247A Notes on Lorentz spaces

Definition 1. For $1 \le p < \infty$ and $f : \mathbb{R}^d \to \mathbb{C}$ we define

$$\|f\|^*_{L^p_{\text{weak}}(\mathbb{R}^d)} := \sup_{\lambda>0} \lambda \big|\{x : |f(x)| > \lambda\}\big|^{1/p} \tag{1}$$

and the *weak L^p space*

$$L^p_{\text{weak}}(\mathbb{R}^d) := \big\{f : \|f\|^*_{L^p_{\text{weak}}(\mathbb{R}^d)} < \infty\big\}.$$

Equivalently, $f \in L^p_{\text{weak}}$ if and only if $|\{x : |f(x)| > \lambda\}| \lesssim \lambda^{-p}$.

Warning. The quantity in (1) does not define a norm. This is the reason we append the asterisk to the usual norm notation.

To make a side-by-side comparison with the usual L^p norm, we note that

$$\begin{aligned}\|f\|_{L^p} &= \left(\iint_{0\le\lambda<|f(x)|} p\lambda^{p-1}\, d\lambda\, dx\right)^{1/p}\\ &= \left(\int_0^\infty |\{|f| > \lambda\}|\, p\lambda^p\, \frac{d\lambda}{\lambda}\right)^{1/p}\\ &= p^{1/p}\big\|\lambda|\{|f| > \lambda\}|^{1/p}\big\|_{L^p\left((0,\infty),\frac{d\lambda}{\lambda}\right)}\end{aligned}$$

and, with the convention that $p^{1/\infty} = 1$,

$$\|f\|^*_{L^p_{\text{weak}}} = p^{1/\infty}\big\|\lambda|\{|f| > \lambda\}|^{1/p}\big\|_{L^\infty\left((0,\infty),\frac{d\lambda}{\lambda}\right)}.$$

This suggests the following definition.

Definition 2. For $1 \le p < \infty$ and $1 \le q \le \infty$ we define the *Lorentz space* $L^{p,q}(\mathbb{R}^d)$ as the space of measurable functions f for which

$$\|f\|^*_{L^{p,q}} := p^{1/q}\big\|\lambda|\{|f| > \lambda\}|^{1/p}\big\|_{L^q\left(\frac{d\lambda}{\lambda}\right)} < \infty. \tag{2}$$

From the discussion above, we see that $L^{p,p} = L^p$ and $L^{p,\infty} = L^p_{\text{weak}}$. Again $\|\cdot\|^*_{L^{p,q}}$ is not a norm in general. Nevertheless, it is positively homogeneous: for all $a \in \mathbb{C}$,

$$\|af\|^*_{L^{p,q}} = \big\|\lambda\big|\{|f| > |a|^{-1}\lambda\}\big|^{1/p}\big\|_{L^q(d\lambda/\lambda)} = |a| \cdot \|f\|^*_{L^{p,q}} \tag{3}$$

(strictly the case $a = 0$ should receive separate treatment). In lieu of the triangle inequality, we have the following:

$$\begin{aligned}\|f+g\|^*_{L^{p,q}} &= \big\|\lambda\big|\{|f+g| > \lambda\}\big|^{1/p}\big\|_{L^q(d\lambda/\lambda)}\\ &\le \big\|\lambda\left(\big|\{|f| > \tfrac{1}{2}\lambda\}\big| + \big|\{|g| > \tfrac{1}{2}\lambda\}\big|\right)^{1/p}\big\|_{L^q(d\lambda/\lambda)}\\ &\le \big\|\lambda\big|\{|f| > \tfrac{1}{2}\lambda\}\big|^{1/p}\big\|_{L^q(d\lambda/\lambda)} + \big\|\lambda\big|\{|g| > \tfrac{1}{2}\lambda\}\big|^{1/p}\big\|_{L^q(d\lambda/\lambda)}\end{aligned}$$

by the subadditivity of fractional powers and the triangle inequality in $L^q(d\lambda/\lambda)$. Thus

$$\|f+g\|^*_{L^{p,q}} \le 2\|f\|^*_{L^{p,q}} + 2\|g\|^*_{L^{p,q}}. \tag{4}$$

Combining (3), (4), and the fact that $\|f\|_{L^{p,q}} = 0$ implies $f \equiv 0$ almost everywhere, we see that $\|\cdot\|^*_{L^{p,q}}$ obeys the axioms of a quasi-norm. When $p > 1$, this quasi-norm is equivalent to an actual norm (see below). When $p = 1$ and $q \ne 1$, there cannot be a norm that is equivalent to our quasi-norm. However there is a metric that generates the same topology. In either case, we obtain a complete metric space.

