Construction of an anticyclotomic Euler system with applications

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Abstract

In this thesis, we construct a new anticyclotomic Euler system (in the sense of Jetchev-Nekovář-Skinner (JNS)) for the Galois representation attached to a newform f of weight $2k$ twisted by an anticyclotomic Hecke character χ of infinity type $(l, -l)$, denoted by $V_f(\chi)$, when the Heegner Hypothesis is not satisfied. The main ingredients for our construction are the Bertolini-Seveso-Venerucci (BSV) diagonal classes and the Lei-Loeffler-Zerbes norm maps. We then show some arithmetic applications of the constructed Euler system, including the rank 0 Bloch-Kato Conjecture for $V_f(\chi)$ when $k \geq l+1$, using the explicit reciprocity law of BSV and the machinery of JNS.

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Chapter 1

Introduction

1.1 Motivation

Classical Iwasawa theory relates special values of the Riemann zeta function $\zeta(n)$ to the ideal class groups of cyclotomic fields by an equality of the ideal generated by the Kubota-Leopoldt p-adic L-function and the characteristic ideal of the ideal class groups. Since then, its theory and philosophy have been generalized to connect the two sides

$$\left\{\begin{array}{c}\textbf{Analytic}\text{ side}\\ \text{Special values of}\\ (p-\text{adic})\ L-\text{functions}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\textbf{Arithmetic}\text{ side}\\ \text{Selmer groups of}\\ \text{Elliptic curves, Galois representations}\end{array}\right\}$$

via an equality of the ideal generated by the p-adic L-function and the characteristic ideal of some corresponding Selmer group over an appropriate Iwasawa algebra. To show such an equality of ideals, it's quite natural to prove that each side divides the other:

$$(\textbf{analytic})|(\textbf{arithmetic})$$
$$(\textbf{arithmetic})|(\textbf{analytic}).$$

In their proof of the classical Iwasawa Main Conjecture (IMC) for GL_1, Mazur-Wiles [MW84] first connect the p-adic L-functions of even Dirichlet characters to the cuspidal subgroups in the Jacobian of modular curves, then use the geometry of the latter to construct sufficiently large quotients of the ideal class group, obtaining (**analytic**)|(**arithmetic**). This so-called 'automorphic approach' is well studied and has been used to prove the (**analytic**)|(**arithmetic**) divisibility for GL_2 using Eisenstein series for $\mathrm{U}(2,2)$ by Skinner-Urban [SU14]. To produce the opposite divisibility for GL_1, Rubin [Rub00] used Kolyvagin's Euler system [Kol90] of cyclotomic units to produce many principal ideals and hence obtained that ideal class groups are as small as expected. Note that showing one divisibility is enough for the IMC for GL_1 due to the existence of the analytic class number formula. For GL_2, though, one does not have such luxury. However, Kato [Kat04] did prove (**arithmetic**)|(**analytic**) and in combination with Skinner-Urban, this resulted in the IMC for (most) elliptic curves at ordinary primes. In the works of Rubin and Kato, $L-$functions enter the arithmetic world, transforming into Euler systems and producing one divisibility (**arithmetic**)|(**analytic**). In this thesis, we will construct a new Euler system for yet another $p-$adic $L-$function.

1.2 The Heegner points story

In this section, we recall the construction of Heegner points [Gro91] together with some of its impactful applications.

Set-up.

1. Let E be a rational elliptic curve of conductor N.

2. Let K be an imaginary quadratic field. Let $K[n]$ be the ring class field of conductor n, for each positive integer n.

3. Assume that all prime factors of N split in K. This condition is normally called the Heegner Hypothesis. From this, we can choose and fix an ideal $\mathfrak{n}$ of $\mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$. Note that this condition implies the root number $w(E/K) = -1$.

4. Let p be an odd prime such that E has good reduction at p, and $\mathrm{Gal}(\mathbb{Q}(E[p]/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

By the modularity theorem, we have the modular parametrization map ϕ_N from the modular curve $X_0(N)$ to E. Combining with results from complex multiplication theory we obtain:

$$\begin{array}{ccc} \mathrm{Pic}(\mathcal{O}_K) & \longrightarrow & X_0(N)(\mathbb{C}), \\ & \searrow & \downarrow{\scriptstyle \phi_N} \\ & & E(\mathbb{C}) \end{array} \qquad \text{explicitly,} \qquad \begin{array}{ccc} [\mathcal{O}_K] & \longmapsto & [\mathbb{C}/\mathcal{O}_K \to \mathbb{C}/\mathfrak{n}^{-1}] \\ & \searrow & \downarrow{\scriptstyle \phi_N} \\ & & y_1 \in E(K[1]). \end{array}$$

We also define

$$y_K = \mathrm{Tr}_{K[1]/K} y_1 \quad \in \quad E(K).$$

Now, in place of $\mathcal{O}_K$, we can do the whole construction over the order of conductor n, $\mathcal{O}_n = \mathbb{Z} + n\mathcal{O}_K$, and get $y_n \in E(K[n])$. A key property of $\{y_n\}$ is that $\mathrm{Tr}_{K[nl]/K[n]} y_{nl} = a_l(E) y_n$ for $(nl, 2Nd_K) = 1$ and l inert in K. The Kummer map $E(K[n]) \otimes \mathbb{Z}_p \to H^1_f(K[n], T_p(E))$ can then be used to construct the (anticyclotomic) Euler system of Heegner points [Gro91]. By focusing on the set of 'nice' primes l (where $\mathrm{Frob}_{K(E[p]/\mathbb{Q})} l$ is conjugate to the complex conjugation, which implies that l is inert in K and splits completely in $\mathrm{Gal}(K(E[p])/K)$, normally called Kolyvagin primes) Kolyvagin can show the **Rank 1** result assuming that $y_K \notin pE(K)$:

$$y_K \text{ has infinite order} \quad \Rightarrow \quad \mathrm{rank}_{\mathbb{Z}} E(K) = 1.$$

The method is robust enough to be called the Kolyvagin system's argument (note that it also implies the finiteness of the Tate-Shafarevich group of E over K).

The Gross-Zagier formula [GZ86], which relates the Néron-Tate height of y_K with $L'(E/K, 1)$ (hence y_K has infinite order iff $\mathrm{ord}_{s=1} L(E/K, s) = 1$), and results of Kolyvagin that we recalled above, then show that the Birch Swinnerton-Dyer conjecture holds for analytic rank 1, i.e.

$$\boxed{\mathrm{ord}_{s=1} L(E, s) = 1) \quad \Rightarrow \quad \mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1.}$$

1.3 About anticyclotomic Euler systems

The Euler system we construct is an 'anticyclotomic' Euler system (in contrast with the 'cyclotomic' Euler system of Kato [Kat04]). The general description of such an Euler system is as follows. More details can be found in Section 4.3.

Notation.

1. p is an odd prime.

2. $K/\mathbb{Q}$ is an imaginary quadratic field.

3. $K(n)$ is the ring class field of K of conductor n.

4. $K[n]$ is the maximal $p-$subextension of $K(n)$.

5. The infinite extension $K(p^\infty) = \bigcup_{n \geq 0} K(p^n)$ contains the **anticyclotomic** $\mathbb{Z}_p$-extension $K_\infty^- = \bigcup_{n \geq 0} K[n]$ that satisfies: $\mathrm{Gal}(K_\infty^-/K) \simeq \mathbb{Z}_p$ and $\mathrm{Gal}(K_\infty^-/\mathbb{Q}) =$

$\mathrm{Gal}(K_\infty^-/K) \rtimes \mathrm{Gal}(K/\mathbb{Q})$, where $\mathrm{Gal}(K/\mathbb{Q}) = \{1, c\}$ and $cgc = g^{-1}$ for all $g \in \mathrm{Gal}(K_\infty^-/K)$. (compare with K_∞^+, the cyclotomic $\mathbb{Z}_p$ extension of K with c acting trivially on $\mathrm{Gal}(K_\infty^+/K)$) (see [MN19]). Denote by $\Lambda_K^- = \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty^-/K)]\!]$ the anticyclotomic Iwasawa algebra.

Set-up.

1. Let $\Phi/\mathbb{Q}_p$ be a finite extension and $\mathcal{O}$ be its ring of integers. Let $\varpi \in \mathcal{O}$ be a uniformizer and denote by $\mathbb{F} = \mathcal{O}/\varpi\mathcal{O}$ the residue field.

2. Let V be a finite-dimensional conjugate self-dual ($V^c \simeq V^\vee(1)$) representation of G_K over Φ, unramified outside a finite set of primes Σ, and let $T \subset V$ be a Galois stable $\mathcal{O}-$lattice.

3. Fix a choice of the Greenberg Selmer group $H^1_{Gr}(K[m], V)$, consisting of elements that are unramfied at $w \nmid p$ and some (well-behaved) conditions at $w|p$.

4. Assume that there exists $\sigma \in G_K$ such that:

   (a) σ fixes $K[1](\mu_{p^\infty})$

   (b) $\dim_\Phi V/(\sigma - 1)V = 1$.

   This condition is called Hyp($\boldsymbol{\sigma}$).

5. For each positive integer n, the set of split-σ Kolyvagin primes level n, denoted $\mathscr{L}_n^\sigma$, is a collection of primes $l \in \mathbb{Q}$ such that:

   (a) $l \nmid 2p$, and l splits in K such that $l = \mathfrak{l}\bar{\mathfrak{l}}$.

   (b) V is unramified at $\mathfrak{l}$ and $\bar{\mathfrak{l}}$.

   (c) Frob_l lies in the G_K conjugacy class of σ in $\mathrm{Gal}(\Omega_n/K)$, where $T_n = T/\varpi^n T$, $\Omega_n = K[1]K(\mu_{p^n})K(T_n)$, and $K(T_n)$ denotes the smallest extension of K such that $G_{K(T_n)}$ acts trivially on T_n.

6. Denote $\mathscr{L}_n^{K,\sigma} = \{\text{primes } \mathfrak{l} \text{ of } K \text{ such that } \mathfrak{l}|l \text{ for some } l \in \mathscr{L}_n^{\sigma}\}$.

7. For $\mathscr{L}$ a set of primes of K, we write $\mathcal{N}(\mathscr{L}) = \{\mathfrak{a} = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_r^{a_r} \subset \mathcal{O}_K$, where $\mathfrak{p}_i \in \mathscr{L}, a_i = 1$ if $\mathfrak{p}_i \nmid p,$ and $\mathfrak{p}_i \neq \mathfrak{p}_j, \bar{\mathfrak{p}}_j\}$.

Notation. Given $\mathscr{L}_i$ a set consisting of primes of K for $i \in \{1, 2\}$, we write $\mathscr{L}_1 \dot{\supset} \mathscr{L}_2$ if the natural density of $(\mathscr{L}_2 \setminus (\mathscr{L}_2 \cap \mathscr{L}_1))$ is 0.

