246B Homework 1 Due TBD

1. Let $z \mapsto M(z)$ be a $n \times n$ matrix valued function on $\mathbb{C}$, such that each entry is an entire function. Show that

$$z \mapsto \|M(z)\|$$

is subharmonic, where $\|M\|$ denotes the operator norm of M induced by the usual Euclidean norm on $\mathbb{C}^n$.

2. Fix $0 < \alpha < \pi$ and let Ω_α denote the sector $\{z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \alpha\}$.
(a) Construct a positive harmonic function $u : \Omega_\alpha \to \mathbb{R}$ so that $u(z) = o(|z|)$ as $\Omega_\alpha \ni z \to 0$.
(b) Deduce that in the Hopf boundary point lemma we cannot replace the interior ball condition by an interior cone condition.

3. From results in class, we know that for any continuous $\phi : \partial\mathbb{D} \to \mathbb{R}$ there is a unique holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $\operatorname{Im} f(0) = 0$ and $\operatorname{Re} f(z)$ extends continuously to $\partial\mathbb{D}$ and agrees with ϕ there. (Indeed, here $\mathbb{D}$ can be replaced by any bounded open simply connected set Ω that is regular for the Dirichlet problem.) However, for such a concrete case there is also a simple formula (a variant of the Poisson integral formula):
(a) Find an explicit kernel $K : \mathbb{D} \times \partial\mathbb{D} \to \mathbb{C}$ so that

$$(*) \qquad f(z) = \int K(z, e^{i\theta})\phi(e^{i\theta})\frac{d\theta}{2\pi} \quad \text{for all } z \in \mathbb{D}.$$

Note that K is unique. This formula is often called the *Herglotz integral formula.*
(b) Relate the Herglotz integral of $\psi(e^{i\theta}) = \frac{d}{d\theta}\phi(e^{i\theta})$ to that of ϕ.

4. For $0 < \alpha < 1$, let $C^\alpha(\partial\mathbb{D})$ denote the space of Holder continuous functions, with the norm

$$\|\phi\|_{C^\alpha} := \sup_{z\in\partial\mathbb{D}} |\phi(z)| + \sup_{z\neq w\in\partial\mathbb{D}} \frac{|\phi(z) - \phi(w)|}{|z-w|^\alpha}$$

and let f denote the Herglotz integral of such a $\phi \in C^\alpha$. Show that

$$|f'(z)| \le A_\alpha (1-|z|)^{\alpha-1}\|\phi\|_{C^\alpha} \quad \text{and} \quad \sup_{z\neq w\in\mathbb{D}} \frac{|f(z)-f(w)|}{|z-w|^\alpha} \le B_\alpha\|\phi\|_{C^\alpha}$$

for some constants A_α and B_α.

Remark: From this problem we see that if $\phi \in C^\alpha(\partial\mathbb{D})$, then f extends continuously to $\partial\mathbb{D}$ and this extension belongs to $C^\alpha(\partial\mathbb{D})$. The function $\tilde\phi(e^{i\theta}) := \operatorname{Im} f(e^{i\theta})$ is traditionally known as the *conjugate function* of ϕ. Combining this with part (b) of the previous problem we obtain the following theorem of Privalov:

Fix $k \ge 0$ an integer and $0 < \alpha < 1$. If $\phi \in C^{k+\alpha}(\partial\mathbb{D})$, then $\tilde\phi \in C^{k+\alpha}(\partial\mathbb{D})$.
This result is not true for any such k if $\alpha = 0$.

5. Let us extend the notion of conjugate function from real-valued functions to complex-valued functions by linearity.
(a) Compute the conjugate function of $\theta \mapsto e^{ik\theta}$ for $k \in \mathbb{Z}$.
(b) Let $\{\phi_n\} \subset\in C^\infty(\partial\mathbb{D})$ be Cauchy in the $L^2(\frac{d\theta}{2\pi})$ norm. Show that $\tilde\phi_n$ is also

Cauchy in the $L^2(\frac{d\theta}{2\pi})$ norm. (Thus $\phi \mapsto \tilde{\phi}$ extends uniquely to a continuous linear map on $L^2(\frac{d\theta}{2\pi})$.)

Remark: M. Riesz proved that $\phi \mapsto \tilde{\phi}$ extends uniquely to a bounded operator on L^p for $1 < p < \infty$. This is not true for $p = 1$ or $p = \infty$.