246A Homework 7 Due Dec. 6

1. Give a direct proof of Montel's Theorem (families uniformly bounded on compacta are normal) from the Cauchy Integral and Arzelà–Ascoli Theorems. (The in-class proof is via the analogue for harmonic functions.)

2. Suppose $\Omega \subset \mathbb{C}$ is open and connected.
(a) Let $f_n : \Omega \to \mathbb{C}$ be univalent (=holomorphic and injective) and converge uniformly on compact sets to some (holomorphic) $f : \Omega \to \mathbb{C}$. Show that f is univalent or constant.
(b) Suppose further that Ω has compact closure and $z_0 \in \Omega$. Show that among all univalent maps $f : \Omega \to \mathbb{D}$ that obey $f(z_0) = 0$, there is (at least) one that achieves the maximal value of $\operatorname{Re} f'(z_0)$. [*Note:* you will need to verify that the set of maps is non-empty and that $\operatorname{Re} f'(z_0)$ is bounded.]
(c) Explain why any such maximal f has $f'(z_0) > 0$.

3. Suppose that $\Omega \subset \mathbb{C}$ is open, simply connected and has compact closure. Let $f : \Omega \to \mathbb{D}$ be one of the univalent maps found in Problem 2 above. To obtain a proof of the Riemann Mapping Theorem, we need only show that f is onto. Suppose not and let $w_0 \in \mathbb{D}$ be a point not in the range of f.
(a) Show that there is a univalent function $r : \Omega \to \mathbb{D}$ so that

$$r(z)^2 = M_1 \circ f(z)$$

where M_1 is a disk automorphism taking w_0 to 0.
(b) Choose a disk automorphism M_2 so that

$$g(z) = M_2 \circ r(z)$$

obeys $g(z_0) = 0$ and $g'(z_0) \geq 0$.
(c) Rearrange the above to find $\phi : \mathbb{D} \to \mathbb{D}$ so that $f(z) = \phi \circ g(z)$.

4. Continuing from the preceding problem:
(a) Use Schwarz Lemma to see that $|\phi'(0)| < 1$.
(b) Now get your hands dirty and actually compute $\phi'(0)$.
(c) Conclude that any maximizing f from Problem 2 must be onto.
(d) Show that f is actually unique.

This completes the proof of the Riemann Mapping Theorem, at least if Ω has compact closure in $\mathbb{C}$. (In class, we will see how to treat more general Ω by reducing them to this case.) This extremal argument is due to Koebe. The alternate argument given in class is closer to Riemann's original vision, relying on the solvability of the Dirichlet problem.

5. (a) Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is continuous and $u(-x_1, x_2, \ldots, x_d) = -u(x_1, x_2, \ldots, x_d)$. Show that if u is harmonic where $x_1 > 0$, then it is harmonic throughout $\mathbb{R}^d$.
(b) Fix $d \geq 2$ and let $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ be bounded and continuous. Show that there is a unique bounded continuous function $u : [0, \infty) \times \mathbb{R}^{d-1} \to \mathbb{R}$ that is harmonic on $(0, \infty) \times \mathbb{R}^{d-1}$ and obeys $u(0, y) = \phi(y)$ for all $y \in \mathbb{R}^{d-1}$.

(c) Give an example of an unbounded harmonic $u : \mathbb{R}^d \to \mathbb{R}$ that vanishes on $\{x : x_1 = 0\}$, thus showing that boundedness is necessary for uniqueness in part (b).

6. Use the Poisson integral formula to prove the following version of Harnack's inequality for the unit ball $B(0,1) \subseteq \mathbb{R}^d$ (and $d \geq 2$): If $u : B(0,1) \to [0,\infty)$ is harmonic, then

$$\frac{1-|x|}{(1+|x|)^{d-1}}u(0) \leq u(x) \leq \frac{1+|x|}{(1-|x|)^{d-1}}u(0)$$

for each $|x| < 1$. Moreover, equality can occur, so these estimates are best possible.

7. (a) Given $a, b \in \mathbb{R}$ and $0 < r < R < \infty$ find the solution to the Dirichlet problem in the region $\Omega := \{x \in \mathbb{R}^2 : r < |x| < R\}$ with boundary values $u(x) = a$ when $|x| = r$ and $u(x) = b$ when $|x| = R$.
(b) Show that if $u : \mathbb{R}^2 \to \mathbb{R}$ is *sub*harmonic and bounded from above then it is constant. (This is false for $\mathbb{R}^d$ when $d \geq 3$ as $u(x) = \max\{-1, -|x|^{2-d}\}$ shows.)

8. Show that if $u : \mathbb{R}^d \to \mathbb{R}$ is harmonic (with $d \geq 3$) and bounded from above then it is constant.