1. Suppose $f : \Omega \to \mathbb{D}$ is holomorphic with $\Omega \subset \mathbb{D}$ convex and $\partial\Omega \cap \partial\mathbb{D}$ consisting of a single (closed) arc $\bar{I}$. Let I denote the interior of $\bar{I}$ viewed as a subset of $\partial\mathbb{D}$. Show the following: if $z \mapsto |f(z)|$ extends continuously to I and is $\equiv 1$ there, then f extends holomorphically to an open neighbourhood of I in $\mathbb{C}$.

2. Suppose $1 < R_2 < R_1$ and let $A(R) := \{z \in \mathbb{C} : 1 < |z| < R\}$. Show that there is no biholomorphism $f : A(R_1) \to A(R_2)$ by filling in the details of each of the following sketches:
(a) By repeated use of the reflection principle, f can be extended to a automorphism (=biholomorphism onto itself) of $\mathbb{C} \cup \{\infty\}$. This leads to a contradiction.
(b) A subsequence of the n-fold iterates of f, that is, $f \circ f \circ \cdots \circ f$, converges uniformly on compact subsets of $A(R_1)$. Show, using the Hadamard Three Circle/Line Theorem, for example, that the limit is constant. Derive a contradiction.
(c) A biholomorphism of Riemann surfaces induces a biholomorphism of their universal covers. Verify that there is no suitable biholomorphism of the universal covers.

Remarks: (i) For part (a) we'd like to know that f extends continuously to the boundary. Use Problem 1 and bit of the argument from Carathéodory's Theorem. There are other approaches, but I suspect this is the simplest.
(ii) A complete list (up to biholomorphism) of Riemann surfaces that are topological annuli is the following: $\mathbb{C} \setminus \{0\}$, and $\{z : r < |z| < 1\}$ for $0 \le r < 1$. Moreover no two members of the list are biholomorphic. The number r is called the conformal modulus of the annulus.

3. Show that $\mathbb{C} \cup \{\infty\}$ is the only elliptic Riemann surface up to biholomorphism. Hint: look for possible deck transformations.

4. Let $\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0, 1\}$ be the universal cover constructed in class; here $\mathbb{H}$ denotes the upper half-plane.
(a) Express $\lambda(z + 1)$ and $\lambda(-1/z)$ in terms of $\lambda(z)$ and so deduce that

$$J(\tau) := \frac{4}{27} \frac{\big[1 - \lambda(\tau) + \lambda(\tau)^2\big]^3}{\lambda(\tau)^2\big[1 - \lambda(\tau)\big]^2} \quad \text{is invariant under} \quad z \mapsto z + 1 \quad \text{and} \quad z \mapsto \frac{-1}{z}.$$

(b) Show that every Mobius transformation

$$z \mapsto \frac{az + b}{cz + d} \quad \text{with} \quad a, b, c, d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1$$

can be written as a composition of $z \mapsto z + 1$ and $z \mapsto -1/z$. *Hint:* Proceed by induction in $|c|$, treating $c = 0$ and $c = 1$ by direct means. Observe that when $c \ge 2$ we can write $d = qc + r$ with $q \in \mathbb{Z}$ and $0 < r < c$.
(c) Suppose $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$ and ordered so that $\operatorname{Im}(\omega_1/\omega_2) > 0$. The quantity $J(\omega_1/\omega_2)$ is called the Klein's invariant of the torus $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$. Show that if two torii are biholomorphic, then they have the same Klein invariant.

Remark: It is also true that if two torii have the same Klein invariant then they are biholomorphic.

5. Fix $0 < a < 1$ and let $\mathcal{T} := \{(x, y) \in \hat{\mathbb{C}}^2 : x^2 = y(y - a)(y - 1)\}$; here $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
(a) Show that $\mathcal{T}$ can be seen as a Riemann surface. Specifically, show that $\mathcal{T}$ can be

covered by open sets on which $\phi_1 : (x, y) \mapsto x$, or $\phi_2 : (x, y) \mapsto y$, or

$$\phi_3^{-1} : t \mapsto \left(t^{-3}\sqrt{[1 - t^2][1 - at^2]}, t^{-2}\right)$$

can be used as a local chart.

(b) By comparing the singularities at $z = 0$, show that the Weierstrass $\wp$ function solves an ODE of the form

$$[\wp'(z)]^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

and identify the coefficients g_2 and g_3 as sums over the period lattice.

Remarks: For part (a), I recommend deducing a holomorphic implicit function theorem from the usual real-variable version. The connection between the two parts of the problem is obscured by having two different cubic equations appearing in the parts; however, it makes for a clearer treatment of part (a) if we know the roots of the cubic (and that they are distinct). We also chose them to be real for the sake of simplicity. In general, the mapping $z \mapsto (\wp'(z), \wp(z))$ provides a covering map of the torus $x^2 = 4y^3 - g_2y - g_3$. Doubly periodic functions first arose in the study of certain 'elliptic integrals' (a class that includes the appropriate integral for determining the length of an arc on an ellipse, hence the name). We can see a connection to part (b): endeavouring to solve the differential equation via separation of variables one discovers that

$$z = z_0 + \int_{\wp(z_0)}^{\wp(z)} \frac{dy}{\sqrt{4y^3 - g_2y - g_3}}$$