(1) Suppose $X \sim \text{Exponential}(\lambda)$. Find the PDF of $Y = \sin^2(X)$.
Note: While the solution involves an infinite sum, it is an example of a geometric series and so can be evaluated.

(2) Let X_1 and X_2 be independent and uniformly distributed on the interval $[0, 1]$. Given $0 < \alpha < 1$, we generate a random variable Y as follows: If $X_1 > \alpha$ then $Y = X_1$, otherwise $Y = X_2$.
(a) What are the CDF and PDF of Y.
(b) What is $\mathbb{E}(Y)$.
(c) For which value of α is $\mathbb{E}(Y)$ largest.

(3) Suppose $X \sim \text{Exponential}(\lambda)$ and define a function floor : $\mathbb{R} \to \mathbb{Z}$ as follows:

$$\text{floor}(x) = n \qquad \iff \qquad x \in [n, n+1)$$

Determine the law of floor(X).

(4) Suppose U and V are independent and follow a Uniform(0,1) law. Show that

$$X = \sqrt{-2\ln(U)}\cos(2\pi V) \quad \text{and} \quad Y = \sqrt{-2\ln(U)}\sin(2\pi V)$$

define independent random variables each with a $N(0, 1)$ law. *Notes:* This is known as the Box–Muller transform. Before attempting this problem, please review book problem 16 from Chapter 4.

(5) Fix $\beta \in (0, 1)$ and suppose $U, V \sim \text{Uniform(0,1)}$ are independent.
(a) Find the pdf of $X = UV$.
(b) Find the conditional pdf of $X = UV$ conditioned on $U \geq \beta$.
(c) What is the probability that $U \geq \beta$ and $UV < \beta$.

(6) Suppose X_1, X_2, and X_3 are independent and follow a $N(0, 1)$ law. Find the PDF of $Y = \sqrt{X_1^2 + X_2^2 + X_3^2}$.

(7) Suppose X_1, X_2, and X_3 are independent and uniformly distributed on the interval $[0, 1]$.
(a) What is the probability of the event $X_1 \leq X_2 \leq X_3$?
(b) What is the joint pdf of X_1 and X_2 conditioned on the event $X_1 \leq X_2$?
(c) What is the joint pdf of X_1, X_2, and X_3 conditioned on $X_1 \leq X_2 \leq X_3$?
(d) Find the PDF for the median of these three random variables.