Definition. (**Euler system**) Let $\mathscr{L}$ be a set consisting of primes of K such that $\mathscr{L} \dot{\supset} \mathscr{L}_n^{K,\sigma}$ for some $n \geq 1$. A (split-σ) anticyclotomic Euler system for $(T, \mathscr{L})$ (in the sense of Jetchev-Nekovář-Skinner) [JNS] is a collection of cohomology classes $\mathbf{c} = \{c_{\mathfrak{m}}, \text{ where } \mathfrak{m} \in \mathcal{N}(\mathscr{L})\}$ such that:

1. $c_{\mathfrak{m}} \in H^1_{Gr}(K[m], T)$, where $m = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{m})$

2. For $\mathfrak{m}\mathfrak{l} \in \mathcal{N}(\mathscr{L})$, where $\mathfrak{l}$ is a prime of K with $l = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{l})$, we have the following norm relation:

$$\mathrm{cores}_{K[ml]/K[m]}(c_{\mathfrak{m}\,\mathfrak{l}}) = P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}^{-1}) c_{\mathfrak{m}}$$

where $P_{\mathfrak{l}}(X) = \det(1 - \mathrm{Frob}_{\mathfrak{l}}^{-1} X | T^{\vee}(1))$.

Example 1.3.1. Alonso-Castella-Rivero in [ACR21] obtained such an anticyclotomic Euler system for $V_f \otimes V_g(\chi)$ (a $4-$dimensional Galois representation where f, g are newforms, and χ is some Hecke character).

Example 1.3.2. Kolyvagin's famous Euler system of Heegner point [Gro91] is also an anticyclotomic one. Though the cohomology classes lie in $K[n]$ for n divisible by only inert primes (not split!) in K.

In this thesis, we will construct an Euler system originating from algebraic cycles, which cannot give a full Euler system but only an anticyclotomic one. The following lines are a heuristic explanation for this phenomenon given by Loeffler. By Shapiro's lemma, one can think of an Euler system as a collection of classes $c_\chi \in H^1(K, V(\chi))$, where χ runs on some finite order characters of G_K. Either χ varies over all finite order characters, which gives us a full Euler system, or χ varies over only anticyclotomic characters (assume that K is CM) which give us an anticyclotomic Euler system. Coming from an algebraic cycle, i.e. by a geometric construction, c_χ will likely land in the Bloch-Kato subspace H^1_f. Now if $0 \neq c_\chi \in H^1_f(K, V(\chi))$, the Bloch-Kato conjecture tells us that $L(V^\vee(1) \otimes \chi^{-1}, 0) = 0$. The only way to force many $L-$values to vanish systematically at $s = 0$ is by 'sign reasons' (not because of poles of $\Gamma-$factors as our classes come from algebraic cycles, and should correspond to central $L-$values). This happens only when $\mathrm{Ind}_K^{\mathbb{Q}} V(\chi)$ is self-dual, forcing χ to be an anticyclotomic character (at least if V is self-dual).

1.4 Construction of an anticyclotomic Euler system and the main theorems

We now describe our construction. Let f be a modular newform of weight k level $\Gamma_0(N_f)$. Let K be an imaginary quadratic field. Let ψ_1, ψ_2 be two Hecke characters of K with infinity types $(1-l_1, 0)$ and $(1-l_2, 0)$ respectively and such that $2|k+l_1+l_2$ and the central characters satisfy: $\chi_{\psi_1}\chi_{\psi_2} = 1$. Corresponding to these Hecke characters we have theta series $\theta_{\psi_i} \in S_{l_i}(N_{\psi_i}, \chi'_{\psi_i})$ [Miy89]. Let $N = \mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})$ and denote by $Y(m)$ the open modular curve of level $\Gamma_1(Nm)$ for each integer m coprime with Np. From the diagonal embedding

$$Y(m) \overset{\Delta}{\hookrightarrow} Y(m) \times Y(m) \times Y(m),$$

Bertolini-Seveso-Venerucci (BSV) constructed a diagonal class [BSV21], which can be pushed-forward to a class

$$\kappa_{m,\underline{r}} \in H^1(\mathbb{Q}, H^3_{\text{ét}}(Y(1) \times Y(m) \times Y(m)_{\bar{\mathbb{Q}}}, \mathscr{L}_{\underline{r}}) \otimes \mathbb{Q}_p(2-r)),$$

where $\underline{r} = (k-2, l_1-2, l_2-2) \in \{(r_1, r_2, r_3) \in \mathbb{Z}^3_{\geq 0}$ such that $r = (r_1+r_2+r_3)/2 \in \mathbb{Z}$ and $r_i + r_j \geq r_k$ for all $i, j, k\}$.

For a triple of cuspidal eigenforms of weight 2, the $p-$adic Abel-Jacobi image of the *generalized* Gross-Kudla-Schoen (GKS) cycle [DR14] (under a comparison isomorphism) equals (up to sign) the BSV diagonal class (see Proposition 2.5.1). The GKS cycle is essentially the diagonal $X_{123} = \{(x,x,x), x \in X\}$ in X^3, where $X = X_1(N)$, modified to make it null-homologous. More precisely, fix the cusp $\infty \in X$ at infinity as base point and follow Gross-Kudla and Gross-Schoen [GK92], [GS95], define Δ to be the class in the Chow group $\mathrm{CH}^2(X^3)$ of codimension 2 cycles in X^3 up to rational equivalence of the formal sum

$$X_{123} - X_{12} - X_{13} - X_{23} + X_1 + X_2 + X_3,$$

where $X_1 = \{(x, \infty, \infty), x \in X\}$, $X_{12} = \{(x, x, \infty), x \in X\}$ and likewise for the remainings. The GKS Δ cycle appears in a $p-$adic Gross-Kudla formula of Darmon-Rotger [DR14] and also in a complex one by Yuan-Zhang-Zhang [YZZ], relating the first derivative $L'(f,g,h,2)$ to the Beilinson-Bloch height of Δ.

Now, we look at the case $k = l_1 = l_2 = 2$. First we project onto the $H^1_{\text{ét}} \otimes H^1_{\text{ét}} \otimes H^1_{\text{ét}}$ component of the Künneth decomposition of $H^3_{\text{ét}}$, and then project each piece to the corresponding geometric realization $V_g^\vee$ of the two-dimensional Galois representation attached to a newform g (where g is one of $f, \theta_{\psi_1}, \theta_{\psi_2}$). Here $V_g^\vee$ is the maximal quotient of $H^1_{\text{ét}}(Y_1(N_g), \mathbb{Z}_p(1)) \otimes \Phi$ on which the Hecke operators $T_l', \langle d \rangle'$

acts as multiplication by $a_l(g)$ and $\chi_g(d)$ respectively, where $(ld, N_g) = 1$ and l is a prime ($\Phi/\mathbb{Q}_p$ is some finite extension containing the Fourier coefficients of g). Fix a $G_{\mathbb{Q}}-$stable lattice $T_f^\vee \subset V_f^\vee$. After pushing forward the cohomology class, we essentially use the decomposition

$$T^\vee_{\theta_{\psi_1}} \otimes T^\vee_{\theta_{\psi_1}} = \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_1{}^{-1}} \otimes \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_2{}^{-1}} = \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_1{}^{-1}\,\psi_2{}^{-1}} \oplus \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_1{}^{-1}\,\psi_2{}^{-c}}$$

and then project to the first direct summand. Finally, we use Shapiro's lemma to obtain:

$$\kappa_{f,\chi,m} \in H^1(K[m], V_f^\vee(1-\frac{k}{2})(\chi))$$

where $\chi = \psi_1{}^{-1}\,\psi_2{}^{-1}\,\mathbf{N}^{-1}$ is anticyclotomic of infinity type $(1,-1)$. Due to the geometric nature of the construction, it can be shown that $\kappa_{f,\chi,m}$ lands in the Bloch-Kato Selmer group $H^1_f(K[m], V_f^\vee(1-\frac{k}{2})(\chi))$.

The results in [BSV21] not only construct diagonal classes attached to a triple (f, g, h) but also to a triple of Hida familes, so we can substitute one theta series with a CM family passing through it. It turns out that by doing a similar analysis for the case $(2, 2, 2)$ but over the anticyclotomic tower for the weight $(k, l, 2)$, one can show the following result (see Theorem 4.3.4):

Theorem 1.4.1. *Let f be a $p-$ordinary newform of even weight k. Let ψ_1, ψ_2 be two Hecke characters over K of infinity types $(1-l, 0)$, $(-1, 0)$ and conductors $\mathfrak{f}_1$, $\mathfrak{f}_2$, respectively, with l even. Assume that $p = \mathfrak{p}\,\bar{\mathfrak{p}}$ splits in K and $(p, h_K\mathfrak{f}_1\mathfrak{f}_2) = 1$. For $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_1^{K,\sigma})$ such that $m = Norm_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to $pN_fD_KNorm_{K/\mathbb{Q}}(\mathfrak{f}_1\mathfrak{f}_2)$, the classes constructed in Theorem 3.2.2*

$$\kappa^\infty_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1_{Gr}(K[mp^\infty], T_f^\vee(1-k/2)\chi_{12\mathfrak{P}})$$

form an anticyclotomic Euler system for $(T_f^\vee(1-k/2)\chi_{12}, \mathscr{L}_1^{K,\sigma})$*, where*

$$\chi_{12} = \psi_1^{-1}\psi_2^{-1}\mathbf{N}^{-l/2}$$

is an anticyclotomic of infinity type $(l/2, -l/2)$*, and the Greenberg Selmer group is defined by* $F_{\mathfrak{p}}^+(V) = V$ *and* $F_{\bar{\mathfrak{p}}}^+(V) = 0$.

Remark 1.4.1. By Shapiro's Lemma, we can think of $\kappa_{f,\psi_1,\psi_2,\mathfrak{m}}^\infty$ as an element of the space $H^1(K[m], V_f^\vee(1-k/2)(\chi_0)\otimes\Lambda_K^-)$. Absorbing the χ_0, the inside module is a deformation of $V_p(f)(1-k/2)$ which p-adically interpolates the twists of $V_p(f)(1-k/2)$ by the *anticyclotomic* Hecke characters χ, and hence the class will admit specialisations to $H^1(K[m], V_f^\vee(1-k/2)(\chi))$.

1.5 Applications of the main theorems

The novelty of our work is that we construct an Euler system even when the Heegner Hypothesis does not hold. In particular, throughout this section, we assume that:

Assumption 1.5.1. (**non-Heeg**): $N_f = N_f^+ N_f^-$ where N_f^+ and N_f^- are the product of split and inert primes in K, respectively, and that N_f^- is a squarefree product of an odd number of inert primes.

