Proposition. *Suppose* $X_1 \sim N(\mu_1, \sigma_1^2)$ *and* $X_2 \sim N(\mu_2, \sigma_2^2)$ *are independent, then*

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

Proof. Let us write $Z = X_1 + X_2$. Then, as shown previously,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{X_1}(z-y) f_{X_2}(y)\, dy \\ &= \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left\{ -\frac{(z-y-\mu_1)^2}{2\sigma_1^2} - \frac{(y-\mu_2)^2}{2\sigma_2^2} \right\} dy \end{aligned}$$

To proceed, we must complete the square in the exponent. In general we have

$$ay^2 + by + c = a\big(y + \tfrac{b}{2a}\big)^2 + c - \tfrac{b^2}{4a}.$$

In our particular case,

$$\begin{aligned} &\frac{(z-y-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \\ &= \left[\frac{1}{\sigma_1^2} + \frac{1}{\sigma_1^2}\right] y^2 - 2\left[\frac{z-\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2}\right] y + \left[\frac{(z-\mu_1)^2}{\sigma_1^2} + \frac{(\mu_2)^2}{\sigma_2^2}\right] \\ &= \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}(y-y_0)^2 - \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\left[\frac{z-\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2}\right]^2 + \left[\frac{(z-\mu_1)^2}{\sigma_1^2} + \frac{(\mu_2)^2}{\sigma_2^2}\right] \end{aligned}$$

where

$$y_0 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\left[\frac{z-\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2}\right]$$

As we will see, there is no advantage to simplifying the formula for y_0; however we do need to simplify the last two terms. After some effort, we find

$$\frac{(z-y-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}(y-y_0)^2 + \frac{(z-\mu_1-\mu_2)^2}{\sigma_1^2+\sigma_2^2}.$$

Consequently,

$$f_Z(z) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} \exp\left\{ -\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}(y-y_0)^2 - \frac{(z-\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)} \right\} dy$$

Now, recalling the general result

$$\int_{-\infty}^{\infty} \exp\left\{ -\frac{(y-y_0)^2}{2\sigma^2} \right\} dy = \sqrt{2\pi\sigma^2},$$

which was shown in class, we deduce that

$$\begin{aligned} f_Z(z) &= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{ -\frac{(z-\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)} \right\} \sqrt{2\pi\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}} \\ &= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left\{ -\frac{(z-\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)} \right\}. \end{aligned}$$

This then shows that $Z \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. □