example of a free scalar field, and manipulate the generating functional (7.1) until it is finite dimensional, and directly from this form we will see that the formula (7.8) holds.

7.1.2 Scalar field functional integral manipulations

As an example, consider a free scalar field with the action

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^n x\,\left[\mathscr{L}_0(x) + \frac{i\epsilon}{2}\phi^2(x)\right], \qquad \mathscr{L}_0 = -\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - m^2\phi,$$

for a n-dimensional Minkowski spacetime. This is identical to the flat space QFT action (2.2), except we have added the infinitesimal factor proportional to ϵ. The choice of this factor is related to the boundary conditions on ϕ, and can be used to make the functional integral convergent. Integration by parts yields

$$\begin{aligned} S[\phi] &= \frac{1}{2}\int \mathrm{d}^n x \left[-\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - m^2\phi + \frac{i\epsilon}{2}\phi^2\right] \\ &= \frac{1}{2}\int \mathrm{d}^n x \left[\phi\eta^{\alpha\beta}\partial_\alpha\partial_\beta\phi - m^2\phi + \frac{i\epsilon}{2}\phi^2\right] = \frac{1}{2}\int \mathrm{d}^n x\, \phi(\Box - m^2 + i\epsilon)\phi. \end{aligned} \tag{7.9}$$

Using this, we may rewrite the exponent in the functional integral (7.1) as

$$\begin{aligned} S[\phi] + \int \mathrm{d}^n x\, J(x)\phi(x) &= \frac{1}{2}\int \mathrm{d}^n x \mathrm{d}^n y\, \phi(x)(\Box_x - m^2 + i\epsilon)\delta^n(x-y)\phi(y) + \int \mathrm{d}^n x\, J(x)\phi(x) \\ &= \frac{1}{2}\int \mathrm{d}^n x \mathrm{d}^n y\, \phi(x) K_{xy}\phi(y) + \int \mathrm{d}^n x\, J(x)\phi(x), \end{aligned} \tag{7.10}$$

for the symmetric operator

$$K_{xy} = (\Box_x - m^2 + i\epsilon)\delta^n(x-y). \tag{7.11}$$

A quick calculation shows that this operator has the properties

$$\int \mathrm{d}^n y\, K_{xy}^{1/2} K_{yz}^{1/2} = K_{xz}^{1/2}, \tag{7.12}$$

$$\int \mathrm{d}^n y\, K_{xy}^{1/2} K_{yz}^{-1/2} = \delta^n(x-z), \tag{7.13}$$

and from inverting the property of the Feynman propagator (2.45),

$$K_{xy}^{-1} = G_F(x,y). \tag{7.14}$$

In the expression (7.10) for the exponential of the functional, we can change the integration variable from ϕ to

$$\phi'(x) = -\int \mathrm{d}^n y\, K_{xy}^{1/2}\phi(y), \tag{7.15}$$

along with the identities (7.12)–(7.14), one can show that

$$\begin{aligned} &\frac{1}{2}\int \mathrm{d}^n x \mathrm{d}^n y\, \phi(x) K_{xy}\phi(y) + \int \mathrm{d}^n x\, J(x)\phi(x) \\ &\quad = -\frac{1}{2}\int \mathrm{d}^n x \left[\phi'(x) - \int \mathrm{d}^n y\, J(y) K_{yx}^{-1/2}\right]^2 + \frac{1}{2}\int \mathrm{d}^n x \mathrm{d}^n y\, J(x) G_F(x,y) J(y). \end{aligned} \tag{7.16}$$

When this last expression is returned to the exponent in the functional integral (7.1), the second term is independent of ϕ and can be pulled out, leaving an integral of Gaussian type which can be evaluated. Functional integrals are defined so that infinite dimensional analogues of certain identities still hold. In particular, multivariate integrals in terms of matrices that can be evaluated in terms of determinants, traces, and exponentiation are upheld for operators in functional integrals as well. Evaluating the Gaussian integral in this way yields

$$Z[J] \propto (\det K^{1/2})^{-1} \exp\left[-\frac{i}{2}\int \mathrm{d}^n x \mathrm{d}^n y\, J(x) G_F(x,y) J(y)\right] \tag{7.17}$$

The term

$$(\det K^{1/2})^{-1} = [\det(-G_F)]^{1/2} = \exp\left[\frac{1}{2}\operatorname{tr}\ln(-G_F)\right] \tag{7.18}$$

is the Jacobian from the change of variable (7.15). From (7.17), the consistency of the correlator generating formula (7.8) can be readily verified.

As a side note, the functional derivative of the exponent of the exponential term in (7.17) is in fact the classical field $\phi_c(z)$ created by the source $J(z)$:

$$\begin{aligned}\frac{\delta}{\delta J(z)}\frac{1}{2}&\int \mathrm{d}^n x \mathrm{d}^n y\, J(x) G_F(x,y) J(y) = \\ &= \frac{1}{2}\int \mathrm{d}^n x \mathrm{d}^n y\, \left[\delta(z-x) G_F(x,y) J(y) + J(x) G_F(x,y)\delta(z-y)\right] \\ &= \int \mathrm{d}^n x\, G_F(z,x) J(x).\end{aligned}$$

(Recall that the Feynman propagator satisfies the Green function differential equation (2.45).) In the next section, we will see how the path integral allows us to formulate the AdS/CFT correspondence.

7.2 AdS/CFT in Euclidean spacetime

We will now describe the idea of AdS/CFT in Euclidean space. The correspondence is a relation between a CFT in d dimensions and a gravity theory in $(d+1)$-dimensional AdS space. The particular case of interest is $d = 4$, and so we will consider the relation between the strongly coupled $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and classical (super)gravity on $\mathrm{AdS}_5 \times S^5$. The conjecture will ultimately be a statement of the form

$$Z_{\mathrm{CFT}}[\Phi_0] = Z_{\text{String Theory}}[\Phi \to \Phi_0]. \tag{7.19}$$

Here, Φ and Φ_0 denote fields, and $\Phi \to \Phi_0$ means that in the limit r is taken to the boundary of AdS_5, the field $\Phi(x,r)$ in string theory becomes $r^{\Delta}\Phi_0(x)$ for $\Phi_0(x)$ the field restricted to the boundary. The left-hand side denotes the path integral of the conformal field theory for the source Φ_0, and the right-hand side is the supergravity (or string theory) partition function computed by integrating over metrics that have a double pole near the boundary and induce the given field Φ_0 on the boundary. That is, the supergravity (or string) partition function $Z_{\text{String Theory}}$ can be computed as a function of the boundary values of massless scalar fields, and then can be interpreted as a generating functional of conformal field theory correlation functions for operators whose sources Φ_0 are the given boundary values.

