

247A Homework.

The two sources for notes are <http://www.math.ubc.ca/~ilaba/wolff/> and http://www.its.caltech.edu/~schlag/notes_033002.pdf

1. Let us define

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(x \sin(\theta)) d\theta$$

Show that

$$f(x) \mapsto F(\xi) = 2\pi \int J_0(2\pi\xi x) f(x) x dx$$

defines a unitary map from $L^2([0, \infty), r dr)$ to itself. Describe the relation to the Fourier transform of radial functions in two dimensions.

2. Let $d\rho$ be a probability measure on \mathbb{R} with $\int x d\mu(x) = 0$ and $\int x^4 d\rho(x) < \infty$. Prove the central limit theorem for the sum of independent random variables with this distribution.

Specifically, if X_1, X_2, \dots are $d\rho$ -distributed, show that for any Schwartz function f ,

$$\mathbb{E}\left\{f\left(\frac{X_1 + \dots + X_n}{n^{1/2}}\right)\right\} \longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left\{-\frac{x^2}{2\sigma^2}\right\} f(x) dx$$

as $n \rightarrow \infty$. Hint: first show convergence for $f(x) = e^{-2\pi i x \xi}$ uniformly for ξ in a compact set.

3. Show that every continuous (group) homomorphism from \mathbb{T} into \mathbb{C}^* (the non-zero complex numbers under multiplication) takes the form $x \mapsto e^{2\pi i n x}$ with n an integer. What is the analogous statement for continuous homomorphisms $\mathbb{R} \rightarrow \mathbb{C}^*$.

4. Let us define a sequence functions on \mathbb{R} by

$$\psi_n(x) = \left[\frac{d}{dx} - 2\pi x\right]^n e^{-\pi x^2}$$

where $n = 0, 1, \dots$. Show that $\psi_n(x)$ form an orthogonal sequence of eigenfunctions for the Fourier transform on $L^2(\mathbb{R})$.

In fact they are a basis, but this is much harder to prove. One approach to this latter problem is to realize that they are the eigenfunctions of the harmonic oscillator:

$$u(x) \mapsto \left[\frac{d}{dx} - 2\pi x\right] \left[-\frac{d}{dx} - 2\pi x\right] u(x) = -\frac{d^2 u}{dx^2} + (4\pi^2 x^2 - 2\pi) u(x).$$

5. Let G be a finite cyclic group and H a subgroup. For $\chi \in \hat{G}$ we write

$$\hat{f}(\chi) = \sum_g f(g) \bar{\chi}(g).$$

We say $\chi \in \hat{G}^H$ if χ is constant on the cosets of H .

Prove the following analogue of the classical Poisson Summation formula:

$$\frac{1}{|G|} \sum_{\chi \in \hat{G}^H} \hat{f}(\chi) = \frac{1}{|H|} \sum_{h \in H} f(h).$$

(The classical version has $G = \mathbb{R}$ and $H = \mathbb{Z}$, which leads to $\hat{G}^H = \{e^{2\pi i n x} : n \in \mathbb{Z}\}$.)

6. Suppose $f \in L^2(\mathbb{R})$ is supported on $[-\frac{1}{2}, \frac{1}{2}]$ then we know that f can be recovered from the values of $\hat{f}(n)$ for $n \in \mathbb{Z}$ (the characters form an orthonormal basis). Prove the Shannon Sampling Theorem:

$$\hat{f}(\xi) = \sum_n \hat{f}(n) \frac{\sin[\pi(n - \xi)]}{\pi(n - \xi)}$$

(which includes proving convergence of this infinite sum).

Remark: The audible spectrum extends only to about 20kHz. Consequently, as heard by a human, one may regard music as a function whose Fourier transform is supported on a finite interval. The above theorem says that to faithfully reproduce music, one need only sample the signal forty thousand times per second. This is what happens in CD recording.

7. Given $\omega \in \mathbb{R}^d$, show that the following are equivalent:
- For $m \in \mathbb{Z}^d$, $m \cdot \omega = 0$ implies $m = 0$.
 - The curve $t \mapsto t\omega + \mathbb{Z}^d$ is dense in $\mathbb{R}^d/\mathbb{Z}^d$.
 - For any continuous function f on $\mathbb{R}^d/\mathbb{Z}^d$,

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t\omega + \mathbb{Z}^d) dt = \int_0^1 \cdots \int_0^1 f(x + \mathbb{Z}^d) dx.$$

[Hint: prove (a) \Leftrightarrow (c) and then (c) \Rightarrow (b) \Rightarrow (a).]