Notice that (i) if $|f| \ge |g|$ then $\|f\|^*_{L^{p,q}} \ge \|g\|^*_{L^{p,q}}$ and (ii) The quasi-norms are *rearrangement invariant*, which is to say that $\|f\|^*_{L^{p,q}} = \|f \circ \phi\|^*_{L^{p,q}}$ for any measure preserving bijection $\phi : \mathbb{R}^d \to \mathbb{R}^d$.

Proposition 3. *Given $f \in L^{p,q}$, we write $f = \sum f_m$ where*

$$f_m(x) := f(x)\chi_{\{x: 2^m \le |f(x)| < 2^{m+1}\}}.$$

Then

$$\|f\|^*_{L^{p,q}} \approx_{p,q} \Big\| \|f_m\|_{L^p_x(\mathbb{R}^d)} \Big\|_{\ell^q_m(\mathbb{Z})}$$

In particular, $L^{p,q_1} \subseteq L^{p,q_2}$ whenever $q_1 \le q_2$.

Proof. It suffices to consider f of the form $f = \sum 2^m \chi_{E_m}$ with disjoint sets E_m (cf. $E_m = \{2^m \le |f| < 2^{m+1}\}$). Now

$$\begin{aligned}\big(\|f\|^*_{L^{p,q}}\big)^q &= p\int_0^\infty \lambda^q |\{|f| > \lambda\}|^{q/p} \frac{d\lambda}{\lambda} \\ &= p \sum_m \int_{2^{m-1}}^{2^m} \lambda^q \Big(\sum_{n \ge m} |E_n|\Big)^{q/p} \frac{d\lambda}{\lambda} \\ &\approx \sum_m \Big| 2^m \Big(\sum_{n\ge m} |E_n|\Big)^{1/p}\Big|^q .\end{aligned}$$

To obtain a lower bound, we keep only the summand $n = m$; for an upper bound, we use the subadditivity of fractional powers. This yields

$$\big\|2^m|E_m|^{1/p}\big\|_{\ell^q_m} \lesssim \|f\|^*_{L_{p,q}} \lesssim \Big\| \sum_{m \le n} 2^m |E_n|^{1/p}\Big\|_{\ell^q_m}. \tag{5}$$

As $\|2^m\chi_{E_m}\|_{L^p} = 2^m|E_m|^{1/p}$, we have our desired lower bound. To obtain the upper bound, we use the triangle inequality in $\ell^q(\mathbb{Z})$:

$$\mathrm{RHS}(5) = \Big\| \sum_{k=0}^\infty 2^{-k} \|2^{m+k}\chi_{E_{m+k}}\|_{L^p}\Big\|_{\ell^q_m} \le \sum_{k=0}^\infty 2^{-k} \Big\| \|2^m \chi_{E_m}\|_{L^p}\Big\|_{\ell^q_m}$$

This completes the proof of the upper bound. □

Lemma 4. *Given $1 \le q < \infty$ and a finite set $\mathcal{A} \subset 2^{\mathbb{Z}}$,*

$$\sum A^q \le \Big|\sum A\Big|^q \le \Big|2 \max_{A \in \mathcal{A}} A\Big|^q \le 2^q \sum A^q$$

where all sums are over $A \in \mathcal{A}$. More generally, for any subset $\mathcal{A}$ of a geometric series and any $0 < q < \infty$,

$$\sum A^q \approx \Big|\sum A\Big|^q$$

where the implicit constants depend on q and the step size of the geometric series.

Proposition 5. *For $1 < p < \infty$ and $1 \le q \le \infty$,*

$$\sup\big\{ |\textstyle\int fg| : \|g\|^*_{L^{p',q'}} \le 1\big\} \approx \|f\|^*_{L^{p,q}}. \tag{6}$$

Indeed, LHS(6) defines a norm on $L^{p,q}$. Note that by (6), this norm is equivalent to our quasi-norm. Moreover, under this norm, $L^{p,q}$ is a Banach space and when $q \ne \infty$, the dual Banach space is $L^{p',q'}$, under the natural pairing.

