246B Homework 1 Due Fri. Jan. 31

1. Let $z \mapsto M(z)$ be a $n \times n$ matrix valued function on $\mathbb{C}$, such that each entry is an entire function. Show that

$$f : z \mapsto \|M(z)\|$$

is subharmonic, where $\|M\|$ denotes the operator norm of M induced by the usual Euclidean norm on $\mathbb{C}^n$. *Hint:* write f as a pointwise supremum of subharmonic functions.

2. Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $\phi : \partial\Omega \to \mathbb{R}$ be continuous. In class, we defined the Perron solution $u : \Omega \to \mathbb{R}$ of the Dirichlet problem as the pointwise supremum of subsolutions. Alternately one may consider the pointwise infimum of supersolutions (i.e., superharmonic functions that exceed the boundary values in the corresponding sense). It is elementary to see that this function, which we will call $v : \Omega \to \mathbb{R}$ can also be constructed as the negative of the Perron solution corresponding to boundary values $-\phi$. Our goal in this problem is to show that $u \equiv v$. This is a theorem of Wiener.
(a) Show that $\mathcal{R} := \{\phi : u \equiv v\}$ is a non-empty closed subset of $C(\partial\Omega; \mathbb{R})$.
(b) Show that $\mathcal{R}$ is a vector space (over $\mathbb{R}$).
(c) If $w : \mathbb{R}^d \to \mathbb{R}$ is continuous and subharmonic (or superharmonic) then $w|_{\partial\Omega} \in \mathcal{R}$.
(d) If $w : \mathbb{R}^d \to \mathbb{R}$ is smooth, show that the restriction of $w(x) = w(x) - \lambda|x|^2 + \lambda|x|^2$ to $\partial\Omega$ belongs to $\mathcal{R}$ by choosing λ very large.
(e) Verify that every $\phi \in C(\partial\Omega; \mathbb{R})$ belongs to $\mathcal{R}$.

3. (a) Fix $0 \le \alpha \le \pi$. Evaluate the Poisson integral formula (for $\mathbb{D}$) with boundary values

$$\phi(e^{i\theta}) = \begin{cases} 1 & : |\theta| \le \alpha \\ 0 & : \text{ otherwise.} \end{cases}$$

Evidently this yields are harmonic function, say u, on $\mathbb{D}$.
(b) Show that u agrees with the pointwise supremum of subsolutions and the pointwise infimum of supersolutions. Note that the argument given in class still shows that these Perron solutions are harmonic. Indeed, we used continuity of ϕ only to get boundedness on $\partial\Omega$.

4. Fix $0 < \alpha < 2\pi$ and let Ω_α denote the sector $\{z \in \mathbb{C} \setminus \{0\} : 0 < \arg z < \alpha\}$. Construct a positive harmonic function $u : \Omega_\alpha \to (0, \infty)$ that satisfies

$$\lim_{\Omega_\alpha \ni y \to x} u(y) = 0$$

for every $x \in \partial\Omega$.