(1) (a) Compute the moment generating function of a $\chi^2_\nu$ random variable (see Homework 2 for the definition).
(b) Use these moment generating functions give a simpler demonstration that if $X \sim \chi^2_\nu$ and $Y \sim \chi^2_\kappa$ are independent, then $X + Y \sim \chi^2_{\nu + \kappa}$.

(2) Use moment generating functions to verify the following:
(a) The expected value of the sum of independent random variables is the sum of the expected values.
(b) The variance of a sum of independent random variables is the sum of the variances.
(c) If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent, then $X + Y \sim \text{Poisson}(\lambda + \mu)$.

(3) I claim that one can generate a Uniform([0,1]) random variable, by uniformly and independently generating its binary digits. To this end, consider random variables $X_k$ that are independent with $P(X_k = 0) = \frac{1}{2}$ and $P(X_k = 2^{-k}) = \frac{1}{2}$
(a) Find the MGF of each $X_k$ and thence that of $X_1 + X_2 + \cdots + X_n$.
(b) Simplify your answer using an observation such as the following: For $N = 2^n$,
   $$(1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \cdots (1 + z^{2N})$$
   $$= 1 + z + z^2 + z^3 + z^4 + z^5 + \cdots + z^{2N-1}$$
   $$= (1 - z^{2N})/(1 - z)$$
   This is easily checked by induction.
(c) Send $n \to \infty$ and verify that the limiting MGF is that of a Uniform([0,1]).

(4) Suppose $\Lambda \sim \text{Exponential}(\gamma)$ and $X \sim \text{Poisson}(\Lambda)$. Use generating functions to show that $X + 1 \sim \text{Geometric}(p)$ and determine $p$ in terms of $\gamma$.

(5) Recall that $X \sim \text{Uniform}([0, 1, 2, \ldots, n - 1])$ if
$$P(X = k) = \begin{cases} \frac{1}{n} & \text{if } k \in \{0, 1, 2, \ldots, n - 1\}, \\ 0 & \text{otherwise} \end{cases}$$
(a) Determine the MGF of such a random variable.
(b) Let $X_1, X_2, X_3$ be independent random variables with
   $X_1 \sim \text{Uniform}([0, 1]) \quad X_2 \sim \text{Uniform}([0, 1, 2]) \quad X_3 \sim \text{Uniform}([0, 1, 2, 3, 4])$.
   Find the laws of both $Y_1 = X_1 + 2X_2 + 6X_3$ and $Y_2 = 15X_1 + 5X_2 + X_3$.
(c) What is the correlation coefficient of $Y_1$ and $Y_2$?