Defn The <u>moment generating function</u> $M_X(s)$ of a random variable X is defined by

$$M_X(s) = \mathbb{E}\{e^{sX}\}$$

Remarks (1) It makes sense to allow s to be a complex number, but in this course we will confine our attention to $s \in \mathbb{R}$.

(2) At least when X is continuous, this may remind you of the Laplace transform; indeed

$$M_X(s) = \int e^{sx} f_X(x)\, dx$$

Moreover, if s was purely imaginary then we would get the Fourier transform. (I wanted to acknowledge the relation in case you have seen Fourier/Laplace transforms before. If not, don't worry — we will do everything from scratch.)

(3) We can also define the <u>joint MGF</u> of several random variables:

$$M_{X_1, X_2, \ldots, X_n}(s_1, \ldots, s_n) = \mathbb{E}\left\{e^{s_1 X_1 + s_2 X_2 + \cdots + s_n X_n}\right\}$$

(4) The sense in which $M_X(s)$ "generates the moments" of X is as follows:

Expanding $e^{sX} = \sum_{m=0}^{\infty} \frac{s^m X^m}{m!}$

we are led to the relation

$$M_X(s) = \sum_{m=0}^{\infty} \frac{s^m}{m!} \mathbb{E}\{X^m\} \qquad (*)$$

↑ these are the 'moments of X'.

Thus $M_X(s)$ is the "exponential generating function" of the moments.

= clever way of encoding a seq. of numbers that can help solve problems, particularly combinatorial problems.

One can recover the moments of X from $M_X(s)$ by differentiating:

$$M_X(0) = \mathbb{E}\{1\} = \mathbb{E}\{X^0\} = 1$$

$$\frac{dM_X}{ds}(s) = \mathbb{E}\left\{\frac{d}{ds} e^{sX}\right\} = \mathbb{E}\left\{X e^{sX}\right\}$$

Thus, $\frac{dM_X}{ds}(s=0) = \mathbb{E}\{X\}$.

More generally,

$$\left[\frac{d^m}{ds^m} M_X\right](s=0) = \mathbb{E}\left\{\left[\frac{d^m}{ds^m} e^{sX}\right](s=0)\right\}$$

$$= \mathbb{E}\{X^m\}.$$

[One can also derive this relation by differentiating the series (*) term by term

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Examples

① If $X \sim \text{Bernoulli}(p)$ then

$$M_X(s) = \mathbb{E}\{e^{sX}\} = 1 \cdot \mathbb{P}(X=0) + e^s \cdot \mathbb{P}(X=1)$$

$$= pe^s + (1-p)$$

② If $X \sim \text{Geometric}(p)$ i.e. $\mathbb{P}(X=k) = \begin{cases} p(1-p)^{k-1} & k \geq 1 \text{ an integer} \\ 0 & \text{otherwise} \end{cases}$

$$M_X(s) = \mathbb{E}\{e^{sX}\} = \sum_{k=1}^{\infty} e^{sk} p (1-p)^{k-1}$$

$$= p \cdot e^s \cdot \sum_{k=1}^{\infty} \left[e^s(1-p)\right]^{k-1}$$

sum geometric series

$$\stackrel{\swarrow}{=} \frac{pe^s}{1-(1-p)e^s}$$

③ $X \sim \text{Poisson}(\lambda)$

$$M_X(s) = \sum_{k=0}^{\infty} e^{sk} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{(\lambda e^s)^k}{k!}}_{\text{Exponential series}}$$

$$= e^{-\lambda} \cdot \exp\{\lambda e^s\} = \exp\{\lambda e^s - \lambda\}.$$

④ $X \sim N(0,1)$ (Normal = Gaussian)

$$M_X(s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{sx - \frac{1}{2}x^2} \, dx$$

Complete the square:

$$sx - \tfrac{1}{2}x^2 = -\tfrac{1}{2}(x-s)^2 + \tfrac{1}{2}s^2$$

$$= \frac{e^{\frac{1}{2}s^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-s)^2} \, dx$$

Change var's to $y = x - s$ and recognise pdf for $N(0,1)$ (which integrates to unity)

$$= e^{\frac{1}{2}s^2}$$

or directly recognise the pdf for $N(s,1)$.

(5) • Uniform & Exponential are easy. Do them yourself.

• χ^2_{ν} is on the HW

• A random variable T with pdf

$$f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{2\pi}\ \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

that showed up on Homework 2 is said to be <u>Student's-t</u> distributed with ν degrees of freedom.* For these r.v. we have

$$M_T(s) = \begin{cases} 1 & s = 0 \\ \infty & s \in \mathbb{R}\setminus\{0\}. \end{cases}$$

because the polynomial decay of the pdf can't compete with the exponential growth of e^{st} when $s \neq 0$.

* Note ν must obey $\nu > 0$. The weird name is explained on Wikipedia.

