247A Notes on Lorentz spaces

Definition 1. For $1 \leq p < \infty$ and $f : \mathbb{R}^d \to \mathbb{C}$ we define

$$\|f\|_{L^p_{\text{weak}}(\mathbb{R}^d)} := \sup_{\lambda > 0} \lambda \left\{ \{x : |f(x)| > \lambda\} \right\}^{1/p}$$

and the weak $L^p$ space

$$L^p_{\text{weak}}(\mathbb{R}^d) := \left\{ f : \|f\|_{L^p_{\text{weak}}(\mathbb{R}^d)} < \infty \right\}.$$ 

Equivalently, $f \in L^p_{\text{weak}}$ if and only if $\|\{x : |f(x)| > \lambda\}\| \lesssim \lambda^{-p}$. 

Warning. The quantity in (1) does not define a norm. This is the reason we append the asterisk to the usual norm notation.

To make a side-by-side comparison with the usual $L^p$ norm, we note that

$$\|f\|_{L^p} = \left( \int \int_{0 \leq \lambda < \{f(x)\}} p\lambda^{p-1} \, d\lambda \, dx \right)^{1/p}$$

and, with the convention that $\|\cdot\|_0 := \int d\lambda/\lambda$,

$$\|f\|_{L^p} = \left( \int \int_{0 \leq \lambda < \{f(x)\}} p\lambda^{p-1} \, d\lambda \, dx \right)^{1/p} = p^{1/p} \|\lambda \{\{f\} > \lambda\}\|_{L^p((0,\infty), \frac{d\lambda}{\lambda})}$$

and, with the convention that $p^{1/\infty} = 1$,

$$\|f\|_{L^p_{\text{weak}}} = p^{1/\infty} \|\lambda \{\{f\} > \lambda\}\|_{L^\infty((0,\infty), \frac{d\lambda}{\lambda})}.$$ 

This suggests the following definition.

Definition 2. For $1 \leq p < \infty$ and $1 \leq q \leq \infty$ we define the Lorentz space $L^{p,q}(\mathbb{R}^d)$ as the space of measurable functions $f$ for which

$$\|f\|_{L^{p,q}} := \|p^{1/q} |\lambda\{f > \lambda\}|^{1/p} \|_{L^q(d\lambda/\lambda)} < \infty.$$ 

From the discussion above, we see that $L^p L^p = L^p$ and $L^{p,\infty} = L^{p,\text{weak}}$. Again $\lambda \cdot \|f\|_{L^{p,q}}$ is not a norm in general. Nevertheless, it is positively homogeneous: for all $a \in \mathbb{C},$

$$\|af\|_{L^{p,q}} = \|\lambda \{\{f\} > |a|^{-1}\lambda\}\|_{L^q(d\lambda/\lambda)} = |a| \cdot \|f\|_{L^{p,q}}$$

(strictly the case $a = 0$ should receive separate treatment). In lieu of the triangle inequality, we have the following:

$$\|f + g\|_{L^{p,q}} = \|\lambda \{\{f + g\} > \lambda\}\|_{L^q(d\lambda/\lambda)}^{1/p}$$

$$\leq \|\lambda \{\{f\} > \frac{1}{2}\lambda\} + \{\{g\} > \frac{1}{2}\lambda\}\|_{L^q(d\lambda/\lambda)}^{1/p}$$

$$\leq \|\lambda \{\{f\} > \frac{1}{2}\lambda\}\|_{L^q(d\lambda/\lambda)}^{1/p} + \|\lambda \{\{g\} > \frac{1}{2}\lambda\}\|_{L^q(d\lambda/\lambda)}^{1/p}$$

by the subadditivity of fractional powers and the triangle inequality in $L^q(d\lambda/\lambda)$. Thus

$$\|f + g\|_{L^{p,q}}^{1/p} \leq 2 \|f\|_{L^{p,q}} + 2 \|g\|_{L^{p,q}},$$

Combining (3), (4), and the fact that $\|f\|_{L^{p,q}} = 0$ implies $f \equiv 0$ almost everywhere, we see that $\lambda \cdot \|f\|_{L^{p,q}}$ obeys the axioms of a quasi-norm. When $p > 1$, this quasi-norm is equivalent to an actual norm (see below). When $p = 1$ and $q \neq 1$, there cannot be a norm that is equivalent to our quasi-norm. However there is a metric that generates the same topology. In either case, we obtain a complete metric space.