Remark. When $p = 1$ (and $q \ne 1$), the LHS(6) is typically infinite; indeed, $\int_E |f|$ may well be infinite even for some set E of finite measure. In fact, there there cannot be a norm on $L^{p,q}$ equivalent to our quasi-norm. For example, the impossibility of finding an equivalent norm for $L^{1,\infty}(\mathbb{R})$ can be deduced by computing

$$\Big\| \sum_{n=0}^N |x-n|^{-1}\Big\|^*_{L^{1,\infty}} \approx N \log(N) \quad \text{and} \quad \sum_{n=0}^N \big\| |x-n|^{-1}\big\|^*_{L^{1,\infty}} \approx N.$$

Proof. Because the quasi-norm is positively homogeneous, we need only verify (6) in the case that f and g have quasi-norm comparable to one. We may also assume that $f = \sum 2^n \chi_{F_n}$ and $g = \sum 2^m \chi_{E_m}$. By the normalization just mentioned,

$$\sum_n \big(2^n |F_n|^{1/p}\big)^q \approx 1 \approx \sum_m \big(2^m |E_m|^{1/p'}\big)^{q'} \tag{7}$$

Combining the above with Lemma 4, we obtain

$$\sum_{A\in 2^{\mathbb{Z}}} \Big| \sum_{n:|F_n|\approx A} 2^n A^{1/p} \Big|^q \approx \sum_{A\in 2^{\mathbb{Z}}} \sum_{n:|F_n|\approx A} \big(2^n |F_n|^{1/p}\big)^q \approx 1. \tag{8}$$

and similarly for g. Now we compute:

$$\begin{aligned}
\int |fg|\, dx &= \sum_{n,m} 2^n 2^m |F_n \cap E_m| \\
&\leq \sum_{A,B\in 2^{\mathbb{Z}}} \Big| \sum_{n:|F_n|\sim A} 2^n \Big| \cdot \min(A,B) \cdot \Big| \sum_{m:|E_m|\sim B} 2^m \Big| \\
&\leq \sum_{A,B\in 2^{\mathbb{Z}}} \Big| \sum_{n:|F_n|\sim A} 2^n A^{1/p} \Big| \cdot \min\Big(\big[\tfrac{A}{B}\big]^{\frac{1}{p'}},\ \big[\tfrac{B}{A}\big]^{\frac{1}{p}} \Big) \cdot \Big| \sum_{m:|E_m|\sim B} 2^m B^{1/p'} \Big|.
\end{aligned}$$

Notice that this has the structure of a bilinear form: two vectors (indexed over $2^{\mathbb{Z}}$) with a matrix sitting between them. Moreover, by Schur's test, the matrix is a bounded operator on $\ell^q(2^{\mathbb{Z}})$. Thus,

$$\int |fg|\, dx \lesssim \Big\| \sum_{n:|F_n|\sim A} 2^n A^{1/p} \Big\|_{\ell^q(A\in 2^{\mathbb{Z}})} \cdot \Big\| \sum_{m:|E_m|\sim B} 2^m B^{1/p'} \Big\|_{\ell^{q'}(B\in 2^{\mathbb{Z}})} \approx 1$$

by (8) and the corresponding statement for g. This completes proof of the $\lesssim$ part of (6). We turn now to the opposite inequality. Given $f = \sum 2^n \chi_{F_n} \in L^{p,q}$, we choose

$$g = \sum_n \Big(2^n |F_n|^{\frac{1}{p}}\Big)^{q-1} |F_n|^{-\frac{1}{p'}} \chi_{F_n} = \sum_n 2^{n(q-1)} |F_n|^{\frac{q-p}{p}} \chi_{F_n}.$$

Then

$$\int fg = \sum_n \Big(2^n |F_n|^{\frac{1}{p}}\Big)^{q-1} 2^n |F_n|^{1-\frac{1}{p'}} = \sum_n \Big(2^n |F_n|^{\frac{1}{p}}\Big)^q \approx \Big(\|f\|^*_{L^{p,q}}\Big)^q \approx 1.$$

It remains to show that $\|g\|^*_{L^{p',q'}} \lesssim 1$. By Proposition 3,

$$\big(\|g\|^*_{L^{p',q'}}\big)^{q'} \approx \sum_{A\in 2^{\mathbb{Z}}} A^{q'} \Big| \sum_{n\in N(A)} |F_n| \Big|^{q'/p'} \quad \text{where} \quad n \in N(A) \Leftrightarrow 2^{n(q-1)} |F_n|^{\frac{q-p}{p}} \approx A.$$

Notice that for each A, the sum in n is over part of a geometric series; indeed,

$$n \in N(A) \iff |F_n| \approx A^{\frac{p}{q-p}} 2^{-n\frac{p(q-1)}{q-p}}.$$

Thus Lemma 4 applies and yields

$$\big(\|g\|^*_{L^{p',q'}}\big)^{q'} \approx \sum_{A\in 2^{\mathbb{Z}}} A^{q'} \sum_{n\in N(A)} |F_n|^{q'/p'} \approx \sum_n 2^{nq} |F_n|^{q/p} \approx 1.$$

This provides the needed bound on g and so completes the proof of (6).

