246B Homework 2 Due Feb. 13

1. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $\varepsilon > 0$. Show that

$$F_\varepsilon : z \mapsto \log\bigl[\max(|f(z)|, \varepsilon\bigr]$$

is subharmonic.

Remark: Putting the $\varepsilon > 0$ in the previous problem made the function F_ε continuous, so we can apply all we know about continuous subharmonic functions. To get results about the original function f we simply send $\varepsilon \to 0$ at the end (at least if $f \not\equiv 0$).

2. (a) Let $\Omega = \{z : 0 < \operatorname{Re} z < 1\}$. Suppose $u : \bar{\Omega} \to \mathbb{R}$ is continuous and subharmonic on Ω and obeys

$$u(z) \leq A\cosh(\alpha \operatorname{Im} z) \qquad \text{for some } A > 0 \text{ and } 0 < \alpha < \pi.$$

Show that

$$\sup_{z\in\Omega} u(z) \leq \sup_{x\in\{0,1\}} \sup_{y\in\mathbb{R}} u(x+iy),$$

which is a form of maximum principle for this unbounded domain — a form of Phragmén–Lindelöf principle.

(b) Upgrade the conclusion of part (a) to read as follows: for each $0 < \theta < 1$,

$$\sup_{y\in\mathbb{R}} u(\theta+iy) \leq (1-\theta)\bigl[\sup_{y\in\mathbb{R}} u(0+iy)\bigr] + \theta\bigl[\sup_{y\in\mathbb{R}} u(1+iy)\bigr].$$

(c) Deduce the Three Lines Theorem of Hadamard: Suppose $f : \bar{\Omega} \to \mathbb{C}$ is bounded and continuous on $\bar{\Omega}$ and holomorphic on Ω, then

$$M(\theta) \leq M(0)^{1-\theta} M(1)^{\theta} \quad \text{where} \quad M(x) := \sup_{y\in\mathbb{R}} |f(x+iy)|.$$

Remark: Naturally this argument allow us to relax the assumption that f is bounded; however the version of the Three Lines Theorem stated above invariably suffices for applications. An alternate approach to this theorem is to conformally map the strip to the disk and apply the Lindelöf Maximum Principle. *Hint:* For part (a), consider $b(z) := \operatorname{Re}\cos(\alpha[z-\frac{1}{2}])$.

3. Let Ω_α denote the sector $\{z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \alpha\}$. Suppose f is holomorphic in a neighbourood of $\bar{\Omega}_\alpha$ and

$$f(z) = O\bigl(\exp(|z|^\beta)\bigr) \quad \text{for some} \quad \beta < \frac{\pi}{\alpha}.$$

Show that

$$\sup_{z\in\Omega_\alpha} |f(z)| \leq \sup_{z\in\partial\Omega_\alpha} |f(z)|.$$

4. Let $\vec{v} : \Omega \to \mathbb{R}^2$ be a smooth irrotational divergence-free vector field in a simply connected region $\Omega \subset \mathbb{C}$:

$$\operatorname{rot}(\vec{v}) := \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \quad \text{and} \quad \operatorname{div}(\vec{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0.$$

This can represent the steady-state velocity of an incompressible fluid.
(a) Show that there is a holomorphic function $f : \Omega \to \mathbb{C}$ so that $\vec{v} = \nabla \operatorname{Re} f$. (Note that $\operatorname{Re} f$ is called the velocity potential. $\operatorname{Im} f$ is called the stream function and its level curves depict particle trajectories.)
(b) Use a computer to draw the level curves of $\operatorname{Im}(z+\frac{1}{z})$ and $\operatorname{Im}(z+\frac{1}{z}+\frac{i}{2}\log(z))$ in the region $\mathbb{C} \setminus \mathbb{D}$. Note that both represent idealized fluid flow around a cylinder. In the later case the cylinder is rotating; note that the flow is still irrotational (curl-free) — the circulation is hiding inside the cylinder. The second example also demonstrates the necessity of assuming that Ω is simply connected in part (a).
(c) Use the conformal mapping $z \mapsto w = z + e^z$ from the strip $-\pi < \operatorname{Im} z < \pi$ to determine the fluid flow from a submerged pipe (in 2D). In particular, what does the flow look like far from the origin in the exterior of the pipe — give an expansion up to $o(|w|^{-2})$.

5. Show that no two of the following are biholomorphic: $\mathbb{C}$, $\mathbb{C} \cup \{\infty\}$, $\mathbb{D}$.
Note: The proof should be rather short!

6. Find a uniformizing map for $\mathbb{D} \setminus \{0\}$ and the corresponding group of deck transformations.