8. Let $d\mu$ be a finite complex measure on \mathbb{R} .
- Show that

$$\lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L |\hat{\mu}(\xi)|^2 d\xi = \sum_{x \in \mathbb{R}} |\mu(\{x\})|^2$$

(finiteness of the measure implies that only countably many terms in the sum are non-zero).

(b) Suppose that $d\mu$ is purely atomic, that is, $d\mu$ is a (countable) linear combination of delta measures. Show that $\hat{\mu}$ is almost periodic.

A function on f on \mathbb{R} is said to be almost periodic if for any $\epsilon > 0$, there exists $L > 0$ so that any interval of length L contains an ϵ -almost period:

$$\forall a \in \mathbb{R} \quad \exists p \in [a, a + L] \quad \text{such that} \quad \sup_x |f(x) - f(x + p)| < \epsilon.$$

Hint: For part (b) begin by considering the case $\hat{\mu}(\xi) = e^{i\xi} + e^{2\pi i\xi}$.

9. The dyadic cubes in \mathbb{R}^d are the sets of the form

$$Q_{n,k} = [k_1 2^n, (k_1 + 1)2^n) \times \cdots \times [k_d 2^n, (k_d + 1)2^n)$$

where n ranges over \mathbb{Z} and $k \in \mathbb{Z}^d$.

(a) Given a collection of dyadic cubes whose diameters are bounded, show that one may find a sub-collection which covers the same region of \mathbb{R}^d but with all cubes disjoint.

(b) Define the (uncentered) dyadic maximal function by

$$[M_D f](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(y) dy$$

where the supremum is over all dyadic cubes that contain x . Show that this operator is of weak type (1,1).

(c) Deduce boundedness of the Hardy-Littlewood maximal function from the above.

Remarks: Part (a) provides a replacement for the Vitali Covering Lemma. I propose you address (c) ‘geometrically’; draw some pictures in the planar ($d = 2$) case.

10. (a) Evaluate

$$D_N(x) = \sum_{n=-N}^N e^{2\pi i n x}$$

and show that it is not an approximate identity on \mathbb{T} .

(b) Show that $\frac{1}{2N+1}|D_N(x)|^2$ is an approximate identity and derive its relation to the Fejer kernel.

(c) Calculate

$$\sum_{n \in \mathbb{Z}} r^{|n|} e^{2\pi i n x}$$

for $0 < r < 1$ and show that for $r \rightarrow 1$ it gives rise to an approximate identity.

(d) Suppose ϕ_n is an approximate identity and $d\mu$, a finite complex measure on \mathbb{T} . Show that $\phi_n * d\mu$ converges weak-* to $d\mu$.

Note: $d\mu_n$ converges weak-* to $d\mu$ iff for every bounded continuous function, f , $\int f d\mu_n \rightarrow \int f d\mu$.

11. (a) Given $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, show that $t \mapsto f(x+t)$ defines a continuous map of \mathbb{R} into $L^p(\mathbb{R}, dx)$.
 (b) Show that it is not equi-continuous as f varies over the set of f with $\|f\|_{L^p} \leq 1$. (That is, ϵ cannot be chosen from δ independently of f .)
 (c) Show that part (a) is false for L^∞ and $M(\mathbb{R})$.

12. (From Wolff §4.) Find a sequence of Schwartz functions ϕ_n such that (a) $\|\phi_n\|_{L^p}$ and $\|\hat{\phi}_n\|_{L^{p'}}$ are constant. The supports of $\hat{\phi}_n$ are disjoint and those of ϕ_n are almost disjoint. Use $\sum_{n=1}^N \phi_n$ to show that if $\|\hat{f}\|_{L^{p'}} \lesssim \|f\|_{L^p}$ then $p \leq 2$.

By almost disjoint we mean $\|\sum_{n=1}^N \phi_n\|_{L^p}^p \leq \frac{100}{99} \sum_{n=1}^N \|\phi_n\|_{L^p}^p$. Notice that if the supports were actually disjoint, then 100/99 could be replaced by 1.

Hint: Take a single C_c^∞ function and modify it by translation and multiplication by characters.

13. Prove the Rising Sun Lemma: Given a non-negative $f \in L^1(\mathbb{R})$, define

$$[M_R f](x) = \sup_{t>0} \frac{1}{t} \int_0^t f(x+s) ds.$$

If $S = \{x : M_R f > \lambda\}$ then $|S| = \lambda^{-1} \int_S f(x) dx$. [Hint: S is open.]