The fact that LHS(6) is indeed a norm is a purely abstract statement about vector spaces and (separating) linear functionals. The proof that $L^{p,q}$ is complete in this norm differs little from the usual Riesz–Fischer argument.

Let ℓ be a continuous linear functional on $L^{p,q}$. By definition, $|\ell(\chi_E)| \lesssim |E|^{1/p}$ and so the measure $E \mapsto \ell(\chi_E)$ is absolutely continuous with respect to Lebesgue measure and so is represented by some locally L^1 function g. This is the Radon–Nikodym Theorem. By linearity this representation of the functional extends to simple functions. Boundedness when tested against simple functions suffices to show that $g \in L^{p',q'}$. When $q \neq \infty$, the simple functions are dense in $L^{p,q}$ and so our linear functional admits the desired representation.

When $q = \infty$ the simple functions are not dense. For example, one cannot approximate $|x|^{-d/p} \in L^{p,\infty}(\mathbb{R}^d)$ by simple functions. Indeed, inspired by the Banach limit linear functionals on $\ell^\infty(\mathbb{Z})$ we can construct a non-trivial linear functional on $L^{p,\infty}$ that vanishes on simple functions. Let $\mathcal{L}$ denote the vector space of $f \in L^{p,\infty}$ such that

$$\ell(f) := \lim_{x\to 0} |x|^{d/p} f(x) \quad \text{exists.}$$

Notice that $\mathcal{L}$ contains the simple functions and that ℓ vanishes on these. By the Hahn–Banach theorem, we can extended ℓ to a linear functional on all of $L^{p,q}$. $\square$

Definition 6. We say that a mapping T on (some class of) measurable functions is *sublinear* if it obeys

$$\big|T(cf)(x)\big| \leq |c|\big|Tf(x)\big| \quad \text{and} \quad \big|T(f+g)(x)\big| \leq \big|[Tf](x)\big| + \big|[Tg](x)\big|$$

for all $c \in \mathbb{C}$ and measurable functions f and g (in the domain of T).

Linear maps are obviously sublinear. Moreover, if $\{T_t\}$ is a family of linear maps then

$$[\mathcal{T}f](x) := \big\|[T_t f](x)\big\|_{L^q_t}$$

is sublinear. The case $q = \infty$ yields a kind of 'maximal function', while $q = 2$ gives a kind of 'square function'.

Theorem 7 (Marcinkiewicz interpolation theorem)**.** *Fix $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ with $p_0 \neq p_1$ and $q_0 \neq q_1$. Let T be a sublinear operator that obeys*

$$\int \big|\chi_E(x)[T\chi_F](x)\big|\, dx \lesssim |E|^{1/q_j'}|F|^{1/p_j} \qquad j \in \{0,1\} \tag{9}$$

uniformly for finite-measure sets E and F. Then for any $1 \leq r \leq \infty$ and $\theta \in (0,1)$,

$$\big\|Tf\big\|^*_{L^{q_\theta,r}} \lesssim \big\|f\big\|^*_{L^{p_\theta,r}}$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and similarly, $q_\theta = (1-\theta)/q_0 + \theta/q_1$.

Remarks. 1. This form of the result is actually due to Hunt. The original version is Corollary 8 below.

2. Inequalities of the form (9) are known as *restricted weak type estimates.* Note

$$\int \big|\chi_E[T\chi_F]\big|\, dx \lesssim |E|^{1/q'}|F|^{1/p} \;\Leftrightarrow\; \big\|T\chi_F\big\|_{L^{q,\infty}} \lesssim |F|^{1/p} \;\Leftarrow\; \big\|Tf\big\|_{L^{q,\infty}} \lesssim \|f\|_{L^p}$$

as can be shown using Propositions 3 and 5. The rightmost inequality here is called a *weak type estimate.* At the top of the food chain sits the *strong type estimate:* $\|Tf\|_{L^q} \lesssim \|f\|_{L^p}$. If $p_\theta \leq q_\theta$ we then we can choose $r = q_\theta$ and so (using the nesting of Lorentz spaces) obtain a strong type estimate as the conclusion of the theorem.