Let $\kappa_{f,\chi} = \mathrm{cores}_{K[1]/K}(\mathrm{proj}_{K[1]}(\kappa_{f,\psi_1,\psi_2,1}^\infty))$, i.e. first taking the Iwasawa cohomology class of conductor 1 from Theorem 1.4.1 and then taking the norm down to $H^1(K, V_f^\vee(1-k/2)(\chi))$, where χ has infinity type $(l/2, -l/2)$ for l even. Feeding our Euler system into the JNS machinery, we show the following result, (see Corollary 5.1.2) :

Theorem 1.5.1. *(**Rank 1** result) Under the same hypotheses as Theorem 1.4.1, if f is not of CM type and $l \geq k$:*

$$\boxed{\kappa_{f,\chi} \neq 0 \quad \Longrightarrow \quad \dim_{\Phi} H^1_f(K, V_f^{\vee}(1-k/2)(\chi)) = 1.} \tag{1.5.0.1}$$

Next we use global duality to compare local conditions of the Bloch-Kato Selmer group with the Greenberg Selmer goup, the reciprocity law of [BSV21] that relates the cohomology class with the triple product $p-$adic $L-$function, and a non-vanishing of central $L-$values with anticyclotomic twists result of Chida-Hsieh [CH18b] to obtain (see Theorem 5.1.3):

Theorem 1.5.2. *(**Rank 0** Bloch-Kato) Assume the same hypotheses with Theorem 1.4.1 together with:*

1. *$k \geq 4$, $(N_f, D_K) = 1$ and $T_f^{\vee}$ is residually absolutely irreducible ,*
2. *the local sign $\epsilon_q(\mathbb{V}^{\dagger}_{f\theta_{\psi_1}\theta_{\psi_2}}) = 1$ for all primes $q|N$ (see details in Assumption 4.1.1),*
3. *$p \geq k+2$ and $p \nmid N_f D_K$.*

If χ is an anticyclotomic Hecke character of infinity type $(l/2, -l/2)$ such that

$$(pN_f D_K, cond(\chi)) = 1,$$

then

$$\boxed{L(f, \chi, \frac{k}{2}) \neq 0 \quad \Longrightarrow \quad H^1_f(K, V_f^{\vee}(1-k/2)(\chi)) = 0.} \tag{1.5.0.2}$$

*In other words, the **Bloch-Kato** conjecture holds in this analytic rank 0 case.*

Note that the (**non-Heeg**) condition combining with $L(f, \chi, \frac{k}{2}) \neq 0$ force $k \geq l+2$. The case $k = 2$ and $l = 0$ was already worked out by Bertolini-Darmon in [BD05] and generalized by Longo-Vigni in [LV10]. The case $k \geq 4$ and $l = 0$ was obtained by Chida in [Chi17], using the same methods as [BD05]. Kings-Loeffler-Zerbes [KLZ17] achieved a similar result in the case of the Rankin-Selberg product of two modular forms f and g (i.e. for a twist of f by a ray class character, not just a ring class character). As they require $\chi_f \chi_g \neq 1$ while we require $\chi_f = \chi_g = 1$, we do obtain a new case for the Rank 0 Bloch-Kato Conjecture. Moreover, our method has the advantage of being generalizable to totally real fields, which is not known for the Euler system of Rankin-Eisenstein classes that was used in [KLZ17].

1.6 Main motivation

By the modularity theorem, we can associate a newform f of weight 2 to each rational elliptic curve. Let χ be an anticyclotomic Hecke character of K of infinity type $(n, -n)$. We have the following table Ⓡ:

Ⓡ	$\omega(E/K) = -1$	$\omega(E/K) = 1$
$n = 0$	1st quadrant $\epsilon(E, \chi) = -1$ Euler system of Heegner points	2nd quadrant $\epsilon(E, \chi) = 1$??
$n \geq 1$	3rd quadrant $\epsilon(E, \chi) = 1$??	4th quadrant $\epsilon(E, \chi) = -1$ My diagonal Euler system

The 1st quadrant is the classical Euler system of Heegner points [Gro91] constructed by Kolyvagin. It also comes with the formula of Gross-Zagier [GZ86] relating the Néron-Tate canonical height of the Heegner point y_K to the first derivative at $s = 1$ of the Rankin L-function $L(f \otimes \theta_K, s)$.

We put ?? into the 2nd and 3rd quadrants to indicate the absence of a geometric construction of an anticyclotomic Euler system. There have been efforts to go from the 1^{st} to the 3^{rd} using a $p-$adic Gross-Zagier formula by Bertolini-Darmon-Prasanna [BDP13] that relates special values of an anticyclotomic $p-$adic $L-$function to image of some Heegner cycles under the $p-$adic Abel-Jacobi map. To go from the 1^{st} to the 2^{nd}, Bertolini-Darmon [BD05] used the theory of congruences between modular forms on quaternions algebras and the Cerednik-Drinfeld interchange of invariants to realise the Galois representation $E[p^n]$ in the p^n-torsion of the Jacobian of certain Shimura curves for which the Heegner point construction becomes available, leading to results similar to Theorem 1.5.2 but with some '**level raising**' conditions (see Remark 5.1.3 for details).

What I do in my thesis is to fill in the 4^{th} quadrant with a new diagonal Euler system, then pass to the 2^{nd} quadrant by $p-$adic methods and the reciprocity law of [BSV21]. It is worth mentioning that building on the Bertolini-Darmon-Prasanna formula for Heegner cycles [BDP13], Castella-Hsieh obtained similar results to Theorem 1.5.1 and 1.5.2 but in the $\omega(E/K) = -1$ setting by using $p-$adic methods [CH18a]. Note that analogously to the Gross-Zagier formula, a result of Yuan-Zhang-Zhang [YZZ] relates the first derivative $L'(f \otimes g \otimes h, 2)$ to the Beilinson-Bloch height of Δ, whereas our diagonal Euler system should be thought as a project of the image of this Δ under the $p-$adic Abel-Jacobi map.

1.7 Future research directions

We recall possible directions for future research.

1. The first project is to upgrade the divisibility of the Iwasawa Main Conjecture **without** $L-$function (obtained by the [JNS] machinery) to the divisibility of the Iwasawa Main Conjecture **with** $L-$function when weight$(\chi) \geq$weight$(f)/2$, in the spirit of Perrin-Riou [PR87].

2. Another direction is to go from the 4^{th} quadrant to the 3^{rd} quadrant using

congruences like in [BD05] and [Chi17], by substituting our modular curves with Shimura curves over totally real fields [Dis17].

3. We can also apply similar ideas from my thesis to other situations of special cycles on certain Shimura varieties, such as those appearing naturally in the Gan-Gross-Prasad setups. One direction would be to specialize the Euler system of cycles, constructed by Jetchev et al. [Jet], [BBJ20] at points where the Galois representation decomposes (analogously to $\mathrm{Ind}_K^{\mathbb{Q}}\mathcal{O}_{\psi_1}\otimes\mathrm{Ind}_K^{\mathbb{Q}}\mathcal{O}_{\psi_2} = \mathrm{Ind}_K^{\mathbb{Q}}\mathcal{O}_{\psi_1\,\psi_2}\oplus \mathrm{Ind}_K^{\mathbb{Q}}\mathcal{O}_{\psi_1\,\psi_2{}^c}$). These cycles are higher-dimensional counterparts of Heegner points, getting from the diagonal embedding $\mathbf{U}(1,1)\hookrightarrow\mathbf{U}(2,1)\times\mathbf{U}(1,1)$.

Chapter 2

Preliminaries

2.1 Modular curves and Hecke operators

We give a precise description of the modular curves and Hecke operators that will appear in our construction. This section follows [Kat04, §2], [BSV21, §2], and [ACR21, §2].

2.1.1 Modular curves

Let M, N, u, v be positive integers such that $M + N \geq 5$. Define $Y(M, N)$ to be the affine modular curve over $\mathbb{Z}[1/MN]$, that represents the functor:

$$S \mapsto \begin{cases} \text{isomorphism classes of triples } (E, P, Q) \text{ where } E \text{ is an elliptic curve over } S, \\ P,\ Q \text{ are sections of } E \text{ over } S \text{ such that } M \cdot P = N \cdot Q = 0; \text{ and the map} \\ \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to E,\ \text{ mapping } (a, b) \mapsto a \cdot P + b \cdot Q \text{ is injective} \end{cases}$$

for S a $\mathbb{Z}[1/MN]-$scheme. More generally, we also define the affine modular curve $Y(M(u), N(v))$ over $\mathbb{Z}[1/MNuv]$ that represents the functor:

$$S \mapsto \begin{cases} \text{isomorphism classes of quintuples } (E,P,Q,C,D) \text{ where } (E,P,Q) \text{ is as above,} \\ P \in C \text{ is a cyclic subgroup of } E \text{ of order } Mu, \\ Q \in D \text{ is a cyclic subgroup of } E \text{ of order } Nv \text{ such that} \\ C \text{ is complementary to } Q \text{ and } D \text{ is complementary to } P \end{cases}$$

for S a $\mathbb{Z}[1/MuNv]-$scheme.

Let $\mathbf{H}$ be the Poincaré upper half-plane and define the modular group:

$$\Gamma(M(u), N(v)) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \text{ such that } \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod \begin{pmatrix} M & Mu \\ Nv & N \end{pmatrix} \right\}.$$

When either $u = 1$ or $v = 1$, we drop the (1) from the notation. The Riemann surface $Y(M, N)(\mathbb{C})$ admits a complex uniformisation:

$$\begin{array}{ccc} (\mathbb{Z}/M\mathbb{Z})^\star \times \Gamma(M,N)\backslash\mathbf{H} & \xrightarrow{\sim} & Y(M,N)(\mathbb{C}) \\ (m, z) & \mapsto & (\mathbb{C}/\mathbb{Z}+\mathbb{Z}z, mz/M, 1/N), \end{array}$$

and similarly for $Y(M(u), N(v))(\mathbb{C})$.

Let $l \geq 2$ be a prime. There is an isomorphism of $\mathbb{Z}[1/lMN]-$schemes:

$$\begin{array}{rcl} \varphi_l : Y(M, N(l)) & \to & Y(M(l), N) \\ (E, P, Q, C) & \mapsto & (E/NC, P + NC, l^{-1}(Q) \cap C + NC, (l^{-1}(\mathbb{Z} \cdot P + NC)/NC)), \end{array}$$

which under the complex uniformisation is induced by the map $(m, z) \mapsto (m, l \cdot z)$.

2.1.2 Degeneracy maps

We have the natural degeneracy maps:

$$\begin{array}{ccccc} Y(M,Nl) & \xrightarrow{\mu_l} & Y(M,N(l)) & \xrightarrow{\nu_l} & Y(M,N) \\ & & \downarrow{\scriptstyle \varphi_l} & & \\ Y(M,Nl) & \xrightarrow[\tilde{\mu}_l]{} & Y(M(l),N) & \xrightarrow[\tilde{\nu}_l]{} & Y(M,N) \end{array}$$

where $\mu_l(E,P,Q) = (E,P,l\cdot Q, \mathbb{Z}\cdot Q)$, $\nu_l(E,P,Q,C) = (E,P,Q)$, and $\tilde{\mu}_l$, $\tilde{\nu}_l$ are defined similarly. We also denote:

$$\begin{aligned} \mathrm{pr}_1 &:= \nu_l \circ \mu_l && \text{i.e. } \mathrm{pr}_1 : Y(M,Nl) \to Y(M,N) \\ & && \qquad (E,P,Q) \mapsto (E,P,l\cdot Q) \\ \mathrm{pr}_l &:= \tilde{\nu}_l \circ \varphi_l \circ \mu_l && \text{i.e. } \mathrm{pr}_l : Y(M,Nl) \to Y(M,N) \\ & && \qquad (E,P,Q) \mapsto (E/N\,\mathbb{Z}\cdot Q, P+N\,\mathbb{Z}\cdot Q, Q+N\,\mathbb{Z}\cdot Q) \end{aligned}$$

On the complex upper half plane $\mathbf{H}$, the map pr_1, pr_l are induced by the identity and multiplication by l respectively. Moreover, the degeneracy maps $\mu_l, \tilde{\mu}_l, \nu_l, \tilde{\nu}_l, \mathrm{pr}_1, \mathrm{pr}_l$ are all finite étale morphisms of $\mathbb{Z}[1/MNl]$ schemes.

2.1.3 Relative Tate modules and Hecke operators

Fix an integer $r \geq 0$. Let S be a $\mathbb{Z}[1/MNlp]-$scheme where p is a fixed prime. For each $\mathbb{Z}[1/MNlp]-$ scheme X, denote the base change $X_S = X \times_{\mathbb{Z}[1/MNlp]} S$. Notate $A = A_X$ to be either the locally constant sheaf $\mathbb{Z}/p^m\mathbb{Z}(j)$ or the locally constant $p-$adic sheaf $\mathbb{Z}_p(j)$ on $X_{\text{ét}}$ (see [FK88, Def 12.6]) for some fixed $m \geq 1$ and $m, j \in \mathbb{Z}$.

