(1) (a) Argue using the Central Limit Theorem that one may approximate $X \sim \chi_n^2$ by a normal law when n is large.
(b) Under the CLT approximation, find a so that $\mathbb{P}(X > a) = \frac{1}{10}$.

(2) (a) Argue using the Central Limit Theorem that one may approximate $X \sim$ Poisson(n) by a normal law when the integer n is large.
(b) Compare the true values of $\mathbb{E}(X^3)$ and $\mathbb{E}(X^4)$ with those based on the CLT approximation. Note: the requisite moments of normal and Poisson random variables can be extracted from their MGFs.

(3) Observe that any integer $n \geq 1$ can be written uniquely as $n = 2^m + k$ with $m \geq 0$ and $0 \leq k < 2^m$. We define random variables X_n on the probability space $\Omega = [0, 1]$ via

$$X_n(\omega) = \begin{cases} 1 & : k2^{-m} \leq \omega < (k+1)2^{-m} \\ 0 & : \text{otherwise} \end{cases}$$

We define the probability of an event $A \subseteq [0, 1]$ as its length.
(a) Plot X_1. Plot X_2 and X_3. Plot $X_4, \ldots, X_7$. Use the same axes for each group.
(b) Compute the PMF of X_n and so deduce that $X_n \to 0$ in probability.
(c) Observe that X_n do not converge with probability one; indeed X_n does not converge at any point $0 \leq \omega < 1$.

(4) Let X_i be a sequence of i.i.d. $N(0, 1)$ random variables and let Y_j be a sequence of i.i.d. Poisson(1) random variables (that are also independent of the X_i). Now define

$$N_n = Y_1 + Y_2 + \cdots + Y_n \qquad \text{and} \qquad Z_n = \sum_{i=1}^{N_n} X_i$$

(a) Find the moment generating functions of N_n and Z_n.
(b) Compute the variance of Z_n.
(c) Show that $Z_n/n \to 0$ in probability. Mimic the proof of the weak law of large numbers (for random variables with finite second moment) given in class (and in the book).
(d) Show that the moment generating function of $Z_n/\sqrt{n}$ converges for each $s \in \mathbb{R}$ as $n \to \infty$. Identify the limit. (This proves an analogue of the CLT for our sequence.)
(e) Compute $\mathbb{E}(Z_n^4)$. Deduce (as in the proof of the strong law of large numbers) that $Z_n/n \to 0$ with probability one.