246A Homework 5 Due Nov. 8.

1. (a) Evaluate

$$\int_{\mathbb{R}} e^{-t^2/2} e^{i\xi t}\, dt$$

for all $\xi \in \mathbb{C}$. [Hint: compete the square, then move the contour to reduce to that case $\xi = 0$.]

(b) Deduce *Wick's Theorem* (as it is called in the physics literature):

$$\tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^{2k} e^{-t^2/2}\, dt = (2k-1)!! = (2k-1)(2k-3)\cdots(3)(1).$$

Convince yourself that this is the number of ways of pairing off $2k$ objects.

2. Show that for $\xi \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{\exp(i\xi t)}{1+t^2}\, dt = \pi \exp(-|\xi|).$$

3. Show that a Möbius transformation can be uniquely identified by where it maps three points.

4. Let $\gamma, \sigma : [0,1] \to \mathbb{C}$ be rectifiable loops and let $z \in \mathbb{C}$. Suppose

$$|\sigma(t) - \gamma(t)| < |z - \gamma(t)|.$$

Show that $\mathrm{Wind}_\gamma(z)$=$\mathrm{Wind}_\sigma(z)$

5. (a) Fix an open set $\Omega \subseteq \mathbb{C}$. Show that every (continuous) loop $\gamma : [0,1] \to \Omega$ is freely homotopic (in Ω) to a smooth path. [Hint: regard γ as a periodic function on $\mathbb{R}$ and then convolve with a well-chosen mollifier.]

(b) Show that the notion of winding number can be extended from rectifiable loops to merely continuous loops so that $\mathrm{Wind}_\gamma(z) = \mathrm{Wind}_\sigma(z)$ whenever γ and σ are freely homotopic in $\mathbb{C} \setminus \{z\}$.