The Riemann Zeta Function is initially defined for $\mathrm{Re}\, s > 1$ via

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \text{(Dirichlet series)}$$

or

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \qquad \text{(Euler product).}$$

The equivalence of these definitions is a expression of unique factorization.

Last quarter we use the Euler summation technique to see that $\zeta(s)$ extends to a meromorphic function on the larger region $\mathrm{Re}\, s > 0$: First

$$\zeta(s) = \int_{1/2}^{\infty} x^{-s}\, dx + \sum_{n=1}^{\infty} \int_{n-1/2}^{n+1/2} (x^{-s} - n^{-s})\, dx$$

$$= \frac{2^{s-1}}{s-1} + \sum_{n=1}^{\infty} \int_{n-1/2}^{n+1/2} (x^{-s} - n^{-s})\, dx$$

at least for $\mathrm{Re}\, s > 1$. But the first summand clearly extends as a meromorphic function to all of $\mathbb{C}$. Regarding the second summand, we note that for any smooth function f,

$$\int_{n-1/2}^{n+1/2} f(x)\, dx = f(n) + \int_{-1/2}^{1/2} \frac{(1-2|x|)^2}{8} f''(n+x)\, dx$$

So

$$\left| \int_{n-1/2}^{n+1/2} x^{-s}\, ds - n^{-s} \right| \lesssim |s(s+1)| \cdot n^{-Re(s)-2}$$

which is summable whenever $Re(s) > 0$.

Thus

$$\zeta(s) = \frac{1}{s-1} + f(s)$$

with f holomorphic in the region $Re\, s > 0$. In fact, by the functional equation (cf. HW4 last quarter) f is entire.

Corollary (Euler) $\displaystyle\sum_{p \text{ prime}} \frac{1}{p} = \infty$

Pf The product $\prod \left(\frac{1}{1-p^{-s}} \right)$ diverges as $s \downarrow 1$.

Taking a logarithmic derivative of the Euler product, we find

$$-\frac{\zeta'(s)}{s \cdot \zeta(s)} = \frac{1}{s} \cdot \sum_{n=1}^{\infty} \Lambda(n) \cdot n^{-s}$$

where Λ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k, \ p \text{ prime}, k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Continuing, we find

$$-\frac{\zeta'(s)}{s\cdot\zeta(s)} = \sum_{n=1}^{\infty}\int_{n}^{\infty}\Lambda(n)\, x^{-s-1}\, dx$$

$$= \int_{1}^{\infty}\left[\frac{1}{x}\sum_{n\le x}\Lambda(n)\right] x^{-s}\, dx$$

$$= \int_{1}^{\infty}\frac{\psi(x)}{x}\cdot x^{-s}\, dx$$

where $\psi(x) = \sum_{n\le x}\Lambda(n)$. Analogously, one may write $\Theta(x) := \sum_{p\le x}\log(p)$. On the homework, you show

(a) PNT $\Longleftrightarrow$ $\frac{\Theta(x)}{x} \longrightarrow 1$

(b) $$\int_{1}^{\infty}\frac{\Theta(x)}{x}\cdot x^{-s}\, dx = -\frac{\zeta'(s)}{s\cdot\zeta(s)} - \frac{F(s)}{s}$$

with $F(s)$ holomorphic for $\mathrm{Re}\, s > \frac{1}{2}$.

The operation

$$f(x) \longmapsto \int_1^\infty f(x) \cdot x^{-s}\, dx =: F(s)$$

is known as a Mellin transform and can be inverted in much the same fashion as a Dirichlet series:

If f is cts at y then (formally at least)

$$f(y)\, y^{-\sigma} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(\sigma+it)\, y^{it}\, dt$$

Just as the singularities nearest the origin dictate the asymptotic behavior of a Taylor series, so the singularities on the right-most vertical line dictate the asymptotic behavior of a Dirichlet series or Mellin transform.

To understand the behavior of $\frac{\Theta(x)}{x}$ as $x \to \infty$, we must understand the right-most singularities of

$$-\frac{\zeta'(s)}{s\,\zeta(s)}$$

We know from the Euler product that $\zeta(s)$ is holomorphic and non-vanishing for $\operatorname{Re} s > 1$.