14. Prove the following theorem of Milicer-Gruzewska: Let $d\mu$ be a complex measure on \mathbb{T} with the property that $\hat{\mu}(n) \rightarrow 0$ as $n \rightarrow \infty$ (μ is called a Rajchman measure). If $f \in L^1(d|\mu|)$ and $d\nu = f d\mu$ then $\hat{\nu}(n) \rightarrow 0$. [Hint: mimic the proof of the Riemann–Lebesgue Lemma from Schlag’s notes.]
15. Let $R(k)$ be the smallest number such that in any colouring of the edges of the complete graph on $R(k)$ vertices by two colours, one can find a monochromatic complete graph on k vertices. These are known as Ramsey numbers; it is not difficult to show that $R(k) \leq 2^{2^k}$. The problem here is to prove that $2^{k/2} \leq R(k)$, which is due to Erdős.
 (a) Determine the expected number of monochromatic complete graphs on k vertices contained within a random colouring of the complete graph on n vertices.
 (b) Show that this is less than one when $n = 2^{k/2}$ and so complete the problem.

16. Let $f \in C^\alpha$ with $\alpha < 1$, and let K_n denote the Fejér kernel.

(a) Show that

$$\|f * K_n - f\|_{C^0} \lesssim n^{-\alpha} \|f\|_{C^\alpha}.$$

(b)[Optional] Show that $\|f * K_n - f\|_{C^\alpha} \rightarrow 0$ may fail. However, it is true if one restricts to those f with

$$(1) \quad \sup_{|x-y|<\delta} |f(x) - f(y)| = o(\delta^\alpha).$$

(c)[Optional] Show that the set of $f \in C^\alpha$ that obey (1) is exactly the closure of C^∞ in C^α .

17. Let f be a continuous function on \mathbb{T} . Suppose that for each $n > 0$ there is a trigonometric polynomial p_n of degree n (or less) such that

$$\|f - p_n\|_{C^0} \lesssim n^{-\alpha}$$

where $\alpha < 1$. Show that f is α Hölder continuous. Hint: write

$$f = p_1 + \sum_{k=1}^{\infty} (p_{2^k} - p_{2^{k-1}}).$$

18. Let Ω be a simply-connected open domain bounded by a Jordan curve. By a theorem of Carathéodory, any conformal map f of \mathbb{D} onto Ω can be extended to a homeomorphism of $\bar{\mathbb{D}}$ onto $\bar{\Omega}$.

We say that a curve $\gamma : S^1 \rightarrow \mathbb{C}$ is rectifiable if there exists a constant L so that for any $0 \leq \theta_0 < \theta_1 < \dots < \theta_n < 2\pi$,

$$\sum_{k=0}^{n-1} |\gamma(e^{i\theta_k}) - \gamma(e^{i\theta_{k+1}})| \leq L$$

where $\theta_{n+1} = \theta_0$.

Prove the following theorem of F. and M. Riesz: $f' \in H^1$ if and only if $\partial\Omega$ is rectifiable. [Hint: the function $z \mapsto \sum |f(ze^{i\theta_k}) - f(ze^{i\theta_{k+1}})|$ is continuous and sub-harmonic on \mathbb{D} .]

19. Prove the following result of Privalov: For $0 < \alpha < 1$, $f \in C^\alpha$ implies $\tilde{f} \in C^\alpha$.

20. (a) Suppose T is a rotation invariant operator on $L^2(\mathbb{R}/\mathbb{Z})$, that is, $R_y T = T R_y$ for any rotation $[R_y f](x) = f(x - y)$. Show that $e^{2\pi i n x}$, $n \in \mathbb{Z}$, are eigenfunctions of T .

(b) Let T be a bounded operator on $L^2(\mathbb{R}^n)$ such that there is a function K obeying $|K(x, y)| \lesssim |x - y|^{-n}$ so that whenever f and g have disjoint supports,

$$\langle g, T f \rangle = \int \int \bar{g}(x) K(x, y) f(y) dy dx.$$

Show that if T is translation invariant, then $K(x, y) = F(x - y)$, which means that T is a convolution operator. [Hint: Treat (a) and (b) independently.]

21. (a) Let $I \subseteq \mathbb{R}$ be an interval and let $z \in \mathbb{C}^+ = \{z : \text{Im } z > 0\}$. Show that the harmonic measure of $I \subseteq \partial\mathbb{C}^+$ with respect to z is equal to the angle subtended by I at z divided by π . Deduce that the harmonic measure of I is constant on arcs of circles.

