246A Homework 3 Due Oct. 25.

1. (a) Deduce the Fundamental Theorem of Algebra (that polynomials over $\mathbb{C}$ can be completely factored) from Liouville's Theorem.
(b) Let $P \in \mathbb{R}[x]$ be monic and non-constant (that is, a non-constant polynomial in with real coefficients whose leading coefficient is unity) with real coefficients. Show that P can be written uniquely as a product of linear and irreducible quadratic monic polynomials with real coefficients.

2. Prove the *Casorati–Weierstrass Theorem:* Suppose f is defined in a deleted neighbourhood of $z_0 \in \mathbb{C}$, say $\{z : 0 < |z-z_0| < r\}$, and is holomorphic there. Suppose also that z_0 is neither a removable singularity nor a pole. Prove that for each $0 < \epsilon < r$ the set $f(\{z : 0 < |z - z_0| < \epsilon\})$ is dense in $\mathbb{C}$.

3. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose f and g are meromorphic on Ω and $g \not\equiv 0$. Show that f/g is meromorphic (after removing removable singularities).

Remark: With a little extra work, this shows that the set of meromorphic functions on Ω is an infinite-dimensional (commutative) division algebra (over $\mathbb{R}$ and $\mathbb{C}$).

4. For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, let us define

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$$

(as an improper Riemann integral).
(a) Prove that Γ is holomorphic on this region.
(b) Show that $z\Gamma(z) = \Gamma(z+1)$ when $\mathrm{Re}(z) > 0$.
(c) Deduce that $\Gamma(n+1) = n!$ when $n \geq 0$ is an integer.
(d) Argue that there is an extension of Γ to a holomorphic function on $\mathbb{C}\backslash\{0, -1, -2, \dots\}$ that obeys $z\Gamma(z) = \Gamma(z+1)$. Show that the omitted points are polar singularities and determine the principal part.

5. (a) Show that

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

defines a holomorphic function on $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$.
(b) Show that

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

You may take the fundamental theorem of arithmetic for granted, but you must address the issue of (unconditional) convergence.
(c) Identify a function $g : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ so that

$$f(n) = \int_{-1/2}^{1/2} f(n+x)\, dx + \int_{-1/2}^{1/2} f''(n+x) g(x)\, dx$$

for all C^∞ functions f defined in a neighbourhood of $[n - \frac{1}{2}, n + \frac{1}{2}]$.
[*Remark:* one may view this formula as giving the error made when using the mid-point rule of numerical integration. Euler used this technique for accurate numerical evaluation of $\sum \frac{1}{n^2}$ as part of his work on the Basel Problem.]
(d) Use part (c) to show that we can extend the definition of ζ to make it meromorphic in the region $\mathrm{Re}(s) > -1$. Identify the (single) pole and its residue.

6. Let us say that meromorphic function f on $\mathbb{C}$ is *doubly periodic* if there exists $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$ and

$$f(z + \tau) = f(z + 1) = f(z).$$

Notice that we have fixed one of the periods to be 1. This is no real loss of generality since other cases can be reduced to this via a change of variable $z \mapsto az$. The number τ is far from unique as we will discuss later in the course. For now, show that
(a) Holomorphic doubly periodic functions are constant. Indeed,
(b) Nonconstant doubly periodic functions are onto $\mathbb{C} \cup \{\infty\}$, that is, they have a pole and achieve every (finite) value in $\mathbb{C}$.

Remark: It has been reported (though documentary evidence is limited) that Liouville lectured on the result (a), Cauchy objected that this would imply that bounded holomorphic functions are constant, and so what we now call Liouville's Theorem was born. Note that obvious candidate for Cauchy's argument uses the fact that there is a doubly periodic function with property (b), which was known by direct construction (we'll do this later too).