We also know that $\zeta(s) \sim \frac{1}{s-1}$ as $s \to 1$

In fact this is the only singularity on the line $\operatorname{Re} s = 1$:

<u>Theorem</u> (Hadamard/de la Vallée Poussin)

$\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$

<u>Pf</u> (following de la Vallée Poussin).

First note that

$$\cos(2\varphi) + 4\cos(\varphi) + 3 = 2(1+\cos\varphi)^2 \geq 0$$

for every $\varphi \in \mathbb{R}$.

From the Euler product, we find that $\forall \sigma > 1$ & $t \in \mathbb{R}$,

$$\log\left|\zeta(\sigma)^3 \cdot \zeta(\sigma+it)^4 \cdot \zeta(\sigma+2it)\right|$$

$$= \sum_{p \text{ prime}} -3\log(1-p^{-\sigma}) - 4\log\left|1-p^{-\sigma-it}\right| - \log\left|1-p^{-\sigma-2it}\right|$$

$$= \sum_{p} \sum_{m} \frac{1}{m} p^{-m\sigma}\left[3 + 4\cos(\log(t)) + \cos(2\log(t))\right]$$

$$\geq 0.$$

[If $r<1$, $-\log|1-re^{i\theta}| = \operatorname{Re} -\log[1-re^{i\theta}] = \operatorname{Re} \sum_{m=1}^{\infty} \frac{r^m e^{im\theta}}{m} = \sum_{m=1}^{\infty} \frac{r^m}{m}\cos(m\theta)$

Thus, for $\sigma > 1$ and $t \in \mathbb{R}$, we have

$$\left| \zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it) \right| \geq 1. \qquad (*)$$

Now imagine for a moment that $\zeta(1+it) = 0$ then

$$|\zeta(\sigma + it)|^4 \lesssim (\sigma - 1)^4 \quad \text{as } \sigma \downarrow 1$$

while

$$|\zeta(\sigma)|^3 \lesssim \frac{1}{(\sigma - 1)^3} \quad \& \quad |\zeta(\sigma + 2it)| \lesssim 1$$

But this contradicts $(*)$ as $\sigma \downarrow 1$. $\blacksquare$

To finish the proof that $\frac{\theta(x)}{x} \to 1$, we need two Tauberian arguments:

Newman's Tauberian (= unsmudging) Theorem

Suppose $\varphi : [1, \infty) \to \mathbb{R}$ is $O(x)$ and let

$$\Phi(s) := \int_1^\infty \varphi(x) \frac{dx}{x^{s+1}}$$

which is (absolutely convergent and so) holomorphic for $\mathrm{Re}(s) > 1$.

If $\Phi(s)$ admits a holomorphic extension to an open nbd of $\mathrm{Re}(s) \geq 1$, then

$$\lim_{X \to \infty} \int_1^X \varphi(x) \frac{dx}{x^2} = \Phi(1).$$

Remarks 1. We'd like to set $\varphi(x) = \Theta(x)$, but we know that

$$\int_1^\infty \Theta(x) \frac{dx}{x^{s+1}} = - \frac{\zeta'(s)}{s\,\zeta(s)} - \frac{F(s)}{s}$$ has a pole at $s = 1$ (there would be others on $\mathrm{Re}(s) = 1$ if $\zeta(s) = 0$ there). The pole at $s = 1$ is easily removed by working with $\varphi(x) = \Theta(x) - x$ for $\int_1^\infty x \frac{dx}{x^{s+1}} = \frac{1}{s-1}$.

2. If we set $\varphi(x) = x \cdot \sin(\log(x))$ then

$$\Phi(s) = \int_0^\infty \sin(u)\, e^{-(s-1)u}\, du = \cdots = \frac{1}{(s-1)^2 + 1}$$ (poles at $s = 1 \pm i$)

while $$\int_1^X \varphi(x) \frac{dx}{x^2} = \int_0^{\log(X)} \sin(u)\, du$$ which does not converge.