String theory in the Poincaré patch of AdS space is related to $\mathcal{N} = 4$ SYM living the four-dimensional Minkowski space at the boundary. In section 3.2, we saw that the boundary of global AdS_5 space is $\mathbb{R}_t \times S^3$ (cf. figure 2b), and so this means string theory in global AdS_5 is related to gauge theory on the $\mathbb{R}_t \times S^3$ space at its boundary.

The Euclidean version of the metric of global $\mathrm{AdS}_5 \times S^5$ is

$$\mathrm{d}s^2 = \frac{R^2}{\cos^2\theta}\left(\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_3^2\right) + R^2\,\mathrm{d}\Omega_5^2 \tag{7.20}$$

(cf. the metric (3.16)). From section 3.2, we know that the boundary is at $\theta = \pi/2$ (cf. figure 2b), at which the metric (7.20) becomes

$$\mathrm{d}s^2 = \frac{R^2}{\cos^2\theta}\left(\mathrm{d}\tau^2 + \sin^2\theta\,\mathrm{d}\Omega_3^2\right) \tag{7.21}$$

in the limit $(\cos\theta)^{-2} \to \infty$. The Euclidean version of the metric for classical (super)gravity on $\mathrm{AdS}_5 \times S^5$ has the form

$$\mathrm{d}s^2 = \frac{R^2}{z^2}\left(\mathrm{d}\vec{x}^2 + \mathrm{d}z^2\right) + R^2\,\mathrm{d}\Omega_5^2. \tag{7.22}$$

As already mentioned, the four-dimensional CFT lives on the boundary of the AdS_5 space at $z = 0$, for which the metric becomes

$$\mathrm{d}s^2 = \frac{R^2}{z^2}\,\mathrm{d}\vec{x}^2 \tag{7.23}$$

in the limit $z \to 0$. Comparing the two boundary metrics (7.21) and (7.23), we see that they are conformally related via

$$\mathrm{d}\vec{x}^2 = \mathrm{d}x^2 + x^2\,\mathrm{d}\Omega_3^2 = x^2\left[(\mathrm{d}\ln x)^2 + \mathrm{d}\Omega_3^2\right] = x^2\left(\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2\right). \tag{7.24}$$

Therefore, string theory in global $\mathrm{AdS}_5 \times S^5$ is related to $\mathcal{N} = 4$ SYM on the boundary $\mathbb{R}_t \times S^3$ (which is conformally related to $\mathbb{R}^4$).

Suppose that a bulk field ϕ is coupled to an operator $\hat{\mathcal{O}}$ on the boundary in such a way that the interaction Lagrangian is $\phi\hat{\mathcal{O}}$. In this case, the AdS/CFT correspondence is formally stated as the equality

$$\left\langle e^{\int_{\partial M}\phi_0\hat{\mathcal{O}}}\right\rangle = e^{-S_{\mathrm{cl}}[\phi]}. \tag{7.25}$$

The left-hand side is the generating functional for correlators of $\hat{\mathcal{O}}$ in the boundary field theory, and the exponent on the right-hand side is the action of the classical solution to the equation of motion for ϕ in the bulk metric with the boundary condition $\phi|_{z=0} = \phi_0$. This is heuristically of the form (7.19). From this relation, we can obtain correlation functions as in section 7.1 — in particular, the Green function (2.48) — and for Euclidean space this is successful. For examples of this success, see [14, 15].

7.3 AdS/CFT in Minkowski spacetime

7.3.1 Difficulties with Minkowski AdS/CFT

With (7.25) in mind, we can try to formally write the Minkowski version of the AdS/CFT as

$$\left\langle e^{i\int_{\partial M}\phi_0\hat{\mathcal{O}}}\right\rangle = e^{iS_{\rm cl}[\phi]}, \tag{7.26}$$

but this immediately faces problems. In the Euclidean version, the classical solution ϕ is uniquely determined by its value ϕ_0 at the boundary $z = 0$ and the requirement of regularity at the horizon $z = z_H$. Consequently, the Euclidean correlator (2.48) obtained by using the correspondence is unique. Conversely, in Minkowski space, the requirement of regularity at the horizon is insufficient; to select a solution we need a more refined boundary condition. This is thought to reflect the multitude of real-time Green functions (cf. section 2.4) in finite-temperature field theory.

One boundary condition at the horizon is particularly physically appealing: the incoming-wave boundary condition, where waves can only travel to the region inside the horizon of the black branes, but cannot be emitted from there. We may suspect that this boundary condition corresponds to the retarded Green function, while the outgoing-wave boundary condition gives rise to the advanced Green function. However, even after fixing the boundary condition at the horizon, it is still problematic to get (7.26) to work.

We will now look at the issue of (7.26) explicitly. We start with the metric

$$\mathrm{d}s^2 = \frac{R^2}{z^2}\left(f(z)^2\,\mathrm{d}\tau^2 + \mathrm{d}\vec{x}^2 + \frac{\mathrm{d}z^2}{f(z)^2}\right) + R^2\,\mathrm{d}\Omega_5^2 \tag{7.27}$$

for nonzero-temperature field theory on $\mathrm{AdS}_5\times S^5$. Here, $f(z) = 1 - z^4/z_H^4$, $z_H = 1/\pi T_H$ is the horizon, and T_H denotes the Hawking temperature. The Euclidean time coordinate τ is periodic with period $\sim T^{-1}$, and z runs between 0 and z_H. Break apart the AdS part of this metric (7.27) as

$$\mathrm{d}s^2 = g_{zz}\,\mathrm{d}z^2 + g_{\mu\nu}(z)\,\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{7.28}$$

and consider a scalar field on this background. The action for this is

$$S = K\int \mathrm{d}^4x \int_{z_B}^{z_H}\mathrm{d}z\,\sqrt{g}\left[g^{zz}(\partial_z\phi)^2 + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] \tag{7.29}$$

where K is a normalization constant and m is the scalar mass. The limit of the z-integration is between the boundary z_B and the horizon z_H (for the metric (7.27), $z_B = 0$).

