(1) Suppose X_1, X_2, and X_3 are independent and uniformly distributed in $[0,1]$.
(a) Find the joint pdf for $Y = \min\{X_1, X_2, X_3\}$ and $Z = \max\{X_1, X_2, X_3\}$.
(b) Determine $\mathbb{E}(Z|Y)$.
(c) Verify the law of total variance in this setting by explicitly computing all terms.

(2) Let $\Omega = \{0,1\}^3$, that is, all possible (ordered) triples of zeros and ones. All outcomes have equal probability. We define three random variables X_1, X_2, and X_3 on this space representing the first, second, and third digit, respectively. We also define $X = X_1 + X_2 + X_3$.
(a) Tabulate the values (across Ω) of each of the following random variables:

$$\mathbb{E}\{X|X_1\}, \qquad \mathbb{E}\{\mathbb{E}[X|X_1]|X_2\}, \quad \text{and} \quad \mathbb{E}\{X_2|X\}$$

(b) What is the pmf of $\mathbb{E}\{X_2|X\}$.

(3) Suppose X and Y are discrete random variables. Show that

$$\mathbb{E}(X|Y) = \mathbb{E}(X|Y^3)$$

(4) Given a vector $\vec{\mu} \in \mathbb{R}^2$ and a 2×2 positive definite matrix Σ, we say that $(X_1, X_2) \sim N(\vec{\mu}, \Sigma)$ if they have joint pdf

$$f_{X_1,X_2}(\vec{x}) = [\det(2\pi\Sigma)]^{-1/2} \exp\bigl\{-\tfrac{1}{2}(\vec{x}-\vec{\mu}) \cdot \Sigma^{-1}(\vec{x}-\vec{\mu})\bigr\}.$$

Mimic the analysis of problem 4.28 to resolve the following:
(a) Find the conditional pdf of X_1 given $X_2 = y$ and so determine $\mathbb{E}(X_1|X_2)$.
(b) Find the joint law of X_2 and $X_1 - \mathbb{E}(X_1|X_2)$.

(5) Fix $m \geq 1$, an integer, and suppose $P \sim \text{Uniform}([0,1])$ and $N \sim \text{Binomial}(m, P)$.
(a) Determine $\mathbb{E}\bigl(\chi_k(N)P\bigr)$ where $\chi_k(n)$, $k = 0, 1, 2, \ldots$, are defined as follows:

$$\chi_k(n) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

(b) Determine $\mathbb{E}\bigl(\chi_k(N)h(N)\bigr)$ for a general function $h : \mathbb{R} \to \mathbb{R}$.
(c) Determine $\mathbb{E}(P|N)$.

Warning: $\mathbb{E}(P|N)$ is *not* N/m as you might be tempted to guess.
Hint: Use the law of total probability together with the following result which you showed (in greater generality) on the previous homework:

Lemma. *For any pair of integers $n, m \geq 0$,*

$$\int_0^1 x^m (1-x)^n \, dx = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(n+m+2)} = \frac{m!\, n!}{(m+n+1)!}$$