Our next three results you may have seen in 170A. The following is Problem 13 in Chapter 1 of the textbook.

<u>Monotone Convergence Theorem</u>

(a) Suppose the events A_n are increasing, that is

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

Then $$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \mathbb{P}(A_n)$$

(b) Suppose the events B_n are decreasing, that is

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$$

Then $$\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n \right) = \lim_{n \to \infty} \mathbb{P}(B_n)$$

<u>Proof</u>. (a) We need the notation

$$A \setminus B = A \cap B^c = \{a \in A : a \notin B\}.$$

Now we observe that the sets

$$A_1, \quad A_2 \setminus A_1, \quad A_3 \setminus A_2, \quad \ldots$$

are disjoint and that

$$A_n = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2) \cup \cdots \cup (A_n \setminus A_{n-1})$$

$$= A_1 \cup \left(\bigcup_{m=2}^{n} (A_m \setminus A_{m-1}) \right)$$

Thus, by the finite additivity axiom,

$$\mathbb{P}(A_n) = \mathbb{P}(A_1) + \sum_{m=2}^{n} \mathbb{P}(A_m \setminus A_{m-1})$$

This shows that $\mathbb{P}(A_n)$ forms an increasing sequence, bounded above by 1, so $\lim A_n$ exists. Moreover, using the countable additivity axiom,

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A_1) + \sum_{m=2}^{\infty} \mathbb{P}(A_m \setminus A_{m-1}) \quad \text{(by defn of infinite sum)}$$

(countable add. axiom)

$$= \mathbb{P}\left\{ \underbrace{A_1 \cup \left[\bigcup_{m=2}^{\infty} (A_m \setminus A_{m-1}) \right]}_{\text{same set}} \right\}$$

$$= \mathbb{P}\left\{ \bigcup_{n=1}^{\infty} A_n \right\}$$

This proves (a).

Part (b) follows from part (a) by setting

$$A_n = B_n^c$$

Lemma Given events $A_1, A_2, \ldots$ (not necessarily monotone), we have

(a) $\displaystyle\bigcap_{n=1}^{\infty}\Big(\bigcup_{m\geq n} A_m\Big)$ = Event that infinitely many A_n happen.

(b) $\displaystyle\bigcup_{n=1}^{\infty}\Big(\bigcap_{m\geq n} A_m\Big)$ = Event that all but finitely many A_n occur

Pf (a) Notice that $\bigcup_{m\geq n} A_n$ is the event that at least one of the events A_n happens for some n 'after' m. Now to say that infinitely many A_n happen is the same as to say that for any choice of m (no matter how big) at least one event A_n happens 'after' m. (i.e. for some $n \geq m$)

(b) Well, $\bigcap_{m\geq n} A_m$ is the event that every A_m happens for $m \geq n$. But RHS(b) is precisely the event that all A_m happen for every $m \geq n$ and some choice of n, that is, $\displaystyle\bigcup_{n=1}^{\infty}\Big(\bigcap_{m\geq n} A_m\Big)$ $\boxed{\checkmark}$

Remark

$$\left[\bigcap_{n=1}^{\infty}\left(\bigcup_{m\geq n} A_m\right)\right]^c = \text{only finitely many } A_n \text{ happen}$$

$$= \text{all but finitely many } A_n^c \text{ happen}$$

$$= \bigcup_{n=1}^{\infty}\left(\bigcap_{m\geq n} A_m^c\right)$$

Theorem (Borel - Cantelli Lemma)

(a) If events A_n obey $\sum \mathbb{P}(A_n) < \infty$ then

$$\mathbb{P}(\text{infinitely many } A_n \text{ occur}) = 0$$

(b) If events B_n obey $\sum \mathbb{P}(B_n^c) < \infty$ then

$$\mathbb{P}(\text{all but finitely many } B_n \text{ occur}) = 1$$

Pf (a) & (b) are the same — just take $B_n^c = A_n$.

Now to prove (a) we argue as follows:

$$\mathbb{P}(\text{infinitely many } A_n) = \mathbb{P}\Big(\bigcap_{n=1}^{\infty} \underbrace{\Big(\bigcup_{m \geq n} A_m \Big)}_{\text{these are decreasing events}} \Big)$$

$$\overset{\text{monotone convergence thm}}{=} \lim_{n \to \infty} \mathbb{P}\Big(\bigcup_{m \geq n} A_m \Big)$$

$$\leq \lim_{n \to \infty} \sum_{m \geq n} \mathbb{P}(A_m)$$

But the convergence of $\sum_{m=1}^{\infty} \mathbb{P}(A_m)$ guaratees that the 'tail' converges to zero as $n \to \infty$.

$\blacksquare$

Remark There is also a "second Borel-Cantelli lemma" which says

If the A_n are *independent* then

$$\mathbb{P}(\text{infinitely many } A_n) = 0 \iff \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$$

See Problem 48 in Chapter 1 for a proof.

Evidently this material can be covered early in 170A, our reason for discussing it now is that it will help us understand our last notion of convergence of random variables:

Definition A sequence of random variables X_n converge to another random variable X *almost surely* (= with probability one) if

$$\mathbb{P}\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$