Notice that (i) if $|f| \geq |g|$ then $\|f\|_{L^{p,q}} \geq \|g\|_{L^{p,q}}$ and (ii) The quasi-norms are rearrangement invariant, which is to say that $\|f\|_{L^{p,q}} = \|f \circ \phi\|_{L^{p,q}}$ for any measurable preserving bijection $\phi : \mathbb{R}^d \to \mathbb{R}^d$. 

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Proposition 3. Given \( f \in L^{p,q} \), we write \( f = \sum f_m \) where
\[
f_m(x) := f(x) \chi_{\{ |x| > 2^m \}}.
\]

Then
\[
\|f\|^p_{L^{p,q}} \approx_{p,q} \left\| \sum f_m \right\|_{L^q(\mathbb{R}^d)}.
\]

In particular, \( L^{p,q_1} \subseteq L^{p,q_2} \) whenever \( q_1 \leq q_2 \).

Proof. It suffices to consider \( f \) of the form \( f = \sum 2^m \chi_{E_m} \) with disjoint sets \( E_m \) (cf. \( E_m = \{ 2^m \leq |f| < 2^{m+1} \} \)). Now
\[
\left( \|f\|^p_{L^{p,q}} \right)^q = p \int_0^\infty \lambda^q \{ |f| > \lambda \}^{q/p} d\lambda
\]
\[
= p \sum_m \int_{2^{m-1}}^{2^m} \lambda^q \left( \sum_{n \geq m} |E_n| \right)^{q/p} d\lambda
\]
\[
\approx \sum_m \left( \sum_{n \geq m} |E_n| \right)^{1/p} 2^m.
\]

To obtain a lower bound, we keep only the summand \( n = m \); for an upper bound, we use the subadditivity of fractional powers. This yields
\[
\sum 2^m |E_m|^{1/p} \leq \left\| \sum 2^m |E_m|^{1/p} \right\|_{\ell^p}.
\]

As \( \sum 2^m \chi_{E_m} \|_{L^p} = 2^m |E_m|^{1/p} \), we have our desired lower bound. To obtain the upper bound, we use the triangle inequality in \( \ell^q(\mathbb{Z}) \):
\[
\text{RHS}(5) = \left\| \sum_{k=0}^\infty 2^{-k} 2^{m+k} \chi_{E_{m+k}} \right\|_{L^p} \leq \sum_{k=0}^\infty 2^{-k} \left\| 2^m \chi_{E_m} \right\|_{L^p}.
\]

This completes the proof of the upper bound. \( \square \)

Lemma 4. Given \( 1 \leq q < \infty \) and a finite set \( A \subset 2^\mathbb{Z} \),
\[
\sum A^q \leq \left( \sum A \right)^q \leq 2 \max_{A \in A} A^q \leq 2^q \sum A^q
\]
where all sums are over \( A \in A \). More generally, for any subset \( A \) of a geometric series and any \( 0 < q < \infty \),
\[
\sum A^q \approx \left( \sum A \right)^q
\]
where the implicit constants depend on \( q \) and the step size of the geometric series.

Proposition 5. For \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \),
\[
\sup \{ \|f\|_{L^{p,q}} : \|g\|_{L^{p',q'}} \leq 1 \} \approx \|f\|_{L^{p,q}}.
\]

Indeed, LHS(6) defines a norm on \( L^{p,q} \). Note that by (6), this norm is equivalent to our quasi-norm. Moreover, under this norm, \( L^{p,q} \) is a Banach space and when \( q \neq \infty \), the dual Banach space is \( L^{p',q'} \), under the natural pairing.