(b) Calculate the conjugate function of $\chi_{[0,a]}(\theta) \in L^2(S^1; \frac{d\theta}{2\pi})$.

22. (a) Suppose $f : \mathbb{D} \rightarrow \mathbb{C}^+ = \{z : \text{Im } z > 0\}$ is analytic. Show that there exists a finite positive measure $d\mu$ and a real constant a so that

$$f(z) = a + i \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

This result is due to Herglotz. [Hint: First look at $\text{Im}(f)$.]

- (b) Deduce that any holomorphic mapping of \mathbb{C}^+ into itself admits the representation

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1 + tz}{t - z} d\rho(t).$$

where $a \in \mathbb{R}$, $b \geq 0$, and $d\rho$ is a positive measure.

23. Prove the following theorem of Kolmogorov: suppose $0 < p < 1$ then

$$f(z) = \int_0^{2\pi} \frac{d\mu(\theta)}{e^{i\theta} - z} \quad \text{implies} \quad \sup_{0 < r < 1} \int |f(re^{i\theta})|^p d\theta < \infty$$

for any finite complex measure $d\mu$.

24. Let ℓ_μ^p denote the weighted ℓ^p space

$$\|c\|^p = \sum (|n| + 1)^{-2} |c_n|^p.$$

Let ϕ_n , $n \in \mathbb{Z}$, be an orthonormal basis for $L^2(\mathbb{R}/\mathbb{Z})$ which obeys $\|\phi_n\|_{L^\infty} \lesssim 1$ and define $T : L^2 \rightarrow \ell_\mu^2$ by

$$[Tf](n) = (|n| + 1) \langle \phi_n(x), f(x) \rangle.$$

- (a) Show that T extends to a bounded map of L^p into ℓ_μ^p for all $1 < p \leq 2$. This result is due to Hardy and Littlewood. [Hint: Prove a weak-type bound and use Marcinkiewicz.]

- (b) Given a sequence c_j indexed by $j \in \mathbb{Z}$, define the rearrangement c_j^* as follows: For $j \geq 0$, c_j^* is the $(j+1)$ th largest element of the set $\{|c_0|, |c_1|, \dots\}$ while for $j < 0$, it is the $|j|$ th largest element of $\{|c_{-1}|, |c_{-2}|, \dots\}$. Derive the following inequality of Payley:

$$\sum (1 + |j|)^{p-2} |c_j^*|^p \lesssim \|f\|_{L^p}^p$$

where $c_j = \hat{f}(j)$.

- (c) By splitting the sum dyadically, show that this implies the usual Hausdorff–Young inequality for $1 < p \leq 2$.

25. Suppose $f \in L^1(\mathbb{R}/\mathbb{Z})$ and let Mf denote its (uncentred) dyadic maximal function.

- (a) Show that for $\lambda > \int |f|$,

$$\frac{1}{\lambda} \int_{|f| > \lambda} |f(x)| dx \lesssim |\{x : Mf > \lambda\}|.$$

[Hint: Do a Calderón–Zygmund style decomposition.]

- (b) Deduce that if $Mf \in L^1$, then $|f| \log[1 + |f|] \in L^1$. This result is due to Stein.

- (c) Use the fact that $M : L^\infty \rightarrow L^\infty$ and $L^1 \rightarrow L_w^1$ to show

$$|\{x : Mf > \lambda\}| \lesssim \frac{1}{\lambda} \int_{|f| > c\lambda} |f(x)| dx.$$

for some small constant c .

- (d) Deduce that if $|f| \log[1 + |f|] \in L^1$ then $Mf \in L^1$.

26. For $1 \leq p < \infty$, let $L_w^p(\mathbb{R})$ denote the set of measurable functions on \mathbb{R} for which

$$\|f\|_p^* = \sup_{\lambda > 0} \{\lambda^p |\{x : |f| > \lambda\}|\}^{1/p}$$

is finite. The $*$ is to warn that this isn't a norm; however,

(a) For $1 < p < \infty$, the following defines a norm on $L_w^p(\mathbb{R})$:

$$\|f\|_{p,w} = \sup_E \frac{1}{|E|^{(p-1)/p}} \int_E |f(x)|.$$

Moreover, $\|f\|_p^* \lesssim \|f\|_{p,w} \lesssim \|f\|_p^*$. [*Remark:* with this norm, $L_w^p(\mathbb{R})$ is actually a Banach space.]

