Name: ______________________________ Student ID: ____________ Section: 2___

Prof. Alan J. Laub June 11, 2009

Math 33A – FINAL EXAMINATION
Spring 2009

Instructions:

(a) The exam is closed-book (except for one page of notes) and will last two and a half (2.5) hours.

(b) Notation will conform as closely as possible to the standard notation used in the lectures.

(c) Do all 6 problems. Problems 1–5 are worth 15 points each; problem 6 is worth 25 points (5 subparts, each worth 5 points). Some partial credit may be assigned if warranted.

(d) Label clearly the problem number and the material you wish to be graded.

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1. In a certain experiment, the following measurements (t_i, y_i) are taken:

$$(-2,7),\ (-1,2),\ (0,1),\ (1,6),\ (2,15)\ .$$

It is desired to "fit" this data set with a function of the form

$$y = \alpha t + \beta t^2$$

where α and β are parameters to be determined. Compute the best parameters α and β (in the 2-norm sense) using the method of linear least squares.

2. Consider the initial-value problem

$$\dot{x}(t) = \frac{dx(t)}{dt} = Ax(t); \quad x(0) = x_0$$

for $t \geq 0$. Let $A = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$ and $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Is A asymptotically stable? Why or why not?

(b) Determine an explicit expression for $x(t)$, i.e., solve the initial value problem.

3. For what values of the scalars a and b, if any, is the matrix $A = \begin{bmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{bmatrix}$ positive definite? Show your reasoning carefully, i.e., state explicitly which criteria or criterion for positive definiteness you are using.

4. In class we saw that if $A \in \mathbb{R}_r^{m \times n}$, $m \geq n$, was factored as $A = QR$ where Q had orthonormal columns and $R \in \mathbb{R}_n^{n \times n}$ was upper triangular (for example, via Gram-Schmidt orthogonalization), then the solution of the linear least squares problem was given by

$$x = (A^T A)^{-1} A^T b = (R^T Q^T Q R)^{-1} R^T Q^T b = R^{-1} Q^T b.$$

In other words, $A^+ = R^{-1} Q^T$. Show that this must be so by verifying the four Penrose conditions.

5. Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -4 \\ 0 & 6 & -5 \end{bmatrix}$.

(a) Find a nonsingular matrix S that makes A similar to a diagonal matrix (and find the diagonal matrix) or, alternatively, show that such a matrix S does not exist.

(b) Verify the formulas $\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det(A) = \lambda_1\lambda_2\lambda_3$.

6. Let $A = \frac{1}{10}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ and suppose you compute (e.g., using MATLAB) an SVD $U\Sigma V^T$ of A to produce the following matrices:

$$U = \begin{bmatrix} -0.3147 & -0.7752 & 0.5000 & 0.2236 \\ -0.4275 & -0.3424 & -0.8333 & 0.0745 \\ -0.5402 & 0.0903 & 0.1667 & -0.8199 \\ -0.6530 & 0.5231 & 0.1667 & 0.5217 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.6922 & 0 & 0 \\ 0 & 0.1445 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.2796 & -0.3490 & 0.8944 \\ -0.5592 & -0.6981 & -0.4472 \\ -0.7805 & 0.6252 & 0.0000 \end{bmatrix}.$$

Suppose, furthermore, that you also compute, from this data, that

$$A^+ = \begin{bmatrix} 2 & 1 & 0 & -1 \\ 4 & 2 & 0 & -2 \\ -3 & -1 & 1 & 3 \end{bmatrix}.$$

As you know, many things about A as a linear transformation are available immediately (or via a brief calculation) from its SVD. In some of the following, you are asked to determine a few of them. Any attempt at any other method of determining them will receive zero credit.

(a) Determine the rank of A.

(b) Draw a diagram of the four fundamental subspaces of A and indicate the dimensions of each subspace.

(c) Determine orthonormal bases for each of the four fundamental subspaces of A.

(d) Determine the orthogonal projections $P_{\mathcal{R}(A)}$ and $P_{\mathcal{N}(A)}$.

(e) Find the minimum (2-)norm solution of the linear least squares problem

$$\min_x \|Ax - b\|_2$$

where $b = [\frac{3}{20}, \frac{3}{20}, \frac{3}{10}, \frac{3}{10}]^T$.