HOMEWORK FOR 255A

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Exercise 1. (a) Show that the norm in a Banach space X is weakly lower-semicontinuous.
(b) Deduce the corresponding property of sequences:

$$\big\| \operatorname*{w-lim}_{n\to\infty} x_n \big\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Remark. Similarly, the norm on X^* is lower-semicontinuous in the weak-* topology. The norm is not weakly continuous in any infinite dimensional Banach space. For example, given any $f \in C_c^\infty(\mathbb{R})$, then each of the following sequences converge weakly to zero in $L^2(\mathbb{R})$:

$$f_n(x) = n^{\frac{1}{2}} f(nx), \quad f_n(x) = n^{-\frac{1}{2}} f(x/n), \quad f_n(x) = f(x-n), \quad f_n(x) = e^{inx} f(x).$$

Exercise 2. Prove the following (strictly, Radon and F. Riesz treated the particular case of L^p, $1 < p < \infty$):

Theorem (Radon–Riesz)**.** *Let X be a uniformly convex Banach space. If x_n converges weakly to x and $\|x_n\| \to \|x\|$, then $x_n \to x$ in norm.*

Exercise 3. Prove the following result of M. Riesz (and others):

Theorem. *A subset $\mathcal{F}$ of $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, is precompact if and only if it is*
(*a*) *uniformly bounded, i.e., there exists $M > 0$ so that $\|f\| \leq M$ for all $f \in \mathcal{F}$;*
(*b*) *equicontinuous, i.e., for each $\epsilon > 0$ there is a $\delta > 0$ so that*

$$|y| < \delta \implies \int |f(x+y) - f(x)|^p \, dx < \epsilon^p; \quad \textit{and}$$

(*c*) *tight, i.e., for each $\epsilon > 0$ there is an $R > 0$ so that*

$$\int_{|x|>R} |f(x)|^p \, dx < \epsilon^p.$$

Exercise 4. Let $T : [0,1] \to [0,1]$ be continuous. Show that the extreme points of the set of invariant (regular Borel) probability measures are precisely the ergodic measures.

Exercise 5. Compute the norm of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Exercise 6. Let $K(t, s)$ be continuous on $[0, 1] \times [0, 1]$ and define $T : C([0, 1]) \to C([0, 1])$ via

$$Tf(t) = \int_0^t K(t, s)f(s)\, ds.$$

Such "lower-triangular" operators are called Volterra operators. Show that T is quasi-nilpotent, that is, $\|T^n\|^{1/n} \to 0$ as $n \to \infty$.

Remark. This sort of integral operator shows up with reasonable regularity; typically, the lower triangularity condition reflects the fact causal nature of time (effects do not precede causes). The moral to this problem is 'factorials beat powers'.

Exercise 7. Let A be an $n \times n$ Jordan block with eigenvalue λ. Compute $(z - A)^{-1}$ and thence $f(A)$ for arbitrary analytic f.

Remark. The special case $f(z) = e^{tz}$ should remind you of solving linear constant coefficient ODEs. Many (myself included) are first exposed (implicitly) to Jordan normal form as an algorithm for solving such ODEs.

When $\lambda = 0$, notice that the norm of the resolvent diverges as the size of the matrix grows for any $z \in \bar{\mathbb{D}}$, which is a reflection of the fact that $\bar{\mathbb{D}}$ is the spectrum of the (semi-infinite) unilateral shift. This behaviour, where the resolvent of a non-normal matrix has very large norm away from the spectrum, is known as the *pseudospectral* phenomenon. The region of the complex plane where it occurs is called the *pseudospectrum*.

Exercise 8. Let $A : X \to Y$ and let $B : Y \to X$ be bounded operators between Banach spaces. Show that

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$$

by finding a formula for $(z - AB)^{-1}$ in terms of $(z - BA)^{-1}$.

Remark. Note the special case when A and B are $n \times m$ and $m \times n$ matrices, respectively. When $m \neq n$, this already shows the necessity of taking the union with $\{0\}$.

Exercise 9. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Show that

$$\|A^{-1}\| = (\mu_n)^{-1}$$

where μ_n denotes the *least* singular value.

Exercise 10. $A : H \to H$ be a compact operator and $\lambda_1, \ldots, \lambda_k$ non-zero eigenvalues with repetition not exceeding the *algebraic* multiplicity. Show that the product $\lambda_1 \cdots \lambda_k$ is an eigenvalue of $\wedge^k A$.