246B Homework 2 Due Fri. Feb. 21

1. Let Ω_α denote the sector $\{z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \alpha\}$. Suppose f is holomorphic in a neighbourood of $\bar{\Omega}_\alpha$ and

$$f(z) = O\big(\exp(|z|^\beta)\big) \quad \text{for some} \quad \beta < \frac{\pi}{\alpha}.$$

Show that

$$\sup_{z\in\Omega_\alpha} |f(z)| \le \sup_{z\in\partial\Omega_\alpha} |f(z)|.$$

2. Let $\Omega = \{z : 0 < \operatorname{Re} z < 1\}$. Find explicit functions $K_0(z; y)$ and $K_1(z; y)$ so that

$$u(z) = \int K_0(z, y)u(0 + iy)\, dy + \int K_1(z, y)u(1 + iy)\, dy$$

for every $z \in \Omega$ and every bounded and continuous $u : \overline{\Omega} \to \mathbb{R}$ that is harmonic in Ω.
Remark: This is an analogue of the Poisson integral formula and I would recommend using a conformal mapping. Remember we have not proved the Poisson integral formula for the disk in sufficient generality for this problem (we only did it for harmonic functions continuous continuous up to the boundary) — imagine if $u(x + iy) = x$. Thus you will need to up augment the classical Poisson integral formula a little bit, which can readily be done using the tools of this quarter.

3. Find a conformal mapping of the slit disk $\Omega = \mathbb{D}\setminus(-1, 0]$ to the unit disk, writing it as a composition of simpler maps. One way to do this is to first map Ω to a D-shape, then that to a quadrant, then that to a half-plane and finally that to a disk.

4. Let $f : \overline{\mathbb{D}} \to \mathbb{D}$ be continuous and holomorphic on $\mathbb{D}$.
(a) Find an explicit kernel $K : \mathbb{D} \times \partial\mathbb{D} \to \mathbb{C}$ so that

$$f(z) = i \operatorname{Im} f(0) + \int K(z, e^{i\theta})[\operatorname{Re} f(e^{i\theta})]\tfrac{d\theta}{2\pi} \quad \text{for all } z \in \mathbb{D}.$$

Note that K is unique.
(b) Deduce the following variant of the Borel–Caratheodory inequality:

$$|f(z)| \le \big|\operatorname{Im} f(0)\big| + \frac{1 + |z|}{1 - |z|} \sup_{w\in\partial\mathbb{D}} \big|\operatorname{Re} f(w)\big|.$$

5. Show that the Joukowsky transform $z \mapsto z + 1/z$ is a biholomorphism from $\mathbb{C} \setminus \bar{\mathbb{D}}$ to $\mathbb{C} \setminus [-2, 2]$.
(b) Find the Riemann mapping of $[0, 1] \times [0, \infty)$ to the unit disk.

6. Let $\vec{v} : \Omega \to \mathbb{R}^2$ be a smooth irrotational divergence-free vector field in a simply connected region $\Omega \subset \mathbb{C}$:

$$\operatorname{rot}(\vec{v}) := \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \quad \text{and} \quad \operatorname{div}(\vec{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0.$$

This can represent the steady-state velocity of an incompressible fluid.
(a) Show that there is a holomorphic function $f : \Omega \to \mathbb{C}$ so that $\vec{v} = \nabla \operatorname{Re} f$. (Note that $\operatorname{Re} f$ is called the velocity potential. $\operatorname{Im} f$ is called the stream function and its

level curves depict particle trajectories.)
(b) Use a computer to draw the level curves of $\mathrm{Im}(z+\frac{1}{z})$ and $\mathrm{Im}(z+\frac{1}{z}+\frac{i}{2}\log(z))$ in the region $\mathbb{C}\setminus\mathbb{D}$. Note that both represent idealized fluid flow around a cylinder. In the later case the cylinder is rotating; note that the flow is still irrotational (curl-free) — the circulation is hiding inside the cylinder. The second example also demonstrates the necessity of assuming that Ω is simply connected in part (a).