Lemma If $Y = aX + b$, with $a, b \in \mathbb{R}$, then

$$M_Y(s) = \mathbb{E}\{e^{s(aX+b)}\}$$

$$= e^{sb} \cdot \mathbb{E}\{e^{as X}\}$$

$$= e^{sb} M_X(as)$$

In particular, if $Y \sim N(\mu, \sigma^2)$ then

$$M_Y(s) = e^{\mu s + \frac{1}{2}\sigma^2 s^2} \, .$$

Prop If X & Y are independent, then

(a) $$M_{X,Y}(s,t) = \mathbb{E}\{e^{sX+tY}\}$$

$$= \mathbb{E}\{e^{sX}\} \cdot \mathbb{E}\{e^{tY}\} \quad \text{(Independence)}$$

$$= M_X(s) \cdot M_Y(t)$$

(b) $$M_{X+Y}(s) = \mathbb{E}\{e^{s(X+Y)}\} = M_X(s) \cdot M_Y(s)$$

(as above)

This proposition can be used to give elegant solutions to some problems that were previously really messy (see below & Homework 4); however this relies on being able to go back from the MGF to the law of the random variable:

<u>Theorem</u> If there is some $a < b$ so that

$$M_X(s) < \infty \quad \text{for every } s \in (a,b)$$

the $M_X(s)$ uniquely determines the law of X.

More generally, if there are $a_i < b_i$ so that

$$M_{X_1,\dots,X_n}(s_1,\dots,s_n) < \infty \quad \text{for all } s_1 \in (a_1,b_1) \text{ and } s_2 \in (a_2,b_2) \text{ and } \dots \text{ and } s_n \in (a_n,b_n)$$

Then $M_{X_1,\dots,X_n}(s_1,\dots,s_n)$ uniquely determines the joint law of $X_1,\dots,X_n$.

Remark The student-t example above shows that the finiteness assumption is necessary — the MGF was the same for every value of ν, but the law (expressed through the pdf) was not.

Eg 1 If X & $Y \sim N(0,1)$ are independent, find the joint law of $Z = X+Y$ & $W = X-Y$

$$M_{Z,W}(s,t) = \mathbb{E}\left\{ e^{s(X+Y)+t(X-Y)} \right\}$$

$$= \mathbb{E}\left\{ e^{(s+t)X} \cdot e^{(s-t)Y} \right\}$$

↓ Independence (cf. Proposition above)

$$= \mathbb{E}\left\{ e^{(s+t)X} \right\} \cdot \mathbb{E}\left\{ e^{(s-t)Y} \right\}$$

↓ earlier computation

$$= \exp\left\{ \frac{(s+t)^2}{2} \right\} \cdot \exp\left\{ \frac{(s-t)^2}{2} \right\}$$

↓ boring algebra

$$= \exp\left\{ s^2 + t^2 \right\}$$

$$= \underbrace{e^{\frac{1}{2}\cdot 2s^2}} \cdot \underbrace{e^{\frac{1}{2}\cdot 2t^2}}$$

these are the MGFs of $N(0,2)$ r.v.s.

Thus Z, W have the same joint MGF as a pair of independent $N(0,2)$ r.v.s. But then by the uniqueness theorem, the joint law of Z & W must be that of independent $N(0,2)$ r.v.s.

Eg 2 If $X_1, \dots, X_n$ are independent Bernoulli(p), then we learned long ago that $Y = X_1 + \cdots + X_n \sim \text{Binomial}(n,p)$. But by the proposition & earlier computation,

$$M_Y(s) = M_{X_1}(s) M_{X_2}(s) \cdot \dots \cdot M_{X_n}(s)$$
$$= (1 - p + pe^s)^n$$

Thus we find the MGF of binomial r.v.s.

Eg 3 A turtle lays $Y \sim \text{Poisson}(\lambda)$ many eggs. Each such individual survives to adulthood with probability p independent of all others. Find the law, mean, and variance of the number of adult offspring.
X

Soln The key is to observe that X is a sum of Y many independent Bernoulli(p) random variables. (Or, equivalently, that the law of X conditioned on Y is Binomial(Y, p).) Now

law of iterated expectation.

$$M_X(s) = \mathbb{E}\{e^{sX}\} = \mathbb{E}\{\mathbb{E}\{e^{sX} | Y\}\}$$

$$= \mathbb{E}\{(1-p+pe^s)^Y\}$$

$$= \mathbb{E}\{e^{\ln(1-p+pe^s)\cdot Y}\}$$

$$= M_Y(\ln(1-p+pe^s))$$

MGF of Poisson →

$$\stackrel{*}{=} \exp\{\lambda e^{\ln(1-p+pe^s)} - \lambda\}$$

$$= \exp\{\lambda(1-p+pe^s) - \lambda\}$$

$$= \exp\{\lambda p e^s - p\lambda\}$$

But this is the MGF of a Poisson(λp) r.v.

thus, by the uniqueness theorem,

$$X \sim \text{Poisson}(\lambda p)$$

We could now simply remember that this implies

$$\mathbb{E}(X) = \lambda p \quad \& \quad var(X) = \lambda p$$

However if we forgot this we could also recover them from the MGF:

$$M_X'(s) = \lambda p e^s \cdot \exp\{\lambda p e^s - \lambda p\}$$

so $\mathbb{E}\{X\} = M_X'(0) = \lambda p$ and

$$\mathbb{E}\{X^2\} = M_X''(0) = \cdots = \lambda p + (\lambda p)^2$$