246A Homework 2 Due Oct. 18.

1. (a) Let $\Omega \subseteq \mathbb{C}$ be open. Suppose $f : \Omega \to \mathbb{C}$ is smooth; show that

$$\Delta f = 4\partial_z \partial_{\bar{z}} f,$$

where Δ is the Laplacian.
(b) Let $f : \Omega \to \mathbb{C}$ be continuous and a distributional solution of $\partial_{\bar{z}} f = 0$. Show that f is distributionally harmonic:

$$\int_{\mathbb{C}} f(x+iy)[\Delta\phi](x+iy)\,dx\,dy = 0 \quad \text{for all} \quad \phi \in C_c^\infty(\Omega).$$

(c) Show that a smooth function $f : \Omega \to \mathbb{C}$ is harmonic (i.e. $\Delta f \equiv 0$) if and only if it is distributionally harmonic.

2. (a) Using just the one-variable Fundamental Theorem of Calculus (and the equality of double Riemann integrals with iterated integrals), prove Green's theorem in the following form:

$$2i \iint_T [\partial_{\bar{z}} f](x+iy)\,dx\,dy = \int_{\partial T} f(z)\,dz$$

where $f : \mathbb{C} \to \mathbb{C}$ is C^1 and T is the triangle with vertices $0 + i0$, $a + i0$, and $0 + ib$, where $a, b > 0$. Be explicit about how you parameterize (and orient) the curve ∂T.
(b) Extend part (a) to all triangles.
(c) Show further that if $\bar{\partial} f \equiv 0$, then $\int_\gamma f\,dz = 0$ for any rectifiable closed curve γ.
(d) Explain where your solution of (c) used that f is defined (and satisfies $\bar{\partial} f \equiv 0$) globally on $\mathbb{C}$, rather than merely near γ.
(e) Exhibit an example of a smooth function $f : \mathbb{C} \to \mathbb{C}$ that satisfies $\bar{\partial} f(z) = 0$ for all $\frac{1}{2} < |z| < 2$ but for which the line integral around the unit circle does not vanish.

2. Let $\Omega \subseteq \mathbb{C}$ be open and let $f_n : \Omega \to \mathbb{C}$ be holomorphic. Suppose that for each compact set $K \subset \Omega$ the functions f_n converge uniformly to some $f : \Omega \to \mathbb{C}$. Show that f must me holomorphic. Further, show that $f_n'(z) \to f'(z)$ uniformly on compact subsets of Ω.

3. (a) Prove *Liouville's Theorem*: Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and

$$|f(z)| \le C(1 + |z|)^n$$

for some $C > 0$ and integer $n \ge 0$, then f is a polynomial of degree not exceeding n.
(b) Let Ω be an open neighbourhood of $0 \in \mathbb{C}$. Suppose g is holomorphic on $\Omega \setminus \{0\}$ and obeys

$$|g(z)| \le C|z|^{-n}$$

there (with n and C as before). Show that there is a holomorphic function $h : \Omega \to \mathbb{C}$ and coefficients $a_1, \ldots, a_n$ in $\mathbb{C}$ so that

$$g(z) = h(z) + \sum_{k=1}^{n} a_k z^{-k}.$$

The sum here is called the *principal part* of g at the point 0.
Hint: First treat the case $n = 0$, for which the sum is empty (and so zero). The $n = 0$

result is known as *Riemann's removable singularity theorem.* For general n consider $f(z) = z^n g(z)$.

4. Prove the following stronger versions of the classical Liouville/Riemann Theorems:
(a) If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and

$$f(z) = o(|z|) \quad \text{as } z \to \infty,$$

then f is constant.
(b) Let Ω be an open neighbourhood of $0 \in \mathbb{C}$. Suppose g is holomorphic on $\Omega \setminus \{0\}$ and obeys

$$g(z) = o(|z|^{-1}) \quad \text{as } z \to 0,$$

then g can be extended (uniquely) to a holomorphic function on Ω.
(c) If g is holomorphic on $\Omega \setminus \{0\}$ and obeys

$$\liminf_{r \searrow 0} \iint_{r<|x+iy|<2r} \frac{|g(x+iy)|}{r}\, dx\, dy = 0,$$

then g can be extended (uniquely) to a holomorphic function on Ω.