1. Let us define the Hardy space $H^1(\mathbb{D})$ as those holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\|f\|_{H^1(\mathbb{D})} := \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})| \tfrac{d\theta}{2\pi} < \infty.$$

(a) Show that $|f(z)|$ is subharmonic whenever $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Deduce that

$$r \mapsto \int_0^{2\pi} |f(re^{i\theta})| \tfrac{d\theta}{2\pi}$$

is increasing on $[0, 1)$. (b) Show that if $f \in H^1(\mathbb{D})$ and $f \not\equiv 0$ then

$$\sum 1 - |z_j| < \infty$$

where z_j enumerates the zeros of f, repeated according to multiplicity.
(b) Let B be the Blaschke product formed from the zeros of $0 \not\equiv f \in H^1(\mathbb{D})$. Show that

$$\Big\|\frac{f(z)}{B(z)}\Big\|_{H^1(\mathbb{D})} = \big\|f(z)\big\|_{H^1(\mathbb{D})}.$$

[*Hint:* Show it is true with B replaced by a partial product.]

2. (a) By changing variables to $s = uv$ and $t = u(1 - v)$ in

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty t^{x-1} s^{y-1} e^{-s-t}\, ds\, dt$$

deduce *Euler's Beta Integral:* For $x, y > 0$,

$$B(x, y) := \int_0^1 v^{x-1}(1 - v)^{y-1}\, dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

(b) Use basic properties of Γ to show

$$1 = xB(x, 1) = xe^{-x\log(n)} \prod_{k=1}^{n} \Big(1 + \frac{x}{k}\Big) \int_0^n u^{x-1}\big(1 - \tfrac{u}{n}\big)^n\, du$$

and so derive the *Schlömilch formula*:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \Big(1 + \frac{z}{k}\Big) e^{-z/k}$$

for all $z \in \mathbb{C}$.
(c) Prove *Euler's Reflection Formula:*

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}$$

as meromorphic functions on $\mathbb{C}$.

3. Let Ω be a simply-connected open domain bounded by a Jordan curve. As we know, any conformal map f of $\mathbb{D}$ onto Ω can be extended to a homeomorphism of $\bar{\mathbb{D}}$ onto $\bar{\Omega}$.

Recall that a curve $\gamma : \partial\mathbb{D} \to \mathbb{C}$ is rectifiable if there exists a constant L so that for any $0 \le \theta_0 < \theta_1 < \cdots < \theta_n \le 2\pi$,

$$\sum_{k=0}^{n} |\gamma(e^{i\theta_k}) - \gamma(e^{i\theta_{k+1}})| \le L$$

where $\theta_{n+1} = \theta_0$. The minimal such constant L is called the *length* of γ.

Prove the following theorem of F. and M. Riesz: $f' \in H^1$ if and only if $\partial\Omega$ is rectifiable. [*Hint*: the function $z \mapsto \sum |f(ze^{i\theta_k}) - f(ze^{i\theta_{k+1}})|$ is continuous and subharmonic on $\mathbb{D}$.]

4. Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

extends meromorphically to the whole complex plane.
(a) Show that if

$$(*) \qquad \frac{r^n a_n}{n^k} \to c \neq 0 \quad \text{as} \quad n \to \infty$$

for some integer $k \ge 0$ and some $r > 0$, then $f(z)$ has a pole of order $k+1$ at $z = r$.
(b) Now suppose $f(z)$ has a pole of order $k+1$ at $z = r$ but no other singularities in the closed disk $|z| \le r$. Show that $(*)$ holds.

Remark: Part (a) is an example showing how the asymptotic behaviour of coefficients is passed to the corresponding power series. This type of result is known as an Abelian Theorem. Part (b) goes the other way: understanding the behaviour of the power series tells us about the asymptotics of the coefficients. Such theorems are known as Tauberian Theorems; they are of a more subtle nature (and have stronger hypotheses) since the power series clearly captures strongly averaged information about the coefficients. The Abelian/Tauberian names were introduced by Hardy and Littlewood.

5. In this problem p is always restricted to lie among the (positive) prime numbers. Let us define functions π and θ of $x \in [0, \infty)$ by

$$\pi(x) = \#\{p : p \le x\} \quad \text{and} \quad \theta(x) = \sum_{p \le x} \log(p).$$

(a) Show (following Chebyshev) that $\theta(x) \le 4\log(2)x$ by using the divisibility properties of $\binom{2n}{n} \le 2^{2n}$ to estimate the product of primes in $(n, 2n]$.
(b) Show that as $x \to \infty$,

$$\frac{\theta(x)}{x} \to 1 \quad \iff \quad \frac{\pi(x)\log(x)}{x} \to 1.$$

(c) Show that for $s > 1$

$$\int_1^\infty \theta(x) \frac{dx}{x^{s+1}} = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{F(s)}{s}$$

where $F(s)$ is holomorphic in the half-plane $\operatorname{Re} s > \frac{1}{2}$. Sending $s \downarrow 1$, use this to show that if $\theta(x)/x$ converges as $x \to \infty$, then the limit must be 1.

Remark: Everything you need to know about $\zeta(s)$ was in HW2 last quarter. Note that to prove the Prime Number Theorem, we are faced with a Tauberian type problem — we know that $\theta(x)/x \to 1$ in some averaged sense, but we need to show convergence in the usual sense.