The field equation (2.25) is

$$\frac{1}{\sqrt{g}}\partial_z\left(\sqrt{g}g^{zz}\partial_z\phi\right) + g^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0. \tag{7.30}$$

This has to be solved with a fixed boundary condition at z_B. Taking the Fourier transform

$$f(z,k) = \int \mathrm{d}^4x\, e^{-ik\cdot x} f(z,x) \tag{7.31}$$

of the field equation (7.30), we get

$$0 = \frac{1}{\sqrt{g}}\partial_z\left(\sqrt{g}g^{zz}\partial_z\phi(z,k)\right) - g^{\mu\nu}k_\mu k_\nu\phi(z,k) - m^2\phi(z,k).$$

The solution is

$$\phi(z,x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{ik\cdot x} f_k(z)\phi_0(k), \tag{7.32}$$

where $\phi_0(k)$ is determined by the boundary condition

$$\phi(z_B,x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{ik\cdot x}\phi_0(k). \tag{7.33}$$

$f_k(z)$ denotes the solution to the mode equation

$$\frac{1}{\sqrt{g}}\partial_z\left(\sqrt{g}g^{zz}\partial_z f_k\right) - \left(g^{\mu\nu}k_\mu k_\nu + m^2\right) f_k = 0 \tag{7.34}$$

with unit boundary value at the boundary ($f_k(z_B) = 1$), and satisfying the incoming-wave boundary condition at $z = z_H$. Plugging the solution (7.32) into the action (7.29) and integrating, we find that the action on shell (i.e. for classical solutions) reduces to the surface terms

$$S = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \phi_0(-k)\mathcal{F}(k,z)\phi_0(k)\Big|_{z=z_B}^{z=z_H}, \tag{7.35}$$

where we define

$$\mathcal{F}(k,z) = K\sqrt{g}g^{zz} f_{-k}(z)\partial_z f_k(z). \tag{7.36}$$

Now, if we try to believe to correspondence (7.26), then functionally differentiating the action (7.35) twice would give us the retarded Green function

$$G(k) = -\mathcal{F}(k,z)\Big|_{z=z_B}^{z=z_H} - \mathcal{F}(-k,z)\Big|_{z=z_B}^{z=z_H}. \tag{7.37}$$

This cannot be, however, because we will show this quantity is completely real — the retarded Green function is in general complex. Looking at the mode equation (7.34), we see that $f_k^*(z) = f_{-k}(z)$ is also a solution. Using this, we can write the imaginary part of $\mathcal{F}(k,z)$ as

$$\operatorname{Im}\mathcal{F}(k,z) = \frac{1}{2i}\left[\mathcal{F}(k,z) - \mathcal{F}^*(k,z)\right] = \frac{K}{2i}\sqrt{g}g^{zz}\left[f_k^*(z)\partial_z f_k(z) - f_k(z)\partial_z f_k^*(z)\right] \tag{7.38}$$

This is proportional to quantum mechanical flux, and so it is conserved. In particular, $\partial_z \operatorname{Im}\mathcal{F} = 0$, and so the imaginary parts in (7.37) from $z = z_H$ and $z = z_B$ completely cancel out. Therefore, (7.37) cannot be the desired Green function. We could try to avoid this problem by throwing away the contribution from the horizon, keeping only the boundary term at $z = z_B$, but since $\mathcal{F}(-k,z) = \mathcal{F}^*(k,z)$ then the imaginary parts would still cancel each other. In the next section, we will present a solution to this problem.

7.3.2 The Minkowski prescription

The following solution has been conjectured in [9] for the retarded Green function:

$$G_R(k) = -2\mathcal{F}(k,z)\Big|_{z=z_B}, \tag{7.39}$$

on the grounds that it holds in all cases where G_R can be calculated by other means. Although this prescription resembles (7.37), it does not follow directly from an identity of the form (7.26). The steps to calculating the retarded Green function (7.39) are as follows:

1. Find a solution $f_k(z)$ to the mode equation (7.34) that satisfies:
   - The solution equals 1 at $z = z_B$;
   - For timelike momenta, the solution has an asymptotic expression corresponding to the incoming (outgoing) wave at the horizon. For spacelike momenta, the solution is regular at the horizon.
2. The retarded Green's function is then given by (7.39), with $\mathcal{F}$ defined in (7.36). Only the contribution from the boundary has to be taken — surface terms coming from the horizon or from the "infrared" part of the background geometry must be dropped. This part of the metric influences the correlators only through the boundary condition imposed on the bulk field ϕ.

Since the imaginary part of $\mathcal{F}(k,z)$ is still independent of the radial coordinate, then $\operatorname{Im} G^{R,A}$ can be computed by evaluating $\operatorname{Im}\mathcal{F}(k,z)$ at any convenient value of z (e.g. the horizon z_H). Next, we will demonstrate that this conjecture returns the correct answer for zero-temperature field theory.

7.3.3 Example: zero-temperature field theory

As an example, we use the conjecture (7.39) to compute the retarded two-point Green function of the composite operators $\hat{\mathcal{O}} = F^2/4$ at zero temperature. In this case the action (7.29) is that for a minimally coupled massless scalar on the background (3.1):

$$S = -\frac{N^2}{16\pi^2}\int \mathrm{d}^4x \int_{z_B}^{z_H} \mathrm{d}z\, \frac{1}{z^3}\left[(\partial_z\phi)^2 + \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \tag{7.40}$$

The horizon is at $z_H = \infty$ and the boundary at $z_B = \epsilon \to 0$. The equation of motion reads

$$0 = z^3\partial_z\left(\frac{1}{z^3}\partial_z\phi\right) + \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = \partial_z^2\phi - \frac{3}{z}\partial_z\phi + \eta^{\mu\nu}\partial_\mu\partial_\nu\phi,$$

which yields the mode equation

$$0 = f_k''(z) - \frac{3}{z}f_k'(z) - k^2 f_k(z). \tag{7.41}$$

For spacelike momenta ($k^2 > 0$), we have the general solution

$$g_k(z) = Az^2 I_2(kz) + Bz^2 K_2(kz)$$

for I, K the modified Bessel functions. The solution regular at $z = \infty$ and equal to 1 at $z = \epsilon$ is given by

$$f_k(z) = \frac{z^2 K_2(kz)}{\epsilon^2 K_2(k\epsilon)}.$$

Computing the prescription (7.39) via (7.36), we get

$$G_R(k) = \frac{N^2 k^4}{64\pi^2} \ln k^2, \quad k^2 > 0. \tag{7.42}$$

For timelike momenta, we define $q = \sqrt{-k^2}$. The linear combination (7.3.3) that satisfies the boundary conditions is

$$f_k(z) = \begin{cases} \dfrac{z^2 H_2^{(1)}(qz)}{\epsilon^2 H_2^{(1)}(q\epsilon)} & \omega > 0 \\ \dfrac{z^2 H_2^{(2)}(qz)}{\epsilon^2 H_2^{(2)}(q\epsilon)} & \omega < 0 \end{cases} \tag{7.43}$$

Notice that $f_{-k}(z) = f_k^*(z)$. Computing the prescription (7.39) via (7.36), we get

$$G_R(k) = \frac{N^2 k^4}{64\pi^2} \left(\ln k^2 - i\pi \operatorname{sgn} \omega\right), \quad k^2 < 0. \tag{7.44}$$