Remark. When \( p = 1 \) (and \( q \neq 1 \)), the LHS(6) is typically infinite; indeed, \( \int_E |f| \) may well be infinite even for some set \( E \) of finite measure. In fact, there there cannot be a norm on \( L^{p,q} \) equivalent to our quasi-norm. For example, the impossibility of finding an equivalent norm for \( L^{1,\infty}(\mathbb{R}) \) can be deduced by computing
\[
\left\| \sum_{n=0}^N |x-n|^{-1} \right\|_{L^{1,\infty}} \approx N \log(N) \quad \text{and} \quad \sum_{n=0}^N \left\| |x-n|^{-1} \right\|_{L^{1,\infty}} \approx N.
\]
Proof. Because the quasi-norm is positively homogeneous, we need only verify (6) in the case that \( f \) and \( g \) have quasi-norm comparable to one. We may also assume that \( f = \sum 2^n \chi_{F_n} \) and \( g = \sum 2^n \chi_{E_n} \). By the normalization just mentioned,

\[
\sum_n (2^n |F_n|^{1/p})^q \approx 1 \approx \sum_m (2^m |E_m|^{1/p'})^q
\]

Combining the above with Lemma 4, we obtain

\[
\sum_{A \in 2^\mathbb{Z}} \left| \sum_{n:F_n|A} 2^n A^{1/p} \right|^q \approx \sum_{A \in 2^\mathbb{Z}} \sum_{n:F_n|A} (2^n |F_n|^{1/p})^q \approx 1.
\]

and similarly for \( g \). Now we compute:

\[
\int |fg| \, dx = \sum_{n,m} 2^n 2^m |F_n \cap E_m| \\
\leq \sum_{A,B \in 2^\mathbb{Z}} \left| \sum_{n:F_n|A} 2^n \right| \cdot \min(A,B) \cdot \left| \sum_{m:E_m|B} 2^m \right| \\
\leq \sum_{A,B \in 2^\mathbb{Z}} \left| \sum_{n:F_n|A} 2^n A^{1/p} \right| \cdot \min\left(\left[\frac{2}{2p}\right], \left[\frac{2}{2q}\right]\right) \cdot \left| \sum_{m:E_m|B} 2^m B^{1/p'} \right|.
\]

Notice that this has the structure of a bilinear form: two vectors (indexed over \( 2^\mathbb{Z} \)) with a matrix sitting between them. Moreover, by Schur’s test, the matrix is a bounded operator on \( \ell^2(2^\mathbb{Z}) \). Thus,

\[
\int |fg| \, dx \lesssim \left\| \sum_{n:F_n|A} 2^n \right\|_{\ell^p(A \in 2^\mathbb{Z})} \cdot \left\| \sum_{m:E_m|B} 2^m \right\|_{\ell^{q'}(B \in 2^\mathbb{Z})} \approx 1
\]

by (8) and the corresponding statement for \( g \). This completes proof of the \( \lesssim \) part of (6). We turn now to the opposite inequality. Given \( f = \sum 2^n \chi_{F_n} \in L^{p,q} \), we choose

\[
g = \sum_n (2^n |F_n|^{1/p})^{q-1} |F_n|^{1/p'} \chi_{F_n} = \sum_n 2^{n(q-1)} |F_n|^{\frac{q-p}{p}} \chi_{F_n}.
\]

Then

\[
\int fg = \sum_n (2^n |F_n|^{1/p})^{q-1} 2^n |F_n|^{1-p/p'} = \sum_n (2^n |F_n|^{1/p})^q \approx \left( \|f\|_{L^{p,q}}^* \right)^q \approx 1.
\]

It remains to show that \( \|g\|_{L^{p',q'}_*}^* \lesssim 1 \). By Proposition 3,

\[
(\|g\|_{L^{p',q'}_*}^* \right)^q \approx \sum_{A \in 2^\mathbb{Z}} A^q \cdot \left| \sum_{n \in N(A)} |F_n|^{q'/p'} \right| \quad \text{where} \quad n \in N(A) \Leftrightarrow 2^n(q-1)|F_n|^{\frac{q-p}{p}} \approx A.
\]

Notice that for each \( A \), the sum in \( n \) is over part of a geometric series; indeed,

\[
n \in N(A) \quad \Leftrightarrow \quad |F_n| \approx A^{2^{-n\frac{q-p}{q-1}}}.
\]

Thus Lemma 4 applies and yields

\[
(\|g\|_{L^{p',q'}_*}^* \right)^q \approx \sum_{A \in 2^\mathbb{Z}} A^q \cdot \sum_{n \in N(A)} |F_n|^{q'/p'} \approx \sum_n 2^{nq} |F_n|^{q/p} \approx 1.
\]

This provides the needed bound on \( g \) and so completes the proof of (6).

The fact that LHS(6) is indeed a norm is a purely abstract statement about vector spaces and (separating) linear functionals. The proof that \( L^{p,q} \) is complete in this norm differs little from the usual Riesz–Fischer argument.