(b) Show that there is no norm on $L_w^1(\mathbb{R})$ comparable to $\|f\|_1^*$ by considering the following family of functions

$$\sum_{k=0}^N \frac{1}{|x-k|}$$

as $N \rightarrow \infty$.

27. (a) Let c_n denote the surface area of $S^{n-1} \subseteq \mathbb{R}^n$. Show that for $n \geq 3$,

$$G(x) = \frac{1}{(n-2)c_n|x|^{n-2}}$$

is the Green function for the Laplace equation in \mathbb{R}^n : if $f \in \mathcal{S}$, then $-\Delta(G*f) = f$.

(b) For any $f, g \in \mathcal{S}$,

$$\left| \int f(x)g(x) dx \right|^2 \leq \|\nabla f\|_{L^2}^2 \iint g(x)G(x-y)g(y) dx dy.$$

(c) Deduce the following Sobolev inequality:

$$\forall f \in \mathcal{S}, \quad \|f\|_{L^q} \lesssim \|\nabla f\|_{L^2} \quad \text{where } q = 2n/(n-2)$$

by choosing g appropriately.

(d) Show that on \mathbb{R} , one does not have

$$\|f\|_{L^\infty}^2 \lesssim \|f'\|_{L^2}^2$$

however it is true that

$$\|f\|_{L^\infty}^2 \lesssim \|f'\|_{L^2}^2 + \|f\|_{L^2}^2.$$

[*Remark:* In this regard, \mathbb{R}^2 is like \mathbb{R} ; there is no estimate without adding $\|f\|_{L^2}$. However, one has only

$$\|f\|_{L^q}^2 \lesssim \|\nabla f\|_{L^2}^2 + \|f\|_{L^2}^2$$

for all $2 \leq q < \infty$.]

28. Suppose $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ obeys

$$\frac{\partial^{n+m}}{\partial x^n \partial \xi^m} a(x, \xi) \in L^\infty$$

for all $n, m \geq 0$. We then define an operator on $L^2(\mathbb{R})$ by

$$[Tf](x) = \int a(x, \xi) e^{2\pi i x \xi} \hat{f}(\xi) d\xi.$$

(This is the pseudo-differential operator with symbol, a , which belongs to the exotic symbol class $S_{0,0}^0$.) Show that it is bounded. [*Hint:* let ψ_j be a partition of unity

adapted to the partition of \mathbb{R} by $[j, j+1)$, then apply the Cotlar-Stein Lemma using the operators with symbols $a_{i,j}(x, \xi) = \psi(x-i)a(x, \xi)\psi(\xi-j)$.]

29. (a) Prove that

$$\left| \int_{\mathbb{R}^n} f(x)g(x) dx \right| \leq \int_{\mathbb{R}^n} f^*(x)g^*(x) dx$$

(b) Suppose $f \mapsto f * K$ is a bounded operator on $L^2(\mathbb{R}^n)$ and $K(x) \lesssim |x|^{-n}$. Show that there exists C so that

$$\int_{\epsilon < |x| < N} K(x) dx \leq C$$

for all $0 < \epsilon < N < \infty$.

30. Given a measurable function $t : \mathbb{R} \rightarrow (0, \infty)$, let us define

$$[T_t f](x) = \frac{1}{\sqrt{2\pi t(x)}} \int \exp\left\{-\frac{(x-y)^2}{2t(x)}\right\} f(y) dy.$$

(a) Determine the adjoint of the operator T_t ; write it as an integral operator.

(b) Consider $T_t T_t^\dagger$ and show that for $f \geq 0$,

$$[T_t T_t^\dagger f](x) \lesssim [T_{2t} f](x) + [T_{2t}^\dagger f](x).$$

(c) Deduce that maximal operator

$$[Mf](x) = \sup_{t>0} \frac{1}{\sqrt{2\pi t}} \int \exp\left\{-\frac{(x-y)^2}{2t}\right\} f(y) dy.$$

is bounded on $L^2(\mathbb{R})$. [Remark: There is nothing special about the Gaussian, it was just chosen for concreteness.]

31. Given $n \in \mathbb{Z}^3$, let us write $|n|$ for the ℓ^1 norm: $|n| = |n_1| + |n_2| + |n_3|$. Consider the following operator on $\ell^2(\mathbb{Z}^3)$:

$$[Hu](n) = \sum_{|n-m|=1} u(m).$$

Schur's test (or part (b)) shows that this is a bounded operator.