Combining the retarded Green functions (7.42) and (7.44), we obtain the general form

$$G_R(k) = \frac{N^2 k^4}{64\pi^2} \left(\ln |k^2| - i\pi\theta(-k^2) \operatorname{sgn} \omega\right). \tag{7.45}$$

where θ is the unit step function. To check our answer, we obtain the Feynman propagator from the zero-temperature relation

$$G_F(k) = \operatorname{Re} G_R(k) + i \operatorname{sgn}(\omega) \operatorname{Im} G_R(k)$$

which we will take granted. This yields

$$G_F(k) = \frac{N^2 k^4}{64\pi^2} \left(\ln |k^2| - i\pi\theta(-k^2)\right). \tag{7.46}$$

We could separately calculate the Euclidean Green function from (2.48) to find

$$G_E(k_E) = -\frac{N^2 k_E^4}{64\pi^2} \ln k_E^2, \tag{7.47}$$

and then obtain (7.46) by a Wick rotation (see section 2.4 for details). Thus, the prescription reproduces the correct answer for the retarded Green's function at zero temperature.

This concludes our discussion of the AdS/CFT correspondence. We will now continue to examine AdS spacetime through the Unruh effect and Hawking radiation.

8 Unruh effect in anti-de Sitter spacetime

Our presentation of the Unruh effect [5] in flat Minkowski spacetime in section 4 can be readily generalized to any dimension. That is to say, an observer with uniform acceleration a within flat Minkowski spacetime of any dimension measures a thermal distribution of particles corresponding to the temperature

$$T_U = \frac{a}{2\pi}. \tag{8.1}$$

We now would like to find the temperature effects that are caused by acceleration with respect to an inertial observer in AdS spacetime. However, the result (8.1) does not hold in general for a generic path in curved or even flat space. In this section, we will present the global embedding in Minkowski spacetime (GEMS) method in order to obtain an effective temperature for some special orbits. Although effective, it is not all-powerful, in that it fails in some situations when a temperature can be defined by another means (e.g. the surface gravity method; see [10] for details).

As suggested by the name, the idea behind GEMS is to make use of the fact that any curved Lorentzian spacetime can be embedded in some higher-dimensional flat Minkowski space. Any particle motion then in the curved space can be equivalently described by considering motion of the particle in the higher-dimensional flat Minkowski space. For the examination of the Unruh effect, if it happens that the higher-dimensional orbit is a time-like orbit of constant higher-dimensional acceleration, then the Unruh result (8.1) can be applied to define an effective temperature.

The main purpose of this section is the warning contained in section 8.1. Here, we will see by example of an incorrect application of the GEMS method, and consequently that acceleration in AdS is quite different; a stationary observer in AdS can be shown via GEMS to have zero acceleration and temperature with respect to one embedding, while another choice of embedding yields nonzero acceleration and temperature. Section 8.2 then considers the case of uniform acceleration in AdS to demonstrate an example of the Unruh effect shown via the GEMS method. The analysis of this section is adapted from [10].

8.1 Static observers

We will consider a static observer in an empty five-dimensional anti-de Sitter space (AdS_5), and compute their measured temperature via two different GEMS embeddings. Recall from section 3.2 that AdS_5 can be embedded into six-dimensional flat space $\mathbb{R}^{2,4}$ as the hyperboloid (3.10). From the coordinate definitions (3.12) and (3.14), we found the Poincaré and global coordinate metrics,

$$\mathrm{d}s^2 = R^2 \left[u^2 \left(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \right) + \frac{\mathrm{d}u^2}{u^2} \right] \tag{8.2}$$

$$= R^2 \left(-\cosh^2\rho \, \mathrm{d}\tilde{t}^2 + \mathrm{d}\rho^2 + \sinh^2\rho \, \mathrm{d}\Omega_3^2 \right). \tag{8.3}$$

(In the latter, we have replaced τ with $\tilde{t}$ so that there is no confusion later when we introduce the proper time τ.) The coordinates (3.12) and (3.14) are GEMS embeddings.

First, we will examine the scenario from Poincaré coordinates. Taking all of the x_i coordinates along with $u = u_0$ to be constant, we will calculate the six-dimensional velocity and acceleration in the embedding space. The embedding (3.12) tells us that the first five coordinates $X_0, X_1, \ldots, X_4$ of X^μ are all constant, while X_5 is linear in t. To affinely parameterize this trajectory, we can take $X_5(\lambda) = \lambda$ and let g denote the metric (3.11) on $\mathbb{R}^{2,4}$. The six-"velocity" for this choice is simply $v(\lambda) = (0, 0, 0, 0, 0, 1)$, and so we find the proper time via the anti-derivative

$$\tau = \int \mathrm{d}\lambda \sqrt{-g(v(\lambda), v(\lambda))} = \int \mathrm{d}\lambda = \lambda.$$

That is, our λ was already an affine parameter. Therefore, we have $X_5(\tau) = \tau$, and so the velocity and acceleration are

$$v^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \frac{1}{Ru_0}\frac{dX^\mu}{dt} = \frac{1}{Ru_0}\left(-\frac{u_0^2 t}{u_0}, 0, 0, 0, -\frac{u_0^2 t}{u_0}, Ru_0\right) = \left(-\frac{t}{R}, 0, 0, 0, -\frac{t}{R}, 1\right), \quad (8.4)$$

$$a^\mu = \frac{\mathrm{d}^2 X^\mu}{\mathrm{d}\tau^2} = \frac{1}{Ru_0}\frac{\mathrm{d}v^\mu}{\mathrm{d}t} = \left(-\frac{1}{R^2 u_0}, 0, 0, 0, -\frac{1}{R^2 u_0}, 0\right). \quad (8.5)$$

Contracting with the metric $g_{\mu\nu}$ on $\mathbb{R}^{2,4}$, we find that the norm-squared of the acceleration vector vanishes:

$$a^2 = g_{\mu\nu} a^\mu a^\nu = -\left(-\frac{1}{R^2 u_0}\right)^2 + \left(-\frac{1}{R^2 u_0}\right)^2 = 0, \quad (8.6)$$

as one would expect for a stationary observer. Therefore, applying the general flat-spacetime Unruh result (8.1) we have that the temperature measured by this observer is zero.

