170B Killip Homework 9 Due Fri, June 7

(1) (a) Consider a Poisson process of intensity λ indexed over the whole real line. What is the law and mean of total time elapsed Z between the arrival immediately prior to $t = 0$ and that immediately after. Compare $\mathbb{E}(Z)$ with the mean waiting time for the process.
(b) I find myself changing lightbulbs (in some specific location) according to a Poisson process with intensity λ bulbs/day. Show that the expected total lifespan of a typical bulb is less than that of the particular the bulb that is in place now. It seems that using a bulb makes it last longer — an emotive form of the Feller paradox — though once again we are merely experiencing sampling bias.

(2) Let $N : [0, \infty) \to \mathbb{Z}$ be (the counting function associated to) a Poisson process of intensity λ and fix $\alpha \in (0, \infty)$. Show that $t \mapsto N(\alpha t)$ is a Poisson process and determine its intensity.
Hint: Remember that a Poisson process is the unique process with its marginal distributions.

(3) Fix $0 < a < b < 1$ and let $K \sim$ Poisson(λ). Given K, we lay down this many points independently and uniformly at random in the interval $[0, 1]$. Show that the numbers of points that end up in each of the three intervals $[0, a]$, $(a, b]$, and $(b, 1]$ are statistically independent and find their laws.

(4) Let $N : [0, \infty) \to \mathbb{Z}$ be a Poisson process of intensity λ.
(a) Show that $N(n)/n$ converges almost surely (= with probability one) as $n \to \infty$ and determine the value of this limit. Here $n \in \mathbb{Z}$.
(b) Show that $N(t)/t$ converges almost surely as $t \to \infty$; here $t \in \mathbb{R}$. Hint: Look at what happens when one replaces t by its adjacent integers n and $n + 1$.

(5) Fix an integer $n \geq 1$. (a) Show that the joint law of the arrival times $Y_1, Y_2, \ldots, Y_n$ for a Poisson(λ) process has pdf

$$f(y_1, \ldots, y_n) = \begin{cases} \lambda^n e^{-\lambda y_n} & : 0 < y_1 < y_2 < \cdots < y_n \\ 0 & : \text{otherwise.} \end{cases}$$

Hint: Exploit the relation to the waiting times $T_1, T_2, \ldots, T_n$ to compare MGFs.
(b) Find the joint law of $Y_1, Y_2, \ldots, Y_n$ conditioned on $Y_{n+1} = 1$.

(6) Let $N : [0, \infty) \to \mathbb{Z}$ and $M : [0, \infty) \to \mathbb{Z}$ be independent Poisson processes of intensity $\frac{1}{2}$ and consider the process $S(t) = N(t) - M(t)$.
(a) What is the law of the waiting time to the first jump (up or down)?
(b) Compute the joint MGF of the random variables $S(t_1)$ and $S(t_2) - S(t_1)$ for general $0 < t_1 < t_2 < \infty$.
(c) Given general $0 < t_1 < t_2 < \infty$, show that the joint MGF of the random variables

$$\tfrac{1}{\sqrt{n}} S(nt_1) \quad \text{and} \quad \tfrac{1}{\sqrt{n}} \big[S(nt_2) - S(nt_1)\big]$$

converges as $n \to \infty$ and identify the associated limiting law.

Remark: With a little more work in this direction, one can show that the processes $t \mapsto \frac{1}{\sqrt{n}} S(nt)$ converge, as $n \to \infty$, to a Brownian motion.