To ease the notation, we may write $\cdot$ for $M(u), N(v)$ (i.e. $Y(\cdot) = Y(M(u), N(v))$). Denote $E(\cdot)$ the universal elliptic curve over $Y(\cdot)$. Then one obtains a natural degree l isogeny of universal elliptic curves under the base change by $\varphi_l^\star E(M(l),N) \to$

$Y(M, N(l))$:

$$\lambda_l : E(M, N(l)) \to \varphi_l^\star(E(M(l), N).$$

Denote by $v_\cdot : E(\cdot)_S \to Y(\cdot)_S$ the structure map. We also use $\nu_l, \tilde{\nu}_l$ and λ_l for the base change to S of the corresponding degeneracy maps. Set:

$$\mathscr{T}_\cdot(A) = R^1 v_{\cdot\star}\, \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} A \text{ and } \mathscr{T}_\cdot^\star(A) = \mathrm{Hom}_A(\mathscr{T}_\cdot(A), A)$$

where $R^q v_{\cdot\star}$ is the $q-$th right derivative of $v_{\cdot\star} : E(\cdot)_{\text{ét}} \to Y(\cdot)_{\text{ét}}$. When $A = \mathbb{Z}_p$, this gives the relative Tate module of the universal elliptic curve, in this case we will drop the $\mathbb{Z}_p$ in the notation.

The (perfect) cup product pairing combined with the relative trace:

$$\mathscr{T}_\cdot \otimes_{\mathbb{Z}_p} \mathscr{T}_\cdot \to R^2 v_{\cdot\star}\, \mathbb{Z}_p(2) \cong \mathbb{Z}_p(1)$$

allows one to identify $\mathscr{T}_\cdot(-1)$ with $\mathscr{T}_\cdot^\star$. The smooth base change theorem ([Mil80, Chap IV, Cor 4.2]) implies that $\mathscr{T}_\cdot(A)$ and its dual are locally constant $p-$adic sheaves on $Y(\cdot)_S$, of formation compatible with base change along morphisms of $\mathbb{Z}[1/MNlp]-$schemes $S' \to S$. Define:

$$\mathscr{L}_{\cdot,r}(A) = \mathrm{Tsym}^r_A \mathscr{T}_\cdot(A) \text{ and } \mathscr{S}_{\cdot,r}(A) = \mathrm{Symm}^r_A \mathscr{T}_\cdot^\star(A),$$

where given any finite free module M over a profinite $\mathbb{Z}_p-$algebra R, $\mathrm{Tsym}^r_R M$ is the $R-$submodule of the symmetric tensors in $M^{\otimes r}$, and $\mathrm{Symm}^r_R M$ is the maximal symmetric quotient of $M^{\otimes r}$. When the level is clear, we may simplify the notations, writing:

$$\mathscr{L}_r(A) = \mathscr{L}_{M(u),N(v),r}(A),\ \mathscr{L}_r = \mathscr{L}_r(\mathbb{Z}_p),\ \mathscr{S}_r(A) = \mathscr{S}_{M(u),N(v),r}(A),\ \mathscr{S}_r = \mathscr{S}_r(\mathbb{Z}_p). \tag{2.1.3.1}$$

For this rest of this section, let $\mathscr{F}^r_{\cdot}$ be either $\mathscr{L}_{\cdot,r}(A)$ or $\mathscr{S}_{\cdot,r}(A)$. By the proper base change theorem [Mil80, Chap VI, Cor 2.3] and a commutative diagram for the structural maps (see equation (9) in [BSV21]), one has natural isomorphisms of sheaves:

$$\nu_l^\star(\mathscr{F}^r_{M,N}) \cong \mathscr{F}^r_{M,N(l)} \text{ and } \tilde{\nu}_l^\star(\mathscr{F}^r_{M,N}) \cong \mathscr{F}^r_{M(l),N}\,.$$

These induce pullbacks

$$\begin{array}{ccccc} & & H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) & & \\ & \swarrow^{\nu_l^\star} & & \searrow^{\tilde{\nu}_l^\star} & \\ H^i_{\text{ét}}(Y(M,N(l))_S, \mathscr{F}^r_{M,N(l)}) & & & & H^i_{\text{ét}}(Y(M(l),N)_S, \mathscr{F}^r_{M(l),N}) \end{array}$$

and traces

$$\begin{array}{ccccc} & & H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) & & \\ & \nearrow^{\nu_{l\star}} & & \nwarrow^{\tilde{\nu}_{l\star}} & \\ H^i_{\text{ét}}(Y(M,N(l))_S, \mathscr{F}^r_{M,N(l)}) & & & & H^i_{\text{ét}}(Y(M(l),N)_S, \mathscr{F}^r_{M(l),N}) \end{array}$$

The finite étale isogeny λ_l induces morphisms:

$$\lambda_{l\star} : \mathscr{F}^r_{M,N(l)} \to \varphi_l^\star(\mathscr{F}^r_{M(l),N}) \text{ and } \lambda_l^\star : \varphi_l^\star(\mathscr{F}^r_{M(l),N}) \to \mathscr{F}^r_{M,N(l)}$$

which allow one to define a pushforward

$$\Phi_{l\star} := \varphi_{l\star} \circ \lambda_{l\star} : H^i_{\text{ét}}(Y(M,N(l))_S, \mathscr{F}^r_{M,N(l)}) \to H^i_{\text{ét}}(Y(M(l),N)_S, \mathscr{F}^r_{M(l),N})$$

and a pullback

$$\Phi_l^\star := \lambda_l^\star \circ \varphi_l^\star : H^i_{\text{ét}}(Y(M(l),N)_S, \mathscr{F}^r_{M(l),N}) \to H^i_{\text{ét}}(Y(M,N(l))_S, \mathscr{F}^r_{M,N(l)}).$$

We define a Hecke operator and a dual Hecke operator acting on the cohomology group $H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N})$ by:

$$T_l = \tilde{\nu}_{l\star} \circ \Phi_{l\star} \circ \nu_l^\star \text{ and } T_l' = \nu_{l\star} \circ \Phi_l^\star \circ \tilde{\nu}_l^\star$$

By writing $\mathrm{pr}_1^\star = \mu_l^\star \circ \nu_l^\star$, $\mathrm{pr}_l^\star = \mu_l^\star \circ \Phi_l^\star \circ \tilde{\nu}_l^\star$, $\mathrm{pr}_{1\star} = \nu_{l\star} \circ \mu_{l\star}$, $\mathrm{pr}_{l\star} = \tilde{\nu}_{l^\star} \circ \Phi_{l\star} \circ \mu_{l\star}$, one obtains:

$$\deg(\mu_l) T_l = \mathrm{pr}_{l\star} \circ \mathrm{pr}_1^\star, \text{ and } \deg(\mu_l) T_l' = \mathrm{pr}_{1\star} \circ \mathrm{pr}_l^\star$$

Remark 2.1.1. These definitions agree with [BSV21], but differ from [ACR21] where they define $\pi_{1\star} = \nu_{l\star}$, $\pi_{l\star} = \tilde{\nu}_{l^\star} \circ \Phi_{l\star}$ to kill the extra factor $\deg(\mu_l)$ for T_l.

For $d \in (\mathbb{Z}/MN\mathbb{Z})^\star$, one can define on $Y(\cdot)$ the diamond operator $\langle d \rangle$ which is defined on the moduli problem by

$$(E, P, Q, C, D) \mapsto (E, d^{-1} \cdot P, d \cdot Q, C, D).$$

There also exists a unique diamond operator $\langle d \rangle$ on the universal elliptic curve making the following diagram cartesian:

$$\begin{array}{ccc} E(\cdot)_S & \xrightarrow{\langle d \rangle} & E(\cdot)_S \\ {\scriptstyle v.}\downarrow & & \downarrow{\scriptstyle v.} \\ Y(\cdot)_S & \xrightarrow{\langle d \rangle} & Y(\cdot)_S \end{array}$$

This induces automorphisms $\langle d \rangle = \langle d \rangle^\star$ and $\langle d \rangle' = \langle d \rangle_\star$ on $H^i_{\text{ét}}(Y(\cdot)_S, \mathscr{F}_\cdot)$.

For each profinite $\mathbb{Z}_p-$algebra R and finite free $R-$module M, the evaluation map induces a perfect pairing:

$$\mathrm{Tsym}^r_R M \otimes_R \mathrm{Symm}^r_R M^\star \to R$$

where $M^\star = \mathrm{Hom}_R(M, \mathbb{Z}_p)$. This gives us a perfect pairing $\mathscr{L}_r \otimes_{\mathbb{Z}_p} \mathscr{S}_r \to \mathbb{Z}_p$, i.e. a cup product:

$$\langle \cdot, \cdot \rangle_N : H^1_{\text{ét}}(Y(\cdot)_{\bar{\mathbb{Q}}}, \mathscr{L}_r(1)) \otimes_{\mathbb{Z}_p} H^1_{\text{ét},c}(Y(\cdot)_{\bar{\mathbb{Q}}}, \mathscr{S}_r) \to H^2_{\text{ét}}(Y(\cdot)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p,$$

which is perfect by Poincaré duality after inverting p. The Hecke operators T_l, T_l', $\langle d \rangle$, $\langle d \rangle'$ induce endomorphisms on the compactly supported cohomology $H^1_{\text{ét},c}(Y(\cdot)_{\bar{\mathbb{Q}}}, \mathscr{S}_r)$ and from the construction, (T_l, T_l') and $(\langle d \rangle, \langle d \rangle')$ are adjoint to each other under $\langle \cdot, \cdot \rangle_N$. Moreover, the Eichler-Shimura isomorphism [Shi71]

$$H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathscr{L}_r) \otimes_{\mathbb{Z}_p} \mathbb{C} \cong M_{r+2}(N, \mathbb{C}) \oplus \overline{S_{r+2}(N, \mathbb{C})}$$

commutes with the action of the Hecke operator on both sides.

2.2 Bloch-Kato Conjecture

In this section, we state our convention of $L-$functions attached to a Galois representation, and the statement of the Bloch-Kato Conjecture, following [Bel].

Let p be a prime. Let V be a $p-$adic geometric representation of G_K over Φ, where K is a number field, and $\Phi / \mathbb{Q}_p$ is a finite extension. We first define the local Euler factor $L_v(V, s)$ of V at a prime v of K:

$$L_v(V, s) = \begin{cases} \det(1 - \text{Frob}_v \cdot q_v^{-s} | V^{I_v})^{-1} & \text{if } v \nmid p, \\ \det(1 - \varphi \cdot q_v^{-s} | D_{cris}(V|_{G_{F_v}}))^{-1} & \text{if } v | p, \end{cases}$$

where s is a complex argument, Frob_v is the geometric Frobenius, $q_v = p^{f_v}$ is the size of the residue field of K at v, and $\varphi = \phi^{f_v}$ with ϕ the crystalline Frobenius. We then define the $L-$function of V as an Euler product:

$$L(V,s) = \prod_{v \text{ prime}} L_v(V,s).$$

For V geometric and pure of weight w (recall that $H^i(X,\mathbb{Q}_p)(n)$ is pure of weight $i-2n$), $L(V,s)$ is holomorphic on $\Re(s) > w/2+1$. Furthermore, if V is automorphic i.e. we can associate to V a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ where $n = \dim_\Phi V$, then by results of Hecke (for $n=1$), Jacquet-Langlands (for $n=2$), and Jacquet-Shalika (for $n \geq 3$) that $L(V,s) = L(\pi,s)$ admits a meromorphic continuation on the complex plane. For such a V, one can complete the $L-$function by adding the 'Euler factor at infinity', and obtain $\Lambda(V,s)$ together with its functional equation:

$$\Lambda(V,s)\epsilon(V,s) = \Lambda(V^\vee(1),-s),$$

where $\epsilon(V,s)$ is entire and non-vanishing.

