170B Killip | Homework 2 | Due Fri, April 12

Recall (from class) the definition of the χ^2_ν random variable: $X \sim \chi^2_\nu$ if and only if

$$f_X(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\frac{\nu}{2}-1} e^{-x/2} & : x \geq 0 \\ 0 & : \text{otherwise} \end{cases}$$

where $\nu > 0$ and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$ for all $\alpha > 0$.

(1) Suppose $X \sim \chi^2_\nu$ and $Y \sim \chi^2_k$ are independent. Show that $X + Y \sim \chi^2_{\nu+k}$. In this way, also recover the value of

$$\int_0^1 u^{\frac{\nu}{2}-1}(1-u)^{\frac{k}{2}-1}\,du$$

as a ratio of Gamma functions. Hint: mimic the arguments in class that covered the case $k = 1$.

(2) Suppose $X \sim \chi^2_\nu$ and $Z \sim N(0,1)$ are independent. Find the pdf of

$$T = Z\sqrt{\tfrac{\nu}{X}}.$$

Hint: Solving this problem will lead to the discovery (also of Euler) that

$$\int_{-\infty}^{\infty} (1 + \tfrac{1}{\nu}t^2)^{-\frac{\nu+1}{2}}\,dt = \frac{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})}$$

(3) Suppose $\Theta \sim \text{Uniform}([0, 2\pi])$ and let

$$X = \cos(\Theta) \quad \text{and} \quad Y = \sin(\Theta).$$

(a) Determine the correlation coefficient between X and Y.
(b) Prove that X and Y are *not* independent.

(4) Continuing from the previous problem, define events

$$A = \{X \geq 0\} \quad \text{and} \quad B = \{Y \geq 0\}.$$

(a) Show that A and B are independent.
(b) Show that A and B are not independent conditioned on $A \cup B$.
(c) Determine the correlation coefficient between X and Y conditioned on $A \cup B$.

(5) Let A and B be events of probabilities $\mathbb{P}(A) = p$ and $\mathbb{P}(B) = q$ respectively.
(a) What are the largest and smallest possible values that $\mathbb{P}(A \cap B)$ could take if p and q are known.
(b) Express the correlation coefficient ρ of the indicator random variables

$$\chi_A(\omega) = \begin{cases} 1 & : \omega \in A \\ 0 & : \text{otherwise} \end{cases} \quad \text{and} \quad \chi_B(\omega) = \begin{cases} 1 & : \omega \in B \\ 0 & : \text{otherwise} \end{cases}$$

in terms of $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(A \cap B)$.
(c) Apply Cauchy–Schwarz to the product $\chi_A\chi_B$ to obtain a bound on $\mathbb{P}(A \cap B)$ in terms $\mathbb{P}(A)$ and $\mathbb{P}(B)$.