170B Killip | Homework 8 | Due Fri, May 24

(1) Customers at a coffee shop ask for hot, say $X_n = 0$, or cold, say $X_n = 1$, beverages according to a Bernoulli process with parameter p. Let N denote the first time that a customer wants the same kind as their predecessor.
(a) Find the pmf of N.
(b) What is the probability that $X_{N+1} = 1$?
(c) If cold drinks take 30 seconds to prepare and hot drinks take 60 seconds. What is the expected time taken to serve the first one hundred customers?
(d) The shop only has enough ice for m cold beverages. What is the expected total number of customers served when the supply is exhausted?
(e) What is the probability that no hot drinks have been requested at (or before) the moment the ice is exhausted?

(2) Let X_n be a Bernoulli(p) process indexed by $n \in \mathbb{Z}$. Imagine we stand half-way between $n = 0$ and $n = 1$ and look both to the left and to the right for a successful trial. Let Z denote the number of trials between the last success on the left (not including this success) and the first success on the right (including this success). For example, the configuration

$n =$	-4	-3	-2	-1	0	1	2	3	4
$X_n =$	0	1	0	0	0	0	1	0	1

yields $Z = 5$. Find the law of Z and compute its mean. Compare the mean of Z with the mean waiting time for this process.

(3) The previous problem highlights the discrepancy between the mean waiting time and the mean gap containing a fixed point. Some elevate this to the status of a paradox, the Feller Paradox. It is more simply an example of sampling bias. Consider the following: We could sample U.S. Presidents by choosing at random from a list of names, or we could sample by choosing a random date between April 30, 1789 and the last change of office and looking up who was president. For which sampling method is the expected value of the term in office larger? Why?

- The two questions that follow concern the following variant of the Bernoulli process: Fix $k \geq 1$. At each (integer) time $n \geq 1$ the process takes the value X_n, where X_n are i.i.d. random variables each with the uniform distribution on $\{1, 2, 3, \ldots, k\}$.

(4) (a) What is the PMF for the random variable N defined as the smallest $N \geq 2$ so that $X_N \neq X_1$.
(b) Is N a stopping time?
(c) What is the probability that $X_{N+1} = 3$?
(d) What is the probability that $X_{N+1} \notin \{X_1, X_N\}$?
(e) What is the PMF of the random variable N' defined as the smallest $N' \geq 1$ so that X_1, X_N, and $X_{N+N'}$ are all distinct? Are N and N' independent?

(5) What is the expected value of the random variable T defined as the smallest T so that $\{X_1, X_2, \ldots, X_T\} = \{1, 2, \ldots, k\}$. Note that if cereal boxes each contain one of k toys (with equal probability), then T represents the number of boxes one would need to buy to "collect all k".