(1) (a) Compute the moment generating function of a χ^2_ν random variable (see Homework 2 for the definition).
(b) Use these moment generating functions give a simpler demonstration that if $X \sim \chi^2_\nu$ and $Y \sim \chi^2_k$ are independent, then $X + Y \sim \chi^2_{\nu+k}$.

(2) Use moment generating functions to verify the following:
(a) The expected value of the sum of independent random variables is the sum of the expected values.
(b) The variance of a sum of independent random variables is the sum of the variances.
(c) If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent, then $X + Y \sim \text{Poisson}(\lambda + \mu)$.

(3) I claim that one can generate a Uniform([0,1]) random variable, by uniformly and independently generating its binary digits. To this end, consider random variables X_k that are independent with

$$\mathbb{P}(X_k = 0) = \tfrac{1}{2} \quad \text{and} \quad \mathbb{P}(X_k = 2^{-k}) = \tfrac{1}{2}$$

(a) Find the MGF of each X_k and thence that of $X_1 + X_2 + \cdots + X_n$.
(b) Simplify your answer using an observation such as the following: For $N = 2^n$,

$$\begin{aligned} &(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots(1+z^N) \\ &= 1 + z + z^2 + z^3 + z^4 + z^5 + \cdots + z^{2N-1} \\ &= (1 - z^{2N})/(1 - z) \end{aligned}$$

This is easily checked by induction.
(c) Send $n \to \infty$ and verify that the limiting MGF is that of a Uniform([0,1]).

(4) Suppose $\Lambda \sim \text{Exponential}(\gamma)$ and $X \sim \text{Poisson}(\Lambda)$. Use generating functions to show that $X + 1 \sim \text{Geometric}(p)$ and determine p in terms of γ.

(5) Recall that $X \sim \text{Uniform}(\{0, 1, 2, \ldots, n-1\})$ if

$$\mathbb{P}(X = k) = \begin{cases} \frac{1}{n} & \text{if } k \in \{0, 1, 2, \ldots, n-1\}, \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the MGF of such a random variable.
(b) Let X_1, X_2, X_3 be independent random variables with

$$X_1 \sim \text{Uniform}(\{0,1\}) \quad X_2 \sim \text{Uniform}(\{0,1,2\}) \quad X_3 \sim \text{Uniform}(\{0,1,2,3,4\}).$$

Find the laws of both $Y_1 = X_1 + 2X_2 + 6X_3$ and $Y_2 = 15X_1 + 5X_2 + X_3$.
(c) What is the correlation coefficient of Y_1 and Y_2?