Let \( \ell \) be a continuous linear functional on \( L^{p,q} \). By definition, \( |\ell(\chi_E)| \lesssim |E|^{1/p} \) and so the measure \( E \mapsto |\ell(\chi_E)| \) is absolutely continuous with respect to Lebesgue measure and so is represented by some locally \( L^1 \) function \( g \). This is the Radon–Nikodym Theorem. By linearity this representation of the functional extends to
simple functions. Boundedness when tested against simple functions suffices to show that \(g \in L^{p^*} \). When \( q \neq \infty \), the simple functions are dense in \(L^p\) and so our linear functional admits the desired representation.

When \( q = \infty \) the simple functions are not dense. For example, one cannot approximate \(|x|^{-d/p} \in L^{p,\infty}(\mathbb{R}^d)\) by simple functions. Indeed, inspired by the Banach limit linear functionals on \(\ell^\infty(\mathbb{Z})\) we can construct a non-trivial linear functional on \(L^{p,\infty}\) that vanishes on simple functions. Let \(\mathcal{L}\) denote the vector space of \(f \in L^{p,\infty}\) such that

\[
\ell(f) := \lim_{x \to 0} |x|^{d/p} f(x) \quad \text{exists.}
\]

Notice that \(\mathcal{L}\) contains the simple functions and that \(\ell\) vanishes on these. By the Hahn–Banach theorem, we can extended \(\ell\) to a linear functional on all of \(L^{p^*}\).

**Definition 6.** We say that a mapping \(T\) on (some class of) measurable functions is sublinear if it obeys

\[
|T(cf)(x)| \leq |c||Tf(x)| \quad \text{and} \quad |T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|
\]

for all \(c \in \mathbb{C}\) and measurable functions \(f\) and \(g\) (in the domain of \(T\)).

Linear maps are obviously sublinear. Moreover, if \(\{T_i\}\) is a family of linear maps then

\[
[Tf](x) := \left\|T_i f(x)\right\|_{L^q}
\]

is sublinear. The case \(q = \infty\) yields a kind of ‘maximal function’, while \(q = 2\) gives a kind of ‘square function’.

**Theorem 7** (Marcinkiewicz interpolation theorem). Fix \(1 \leq p_0, p_1, q_0, q_1 \leq \infty\) with \(p_0 \neq p_1\) and \(q_0 \neq q_1\). Let \(T\) be a sublinear operator that obeys

\[
\int |\chi_E(x)T|\chi_F||dx \lesssim |E|^{1/q_j}|F|^{1/p_j} \quad j \in \{0, 1\}
\]

uniformly for finite-measure sets \(E\) and \(F\). Then for any \(1 \leq r \leq \infty\) and \(\theta \in (0, 1)\),

\[
\|Tf\|_{L^{q_0,r}} \lesssim \|f\|_{L^{p_0,r}}
\]

where \(1/p_0 = (1 - \theta)/p_0 + \theta/p_1\) and similarly, \(q_0 = (1 - \theta)/q_0 + \theta/q_1\).

**Remarks.** 1. This form of the result is actually due to Hunt. The original version is Corollary 8 below.

2. Inequalities of the form (9) are known as restricted weak type estimates. Note

\[
\int |\chi_E|T|\chi_F||dx \lesssim |E|^{1/q} |F|^{1/p} \Leftrightarrow \|T|\chi_F||_{L^{q,\infty}} \lesssim |F|^{1/p} \Leftrightarrow \|Tf\|_{L^{q,\infty}} \lesssim \|f\|_{L^p}
\]

as can be shown using Propositions 3 and 5. The rightmost inequality here is called a weak type estimate. At the top of the food chain sits the strong type estimate:

\[
\|Tf\|_{L^\infty} \lesssim \|f\|_{L^p}.
\]

If \(p_0 \leq q_0\) we then can choose \(r = q_0\) and so (using the nesting of Lorentz spaces) obtain a strong type estimate as the conclusion of the theorem.