Now we will repeat the same calculation in global coordinates. Let $\rho = \rho_0$ and the angles θ, ϕ_1, ϕ_2 all be constant. The embedding (3.14) tells us that X_1 through X_4 are constant, while X_0 and X_5 satisfy

$$X_0^2 + X_5^2 = R^2 \cosh^2 \rho_0. \quad (8.7)$$

Choosing a candidate parameterization

$$X_0(\lambda) = R \cosh \rho_0 \cos \lambda, \qquad X_5(\lambda) = R \cosh \rho_0 \sin \lambda,$$

we have the six-"velocity" $v(\lambda) = R\cosh\rho_0(-\sin\lambda, 0, 0, 0, 0, \cos\lambda)$. Now we can find the proper time via

$$\tau = \int \mathrm{d}\lambda \sqrt{-g(v(\lambda), v(\lambda))} = R\cosh\rho_0 \int \mathrm{d}\lambda \sqrt{\sin^2\lambda + R^2\cosh^2\rho_0\cos^2\lambda} = R\cosh\rho_0 \lambda.$$

So now we have $\tilde{t} = \lambda = \tau / R\cosh\rho_0$, and we can compute the velocity and acceleration:

$$v^\mu = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \left(-\sin\left(\frac{\tau}{R\cosh\rho_0}\right), 0, 0, 0, 0, \cos\left(\frac{\tau}{R\cosh\rho_0}\right)\right) = \left(-\sin\tilde{t}, 0, 0, 0, 0, \cos\tilde{t}\right), \quad (8.8)$$

$$a^\mu = \frac{\mathrm{d}^2 X^\mu}{\mathrm{d}\tau^2} = \left(-\frac{\cos(\tau/R\cosh\rho_0)}{R\cosh\rho_0}, 0, 0, 0, 0, -\frac{\sin(\tau/R\cosh\rho_0)}{R\cosh\rho_0}\right)$$
$$= \left(-\frac{\cos\tilde{t}}{R\cosh\rho_0}, 0, 0, 0, 0, -\frac{\sin\tilde{t}}{R\cosh\rho_0}\right). \tag{8.9}$$

This time, however, we find the acceleration

$$a^2 = g_{\mu\nu}a^\mu a^\nu = -\left(-\frac{\cos\tilde{t}}{R\cosh\rho_0}\right)^2 - \left(-\frac{\sin\tilde{t}}{R\cosh\rho_0}\right)^2 = -\frac{1}{R^2\cosh^2\rho_0}$$

While the Unruh relation (8.1) correctly yields the result that the temperature of a static observer in Poincaré coordinates is zero, its naive application here leads to a non-zero (and imaginary!) temperature in global coordinates. What happens is that for $a^2 < 0$, the trajectory of the particle in the GEMS cannot be associated to a hyperbolic motion — in fact, the worldline (8.7) describes a closed curve — and therefore this trajectory does not describe constant acceleration in the larger space. The GEMS method does not apply in such cases. On a side note, because AdS is globally non-hyperbolic (cf. section 3.2.2), the measured temperature will generically depend on the boundary conditions at infinity. In the context of the AdS/CFT correspondence in section 7, the appropriate choice is reflective boundary conditions.

8.2 Accelerated observers

Next, let us consider accelerated observers in empty AdS_5 spacetime. As an example, we will apply the GEMS method to observers which move such that the norm of the four-acceleration a_4 in the directions parallel to the boundary of AdS is constant. Explicitly, the trajectory in Poincaré coordinates is

$$x_1^2 - t^2 = \frac{1}{a_4^2}, \qquad u = u_0 = \text{constant}, \qquad x_2 = x_3 = 0. \tag{8.10}$$

Using the same process as in section 8.1 to calculate the norm of the six-acceleration, one could use the Poincaré metric (8.2) verify the proper five-acceleration a_5 for this trajectory (8.10) is indeed constant.

Let us now attempt the GEMS computation of the temperature. we first determine the velocity and acceleration in the embedding space. Referring to the embedding (3.12), we see that the trajectory (8.10) requires X_0 and X_2 through X_4 to be constant, while

$$X_1 = Ru_0x_1, \qquad X_5 = Ru_0t.$$

Together, this yields the worldlines

$$X_1^2 - X_5^2 = R^2u_0^2(x_1^2 - t^2) = \frac{R^2u_0^2}{a_4^2}, \tag{8.11}$$

which we can choose to parameterize as

$$X_1(\lambda) = \frac{Ru_0}{a_4}\cosh\lambda, \qquad X_5(\lambda) = \frac{Ru_0}{a_4}\sinh\lambda.$$

The six-"velocity" of this choice is then $v(\lambda) = Ru_0 a_4^{-1}(0, \sinh\lambda, 0, 0, 0, \cosh\lambda)$, and so the proper time for our choice of parameterization is

$$\tau = \int \mathrm{d}\lambda \sqrt{-g(v(\lambda), v(\lambda))} = \frac{Ru_0}{a_4} \int \mathrm{d}\lambda \sqrt{-\sinh^2\lambda + \cosh^2\lambda} = \frac{Ru_0}{a_4}\lambda.$$

Now we can find the actual six-velocity and -acceleration for the accelerating observer (8.10) within the larger embedding space:

$$\begin{aligned} v^\mu &= \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \left(0, \sinh\left(\frac{a_4}{Ru_0}\tau\right), 0, 0, 0, \cosh\left(\frac{a_4}{Ru_0}\tau\right)\right) \\ &= (0, a_4 t, 0, 0, 0, a_4 x_1) = \left(0, a_4 t, 0, 0, 0, a_4\sqrt{a_4^{-2} + t^2}\right), \\ a^\mu &= \frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} = \left(0, \frac{a_4}{Ru_0}\cosh\left(\frac{a_4}{Ru_0}\tau\right), 0, 0, 0, \frac{a_4}{Ru_0}\sinh\left(\frac{a_4}{Ru_0}\tau\right)\right) \\ &= \left(0, \frac{a_4^2}{Ru_0}\sqrt{a_4^{-2} + t^2}, 0, 0, 0, \frac{a_4^2}{Ru_0}t\right) \end{aligned} \tag{8.12}$$

The norm-squared of the six-acceleration in the embedding space is therefore

$$a_6^2 = g_{\mu\nu}a^\mu a^\nu = \frac{a_4^4}{R^2u_0^2}(a_4^{-2} + t^2) - \frac{a_4^4}{R^2u_0^2}t^2 = \frac{a_4^2}{R^2u_0^2}. \tag{8.13}$$

The flat space Unruh result (8.1) then yields the observed temperature

$$T = \frac{a_4}{2\pi Ru_0}. \tag{8.14}$$

Note that, although AdS spacetime is homogeneous and isotropic, the temperature is dependent on the coordinate u.

We have just showed an example for which the Unruh effect carries over to AdS spacetime. In the next section, we will examine the behavior of Hawking radiation in AdS.