For $\tau \in \mathrm{Aut}(K/\mathbb{Q})$, let V^τ be the representation of G_K over the same vector space V but $g \in G_K$ acts as $\sigma g \sigma^{-1}$, where $\sigma \in G_{\mathbb{Q}}$ such that to $\sigma|_K = \tau$. We say that V is polarized of weight w_0 if $V^\tau(w_0) \cong V^\vee$ for some $\tau \in \mathrm{Aut}(K/\mathbb{Q})$. If that happens, as $V^\tau(w_0)$ and $V^\vee$ are pure of weight $w - 2w_0$ and $-w$, respectively, we must have $w_0 = w$. We record a fact that if an automorphic representation V is self-dual i.e. K is a totally real field and τ is trivial, or V is conjugate self-dual i.e. K is a CM field and τ is the complex conjugate, then V is regular (i.e. distinct Hodge-Tate weights).

For V geometric, polarized, pure of weight w, and automorphic, as:

$$\Lambda(V^\vee(1),-s) = \Lambda(V^\vee, 1-s) = \Lambda(V^\tau(w), 1-s) = \Lambda(V, w+1-s),$$

the functional equation of V relates $L(V, w+1-s)$ and $L(V,s)$. Hence $(1+w)/2$ is called the center of the functional equation (it also forces $\epsilon(V,(1+w)/2) = \pm 1$). If we replace V by $V(\frac{1+w}{2})$ (pure of weight -1) then the center of the functional equation is $s = 0$.

Conjecture 2.2.1. ***(Bloch-Kato)*** *Let V be a $p-$adic geometric irreducible representation of G_K over Φ, where K is a number field, and $\Phi/\mathbb{Q}_p$ is a finite extension. Then one has:*

$$\boxed{\dim_\Phi H^1_f(K, V^\vee(1)) = \mathrm{ord}_{s=0} L(V,s) + \dim_\Phi(V^\vee(1)^{G_K}),}$$

where H^1_f is the Bloch-Kato Selmer group.

Remark 2.2.1. The last term will be zero unless $V = \Phi(1)$.

Example 2.2.1. For $V = \mathbb{Q}_p$, $L(V,s) = \zeta_K(s)$ the Dedekind zeta function, and its order of vanishing at $s = 0$ is $r_1 + r_2 - 1$. The Kummer map induces $\mathcal{O}_K^\times \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_f(K, \mathbb{Q}_p(1))$, which tells us that $\dim_{\mathbb{Q}_p} H^1_f(K, \mathbb{Q}_p(1)) = \mathrm{rank}\ \mathcal{O}_K^\times$. Hence the Bloch-Kato conjecture in this case is just the Dirichlet's unit theorem.

Example 2.2.2. For E an elliptic curve, denote by $V = V_p(E)$ the Tate module of E over K. Assuming the finiteness of the Tate-Shafarevich group of E over K and using the Kummer map $E(K) \otimes \mathbb{Z}_p \hookrightarrow H^1_f(K, V)$, we also obtain that the Bloch-Kato conjecture for $V = V_p(E)$ is equivalent to the Birch Swinnerton-Dyer conjecture.

2.3 Galois representations associated to newforms

Let $f = \sum_{n \geq 1} a_n q^n$ be a normalized newform of weight $k \geq 2$, level $\Gamma_1(N_f)$, and nebentype χ_f. Let $p \nmid N_f$ be a prime. Fix an embedding $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $L/\mathbb{Q}$ be a finite extension containing all values $i_\infty^{-1}(a_n)$ and $i_\infty^{-1} \circ \chi_f$. Let $\mathfrak{p}$ be the prime of L above p with respect to i_p. Denote $S = \{\text{prime } l | pN\} \cup \{\infty\}$. Then Eichler-Shimura (for $k = 2$ in [Eic54], [Shi58]) and Deligne (for $k > 2$ in [Del71]) construct a $p-$adic Galois representation associated to f:

$$\rho_{f,\mathfrak{p}} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(L_{\mathfrak{p}}) \tag{2.3.0.1}$$

that satisfies: for all $l \notin S$

1. $\mathrm{Trace}(\rho_{f,\mathfrak{p}}(\mathrm{Frob}_l)) = i_p(a_l)$
2. $\det(\rho_{f,\mathfrak{p}}(\mathrm{Frob}_l)) = i_p(\chi_f(l)l^{k-1})$
3. $\rho_{f,\mathfrak{p}}$ is irreducible, hence absolutely irreducible as the complex conjugate has ± 1 eigenvalues [Rib77].
4. Here Frob_l is the geometric Frobenius.

2.3.1 Scholl's motives

Scholl [Sch90] constructed geometrically a Grothendieck motive $M \subset h^{k-1}(Z) \otimes L$ where Z is a suitable smooth compactification of the $(k-1)-$dimensional Kuga-Sato variety over $Y(N_f)$ (say $N_f \geq 3$). The $p-$adic realisation of M is $M_p \subset H^{k-1}(\bar{Z}_{\text{ét}}, L \otimes \mathbb{Q}_p)$, free of rank 2 over $L \otimes \mathbb{Q}_p = \prod_{\mathfrak{p}\,|p} L_{\mathfrak{p}}$, with its $\mathfrak{p}$ $-$component M_f being $\rho_{f,\mathfrak{p}}$. Note also that M_f is pure of weight $k-1$.

2.3.2 Deligne's construction

Before Scholl, Deligne [Del71] also constructed a geometric realisation V_f of $\rho_{f,\mathfrak{p}}$ but by using étale cohomology with non-constant coefficients.

Definition/Proposition.

1. The geometric realisation V_f of $\rho_{f,\mathfrak{p}}$, can be defined as the largest subspace of

$$H^1_{\text{ét}}(Y_1(N_f)_{\bar{\mathbb{Q}}}, \mathscr{S}_{k-2}) \otimes L_{\mathfrak{p}},$$

on which T_l acts as multiplication by a_l for all $l \nmid N_f p$ and $\langle d\rangle' = \langle d\rangle_\star$ acts as multiplication by $\chi_f(d)$ for all $d \in (\mathbb{Z}/N_f\mathbb{Z})^\star$.

2. Its dual $V_f^\vee$ can be interpreted as the maximal quotient of

$$H^1_{\text{ét}}(Y_1(N_f)_{\bar{\mathbb{Q}}}, \mathscr{L}_{k-2}(1)) \otimes L_{\mathfrak{p}},$$

on which the dual Hecke operator T_l' acts as multiplication by a_l for all $l \nmid N_f p$ and $\langle d\rangle = \langle d\rangle^\star$ acts as multiplication by $\chi_f(d)$ for all $d \in (\mathbb{Z}/N_f\mathbb{Z})^\star$.

3. Denote the ring of integer for $L_{\mathfrak{p}}$ as $\mathcal{O}_{\mathfrak{p}}$. We obtain $\mathcal{O}_{\mathfrak{p}}-$lattices T_f, $T_f^\vee$, which lies inside V_f and $V_f^\vee$ respectively, as the image of $H^1_{\text{ét}}(Y_1(N_f)_{\bar{\mathbb{Q}}}, \mathscr{S}_{k-2}) \otimes \mathcal{O}_{\mathfrak{p}}$ and $H^1_{\text{ét}}(Y_1(N_f)_{\bar{\mathbb{Q}}}, \mathscr{L}_{k-2}(1)) \otimes \mathcal{O}_{\mathfrak{p}}$.

4. It can be shown directly by the Hochschild-Serre spectral sequence that M_f and V_f are the same (isomorphic representations).

5. In general when $N_f|N$, the subspace (where $T_l, \langle d\rangle$ acts as above) we get from $H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathscr{S}_{k-2}) \otimes L_{\mathfrak{p}}$ is $V_f(N) \cong \oplus_{i=1}^{\sigma_0(N/N_f)} V_f$ non-canonically. And we have a similar story for $V_f^\vee(N)$.

Properties.

1. V_f is $2-$dimensional, irreducible, and a direct summand of the corresponding $H^1_{\text{ét}}$.

2. For $l \nmid pN_f$, V_f is unramified at l and the Euler factor at l with respect to the geometric Frobenius is:

$$\mathrm{P}_l(V_f, t) = 1 - a_l t + l^{k-1}\chi_f(l)t^2$$

3. If f is ordinary at p, i.e. $i_p(a_p) \in L_{\mathfrak{p}}$ is a $\mathfrak{p}-$adic unit, the restriction of V_f to

$G_{\mathbb{Q}_p}$ is reducible and we have the following exact sequence of $L_{\mathfrak{p}}[G_{\mathbb{Q}_p}]-$modules:

$$0 \to V_f^+ \to V_f \to V_f^- \to 0$$

with $\dim(V_f^{\pm}) = 1$. The equation $x^2 - a_p x + \chi_f(p)p^{k-1} = 0$ has two distinct roots: one is α_p the $\mathfrak{p}-$adic unit and the other is β_p: which is 0 if $p|N_f$ and is $\chi_f(p)p^{k-1}/\alpha_p$ if $p \nmid N_f$. The sub-representation V_f^+ is unramified, with $\mathrm{Frob}_p \in G_{\mathbb{Q}_p}/I_p$ acting on V_f^+ by α_p. Poincaré duality shows that $V_f^\vee \simeq V_f(k-1)(\chi_f^{-1})$, i.e.

$$V_f^- \cong (V_f^+)^\vee(1-k)(\chi_f^{-1}).$$

4. $V_f^\vee \simeq V_{\bar{f}}(k-1)$ where $\bar{f} = f \otimes \chi_f^{-1}$.

5. When f is ordinary at p, by duality, we also obtain an exact sequence for $V_f^\vee$ restricted to $G_{\mathbb{Q}_p}$:

$$0 \to V_f^{\vee,+} \to V_f^\vee \to V_f^{\vee,-} \to 0 \qquad (2.3.2.1)$$

with $\dim(V_f^{\vee,\pm}) = 1$. The sub-representation $V_f^{\vee,-}$ is unramified, with $\mathrm{Frob}_p \in G_{\mathbb{Q}_p}/I_p$ acting on $V_f^{\vee,-}$ by α_p. If we adopt the convention that $\mathbb{Q}_p(1)$ has Hodge-Tate (HT) weight -1, then the HT weight of $V_f^{\vee,-}$ is 0 and the HT weight of $V_f^{\vee,+}$ is $1-k$.

6. For an elliptic curve $E/\mathbb{Q}$ corresponding to a newform f by modularity, $V_f \simeq H^1_{\text{ét}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) \simeq V_p(E)(-1)$.

2.4 Lei-Loeffler-Zerbes norm map

In this section, we will explain our conventions on Hecke characters together with their properties. Then we will recall the 'norm map' of Lei-Loeffler-Zerbes (cf. [LLZ15, Sec 4]), which they used to construct a cyclotomic Euler system attached to a weight

2 modular form twisted by a Hecke character over an imaginary quadratic field K.

