246A Homework 0 Due Oct. 4.

1. (a) Fix $\lambda \in \mathbb{R}$ and $a, b \in \mathbb{C}$ with $\lambda > 0$, $\lambda \neq 1$, and $a \neq b$. Use algebraic manipulations to identify

$$\left\{ z \in \mathbb{C} : \left|\tfrac{z-a}{z-b}\right| = \lambda \right\}$$

as a circle.
(b) Show that every circle can be realized in this manner.
(c) Give analogues of (a) and (b) when $\lambda = 1$.

Remark: Part (a) is due to Apollonius of Perga.

2. Show algebraically that for every triple a, b, c of distinct unimodular complex numbers,

$$\frac{b-a}{1-\bar{a}b} = \frac{c-a}{1-\bar{a}c}.$$

Show that with a little further manipulation this expresses the Inscribed Angle Theorem: "the angle at the center is twice the angle at the circumference".

3. Give a proof of the Intersecting Cord Theorem (= Power of a Point Theorem) of Jakob Steiner in the spirit of the previous two problems.

Remark: The power of a point is defined as the product of the two parts of any chord of the circle passing through the point, with the proper interpretation when the point is outside the circle. (Conventionally, the sign is changed to minus when the point is inside the circle.) The Steiner Theorem is that this is independent of the chord we choose through that point. I recommend attacking the problem from this perspective rather than messing around with pairs of chords.

4. Any mapping that can be represented in the form

$$z \mapsto \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$ is called a Möbius transformation.
(a) Show that every such mapping can be realized by coeffients satisfying $ad - bc = 1$ and determine the number of such representations.
(b) Show that every such map is a bijection of the Riemann sphere.
(c) Show that Möbius transformations form a group (under composition of functions) and the the natural mapping from $SL(2, \mathbb{C})$ onto Möbius transformations is an group homomorphism.
(d) Show that Möbius transformations map every circle and every line to either a circle or a line.

5. Let introduce the quaternions for later discussion. This is a four-dimensional vector space over $\mathbb{R}$ with a binary multiplication that is associative and distributive.

We write our elements as $(a_0, \vec{a})$ with $a_0 \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^3$. In this way, multiplication takes the form

$$(a_0, \vec{a}) \star (b_0, \vec{b}) := (a_0 b_0 - \vec{a} \cdot \vec{b},\ a_0 \vec{b} + b_0 \vec{a} - \vec{a} \times \vec{b}).$$

Here $\cdot$ and $\times$ denote the dot- and cross-products from vector calculus.

(a) Show that the usual cross product is not associative, but also verify that quaternion multiplication is associative.

(b) Show that quaternion multiplication is not commutative.

(c) Show that $(1, \vec{0})$ acts as a (two-sided) identity for multiplication and that every non-zero quaternion has a (two-sided) inverse.

(d) We can identify quaternions $(a_0, \vec{a})$ with 4×4 matrices A via

$$(c_0, \vec{c}) = (a_0, \vec{a}) \star (b_0, \vec{b}) \quad \text{if and only if} \quad \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = A \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Find the entries of A, find the characteristic polynomial of A, and find the minimal polynomial of A.