3. The hypothesis $p_\theta \leq q_\theta$ is needed to obtain the strong type conclusion. Consider, for example,

$$f(x) \mapsto x^{-1/2}f(x) \quad \text{which maps} \quad L^p\big([0,\infty), dx\big) \to L^{\frac{2p}{p+2},\infty}\big([0,\infty), dx\big)$$

boundedly for all $2 \leq p \leq \infty$. However

$$f(x) = x^{-1/p}[\log(x + x^{-1})]^{-\frac{p+2}{2p}}$$

shows that T does not map L^p into $L^{2p/(p+2)}$ for any such p.

Proof of Theorem 7. By the duality relations among Lorentz spaces (cf. Proposition 5), it suffices to show that

$$\left|\int g(x)[Tf](x)\,dx\right| \lesssim 1 \quad \text{whenever} \quad \|f\|^*_{L^{p_\theta,r}} \approx 1 \approx \|g\|^*_{L^{q'_\theta,r'}}.$$

Moreover, we can take g to be of the form $\sum 2^m \chi_{E_m}$.

We would like to take f of the same form, but this takes a little more justification. First by splitting a general f into real/imaginary parts and then each of these into its positive/negative parts, we see that it suffices to consider non-negative functions f. This also justifies taking g of the special form. Note that for g we can safely round up to the nearest power of two; however, since T need not have any monotonicity properties we are not able to do this for f.

Now by using the binary expansion of the values of $f(x) \geq 0$ at each point, we see that it is possible to write f as the sum of a sequence functions of the form $\sum 2^n \chi_{F_n}$ in such a way the summands are bounded pointwise by f, $\frac{1}{2}f$, $\frac{1}{4}f$, and so on. Since L^{p_θ,q_θ} is a Banach space (specifically the triangle inequality holds) we can just sum the pieces back together. (A similar decomposition is possible under a quasi-norm, but a little cunning is required to avoid the summability being swamped by the constants from the triangle inequality.)

Now we have reduced to considering $f = \sum 2^n \chi_{F_n}$ and $g = \sum 2^m \chi_{E_m}$, let us compute:

$$\begin{aligned}\int |g(x)[Tf](x)|\,dx &\lesssim \sum_{n,m} 2^n 2^m \min_{j\in\{0,1\}} \big(|F_n|^{1/p_j} |E_m|^{1/q'_j} \big) \\ &\lesssim \sum_{A,B\in 2^{\mathbb{Z}}} \Bigg(\sum_{n:|F_n|\sim A} 2^n A^{1/p_\theta} \Bigg) \min_{j\in\{0,1\}} \big(A^{\frac{1}{p_j}-\frac{1}{p_\theta}} B^{\frac{1}{q'_j}-\frac{1}{q'_\theta}} \big) \Bigg(\sum_{m:|E_m|\sim B} 2^m B^{1/q'_\theta} \Bigg).\end{aligned}$$

Once again we recognize the structure of a bilinear form with vectors indexed over $2^{\mathbb{Z}}$. With a little effort, we see that the matrix has the form

$$\min_{j\in\{0,1\}} \bigg(\Big[A^{\frac{1}{p_1}-\frac{1}{p_0}} B^{\frac{1}{q'_1}-\frac{1}{q'_0}} \Big]^{j-\theta} \bigg)$$

and so is bounded on $\ell^r(2^{\mathbb{Z}})$ by Schur's test. (It is essential here that $p_0 \neq p_1$ and $q_0 \neq q_1$.) On the other hand, by Lemma 4,

$$\sum_{A\in 2^{\mathbb{Z}}} \Bigg(\sum_{n:|F_n|\sim A} 2^n A^{1/p_\theta} \Bigg)^r \approx \sum_n \big(2^n |F_n|^{1/p_\theta} \big)^r \approx \big(\|f\|^*_{L^{p_\theta,r}} \big)^r \approx 1$$

and similarly for g, though we use power r'. Putting these all together completes the proof. $\square$

Corollary 8 (Marcinkiewicz interpolation theorem)**.** *Suppose* $1 \leq p_0 < p_1 \leq \infty$ *and* T *is a sublinear operator that obeys*

$$\big\|Tf\big\|_{L^{p_0,\infty}} \lesssim \|f\|_{L^{p_0}} \quad \textit{and} \quad \big\|Tf\big\|_{L^{p_1,\infty}} \lesssim \|f\|_{L^{p_1}} \tag{10}$$

uniformly for measurable functions f*. Then for any* $\theta \in (0,1)$,

$$\|Tf\|_{L^{p_\theta}} \lesssim \|f\|_{L^{p_\theta}}$$

where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ *and similarly,* $q_\theta = (1-\theta)/q_0 + \theta/q_1$.