2.4.1 Hecke characters and theta series

Definition/Proposition. Let K be an imaginary quadratic field. Fix an embedding $i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. For a prime p that splits in K, let $\mathfrak{p}\,|p$ be the prime of K with respect to i_p, i.e. $p = \mathfrak{p}\,\bar{\mathfrak{p}}$.

1. For a pair $(a, b) \in \mathbb{Z}^2$, an algebraic Hecke character ψ of K with infinity type (a, b) is a continuous homomorphism: $\mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ such that $\psi_\infty(x_\infty) = x_\infty^a \bar{x}_\infty^b$. Such a character ψ is called anticyclotomic if it is trivial on $\mathbb{A}_{\mathbb{Q}}^\times$. The conductor of ψ is the largest integral ideal $\mathfrak{f}$ of K such that $\psi_{\mathfrak{q}}(u) = 1$ for all $u \in (1 + \mathfrak{f}\mathcal{O}_{K,\mathfrak{q}})^\times \hookrightarrow K_{\mathfrak{q}}^\times$.

2. We can identify ψ with a character on the set of ideals on $\mathcal{O}_K$ that is coprime to $\mathfrak{f}$ (i.e. a character of $H_{\mathfrak{f}}$, the ray class group of K with conductor $\mathfrak{f}$) by defining $\psi(\mathfrak{a}) = \prod_{\mathfrak{q}|\mathfrak{a}} \psi_{\mathfrak{q}}(\varpi_{\mathfrak{q}})^{v_{\mathfrak{q}}(\mathfrak{a})}$, where $\varpi_{\mathfrak{q}}$ is a uniformizer at $\mathfrak{q}$, such that $\psi((\alpha)) = \alpha^{-a}\bar{\alpha}^{-b}$ for all principal ideals (α) such that $\alpha \equiv 1 \pmod{\mathfrak{f}}$. By restricting to $\mathbb{A}_{\mathbb{Q}}^\times$, we obtain a Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$ such that $\psi((n)) = n^{-a-b}\chi(n)$ for all integers n coprime with $N_{K/\mathbb{Q}}(\mathfrak{f})$.

3. Denote $\mathrm{rec}_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}}$ the geometrically normalized Artin reciprocity map. The $(0, 0)-$infinity type Hecke character $\psi(x)x_\infty^{-a}\bar{x}_\infty^{-b}$ will be a ray class character, hence it will take value in a finite extension L/K. Denote primes $\mathfrak{P}|\mathfrak{p}|p$ of $L/K/\mathbb{Q}$ respectively with respect to i_p. We attach a $p-$adic Galois representation $\psi_{\mathfrak{P}}$ to ψ as follow: for $g \in G_K$, first denote its image $g' \in G_K^{\mathrm{ab}}$, then we take $x \in \mathbb{A}_K^\times$ such that $\mathrm{rec}_K(x) = g'$ and define

$$\psi_{\mathfrak{P}}(g) = i_p \circ i_\infty^{-1}(\psi(x)x_\infty^{-a}\bar{x}_\infty^{-b})x_{\mathfrak{p}}^a x_{\bar{\mathfrak{p}}}^b. \qquad (2.4.1.1)$$

Such a $\psi_{\mathfrak{P}}$ will be called the $p-$adic avatar of ψ.

Let ψ be a Hecke character of K with infinity type $(-1,0)$, conductor $\mathfrak{f}$, taking values in a finite extension L/K. Denote by χ the unique Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$ such that $\psi((n)) = n\chi(n)$ for all $(n, N_{K/\mathbb{Q}}(\mathfrak{f})) = 1$. The theta series attached to ψ is:

$$\theta_\psi = \sum_{(\mathfrak{a},\mathfrak{f}=1)} \psi(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in S_2(\Gamma_1(N_\psi), \chi\epsilon_K)$$

where ϵ_K is the quadratic Dirichlet character attached to K. The cuspform θ_ψ is new of level $N_\psi = N_{K/\mathbb{Q}}(\mathfrak{f}) \cdot \mathrm{disc}(K/\mathbb{Q})$ [Miy89, Thm 4.8.2].

Fix a prime $p \geq 5$ unramified in K with $(p, \mathfrak{f}) = 1$ and primes $\mathfrak{P}|\mathfrak{p}|p$ of $L/K/\mathbb{Q}$ respectively. Let $\mathcal{O} \subset L_{\mathfrak{P}}$ be the ring of integers. Let $\psi_{\mathfrak{P}}$ be the $p-$adic avatar of ψ, then the $p-$adic representation attached to θ_ψ is:

$$V_{\theta_\psi} \cong \mathrm{Ind}_K^{\mathbb{Q}}(\psi_{\mathfrak{P}}) \text{ and its dual } V_{\theta_\psi}^\vee \cong \mathrm{Ind}_K^{\mathbb{Q}}(\psi_{\mathfrak{P}}^{-1}).$$

2.4.2 Hecke algebras and norm maps

Let $\mathfrak{n}$ be an integral ideal of K such that $\mathfrak{f}|\mathfrak{n}$ and let $N = N_{K/\mathbb{Q}}(\mathfrak{n})\mathrm{disc}(K/\mathbb{Q})$. Let $K_{\mathfrak{n}}$ be the ray class field of K with conductor $\mathfrak{n}$, and let $H_{\mathfrak{n}}$ be the ray class group of K modulo $\mathfrak{n}$. Let $K_{\mathfrak{n}}^p$ be the largest abelian $p-$extension of K of conductor dividing $\mathfrak{n}$, i.e. $\mathrm{Gal}(K_{\mathfrak{n}}^p/K) \cong H_{\mathfrak{n}}^{(p)}$ is the largest $p-$power quotient of $H_{\mathfrak{n}}$. For an ideal $\mathfrak{k}$ of K coprime to $\mathfrak{n}$, let $[\mathfrak{k}]$ be the class of $\mathfrak{k}$ in $H_{\mathfrak{n}}$.

Let $\mathbb{T}'(N)$ be the subalgebra of $\mathrm{End}_{\mathbb{Z}}(H^1(Y_1(N)(\mathbb{C}), \mathbb{Z}))$ generated by the diamond operators $\langle d \rangle'$, T_l' for $l \nmid N$, and U_l' for $l|N$.

Proposition 2.4.1. *[LLZ15, Prop 3.2.1] There exists a homomorphism* $\phi_{\mathfrak{n}} : \mathbb{T}'(N) \to$

$\mathcal{O}[H_\mathfrak{n}]$ *acting on the generators as follows:*

$$\phi_\mathfrak{n}(T_l) = \sum_{\left\{\substack{\text{ideals } \mathfrak{l}\nmid\mathfrak{n}, \\ N_{K/\mathbb{Q}}(\mathfrak{l})=l}\right\}} [\mathfrak{l}]\psi(\mathfrak{l})$$

$$\phi_\mathfrak{n}(\langle d\rangle') = \chi(d)\epsilon_K(d)[(d)].$$

Proof. By specializing at a characters ρ of $H_\mathfrak{n}$, we would want a system of eigenvalues corresponds to $\theta_{\psi\rho}$, which exists, again by [Miy89, Thm 4.8.2]. □

For $\mathfrak{n}' = \mathfrak{n}\mathfrak{l}$, where $\mathfrak{l}$ is a prime ideal and $(\mathfrak{n}', p) = 1$, let $N' = N_{K/\mathbb{Q}}(\mathfrak{n})\mathrm{disc}(K/\mathbb{Q})$ and define the norm map:

$$\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'} : \mathcal{O}[H_{\mathfrak{n}'}^{(p)}] \otimes_{\mathbb{T}'(N')\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}'}} H^1_{\text{ét}}(Y_1(N')_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \to \mathcal{O}[H_\mathfrak{n}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbb{Z}_p,\phi_\mathfrak{n}} H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \tag{2.4.2.1}$$

by the following formulae (see [LLZ15, Def 3.3.1, Prop 5.2.5]):

1. If $\mathfrak{l}|\mathfrak{n}$ then
$$\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'} = 1 \otimes \mathrm{pr}_{1\star}$$

2. If $\mathfrak{l} \nmid \mathfrak{n}$ is split or ramified in $K/\mathbb{Q}$ then
$$\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'} = 1 \otimes \mathrm{pr}_{1\star} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{l} \otimes \mathrm{pr}_{l\star}$$

3. If $\mathfrak{l} \nmid \mathfrak{n}$ is inert i.e. $\mathfrak{l} = (l)$ then
$$\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'} = 1 \otimes \mathrm{pr}_{1\star} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{l^2} \otimes \mathrm{pr}_{ll\star}$$

4. We extend the definition of $\mathcal{N}_\mathfrak{n}^{\mathfrak{n}'}$ to any pair of ideals $\mathfrak{n}|\mathfrak{n}'$ by composition.

Assumption 2.4.1. (†): $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K. If $\mathfrak{p}|\mathfrak{f}$ we assume that $\bar{\mathfrak{p}} \nmid \mathfrak{f}$ and $\psi_{\mathcal{O}_{K,\mathfrak{p}}^\times}$ is not congruent to ω, the Teichmüller character modulo $\mathfrak{P}$.

Remark 2.4.1. This condition is so that the maximal ideal $\mathcal{I}_{\mathfrak{n}}$ of the Hecke algebra associated to $\phi_{\mathfrak{n}}$ is non-Eisenstein, and $p-$distinguished (i.e. $p-$ordinary+ non-Eisenstein) (see [LLZ15, Rem 5.1.3]). Later on, we will assume $(p, \mathfrak{f}) = 1$, hence this condition will be automatically satisfied.

Theorem 2.4.2. *Let $\mathcal{A}$ be the set of integral ideals $\mathfrak{n}$ of K, generated by prime ideals coprime to $\bar{\mathfrak{p}}$, and let $\mathcal{A}_{\mathfrak{f}} = \{\mathfrak{n}\mathfrak{f} : \text{ such that } \mathfrak{n} \in \mathcal{A}\}$. Assume (†) holds, then there is a family of Galois equivariant isomorphisms of $\mathcal{O}[H_{\mathfrak{n}}^{(p)}]$ modules for any $\mathfrak{n} \in \mathcal{A}_{\mathfrak{f}}$:*

$$\upsilon_{\mathfrak{n}} : \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\cong} Ind_{K_{\mathfrak{n}}^p}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1})$$

such that for any $\mathfrak{n}' \in \mathcal{A}$ with $\mathfrak{n}|\mathfrak{n}'$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}[H_{\mathfrak{n}'}^{(p)}] \otimes_{\mathbb{T}'(N')\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}'}} H^1_{\text{ét}}(Y_1(N')_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\cong]{\upsilon_{\mathfrak{n}'}} & Ind_{K_{\mathfrak{n}'}^p}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \\ \downarrow{\scriptstyle \mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'}} & & \downarrow{\scriptstyle \mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{n}'}} \\ \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) & \xrightarrow[\cong]{\upsilon_{\mathfrak{n}}} & Ind_{K_{\mathfrak{n}}^p}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \end{array}$$

where $\mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{n}'}$ is the natural norm map (see the discussion leading to equation (3.1.0.6) below).