Pf of Newman's Tauberian Theorem

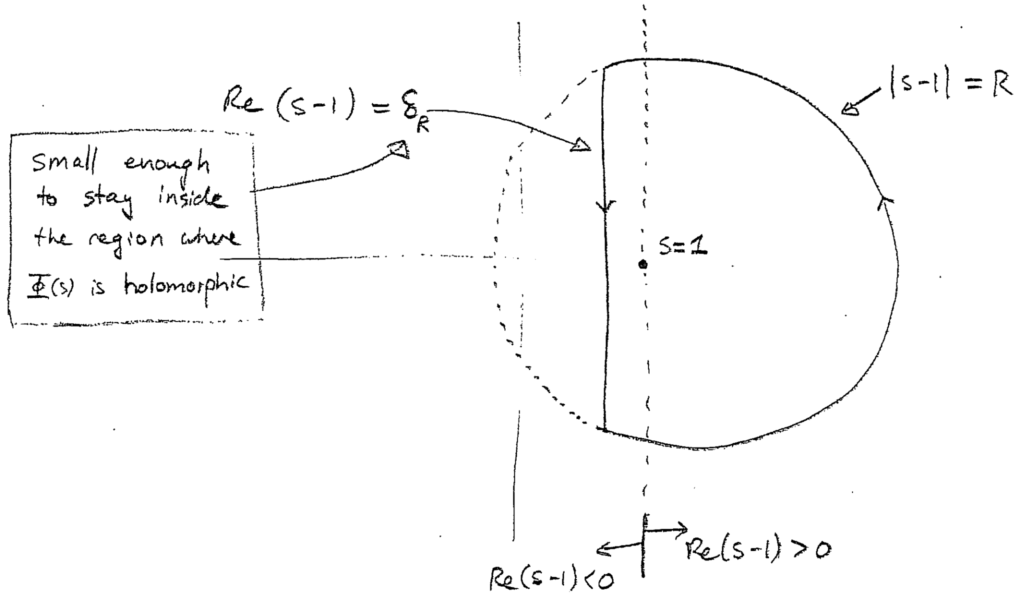
Let $\quad \Phi_X(s) = \int_1^X \varphi(x) \frac{dx}{x^{s+1}} \quad$ which is an entire fn of s.

We wish to show that $\quad \Phi_X(1) \xrightarrow{X \to \infty} \Phi(1)$.

Well, for any $R > 0$,

$$\Phi(1) - \Phi_X(1) = \frac{1}{2\pi i} \int_{\gamma_R} [\Phi(s) - \Phi_X(s)] \underbrace{X^{s-1}\left(1 + \frac{(s-1)^2}{R^2}\right)}_{\text{Two "fudge factors"}} \frac{ds}{s-1}$$

where γ_R is the contour



Note that $\quad s-1 = Re^{i\theta} \quad \Rightarrow \left| \frac{1}{s-1}\left(1 + \frac{(s-1)^2}{R^2}\right) \right| = \frac{1}{R} \left| e^{-i\theta} + e^{i\theta} \right|$

$$= \frac{2|\cos\theta|}{R} \lesssim \frac{|Re(s-1)|}{R^2}$$

Without the "fudge factor" $\left(1 + \frac{(s-1)^2}{R^2}\right)$ we would merely get $O(\frac{1}{R})$ here which is too big to control the part of the contour where $Re(s)$ is close to 1.

Let $\gamma_R^+ = \gamma_R \cap \{\mathrm{Re}(s-1) > 0\}$. On this part of the contour, we use

$$\left| X^{s-1}\left(\Phi(s) - \Phi_X(s)\right)\right| \le X^{\mathrm{Re}(s-1)} \int_X^\infty \left|\frac{\varphi(x)}{x}\right| \frac{dx}{x^{\mathrm{Re}(s)}}$$

$$\lesssim \frac{1}{\mathrm{Re}(s-1)} \quad \text{since} \quad \left|\frac{\varphi(x)}{x}\right| \lesssim 1.$$

Thus

$$\left|\text{contribution of } \gamma_R^+\right| \lesssim \int_{\gamma_R^+} \frac{1}{|\mathrm{Re}(s-1)|} \cdot \frac{|\mathrm{Re}(s-1)|}{R^2} \frac{|ds|}{2\pi} \lesssim \frac{1}{R}$$