(a) Given $n \in \mathbb{Z}^d$, let us write δ_n for the function $k \mapsto \delta_{k,n}$. Show that $\langle \delta_m | H^N \delta_n \rangle$ is equal to the number of paths of length N from n to m in the \mathbb{Z}^3 lattice.

(b) As H is translation invariant, we know that we can write it as a Fourier multiplier. Find the Fourier multiplier.

(c) Determine the leading term in the $t \rightarrow \infty$ asymptotics of

$$\langle \delta_0 | e^{tH} \delta_0 \rangle.$$

(d) [Optional] Use the Borel-Cantelli Lemma to deduce that in three dimensions, a random walker starting at the origin will return to the origin only finitely many times (with probability one).

32. Let Ω denote a hyperplane in \mathbb{R}^d and let $d\sigma$ denote the induced Lebesgue measure. For $s \geq 0$, H^s denotes the Sobolev space of functions $f \in L^2$ for which

$$\|f\|_{H^s}^2 = \int |\hat{f}|^2 (1 + |\xi|^2)^s d\xi$$

is finite.

Show that for $\epsilon > 1/2$, $f \mapsto f|_\Omega$ defines a continuous map from $H^s(\mathbb{R}^d)$ to $H^{s-\epsilon}(\Omega)$. Also show that for $\epsilon \leq 1/2$, it does not.

33. Let Ω denote the cone $|\xi_0|^2 = |\xi_1|^2 + \dots + |\xi_d|^2$ in \mathbb{R}^{d+1} and let $d\sigma$ denote the induced surface measure.

(a) If f is a smooth function supported in a compact subset of $\mathbb{R}^{d+1} \setminus \{0\}$, show that Fourier transform of $f d\sigma$ has a natural interpretation as a solution of the wave equation:

$$\frac{d^2 u}{dt^2} = \sum_j \frac{d^2 u}{dx_j^2}.$$

(b) Calculate the leading term asymptotics of $\widehat{f d\sigma}$ as $|\xi| \rightarrow \infty$ in a fixed direction. For simplicity, just treat the case $d = 2$ with f supported in the region $\{\xi_0 > 0\}$. Warning: the cone does not have non-vanishing Gaussian curvature!

34. (a) Given ψ_0 with $\hat{\psi}_0 \in C_c^\infty(\mathbb{R})$, write the solution of the free Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = - \frac{\partial^2 \psi}{\partial x^2}, \quad \psi(x, t=0) = \psi_0(x)$$

as an integral involving $\hat{\psi}_0$.

(b) Study the asymptotics in the regime $t \rightarrow \infty$ with $x = vt$ and $v \in \mathbb{R}$ fixed. Specifically, prove that

$$\left| \psi(x, t) - \frac{1}{\sqrt{4\pi t}} e^{-i\pi/4} e^{ix^2/4t} \hat{\psi}_0\left(\frac{x}{4\pi t}\right) \right| \lesssim (t^2 + x^2)^{-3/4}$$

for t sufficiently large.

(c) Let us call the map of ψ_0 into the leading asymptotic behaviour $V(t)$. That is, the LHS of the equation above is $|\psi(x, t) - V(t)\psi_0|$. Check that this determines a unitary map and that $\psi(t) - V(t)\psi_0$ converges to zero in L^2 .

(d) Use the fact that for t fixed, $\psi_0 \mapsto \psi(t)$ is also a unitary map to deduce that the above asymptotic holds in L^2 sense for any initial data $\psi_0 \in L^2$.

35. Prove the van der Corput Lemma: (a) Suppose ϕ is real-valued and smooth in (a, b) and that for some $k \geq 1$, $\phi^{(k)}(x) \geq 1$ on $[a, b]$. Show that

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq 3^k \lambda^{-1/k}$$

for $k \geq 2$ and also for $k = 1$ if we assume that ϕ' is monotone. [Hints: Proceed by induction. For $k = 1$, integrate by parts wisely. For the step from k to $k + 1$, treat any interval with $|\phi^{(k)}(x)| \leq \delta(\lambda)$ separately from those where it is bigger than $\delta(\lambda)$.]

(b) Deduce that

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \lesssim \lambda^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right].$$

36. (a) Prove Debye's asymptotics for Bessel functions: given $\alpha \in (0, \infty)$,

$$J_\nu(\nu \operatorname{sech}(\alpha)) = \frac{e^{\nu[\tanh(\alpha) - \alpha]}}{\sqrt{2\pi\nu \tanh(\alpha)}} \left[1 + O(\nu^{-1}) \right]$$

as $\nu \rightarrow \infty$.