9 Hawking radiation for asymptotically anti-de Sitter spacetimes

This final section is dedicated to the investigation of Hawking radiation for asymptotically AdS spacetimes. Although there are other regimes from which we could analyze the radiation, such as an eternal black hole source or as a tunneling process, we will again assume a spherically symmetric collapsing mass (and hence an AdS Schwarzschild black hole) as in section 5 (for more on the other methods, see [11]). The final mathematical result will ultimately be the same — that the black hole emits particles with a thermal spectrum corresponding to the Hawking temperature. However, section 3.2 told us that since this space is not globally hyperbolic, we will need the addition of reflective boundary conditions at infinity in order to use QFT.

After we introduce the preliminaries in 9.1, we will need to discuss the appropriate definition of past vacuum to be compared with the future vacuum and the reflective boundary conditions. This question will be motivated and answered by section 9.2, after which in section 9.3 we will be free to compare the two sets of mode solutions within the geometric optics approximation. Our analysis in this section follows that of [11].

9.1 The set-up

One method to derive Hawking radiation is to use a Bogoliubov transformation between two bases of annihilation and creation operators corresponding to mode expansions of the field operator in two coordinate systems describing the black hole: the asymptotic coordinates and the Kruskal coordinates. In the asymptotic coordinates, the AdS_{d+1} black hole metric is

$$\mathrm{d}s^2 = -F(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{F(r)} + r^2\,\mathrm{d}\Omega_{d-1}^2, \qquad F(r) = 1 - \frac{\mu}{r^{d-2}} + r^2, \tag{9.1}$$

with units so that the AdS radius of curvature is $R = 1$. As in section 5, the parameter μ is proportional to the ADM mass M if the black hole

$$M = \frac{(d-1)\Omega_{d-1}}{16\pi G_{d+1}}\mu \tag{9.2}$$

where G_{d+1} is Newton's constant in $d+1$ dimensions and Ω_{d-1} is the volume of a unit $(d-1)$-sphere. The horizon radius r_H is defined as the largest root of $F(r) = 0$. In five dimensions ($d = 4$), the explicit formula is

$$r_H = \sqrt{\frac{1}{2}\left(\sqrt{1+4\mu} - 1\right)}. \tag{9.3}$$

We still define the Hawking temperature

$$T_H = \frac{1}{4\pi}F'(r_H) \tag{9.4}$$

as before, and we aim to show that this is still the temperature of the black hole's thermal distribution. In five dimensions, we have

$$T_H = \frac{\sqrt{1+4\mu}}{2\pi r_H}. \tag{9.5}$$

Now, we will make the appropriate coordinate changes in order to simplify the metric (and hence the equation of motion (2.25)). First, we want to define the tortoise coordinate,

$$r_* = \int_{r_H}^{r} \frac{\mathrm{d}r'}{F(r')}, \tag{9.6}$$

for our new radial function F. In five dimensions, this integrates out to

$$r_* = \frac{1}{\sqrt{1+4\mu}}\left[r_0 \arctan\left(\frac{r}{r_0}\right) + \frac{1}{2}r_H \ln\left(\frac{r-r_H}{r+r_H}\right)\right], \tag{9.7}$$

where we have introduce the short hand

$$r_0 = \sqrt{\frac{1}{2}\left(\sqrt{1+4\mu}+1\right)}. \tag{9.8}$$

Notice that in (9.7), near the horizon r_H we have the behavior $\ln(r - r_H)$, as we did for the four-dimensional flat Schwarzschild black hole. We also will make use of the null coordinates

$$u = t - r_*, \qquad v = t + r_*, \tag{9.9}$$

which put the metric (9.1) in the form

$$\mathrm{d}s^2 = -F(r)\,\mathrm{d}u\mathrm{d}v + r^2\,\mathrm{d}\Omega_{d-1}^2. \tag{9.10}$$

Together, the null coordinates u, v cover only the region outside the past and future horizons. However, we can regain the entire spacetime by introducing the Kruskal coordinates,

$$U = -\exp(-2\pi T_H u), \qquad V = \exp(2\pi T_H v). \tag{9.11}$$

In these coordinates the metric becomes

$$\mathrm{d}s^2 = -\frac{F(r)\exp(-4\pi T_H r_*)}{(2\pi T_H)^2}\,\mathrm{d}U\mathrm{d}V + r^2\,\mathrm{d}\Omega_{d-1}^2, \tag{9.12}$$

and for the special case of five dimensions,

$$\mathrm{d}s^2 = -\frac{r_H^2}{1+4\mu}\left(1+\frac{r_0^2}{r^2}\right)(r_H+r)^2\exp\left[-\frac{2r_0}{r_H}\arctan\left(\frac{r}{r_0}\right)\right]\mathrm{d}U\mathrm{d}V + r^2\,\mathrm{d}\Omega_3^2. \tag{9.13}$$

On the range of u, v, the Kruskal coordinates are both single-signed. However, by analytic continuation they can be extended to cover the whole spacetime, except for the true curvature singularity at $r = 0$.

Much like we did in section 5, from here we could focus on s-waves, adopt the geometric optics approximation, and truncate to two dimensions. Note that, unlike our flat-space analysis, we have not made the assumption yet of a collapsing mass that is spherically symmetric throughout time. Before doing so, we must pause and take a moment to consider appropriate boundary conditions.

9.2 Boundary conditions

When we compare sets of orthonormal modes, what do we choose for the past vacuum? For a quantum field in the black hole background, there are two canonical choices for a natural vacuum state:

- Unruh vacuum: If one wants to mimic a situation where the black hole is created by collapsing matter, one requires the field to be in a vacuum corresponding absence of positive energy modes in the U and v coordinates near the past horizon $V = 0$. This boundary condition refers only to the past of the asymptotic region of spacetime .

- Hartle-Hawking vacuum: This refers to a mixture of boundary conditions in the past and future horizons. Here, one requires the absence of positive energy U modes near $V = 0$, and the absence of positive energy V modes near $U = 0$. Physically, this mimics a black hole in thermal equilibrium with an external heat bath.

The vacuum we choose will have to be compared Boulware vacuum which is a natural vacuum for a fiducial observer in the asymptotic region, and corresponds to absence of positive energy u and v modes. A justified analysis will also need to take into account the reflective boundary condition at the boundary of the AdS space.