Next consider $\gamma_R^- = \gamma_R \cap \{\mathrm{Re}(s-1) < 0\}$, for which we treat $\Phi(s)$ and $\Phi_X(s)$ separately. In the former case, we note that

$$\left|\Phi(s) \cdot \left(1 + \frac{(s-1)^2}{R^2}\right) \cdot \frac{1}{s-1}\right| \lesssim C(R) \qquad (\text{independent of } X)$$

Note that C depends on R via δ_R and also the size of Φ on γ_R^-. Nevertheless since $|X^{s-1}| = X^{\mathrm{Re}(s-1)} \xrightarrow{X\to\infty} 0$ pointwise on γ_R^- we deduce

$$\overline{\lim_{X\to\infty}} \left| \int_{\gamma_R^-} \Phi(s)\, X^{s-1}\left(1 + \frac{(s-1)^2}{R^2}\right) \frac{ds}{2\pi i (s-1)} \right| = 0.$$

This leaves $\Phi_X(s)$ for which we deform the contour to $\tilde{\gamma}_R^- = \{|s-1| = R,\ \mathrm{Re}(s-1) < 0\}$. and use

$$\left| X^{s-1} \int_1^X \frac{\varphi(x)}{x} \frac{dx}{x^s} \right| \lesssim X^{\mathrm{Re}(s-1)} \int_1^X \frac{dx}{x^{\mathrm{Re}(s)}} \lesssim \frac{1}{\mathrm{Re}(s-1)}$$

to deduce that

$$\left| \text{contribution of } \Phi_X(s) \text{ on } \tilde{\gamma}_R^- \right| \lesssim \int_{\tilde{\gamma}_R^-} \frac{1}{|\mathrm{Re}(s-1)|} \cdot \frac{|\mathrm{Re}(s-1)|}{R^2} \, |ds| \lesssim \frac{1}{R}$$

much as in the case of γ_R^+.

Putting the pieces together we obtain

$$\limsup_{X \to \infty} \left| \Phi(1) - \Phi_X(1) \right| \lesssim R^{-1}$$

as R can be made arbitrarily large, we deduce the Thm $\blacksquare$

From Newman's Tauberian Theorem, we see that

$$(**) \qquad \int_1^X \frac{\Theta(x) - x}{x^2}\, dx \longrightarrow c \in \mathbb{R} \qquad \text{as } X \to \infty.$$

as $X \to \infty$. One could even find c, but thats not important.

We will need a second Tauberian argument to deduce that $\frac{\Theta(x)}{x} \to 1$. The key observation is that $\frac{\Theta(x)}{x}$ changes slowly....

<u>Case 1</u> Suppose (toward a contradiction) that $\limsup_{x \to \infty} \frac{\Theta(x)}{x} > 1$.

Then $\exists\, \varepsilon > 0$ and $x_n \to \infty$ so that

$$\frac{\Theta(x_n)}{x_n} \geq 1 + \varepsilon$$

But then, since $\Theta(x) = \sum_{p \leq x} \log(p)$ is nondecreasing, we see that $\Theta(x) \geq (1+\varepsilon) x_n$ for any $x \geq x_n$. Thus

$$\int_{x_n}^{(1+\varepsilon)x_n} \frac{\Theta(x) - x}{x^2}\, dx \geq \int_{x_n}^{(1+\varepsilon)x_n} \frac{(1+\varepsilon)x_n - x}{x^2}\, dx$$

evaluate exactly

$$\geq \varepsilon - \log(1+\varepsilon) > 0$$

log is strictly concave

uniformly in n. This is inconsistent with the existence of the limit in $(**)$.

<u>Case 2</u> If $\liminf_{x \to \infty} \frac{\Theta(x)}{x} < 1$, then $\exists\, \varepsilon > 0$ and a sequence $x_n \to \infty$ so that $\Theta(x_n) \leq (1-\varepsilon) x_n$. Using monotonicity of Θ once again, we deduce that

$$\int_{(1-\varepsilon)x_n}^{x_n} \frac{\Theta(x) - x}{x^2}\, dx \leq \int_{(1-\varepsilon)x_n}^{x_n} \frac{(1-\varepsilon)x_n - x}{x^2}\, dx$$

$$\leq \varepsilon - \log\left(\frac{1}{1-\varepsilon}\right) < 0$$

which is again inconsistent with (**).