Name: ______________________________ Student ID: ______________

Prof. Alan J. Laub June 9, 2011

Math 33A/1 – FINAL EXAMINATION
Spring 2011

Instructions:

(a) The exam is closed-book (except for one two-sided page of notes) and will last two and a half hours. No calculators, cell phones, or other electronic devices are allowed at any time.

(b) Notation will conform as closely as possible to the standard notation used in the lectures, not the textbook.

(c) Do all 6 problems. Problems 1–5 are worth 15 points each and problem 6 is worth 25 points (5 subparts, each worth 5 points). Hand in the exam with work shown where appropriate. Some partial credit may be assigned if warranted.

(d) Label clearly the problem number and the material you wish to be graded. Make sure your name and student ID are clearly marked above.

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1. In a certain experiment, the following measurements (t_i, y_i) are taken:

$$(-2, 8),\ (-1, 3),\ (0, 1),\ (1, 5),\ (2, 10)\ .$$

It is desired to "fit" this data set with a function of the form

$$y = \alpha + \beta t^2$$

where α and β are parameters to be determined. Compute the best parameters α and β (in the 2-norm sense) using the method of linear least squares.

2. Consider the initial-value problem

$$\dot{x}(t) = \frac{dx(t)}{dt} = Ax(t); \quad x(0) = x_0$$

for $t \geq 0$. Let $A = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}$ and $x_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(a) Is A asymptotically stable? Why or why not?

(b) Determine an explicit expression for $x(t)$, i.e., solve the initial value problem.

3. (a) For what values of the scalar a, if any, is the matrix $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$ positive definite? Show your reasoning carefully, i.e., state explicitly which criterion for positive definiteness you are using.

(b) For what values of the scalar b, if any, is the matrix $B = \begin{bmatrix} b & -4 & -4 \\ -4 & b & -4 \\ -4 & -4 & b \end{bmatrix}$ positive definite? Show your reasoning carefully, i.e., state explicitly which criterion for positive definiteness you are using.

4. The *spectral factorization* or *spectral representation* of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is given by

$$\begin{aligned} A &= \lambda_1 x_1 x_1^+ + \cdots + \lambda_n x_n x_n^+ \\ &= \lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T \\ &= \lambda_1 P_1 + \cdots + \lambda_n P_n \end{aligned}$$

where λ_i denotes the ith eigenvalue of A with corresponding *unit* eigenvector x_i and P_i denotes the ith spectral projector .

(a) Write the spectral factorization of the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ in the form

$$\lambda_1 P_1 + \lambda_2 P_2$$

(b) Show that the matrices P_1 and P_2 are orthogonal to each other, i.e., that $P_1 P_2 = 0$.

5. Let $A = \begin{bmatrix} -2 & -4 & 4 \\ 0 & -10 & 8 \\ 0 & -12 & 10 \end{bmatrix}$.

(a) Find a nonsingular matrix S that makes A similar to a diagonal matrix and compute the similarity $S^{-1}AS$ or, alternatively, show that such a matrix S does not exist.

(b) Verify the formulas $\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det(A) = \lambda_1\lambda_2\lambda_3$.

6. Let $A = \frac{1}{10}\begin{bmatrix} 1 & 4 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ and suppose you compute (e.g., using MATLAB) an SVD $U\Sigma V^T$ of A to produce the following matrices:

$$U = \begin{bmatrix} -0.6530 & 0.5231 & 0.0236 & 0.5472 \\ -0.5402 & 0.0903 & -0.4393 & -0.7120 \\ -0.4275 & -0.3424 & 0.8079 & -0.2176 \\ -0.3147 & -0.7752 & -0.3921 & 0.3824 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.6922 & 0 & 0 \\ 0 & 0.1445 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.2796 & -0.3490 & 0.8944 \\ -0.7805 & 0.6252 & 0.0000 \\ 0.5592 & -0.6981 & -0.4472 \end{bmatrix}.$$

Suppose that you also compute, from this data, that $A^+ = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 3 & 1 & -1 & -3 \\ -2 & 0 & 2 & 4 \end{bmatrix}$.

As you know, many things about A as a linear transformation are available immediately (or via a brief, easy calculation) from its SVD. In some of the following, you are asked to determine a few of them. Any attempt at any other method of determining them will receive zero credit (as well as potentially wasting your time).

(a) Determine the rank of A and A^+.

(b) Draw a diagram of the four fundamental subspaces of A and indicate the dimensions of each subspace.

(c) Determine *orthonormal* bases for each of the four fundamental subspaces of A.

(d) Determine the orthogonal projections $P_{\mathcal{R}(A)}$ and $P_{\mathcal{N}(A)}$ numerically from A and/or A^+.

(e) Find the minimum (2-)norm solution of the linear least squares problem $\min_x \|Ax - b\|_2$ where $b = [1 \quad 2 \quad 1 \quad 1]^T$.