Consider mode solutions to the wave equation for a free scalar field in the AdS black hole spacetime. The solutions are either non-normalizable $\phi^{(-)}_{\omega,k}$ or are normalizable $\phi^{(+)}_{\omega,k}$. In the limit $r \to \infty$ one can check from the metric (9.1) that the radial dependence of these are given asymptotically by

$$\phi^{(\pm)}_{\omega,k}(t,r,\Omega) \simeq r^{2h_\pm}\tilde{\phi}^{(\pm)}_{\omega,k}(t,\Omega), \quad r \to \infty, \tag{9.14}$$

for the parameters

$$2h_\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4m^2}. \tag{9.15}$$

The quantized field ϕ is expanded as a linear combination of the normalizable modes, and these modes have decay behavior at the boundary corresponding to reflection. Specializing briefly to the three-dimensional ($d = 2$) case, it is straightforward to find the exact mode solutions $\phi^{(+)}$ can be found in terms of hypergeometric functions. Near the black hole horizon, the normalizable modes reduce to an asymptotic form

$$\phi^{(+)} \simeq \left(e^{-i\omega u} + e^{-i\omega v + i2\theta_0}\right) e^{-in\varphi}, \quad r \to r_H, \tag{9.16}$$

where θ_0 is a phase shift factor. In the geometric optics approximation (see 5.3 for details), tracing these modes through time means taking the null coordinates u, v to be constant. Therefore, the modes which take into account reflection from the boundary and are appropriate to a fiducial observer are a linear combination of the positive energy u and v modes,

$$\phi_\omega = e^{-i\omega u} + e^{-i\omega v + i2\theta_0}. \tag{9.17}$$

Since we also need reflection at infinity, we see from our two choices for past vacua that the correct choice is the *Hartle-Hawking vacuum*. This way, a fiducial observer sees a thermal spectrum for both infalling and outgoing modes. For an eternal AdS spacetime, the Unruh vaccum is not well-defined with respect to the boundary condition at infinity.

9.3 Particle creation by a collapsing spherical shell

As was assumed for the derivation in section 5, we will now consider particle creation by a collapsing spherical body in AdS_{d+1}. In the infinite past, we assume that the mass is so dilute that we can consider our quantum field is in a vacuum constructed with respect to the global coordinates (3.16),

$$\mathrm{d}s^2 = \sec^2\rho\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2\right) + \tan^2\rho\,\mathrm{d}\Omega_{d-1}^2. \tag{9.18}$$

Noting that

$$\sqrt{g} = \sqrt{|\det g|} = \sqrt{\sec^4\rho\tan^2\rho\cdot(\Omega\text{ dependence})} = \sec^2\rho\tan\rho\cdot(\Omega\text{ dependence}),$$

we can use separation of variables to find the normalizable mode solutions to the corresponding Klein-Gordon equation (2.25):

$$\phi_{n,l}^{(+)} = e^{-i\omega t}Y_{l,\{m\}}(\Omega)(\cos\rho)^{2h_+}(\sin\rho)^l P_n^{(l+d/2-1,\,2h_+-d/2)}(\cos 2\rho). \tag{9.19}$$

Recall that we have defined the shorthand h_+ in (9.15), $Y_{l,\{m\}}(\Omega)$ denotes the spherical harmonics on the $(d-1)$-dimensional sphere (see section 5.2 for details), and $P_n^{(l+d/2-1,\,2h_+-d/2)}$ denotes a Jacobi polynomial. The reflective boundary conditions at the origin and the boundary yield a discrete spectrum

$$\omega_n = 2h_+ + 2n + 2l \quad n = 0, 1, 2, \ldots. \tag{9.20}$$

For the rest of the section, will focus on s-waves and set $l = 0$. As is the case in flat space, the quantum should experience a strong redshift as it propagates across the collapsing body. Therefore, in the remote past we are interested in the high-frequency limit

$$\omega \to \infty, \qquad n \to \infty, \tag{9.21}$$

for which the mode solutions (9.19) become

$$\phi_n^{(+)} \simeq e^{-i\omega_n t}(\cos\rho)^{(d-1)/2}\cos\left[\omega_n\rho - \frac{d\pi}{4}\right] \tag{9.22}$$

up to normalization factors. This is a standing wave — a superposition of an ingoing and outgoing spherical waves with a discrete spectrum. Switching from ρ to the radial coordinate r, the factor $(\cos\rho)^{(d-1)/2} \sim (1/r)^{(d-1)/2}$ as $r \gg 1$, so we recover the expected overall decay factor for the amplitude.

Now, we will add the collapsing body and try to compute the consequent redshift of the wave (9.22) as it passes across it. We will use the geometric optics approximation and truncate to two dimensions in the (t, r)-plane. Assume that the collapsing body is a thin shell of radius R which strictly decreases in time. The truncated metric inside and outside the shell is of the form

$$\mathrm{d}s^2 = -F_\pm(r)\,\mathrm{d}t_\pm^2 + \left[F_\pm(r)\right]^{-1}\mathrm{d}r^2, \tag{9.23}$$

where

$$F_+(r) = 1 - \frac{\mu}{r^{d-2}} + r^2, \qquad F_-(r) = 1 + r^2.$$

That is, F_- describes empty AdS$_{d+1}$ space and F_+ describes the $(d+1)$-dimensional Schwarzschild AdS black hole. Define the tortoise coordinates $r_*^\pm$ by

$$r_*^\pm = \int \frac{\mathrm{d}r'}{F_\pm(r')}. \tag{9.24}$$

Since we are still interested in the trajectories of light, then we will also find it convenient to use the null coordinates

$$\begin{aligned} u &= t_+ - r_*^+ \qquad & v &= t_+ + r_*^+ \\ U &= t_- - r_*^- \qquad & V &= t_- + r_*^- \end{aligned} \tag{9.25}$$

These make both the interior and exterior metrics conformal to flat space. Note that in the interior we have $r_*^- = \arctan r$, and so $r = 0$ corresponds to $r_*^- = 0$ and $V - U = 0$. In the exterior, we have $r_*^- = \rho$.

We would like to determine how the interior and exterior null coordinates are related, via the to-be-determined functions

$$v = \beta(V), \qquad U = \alpha(u). \tag{9.26}$$

In the (t, r)-coordinates, the passage of the wave across the origin turns into reflection at $r = 0$, or

$$v = \beta(V) = \beta(U) = \beta(\alpha(u)). \tag{9.27}$$

So in the asymptotic region (near the boundary), the waves have a phase structure

$$\tilde{\phi}^{(+)} \sim e^{-i\omega_n v} - e^{-i\omega_n \beta(\alpha(u))}. \tag{9.28}$$

To find α, β, we match the interior and exterior metrics across the collapsing shell at $r = R(\tau)$. Here τ denote the shell time, which is related to the time coordinates $t_\pm$ in the two regions via

$$\mathrm{d}s^2 = \left[-F_\pm \,\mathrm{d}t_\pm^2 + \frac{\mathrm{d}r^2}{F_\pm} \right]\Bigg|_{r=R(\tau)} = -\mathrm{d}\tau^2. \tag{9.29}$$