Proof. See Proposition 5.2.5 and Corollary 5.2.6 in [LLZ15] for details. Nevertheless, we will roughly sketch the key ideas for going from level Nl to level N, where $l = \tau\bar{\tau} \nmid pN$ is a prime that splits in K:

1. The main problem is a discrepancy on the Hecke action of the integral cohomology groups of different level modular curves. Concretely, if we write $H^1(Y_1(Nl))$ for $H^1(Y_1(Nl)(\mathbb{C}), \mathbb{Z})$, this has an action of $U_l \in \mathbb{T}_{Nl}$ while $H^1(Y_1(N))$ has an action of $T_l \in \mathbb{T}_N$. But by creating an artificial Hecke algebra $\tilde{\mathbb{T}}_N := \mathbb{T}_N[X]/(X^2 - T_l X + l\langle l\rangle)$ (cf. [Wil95]), one can unify the two Hecke algebras

via the following commutative diagram:

$$\begin{array}{ccc} U_l \notin \check{\mathbb{T}}_{Nl} & \longrightarrow & \mathbb{T}_N \ni T_l \\ \big\downarrow & & \big\downarrow \\ U_l \in \mathbb{T}_{Nl} & \xrightarrow{U_l \mapsto X} & \tilde{\mathbb{T}}_N \end{array}$$

2. Next, we use Ihara's lemma [Iha75]. The version here is borrowed from [DDT97, Lem 4.28], which gives the surjectivity of the following horizontal map:

$$\begin{array}{ccc} \mathrm{pr}_{1\star} \oplus \mathrm{pr}_{2\star} : H^1(Y_1(Nl)) & \twoheadrightarrow & H^1(Y_1(N))^{\oplus 2} \\ & \searrow & \big\downarrow \wr \\ & & \tilde{\mathbb{T}}_N \otimes_{\mathbb{T}_N} H^1(Y_1(N)) \end{array}$$

The vertical isomorphism is just $(a, b) \mapsto a - b(T_l - X)/l$. Here, X is acting on $H^1(Y_1(N))^{\oplus 2}$ via the matrix $\begin{pmatrix} T_l & -\langle l \rangle \\ l & 0 \end{pmatrix}$. By localizing at a non-Eisenstein maximal ideal $\mathcal{I}$ of $\mathbb{T}_N$ of characteristic $p \nmid Nl$, we also get $H^1(Y_1(N))_{\mathcal{I}}$ is a free $(\mathbb{T}_N)_{\mathcal{I}}$ module of rank 2. Hence the following map $\iota = \mathrm{pr}_{1\star} - \mathrm{pr}_{2\star}(T_l - X)/l$ is an isomorphism [LLZ15, Thm 4.2.8]:

$$\iota : (\tilde{\mathbb{T}}_N)_{\mathcal{I}} \otimes_{\mathbb{T}_{Nl}} H^1(Y_1(Nl))_{\mathcal{I}} \xrightarrow{\sim} (\tilde{\mathbb{T}}_N)_{\mathcal{I}} \otimes_{\mathbb{T}_N} H^1(Y_1(N))_{\mathcal{I}}$$

3. Now the isomorphism $v_{\mathfrak{n}}$ is obtained by simply patching all the $V^{\vee}_{\theta_{\psi\rho}}$ for twists of ψ by character ρ of $H^{(p)}_{\mathfrak{n}}$ into one big isomorphism (which can be shown easily by specializing at a finite order character ρ of $H^{(p)}_{\mathfrak{n}}$ and obtain $V^{\vee}_{\theta_{\psi\rho}}$).

4. We obtain the 'norm map' $\mathcal{N}^{\mathfrak{n}\mathfrak{l}}_{\mathfrak{n}}$ via base extension, i.e. extending $\phi_{\mathfrak{n}}$ from $\mathbb{T}_N$ to $\tilde{\mathbb{T}}_N$ by defining $T_l - X \mapsto \psi(\tau)[\tau]$.

In the end, we get a commutative diagram for $\mathfrak{n}' = \mathfrak{n}\mathfrak{l}$ where $\mathfrak{l}$ is a split prime of K coprime with $\mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{n})$. Lei-Loeffler-Zerbes show that such a result is also true when $\mathfrak{l}$ is inert, and also in the Hida theory setting (cf. [LLZ15, Sec 4.3]).

To obtain a diagram for arbitrary $\mathfrak{n}'$ with $\mathfrak{n}|\mathfrak{n}'$, we order the set of ideals in $\mathcal{A}_{\mathfrak{f}}$ by divisibility ($\mathfrak{f} = \mathfrak{n}_0|\mathfrak{f}\mathfrak{p}_1 = \mathfrak{n}_1|\mathfrak{f}\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{n}_2|\cdots$ i.e. adding one prime at a time), acquire a commutative diagram for $\mathfrak{n}_i|\mathfrak{n}_j$ $(i \geq j)$, and then define all the cohomology class for $\mathfrak{m}|\mathfrak{n}_j$ by corestriction. Note that in order to rigidify the system of isomorphisms, there is a fixed choice of units of $\mathcal{O}[H_{\mathfrak{n}}^{(p)}]$ for each level $\mathfrak{n}$, see more in [LLZ15, Cor 5.2.6]. $\square$

Remark 2.4.2. We sketch a proof of the following version of Ihara's lemma: the following map is injective:

$$H^1(\Gamma_0(N), \mathbb{Z}/l\mathbb{Z}) \oplus H^1(\Gamma_0(N), \mathbb{Z}/l\mathbb{Z}) \to H^1(\Gamma_0(qN), \mathbb{Z}/l\mathbb{Z}).$$

By a theorem of Ihara that

$$\Gamma_0(N) *_{\Gamma_0(Nq)} \Gamma_0(N) = \Gamma_0(N, \mathbb{Z}[\frac{1}{q}]),$$

one obtains the Lyndon exact sequence:

$$H^1(\Gamma_0(N, \mathbb{Z}[\frac{1}{q}]), \mathbb{Z}/l\mathbb{Z}) \to H^1(\Gamma_0(N), \mathbb{Z}/l\mathbb{Z})^{\oplus 2} \to H^1(\Gamma_0(qN), \mathbb{Z}/l\mathbb{Z}),$$

and so it suffices to show that $H^1(\Gamma_0(N, \mathbb{Z}[\frac{1}{q}]), \mathbb{Z}/l\mathbb{Z}) = 0$. Now, an element of such cohomology group corresponds to a group homomorphism: $\varphi : \Gamma_0(N, \mathbb{Z}[\frac{1}{q}]) \to \mathbb{Z}/l\mathbb{Z}$. Because $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{q}])$ satisfies the congruence subgroup property, $\ker(\varphi)$ will contain a principal congruence subgroup. Therefore, if $(l, N\phi(N)) = 1$ then $\varphi = 0$. This version is used in the 'level raising' paper of K. Ribet [Rib84].

2.5 Bertolini-Seveso-Venerucci diagonal classes construction

We sketch the construction of the diagonal classes in the triple product of modular curves $Y_1(N)$ using classical invariant theory, following Section 3 in [BSV21].

We recall some notation used in Section 2.1.3. Here, $Y_1(N) = Y_1(N)_{\mathbb{Q}}$, $E_1(N) = E_1(N)_{\mathbb{Q}}$ the universal elliptic curve over $Y_1(N)$ together with the structural map $v : E_1(N) \to Y_1(N)$. The relative Tate module of the universal elliptic curve is $\mathscr{T} = R^1 v_\star \mathbb{Z}_p(1)$, and its dual is $\mathscr{T}^\star = \mathrm{Hom}_{\mathbb{Z}_p}(\mathscr{T}, \mathbb{Z}_p)$. The cup product pairing combined with the relative trace:

$$\mathscr{T} \otimes_{\mathbb{Z}_p} \mathscr{T} \to R^2 v_\star \mathbb{Z}_p(2) \cong \mathbb{Z}_p(1)$$

gives a perfect relative Weil pairing

$$\langle , \rangle_{E_1(N)_{p^\infty}} : \mathscr{T} \otimes_{\mathbb{Z}_p} \mathscr{T} \to \mathbb{Z}_p(1),$$

which allows $\mathscr{T}(-1)$ to be identified with $\mathscr{T}^\star$.

For A either the locally constant sheaf $\mathbb{Z}/p^m\mathbb{Z}(j)$ or the locally constant $p-$adic sheaf $\mathbb{Z}_p(j)$ on $X_{\text{ét}}$ for some fixed $m \geq 1$ and $m, j \in \mathbb{Z}$, recall that

$$\mathscr{L}_r(A) = \mathrm{Tsym}^r_A \mathscr{T}(A) \text{ and } \mathscr{S}_r(A) = \mathrm{Symm}^r_A \mathscr{T}^\star(A),$$

where given any finite free module M over a profinite $\mathbb{Z}_p-$algebra R, $\mathrm{Tsym}^r_R M$ is the $R-$submodule of the symmetric tensors in $M^{\otimes r}$, and $\mathrm{Symm}^r_R M$ is the maximal symmetric quotient of $M^{\otimes r}$.

For a fixed geometric point $\eta : \mathrm{Spec}(\bar{\mathbb{Q}}) \to Y_1(N)$, denote by $\mathcal{G}_\eta = \pi_1^{\text{ét}}(Y_1(N), \eta)$ the fundamental group of $Y_1(N)$ with base point η. The stalk of $\mathscr{T}$ at η, denoted $\mathscr{T}_\eta$, is a free $\mathbb{Z}_p$ −module of rank 2, equipped with a continuous action of $\mathcal{G}_\eta$. Fix a choice of $\mathbb{Z}_p$ −module isomorphism $\zeta : \mathscr{T}_\eta \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ such that $\langle x, y\rangle_{E_1(N)_{p^\infty}} = \zeta(x) \wedge \zeta(y)$ (where we identify $\bigwedge^2 \mathbb{Z}_p^2$ with $\mathbb{Z}_p$ via $(1,0)\wedge(0,1) = 1$). One then obtains a continuous group homomorphism:

$$\rho_\eta : \mathcal{G}_\eta \to \mathrm{Aut}_{\mathbb{Z}_p}(\mathscr{T}_\eta) \cong \mathrm{GL}_2(\mathbb{Z}_p).$$

By [FK88, Prop A I.8], the category of locally constant p−adic sheaves on $Y_1(N)_{\text{ét}}$ is equivalent to the category of p−adic representations of $\mathcal{G}_\eta$ via the map $\mathscr{F} \mapsto \mathscr{F}_\eta$. Using ρ_η, one can associate with every continuous representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ over a free finite $\mathbb{Z}_p$ −module M a smooth sheaf $M^{\text{ét}}$ on $Y_1(N)$ such that $M^{\text{ét}}_\eta = M$.

Let $S_i(A)$ be the set of 2−variable homogeneous polynomials of degree i in $A[x_1, x_2]$ equipped with the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ by $gP(x_1, x_2) = P((x_1, x_2)\cdot g)$ for all $g \in \mathrm{GL}_2(\mathbb{Z}_p)$ and $P \in S_i(A)$. Its A−linear dual $L_i(A)$ is also equipped with a $\mathrm{GL}_2(\mathbb{Z}_p)$−action by $g\tau(P(x_1, x_2)) = \tau(g^{-1}P(x_1, x_2))$ for all $g \in \mathrm{GL}_2(\mathbb{Z}_p)$, $P \in S_i(A)$, and $\tau \in L_i(A)$. As sheaves on $Y_1(N)_{\mathbb{Q}}$, one has:

$$L_i(A)^{\text{ét}} = \mathscr{L}_i(A) \quad \text{and} \quad S_i(A)^{\text{ét}} = \mathscr{S}_i(A). \tag{2.5.0.1}$$

Hence $\mathscr{T}_\eta \cong L_1(\mathbb{Z}_p)$ and $\mathbb{Z}_p(1)_\eta \cong \bigwedge^2 \mathscr{T}_\eta \cong \det^{-1}$. This implies that for any $j \in \mathbb{Z}$, and any p−adic representation M of $\mathrm{GL}_2(\mathbb{Z}_p)$:

$$H^0(\mathrm{GL}_2(\mathbb{Z}_p), M \otimes \det^{-j}) \hookrightarrow H^0(\mathcal{G}_\eta, M \otimes \det^{-j}) \cong H^0_{\text{ét}}(Y_1(N), M^{\text{ét}}(j)). \tag{2.5.0.2}$$

Assumption 2.5.1. Let $\mathbf{r} = (r_1, r_2, r_3)$ such that $r_i \in \mathbb{Z}_{\geq 0}$, $(r_1 + r_2 + r_3)/2 = r \in \mathbb{Z}_{\geq 0}$, and $r_i + r_j \geq r_k$ for all permutation (i, j, k) of $(1, 2, 3)$. We call this the **balanced condtion**.