3. The hypothesis \(p_0 \leq q_0\) is needed to obtain the strong type conclusion. Consider, for example,

\[
f(x) \mapsto x^{-1/2}f(x) \quad \text{which maps} \quad L^p([0, \infty), dx) \to L^{2p/(p+2)}([0, \infty), dx)
\]

boundedly for all \(2 \leq p \leq \infty\). However

\[
f(x) = x^{-1/p} [\log(x + x^{-1})]^{-\frac{p+2}{2p}}
\]

shows that \(T\) does not map \(L^p\) into \(L^{2p/(p+2)}\) for any such \(p\).
Proof of Theorem 7. By the duality relations among Lorentz spaces (cf. Proposition 5), it suffices to show that
\[ \int |g(x)[Tf](x)| \, dx \leq 1 \quad \text{whenever} \quad \|f\|_{L^{p_0,\infty}} \approx 1 \approx \|g\|_{L^{q_0,\infty}}. \]
Moreover, we can take \( g \) to be of the form \( \sum 2^n \chi_{E_n} \).

We would like to take \( f \) of the same form, but this takes a little more justification. First by splitting a general \( f \) into real/imaginary parts and then each of these into its positive/negative parts, we see that it suffices to consider non-negative functions \( f \). This also justifies taking \( g \) of the special form. Note that for \( g \) we can safely round up to the nearest power of two; however, since \( T \) need not have any monotonicity properties we are not able to do this for \( f \).

Now by using the binary expansion of the values of \( f(x) \geq 0 \) at each point, we see that it is possible to write \( f \) as the sum of a sequence functions of the form \( \sum 2^n \chi_{F_n} \) in such a way the summands are bounded pointwise by \( f, \frac{1}{2} f, \frac{1}{4} f, \) and so on. Since \( L^{p_0,q_0} \) is a Banach space (specifically the triangle inequality holds) we can just sum the pieces back together. (A similar decomposition is possible under a quasi-norm, but a little cunning is required to avoid the summability being swamped by the constants from the triangle inequality.)

Now we have reduced to considering \( f = \sum 2^n \chi_{F_n} \) and \( g = \sum 2^n \chi_{E_n} \), let us compute:
\[ \int |g(x)[Tf](x)| \, dx \leq \sum_{n,m} 2^n 2^m \min_{j \in \{0,1\}} \left( |F_n|^{1/p_0} |E_m|^{1/q_0'} \right) \]
\[ \leq \sum_{A,B \in 2^\mathbb{R}} \left( \sum_{n: |F_n| \sim A} 2^n A^{1/p_0} \right) \min_{j \in \{0,1\}} \left( A^{\frac{1}{p_0}} B^{\frac{1}{q_0'}} \right) \left( \sum_{m: |E_m| \sim B} 2^m B^{1/q_0'} \right). \]

Once again we recognize the structure of a bilinear form with vectors indexed over \( 2^\mathbb{R} \). With a little effort, we see that the matrix has the form
\[ \min_{j \in \{0,1\}} \left( \left[ A^{\frac{1}{p_0}} B^{\frac{1}{q_0'}} \right]^{j-\theta} \right) \]
and so is bounded on \( \ell'(2^\mathbb{R}) \) by Schur’s test. (It is essential here that \( p_0 \neq p_1 \) and \( q_0 \neq q_1 \).) On the other hand, by Lemma 4,
\[ \sum_{A \in 2^\mathbb{R}} \left( \sum_{n: |F_n| \sim A} 2^n A^{1/p_0} \right)^r \approx \sum_n \left( 2^n |F_n|^{1/p_0} \right)^r \approx \left( \|f\|_{L^{p_0,\infty}} \right)^r \approx 1 \]
and similarly for \( g \), though we use power \( r' \). Putting these all together completes the proof. \( \square \)

Corollary 8 (Marcinkiewicz interpolation theorem). Suppose \( 1 \leq p_0 < p_1 \leq \infty \) and \( T \) is a sublinear operator that obeys
\[ \|Tf\|_{L^{p_0,\infty}} \lesssim \|f\|_{L^{p_0}} \quad \text{and} \quad \|Tf\|_{L^{p_1,\infty}} \lesssim \|f\|_{L^{p_1}} \]
uniformly for measurable functions \( f \). Then for any \( \theta \in (0,1) \),
\[ \|Tf\|_{L^{p_\theta}} \lesssim \|f\|_{L^{p_0}} \]
where \( 1/p_\theta = (1-\theta)/p_0 + \theta/p_1 \) and similarly, \( q_\theta = (1-\theta)/q_0 + \theta/q_1 \).