Dividing through by $\mathrm{d}\tau^2$ tells us that

$$-1 = -F_\pm(R)\dot{t}_\pm^2 + \frac{1}{F_\pm(R)}\dot{R}^2 \tag{9.30}$$

at the shell $r = R(\tau)$, where $\cdot$ denotes $d/d\tau$. This relation allows us to calculate the derivatives entirely in terms of $\dot{R}$ and $F_\pm(R)$:

$$\alpha'(u) = \frac{dU}{du} = \frac{\dot{U}}{\dot{u}} = \frac{F_+(R)\left[\sqrt{F_-(R) + \dot{R}^2} - \dot{R}\right]}{F_-(R)\left[\sqrt{F_+(R) + \dot{R}^2} - \dot{R}\right]} \tag{9.31}$$

$$\beta'(V) = \frac{dv}{dV} = \frac{\dot{v}}{\dot{V}} = \frac{F_-(R)\left[\sqrt{F_+(R) + \dot{R}^2} + \dot{R}\right]}{F_+(R)\left[\sqrt{F_-(R) + \dot{R}^2} + \dot{R}\right]}. \tag{9.32}$$

As the radius $R \to r_H$ approaches the horizon, we approximate

$$F_+(R) = F_+(r_H) + F'_+(r_H)(R - r_H) + \mathcal{O}((R - r_H)^2) = 4\pi T_H(R - r_H) + \mathcal{O}((R - r_H)^2),$$
$$F_-(R) = F_-(r_H) + \mathcal{O}(R - r_H) \equiv A_H + \mathcal{O}(R - r_H). \tag{9.33}$$

From (9.31), this yields the approximation

$$\frac{dU}{du} \simeq -\frac{2\pi T_H(R - r_H)\left[\sqrt{A_H + \dot{R}^2} - \dot{R}\right]}{A_H\dot{R}} \tag{9.34}$$

Note that we must take $\sqrt{\dot{R}^2} = -\dot{R}$ and $\dot{R} \neq 0$, since the spherical shell is collapsing. In the above, $\dot{R}$ denotes $\dot{R}|_{r_H}$. After Taylor expanding U:

$$U = U(r_H) + \left.\frac{dU}{dR}\right|_{R=r_H}(R - r_H) + \mathcal{O}((R - r_H)^2),$$

and using (9.34):

$$\left.\frac{dU}{dR}\right|_{R=r_H} = \left.\frac{du}{dR}\frac{dU}{du}\right|_{R=r_H} = -\left.\frac{1}{F_+(R)}\frac{dU}{du}\right|_{R=r_H} \simeq \frac{\sqrt{A_H + \dot{R}^2} - \dot{R}}{2A_H\dot{R}},$$

we can flip the Taylor expansion to find

$$R - r_H \approx \frac{(U - U(r_H))}{dU/dR|_{R=r_H}} \simeq (U - U(r_H))\frac{A_H\dot{R}}{\sqrt{A_H + \dot{R}^2} - \dot{R}}.$$

Putting this back into (9.34) yields

$$\frac{dU}{du} \approx -2\pi T_H(U - U(r_H)) = -k_H U + \text{constant} \tag{9.35}$$

where k_H is the surface gravity of the black hole. Integrating yields

$$U = \alpha(u) = e^{-k_H u} + \text{constant}. \tag{9.36}$$

The same exact process for dv/dV yields

$$\frac{dv}{dV} \approx c \quad \text{(constant)}, \tag{9.37}$$

and hence

$$v = \beta(V) = cV + \text{constant}. \tag{9.38}$$

So in the asymptotic region, the wave phase (9.28) becomes

$$\tilde{\phi}^{(+)} \simeq e^{-i\omega_n v} - e^{-i\omega_n c(e^{-k_H u} + \text{constant})}. \tag{9.39}$$

To obtain modes where the outgoing wave is of standard form, we invert functionally and write

$$\tilde{\phi}^{(+)} \simeq e^{i\omega_n k_H^{-1}\ln[(v_0-v)/c]} - e^{-i\omega_n u}, \tag{9.40}$$

valid for $v < v_0$. Now we move back to $(d+1)$ dimensions and compare with the high-frequency limit of the global modes (9.22). In the far asymptotic region, $r \to \infty$, the exterior tortoise coordinate r_*^+ is approximately ρ, and so we may write the global mode as

$$\phi^{(+)} \simeq (\cos\rho)^{(d-1)/2}\left(e^{-i\omega_n v - id\pi/2} - e^{-i\omega_n u}\right). \tag{9.41}$$

We want to compare this with what we found in the collapsing shell geometry (9.40),

$$\tilde{\phi}^{(+)} \simeq (\cos\rho)^{(d-1)/2}\left(e^{i\omega_n k_H^{-1}\ln[(v_0-v)/c]} - e^{-i\omega_n u}\right), \quad v < v_0, \tag{9.42}$$

where we added the overall decay factor. This comparison is of the exact same form as in sections 5 and 4, and so computing the Bogoliubov transformation will again show that the outgoing modes are thermally excited with temperature T_H.

Although there are other methods to derive the Hawking effect both here and in flat spacetime, this analysis only describes the Hawking radiation created as the black hole is formed. The behavior of null geodesics and reflective boundary conditions at infinity make AdS act somewhat like a finite box, in that the outgoing radiation has nowhere to escape. The radiation will therefore accumulate until the black hole comes into thermal equilibrium with the surrounding bath, with exception only if (for $d > 2$) the initial size of the black hole is too small and the black hole evaporates away before this can happen.

Acknowledgments

I would like to thank Professor Leopoldo Pando Zayas for his invaluable guidance and the mentoring of this project. I would also like to thank Marina David for her helpful discussions and insight.

References

[1] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.*, vol. 43, pp. 199–220, 1975. [,167(1975)].

[2] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*. No. 7, Cambridge university press, 1984.

[3] H. Nastase, "Introduction to ads-cft,"

[4] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, "Large n field theories, string theory and gravity,"

[5] W. G. Unruh, "Notes on black hole evaporation," *Phys. Rev.*, vol. D14, p. 870, 1976.

[6] P.-H. Lambert, "Introduction to black hole evaporation,"

[7] T. Harmark, J. Natario, and R. Schiappa, “Greybody factors for d-dimensional black holes,”

[8] A. Neitzke, “Greybody factors at large imaginary frequencies,”

[9] D. T. Son and A. O. Starinets, “Minkowski-space correlators in ads/cft correspondence: recipe and applications,”

[10] A. Paredes, K. Peeters, and M. Zamaklar, “Temperature versus acceleration: the unruh effect for holographic models,”

[11] S. Hemming and E. Keski-Vakkuri, “Hawking radiation from ads black holes,”

[12] R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics.* University of Chicago Press, 1994.

[13] R. d’Inverno, *Introducing Einstein’s Relatvity.* Oxford University Press, USA, 1899.

[14] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,”

[15] E. Witten, “Anti de sitter space and holography,”