170B Killip | Final | March 20th

First Name: ____________________ ID#____________________

Last Name: ____________________

Rules.

- There are **NINE** problems; equal points per problem.
- There are extra pages after each problem. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...

1	2	3	4	5
6	7	8	9	$\sum$

(1) (a) State the Strong Law of Large Numbers.
(b) State the Weak Law of Large Numbers.
(c) Define what it means for a sequence of random variables X_n to converge to a random variable X *in distribution*.
(d) If X follows a Pascal distribution with parameters k and p, what is the PMF of X?

extra paper

(2) (a) State the Cauchy–Schwarz inequality.

(b) State the law of iterated expectation.

(c) **True or False**: If $X_n \to X$ in probability, then $X_n \to X$ with probability one.

(d) Suppose $\mathbb{E}(X) = 1$ and var(X)=2. Use Chebyshev's inequality to give an upper bound on $\mathbb{P}(X \geq 7)$.

$\boxed{\text{extra paper}}$

(3) Suppose $X \sim \text{Exponential}(\lambda)$.

(a) Find the pdf of $Z = e^{-X}$.

(b) Determine the covariance of X and Z.

(c) Determine the (Pearson) correlation coefficient of X and Z.

$\boxed{\text{extra paper}}$

(4) Consider the following random family tree: Let Y_n denote the (random) number of people in the nth generation. Each person in the nth generation produces a random number of offspring, which has a Poisson(λ) distribution. The total number of such children is then denoted Y_{n+1}. The number of offspring produced by any person is (statistically) independent of the number produced by another person. Moreover, $Y_0 = 1$, that is, there is exactly one person in the zeroth generation.

(a) Determine $\mathbb{E}(Y_{n+1}|Y_n)$ and so also $\mathbb{E}(Y_n)$.
(b) Relate the moment generating function of Y_{n+1} to that of Y_n.
(d) Give a formula for the variance of Y_{n+1} in terms of that of Y_n.

$\boxed{\text{extra paper}}$

(5) Suppose $B(t)$ is a Brownian motion.

(a) What is the joint law of $B(t_0), \ldots, B(t_k)$ for given $0 = t_0 < t_1 < \cdots < t_k$, (e.g. as appearing in the definition of a Brownian motion).

(b) Determine $\mathbb{E}(B(2)|B(1))$.

(c) What is the probability that both $B(1) \geq 0$ and $B(2) \geq 0$?

$\boxed{\text{extra paper}}$

(6) Each morning, twenty five police cadets are arranged like the entries of a 5×5 matrix. Each cadet forgets their cap with probability $1-p$, independently of the other cadets. Define random variables

$$X_i = \begin{cases} 1 & : \text{one or more cadets in row } i \text{ forgot their cap} \\ 0 & : \text{otherwise} \end{cases}$$

and

$$Y_j = \begin{cases} 1 & : \text{one or more cadets in column } j \text{ forgot their cap} \\ 0 & : \text{otherwise} \end{cases}$$

Lastly, let $Z = X_1 + X_2 + \cdots + X_5 + Y_1 + Y_2 + \cdots + Y_5$.

(a) What is $\mathbb{E}(X_1)$?
(b) What is $\mathbb{P}(X_3 = 1 \text{ and } Y_2 = 1)$?
(c) Determine $\mathbb{E}(Z)$.
(d) Determine the variance of Z.

$\boxed{\text{extra paper}}$

(7) Starting at time $t = 0$, passengers arrive for a car-pool service according to a Poisson process of intensity λ. When a car accumulates three passengers, it leaves and we start filling the next car, and so on.

(a) What is the expected time that the first car leaves?

(b) What is the probability that the second car is still waiting for passengers one unit of time after the first car leaves?

(c) What is the expected number of people in the second car one unit of time after the first car leaves, conditioned on the event that it is still waiting?

(d) What is the expected number of people in the second car one unit of time after the first car leaves (unconditionally)?

(e) Suppose the first passenger for the third car arrives at $t = 6$ and its second passenger arrives at $t = 9$. Conditioned on this, what is the variance of the time t that the third car departs?

$\boxed{\text{extra paper}}$

(8) Answer the following in the context of a Bernoulli process X_n indexed by $n = 1, 2, 3, \ldots$.

(a) Give an example of a stopping time.

(b) Give an example of a random time that is not a stopping time.

(c) If T_1 and T_2 are stopping times, is $\min\{T_1, T_2\}$ a stopping time?

(d) Describe the 'fresh start' and 'memoryless' properties of X_n with respect to a stopping time.

$\boxed{\text{extra paper}}$

(9) Let X_n be a Bernoulli process indexed by $n = 1, 2, 3, \ldots$ with $\mathbb{P}(X_j = 1) = p$. Let N denote the first time that the process differs from X_1. Thus $X_1 = X_2 = \cdots = X_{N-1}$ but $X_N \neq X_1$.

Find the PMF of N.

$\boxed{\text{extra paper}}$