(b) Prove that

$$\cos[z \sin(\theta)] = J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos(2k\theta)$$

for all $\theta \in \mathbb{R}$. Why does the series converge?

37. Let E be a compact subset of \mathbb{R}^n of non-zero α -capacity. In class we proved the existence of a probability measure $d\nu$ so that

$$\frac{1}{C_\alpha(E)} = \inf_{\text{supp}(\mu) \subseteq E} I_\alpha(\mu) = I_\alpha(\nu).$$

Recall that

$$V_\mu(x) = \int \frac{d\mu(y)}{|x-y|^\alpha}$$

denotes the potential generated by $d\mu$.

(a) Show that $V_\nu(x) \geq 1/C_\alpha(E)$ for p.p. $x \in E$. (Recall that ‘p.p.’ means except for a set of zero capacity.)

(b) Show that for any positive measure μ , $V_\mu(x)$ is upper semi-continuous, that is, for every $a \in \mathbb{R}$, the set $\{x : V_\mu(x) > a\}$ is open. Equivalently,

$$\liminf_{x_n \rightarrow x} V_\mu(x_n) \geq V_\mu(x).$$

(c) From part (a), it follows that $C_\alpha(E)V_\nu(x) \geq 1$ for ν -almost all x . Explain. From this and part (b), show that $C_\alpha(E)V_\nu(x) \leq 1$ for all $x \in \text{supp}(\nu)$.

(d) Show that

$$C_\alpha(E) = \sup\{\|\mu\| : \text{supp}(\mu) \subseteq E \text{ and } \forall x \in \text{supp}(\mu), V_\mu(x) \leq 1\}.$$

38. Let $E \subseteq \mathbb{R}^n$, be a compact set of non-zero α -capacity ($0 < \alpha < n$). Let us define

$$D_n = \inf \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} |x_i - x_j|^{-\alpha}$$

and

$$M_n = \sup \inf_{x \in E} \frac{1}{n} \sum_{1 \leq i \leq n} |x - x_i|^{-\alpha}.$$

The infimum in the definition of D_n and the supremum in the definition of M_n are over $\{x_1, \dots, x_n\} \subset E$.

Show that $D_{n+1} \leq M_n$, that $M_n \leq 1/C_\alpha(E)$, and that $\liminf D_n \geq 1/C_\alpha(E)$. Conclude that $C_\alpha(E) = \lim D_n = \lim M_n$. [You may use the results of Question 3.]

39. Functions defined by

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

are known as Dirichlet series. The most famous example is the Riemann zeta function, where $a_n \equiv 1$. By writing $s = \sigma + it$ we have $n^{-s} = e^{-\sigma \log(n)} e^{-it \log(n)}$ which shows the connection to Fourier integrals.

(a) Given $f(s)$ as above and $g(s) = \sum b_n n^{-s}$, show that $f(s)g(s)$ can also be written as a Dirichlet series and find the formula for the coefficients. In this way, interpret the coefficients of $\zeta(s)^2$. (This operation is the multiplicative analogue of convolution.)

(b) Let f and g be Dirichlet series absolutely convergent for $\text{Re}(s) > \sigma_0$. Show that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\alpha + it)g(\beta - it) = \sum \frac{a_n b_n}{n^{\alpha+\beta}}$$

for $\text{Re}(\alpha)$ and $\text{Re}(\beta)$ larger than σ_0 . This is the analogue of the Plancherel Theorem.

(c) Prove the following simple Abelian theorem: Given $\alpha < 1$,

$$\lim_{n \rightarrow \infty} \log^\alpha(n) a_n = A \implies \lim_{u \downarrow 0} u^{1-\alpha} f(1+u) = C_\alpha A$$

and determine the value of C_α . What if $\alpha = 1$?

40. (a) Let $d(n) = \#\{d > 0 : d|n\}$. Prove that if $n = \prod p_i^{a_i}$ then

$$\frac{d(n)}{n^\delta} = \prod \left(\frac{a_i + 1}{p_i^{\delta a_i}} \right) < \exp \left\{ \frac{2^{1/\delta}}{\delta \log(2)} \right\}$$

[Hint: be wasteful, $\frac{a+1}{p^{\delta a}} \leq 1 + \frac{a}{p^{\delta a}} \leq 1 + \frac{1}{\delta \log(2)}$.]

