1. Prove *Hurwitz Theorem:* Suppose $\{f_n\}$ and f are holomorphic functions on a connected open set $\Omega \subset \mathbb{C}$. If $f_n \to f$ uniformly on compact sets and $f \not\equiv 0$, then the zeros of f_n converge to those of f in the following sense: given a ball B with $\bar{B} \subseteq \Omega$ and f nowhere zero on ∂B, the number of zeros (counting multiplicity) of f_n in B converges to the number for f. [*Remark:* Since the number is an integer, convergence means eventual equality.]

2. Let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a non-constant doubly periodic function with periods 1 and τ as in HW3. (Recall $\tau \in \mathbb{C}$ has $\operatorname{Im} \tau > 0$.)
(a) Show that the sum of the residues in any fundamental parallelogram (i.e., with vertices $\{z, z+1, z+1+\tau, z+\tau\}$) whose sides miss the poles of f is equal to zero.
(b) Show that $f(z) = w$ and $f(z) = \infty$ have an equal number of solutions in each fundamental parallelogram (for any $w \in \mathbb{C}$ and counting with multiplicity). From HW3 we know that this number is non-zero.
(c) Show that

$$\wp(z) := \frac{1}{z^2} + \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(z - n_1 - n_2\tau)^2} - \frac{1}{(n_1 + n_2\tau)^2}$$

defines a meromorphic function and that

$$\wp'(z) := -2 \sum_{n \in \mathbb{Z}^2} \frac{1}{(z - n_1 - n_2\tau)^3}.$$

Once you prove convergence, it is clear that $\wp'(z)$ is doubly periodic.
(d) Use the fact that $\wp$ is even (i.e. $\wp(z) = \wp(-z)$) to deduce that $\wp$ is doubly periodic.

3. (a) Let A be a $n \times n$ matrix and let $f(z) := \det(z\mathrm{Id} - A)$. Show

$$\frac{f'(z)}{f(z)} = \operatorname{Tr}\big\{(z\mathrm{Id} - A)^{-1}\big\}.$$

(b) Apply Rouché's Theorem to f to show that the eigenvalues of A (repeated according to *algebraic* multiplicity) depend continuously on the entries of A. Here, we say that the distance between two multi-sets is the sum of the distances under the shortest matching.
(c) Now suppose $A(t)$ depends holomorphically on a parameter $t \in \mathbb{D}$ and that λ_0 is a simple (i.e. multiplicity one) eigenvalue of $A(0)$. Show that there is a holomorphic function $t \mapsto \lambda(t)$ defined in an open neighbourhood of 0 so that $\lambda(0) = 0$ and $\lambda(t)$ is an eigenvalue of $A(t)$.

Remark: The matrix function $t \mapsto \left[\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right]$ shows that the eigenvalues may only be Hölder continuous as functions of the entries in A. In the $n \times n$ case, they may be merely Hölder $\frac{1}{n}$ continuous.

4. Suppose $f : \mathbb{C} \to \mathbb{C}$ is bijective and holomorphic.
(a) Show that f^{-1} is holomorphic.

(b) Use the continuity and surjectivity of f^{-1} to show that $|f(z)| \to \infty$ as $|z| \to \infty$.
(c) Show that $f(z) = O(|z|)$ as $z \to \infty$. [*Hint:* Look at $g(z) = 1/f(1/z)$.]
(d) Deduce that $f(z) = az + b$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$.

5. Show that every meromorphic bijection f from $\mathbb{C} \cup \{\infty\}$ to itself is of the form

$$f(z) = \frac{az + b}{cz + d}$$

for some quadruplet $a, b, c, d \in \mathbb{C}$ obeying $ad - bc = 1$.

6. (a) Use Möbius transformations to determine the analogue of the Schwarz Reflection Principle for holomorphic functions f defined in an open neighbourhood of an arc of the circle $\{|z| = 1\}$ that obey $|f(e^{i\theta})| = 1$.
(b) Use Möbius transformations to determine the analogue of Schwarz Lemma for mappings f of the half-plane $\{\operatorname{Re} z > 0\}$ to itself that obey $f(1) = 1$.
(c) Use Möbius transformations and Schwarz Lemma to prove the *Borel–Carathéodory Theorem:* Let f be holomorphic in an open neighbourhood of the closed unit disk, $\bar{\mathbb{D}}$. For each $z \in \mathbb{D}$,

$$|f(z)| \le \frac{1 + |z|}{1 - |z|}|f(0)| + \frac{2|z|}{1 - |z|} \sup_{w \in \mathbb{D}} \operatorname{Re} f(w).$$

7. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic with $f(0) = 0$ and $f'(0) = 1$ and let

$$M := \sup_{z \in \mathbb{D}} |f(z)|$$

(a) Explain why $M \ge 1$.
(b) Show that

$$|f(z) - z| \le \frac{1}{12M} \qquad \text{for} \quad |z| = \frac{1}{4M},$$

for example, by estimating the power series.
(c) Deduce that $f(\mathbb{D})$ contains $\{z : |z| < \frac{1}{6M}\}$.

8. Let $\Omega \subseteq \mathbb{C}$ be *convex* and open. Use the sketch below to prove the following: If $f : \mathbb{D} \to \Omega$ is a biholomorphism (onto Ω), then $G_r := f(\{|z| \le r\})$ is convex for each $0 \le r < 1$.
(a) Argue that it suffices to show that for all $|z_1| \le |z_2| < 1$ the line segment joining $f(z_1)$ to $f(z_2)$ is inside the image of the ball $\{z : |z| \le |z_2|\}$.
(b) For $t \in [0, 1]$ fixed, apply Schwarz Lemma to

$$z \mapsto f^{-1}\big(tf(zz_1/z_2) + (1 - t)f(z)\big).$$

(c) Now imagine that Ω is *star-shaped* with respect to the origin and $f : \mathbb{D} \to \Omega$ is a biholomorphism (onto Ω) with $f(0) = 0$. Show that each set $G_r := f(\{|z| \le r\})$ is star-shaped with respect to the origin.

Remark: These are theorems of E. Study, while the proof is that of T. Radó. Later, we will prove the Riemann mapping theorem which guarantees the existence of such a biholomorphism (except in the case $\Omega = \mathbb{C}$, when it is not possible).