(1) Consider the probability space $\Omega = [0, 1]$. We define the probability of an event $A \subseteq \Omega$ to be its length. We define a sequence random variables as follows: When n is odd,

$$X_n(\omega) = \begin{cases} 1 & : 0 \le \omega < \frac{1}{2} \\ 0 & : \text{otherwise} \end{cases}$$

while, when n is even,

$$X_n(\omega) = \begin{cases} 0 & : 0 \le \omega < \frac{1}{2} \\ 1 & : \text{otherwise.} \end{cases}$$

(a) Compute the PMF and CDF of each X_n.
(b) Deduce that X_n converge in distribution.
(c) Show that for any n and any random variable $X : \Omega \to \mathbb{R}$,

$$\big\{\omega : |X_n - X| \ge \tfrac{1}{2} \text{ or } |X_{n+1} - X| \ge \tfrac{1}{2}\big\} = \Omega$$

(d) Deduce that X_n does not converge in probability (to any random variable X).

(2) Let X_n be a sequence of random variables and let X be another random variable. Given $1 \le p < \infty$, we say $X_n \to X$ in L^p if $\mathbb{E}(|X_n - X|^p) \to 0$ as $n \to \infty$. Show that this implies that $X_n \to X$ in probability. (cf. Problem 5.7.)

(3) Let $f : [0, 1] \to \mathbb{R}$ be continuous and let $X_1, X_2, \ldots,$ be independent and uniformly distributed on $[0, 1]$. Show that

$$\tfrac{1}{n} \sum_{i=1}^{n} f(X_i) \to \int_0^1 f(x)\, dx$$

in probability. This method of approximating integrals is known as the Monte Carlo technique.

(4) We wish to use it to compute $\int_0^1 x\, dx$. How large should we choose n to ensure our answer lies between 0.49 and 0.51 with 95% probability. Use a CLT approximation.