(b) Refine the above argument to show that

$$\log[d(n)] \leq \frac{(1+\epsilon) \log(2) \log(n)}{\log \log(n)}$$

for n sufficiently large (depending on ϵ).

(c) By the prime number theorem, $\vartheta(x) = \sum_{p \leq x} \log(p)$ obeys $\vartheta(x)/x \rightarrow 1$. Use this to show that

$$\log[d(n)] \geq \frac{(1-\epsilon) \log(2) \log(n)}{\log \log(n)}.$$

infinitely often.

(d) By counting lattice points under the hyperbola $xy = n$, show that

$$d(1) + d(2) + d(3) + \cdots + d(n) = n \log(n) + O(n).$$

While part (c) shows that $d(n)$ can be enormous, this result shows that it is typically much smaller.

41. Let $\omega = e^{2\pi i/3}$.

(a) Show that $\mathbb{Z}[\omega]$ is a Euclidean domain using the norm $N(a + b\omega) = |a + b\omega|^2 = a^2 - ab + b^2$.

(b) Determine the units (there are six).

(c) Show that the following is a complete list of the primes in $\mathbb{Z}[\omega]$ (without repetition):

- (i) $1 - \omega$ and its associates,
- (ii) the rational primes of the form $3n + 2$ and their associates, and
- (iii) the (non-trivial) factors $a + b\omega$ of rational primes of the form $3n + 1$.

(d) Deduce that the number of solutions $(n, m) \in \mathbb{Z}^2$ to $N = n^2 + 3m^2$ is bounded by $C_\epsilon N^\epsilon$. (You may use results from the previous problem.)

42. (a) Use the previous problem to prove the following result of Bourgain¹:

$$\left\| \sum_{|n| \leq N} a_n e^{2\pi i[nx + n^2 t]} \right\|_{L^6}^2 \lesssim N^\epsilon \sum |a_n|^2$$

where L^6 denotes $L^6(\mathbb{R}^2/\mathbb{Z}^2; dx dt)$.

(b) Gauss proved that if a, b are integers and q is an odd prime with $a, b \in [1, q-1]$, then

$$\left| \sum_{n=0}^{q-1} e^{2\pi i[an^2 + bn]/q} \right| = \sqrt{q}.$$

¹Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations', I. *Geom. Funct. Analysis*, **3** (1993) 107–156.

(Such sums are known as Gauss sums.) From this we may deduce

$$\left| \sum_{n=0}^N e^{2\pi i[an^2+bn]/q} \right| \gtrsim \frac{N}{\sqrt{q}}$$

when $N \geq q^2$, for example. By studying small regions around $x = b/q$ and $t = a/q$ with $q \in [3, \sqrt{N}]$ show that N^ϵ cannot be replaced by an N -independent constant in part (a).

43. Prove the following two Strichartz estimates due to Bourgain²:

$$\left\| \sum a_n e^{2\pi i(nx+n^2t)} \right\|_{L^4}^2 \lesssim \sum |a_n|^2$$

where L^4 denotes $L^4(\mathbb{R}^2/\mathbb{Z}^2; dx dt)$ and

$$\left\| \sum_{n^2+m^2 \leq N^2} a(n, m) e^{2\pi i[nx+my+(n^2+m^2)t]} \right\|_{L^4}^2 \lesssim N^\epsilon \sum |a(n, m)|^2$$

where L^4 denotes $L^4(\mathbb{R}^3/\mathbb{Z}^3; dx dy dt)$. [*Hint*: Use the method from the proof of Zygmund's restriction theorem.]

44. Let $d\mu$ and $d\mu_n$ be probability measures on $[0, \infty)$.

(a) Show that if $d\mu_n$ converges weak-* to $d\mu$, then

$$\limsup_{n \rightarrow \infty} \mu_n(K) \leq \mu(K)$$

for any closed set K . Also show that for any open set, O ,

$$\liminf_{n \rightarrow \infty} \mu_n(O) \geq \mu(O).$$

(b) Give examples that show that the inequalities in part (a) can fail to be equalities.

(c) If we do not assume that $d\mu$ is a probability measure, half of (a) can fail. Which half and why?

(c) Show that if

$$\lim_{n \rightarrow \infty} \int e^{-mx} d\mu_n(x) = \int e^{-mx} d\mu(x)$$

for all $m \in \{0, 1, 2, \dots\}$ then $d\mu_n$ converges weak-* to $d\mu$.

²Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations', I. *Geom. Funct. Analysis*, **3** (1993) 107–156.