Under the assumption 2.5.1, let

$$S_{\mathbf{r}} = S_{r_1}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} S_{r_2}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} S_{r_3}(\mathbb{Z}_p)$$

a $\mathrm{GL}_2(\mathbb{Z}_p)-$representation, and let

$$\mathscr{S}_{\mathbf{r}} = S_{\mathbf{r}}^{\text{ét}} = \mathscr{S}_{r_1}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathscr{S}_{r_2}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathscr{S}_{r_3}(\mathbb{Z}_p).$$

We identify $S_{\mathbf{r}}$ with the module of $6-$variable polynomials $\mathbb{Z}_p[x_1, x_2, y_1, y_2, z_1, z_2]$ which is homogeneous of degree r_1, r_2, and r_3 in the variables (x_1, x_2), (y_1, y_2), and (z_1, z_2) respectively. By the Clebsch-Gordan decomposition of classical invariant theory, the following is a $\mathrm{GL}_2(\mathbb{Z}_p)-$invariant of $S_{\mathbf{r}} \otimes \det^{-r}$ (cf. the balanced condition)

$$\mathrm{Det}_N^{\mathbf{r}} := \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{r-r_3} \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_2} \det \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_1}$$

i.e. $\mathrm{Det}_N^{\mathbf{r}} \in H^0(\mathrm{GL}_2(\mathbb{Z}_p), S_{\mathbf{r}} \otimes \det^{-r})$ and denote its image under (2.5.0.2) as:

$$\mathtt{Det}_N^{\mathbf{r}} \in H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\mathbf{r}}(r)). \tag{2.5.0.3}$$

Let $p_j : Y_1(N)^3 \to Y_1(N)$ for $j \in \{1, 2, 3\}$ be the natural projections and denote

$$\mathscr{S}_{[\mathbf{r}]} := p_1^\star \mathscr{S}_{r_1}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} p_2^\star \mathscr{S}_{r_2}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} p_3^\star \mathscr{S}_{r_3}(\mathbb{Z}_p),$$
$$\mathtt{W}_{N,\mathbf{r}} := H^3_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{S}_{[\mathbf{r}]}(r+2)).$$

As $Y_1(N)_{\bar{\mathbb{Q}}}$ is a smooth affine curve over $\bar{\mathbb{Q}}$, $H^4_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{S}_{[\mathbf{r}]}(r+2)) = 0$. By the Hochschild-Serre spectral sequence,

$$H^p(\mathbb{Q}, H^q_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{S}_{[\mathbf{r}]}(r+2))) \Longrightarrow H^{p+q}_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{S}_{[\mathbf{r}]}(r+2))$$

one obtains

$$\mathtt{HS} : H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\mathbf{r}]}(r+2)) \to H^1(\mathbb{Q}, \mathtt{W}_{N,\mathbf{r}}).$$

If we let $d : Y_1(N) \to Y_1(N)^3$ be the diagonal embedding, then there is a natural isomorphism $d^\star \mathscr{S}_{[\mathbf{r}]} \cong \mathscr{S}_{\mathbf{r}}$ of smooth sheaves on $Y_1(N)_{\text{ét}}$. As d is an embedding of codimension 2, there is a pushforward map:

$$d_\star : H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\mathbf{r}}(r)) \to H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{\mathbf{r}}(r+2)),$$

and we define the class

$$(\mathtt{HS} \circ d_\star)(\mathtt{Det}^{\mathbf{r}}_N) \in H^1(\mathbb{Q}, \mathtt{W}_{N,\mathbf{r}}).$$

A result of Nekovář-Nizioł (see [NN16, Thm 5.9]) then tells us that this class is unramified at all primes different from p and is geometric at p, i.e. lands in $H^1_g(\mathbb{Q}, W_{N,\mathbf{r}})$ where $W_{N,\mathbf{r}} = \mathtt{W}_{N,\mathbf{r}} \otimes \mathbb{Q}_p$ (see the definition of H^1_g in equation (4.3.0.1) below, see more in [BSV21, Prop 3.2]).

Dually, by the bilinear form $\det^\star : L_i(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} L_i(\mathbb{Z}_p) \to \mathbb{Z}_p \otimes \det^{-i}$ defined by $\det^\star(\tau \otimes \sigma) = \tau \otimes \sigma((x_1y_2 - x_2y_1)^i)$ that becomes perfect after inverting p, we can define an isomorphism of $\mathrm{GL}_2(\mathbb{Z}_p)-$modules:

$$\mathbf{s}_i : S_i(\mathbb{Q}_p) \cong L_i(\mathbb{Q}_p) \otimes \det^i, \quad \text{i.e.} \quad \mathbf{s}_i : \mathscr{S}_i(\mathbb{Q}_p) \cong \mathscr{L}_i(\mathbb{Q}_p) \otimes \det^i \tag{2.5.0.4}$$

by the equivalence of categories. We then similarly define the sheaves $\mathscr{L}_{\mathbf{r}}$ on $Y_1(N)$ and $\mathscr{L}_{[\mathbf{r}]}$ on $Y_1(N)^3$. Set

$$\mathtt{V}_{N,\mathbf{r}} := H^3_{\text{ét}}(Y_1(N)^3_{\bar{\mathbb{Q}}}, \mathscr{L}_{[\mathbf{r}]}(2-r)) \quad \text{and} \quad V_{N,\mathbf{r}} = \mathtt{V}_{N,\mathbf{r}} \otimes \mathbb{Q}_p\,. \tag{2.5.0.5}$$

Let $\mathbf{s}_{\mathbf{r}} = \mathbf{s}_{\mathbf{r}_1} \otimes \mathbf{s}_{\mathbf{r}_2} \otimes \mathbf{s}_{\mathbf{r}_3}$, which gives an isomorphism: $W_{N,\mathbf{r}} \to V_{N,\mathbf{r}}$. Finally, we arrive at the following geometric class

$$(\mathbf{s}_{\mathbf{r}\star} \circ \mathtt{HS} \circ d_\star)(\mathtt{Det}_N^{\mathbf{r}}) \in H^1_g(\mathbb{Q}, V_{N,\mathbf{r}}). \tag{2.5.0.6}$$

Remark 2.5.1. Note that by [BSV21, Rem 3.3], going back and forth from $S_i(\mathbb{Q}_p)$ to $L_i(\mathbb{Q}_p)$ introduced an extra factor that divides $i!$. Therefore, we can obtain an integral class by multiplying with $i!$ if necessary.

We record the following fact (see [BSV21, Prop 3.6]) that compare the generalised Gross-Kudla-Schoen diagonal cycles $\Delta_{k,l,m}$ (see the second paragraph of Section 1.4 for the idea of the construction, more details are in [DR14, Def 3.3]) with the class we constructed above in equation (2.5.0.6).

Proposition 2.5.1. *There exists a natural isomorphism that maps the $p-$adic Abel-Jacobi image of* $(\Delta_{r_1+2,r_2+2,r_3+2})$ *to* $(\mathtt{HS} \circ d_\star)(\mathtt{Det}_N^{\mathbf{r}})$ *(up to sign).*

Chapter 3

Main theorems

3.1 Tame norm relation for weight $(2, 2, 2)$

In this section, we will construct cohomology classes using results from [BSV21] and [LLZ15] recalled above, prove that they satisfy the norm relation, and obtain an anticyclotomic Euler system.

Let $f \in S_k(\Gamma_0(N_f))$, $g \in S_l(N_g, \chi_g)$, and $h \in S_m(N_h, \chi_h)$ be three newforms such that $\chi_g \chi_h = 1$. Let L/K be a finite extension that contains the Fourier coefficients of these newforms. Let $N = \mathrm{lcm}(N_f, N_g, N_h)$ and denote

$$Y(m) = Y(1, Nm) = Y_1(Nm)$$

(i.e. level $\Gamma_1(mN)$) for every positive integer m.

Let $\mathbf{r} = (r_1, r_2, r_3)$ be a triple of non negative integers such that the **balanced condition** holds. Denote:

$$\mathscr{L}_{[\mathbf{r}]} = \mathscr{L}_{1,Nm,r_1}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathscr{L}_{1,Nm,r_2}(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathscr{L}_{1,Nm,r_3}(\mathbb{Z}_p).$$

We define a cohomology class:

$$\kappa^1_{m,\mathbf{r}} \in H^1(\mathbb{Q}, H^3_{\text{ét}}(Y(m)^3_{\bar{\mathbb{Q}}}, \mathscr{L}_{[\mathbf{r}]}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(2-r)) \tag{3.1.0.1}$$

which is the BSV class $\mathbf{s}_{\mathbf{r}^\star} \circ \mathtt{HS} \circ d_\star(\mathtt{Det}^{\mathbf{r}}_{Nm})$ in Section 2.5. By Remark 2.5.1, the only possible denominators of $\kappa^1_{m,\mathbf{r}}$ are divisors of $w := (k-2)!(l-2)!(m-2)!$. Multiplying $\kappa^1_{m,\mathbf{r}}$ with $p^{v_p(w)}$, we obtain an integral class, which is also denoted $\kappa^1_{m,\mathbf{r}}$ by an abuse of notation.

Proposition 3.1.1. *For a prime number q and a positive integer m, if $(mq, pN) = 1$ then*

$$(pr_{i\star}, pr_{j\star}, pr_{k\star})\kappa^1_{mq,\boldsymbol{r}} = (\bigstar)\kappa^1_{m,\boldsymbol{r}}$$

where

(i,j,k)	$\bigstar$
$(q,1,1)$	$(q-1)(T_q,1,1)$
$(1,q,1)$	$(q-1)(1,T_q,1)$
$(1,1,q)$	$(q-1)(1,1,T_q)$
$(1,q,q)$	$q^{r-r_1}(q-1)(T'_q,1,1)$
$(q,1,q)$	$q^{r-r_2}(q-1)(1,T'_q,1)$
$(q,q,1)$	$q^{r-r_3}(q-1)(1,1,T'_q)$

If $(q,m) = 1$ then we also have

(i,j,k)	$\bigstar$
$(1,1,1)$	(q^2-1)
(q,q,q)	$(q^2-1)q^r$

Proof. See equation (174) and (176) in [BSV21]. □

Define:

$$\kappa^2_{m,\mathbf{r}} = (\mathrm{pr}_{m\star}, 1, 1)\kappa^1_{m,\mathbf{r}} \in H^1(\mathbb{Q}, H^3_{\text{ét}}(Y(1) \times Y(m)^2_{\bar{\mathbb{Q}}}, \mathscr{L}_{[\mathbf{r}]}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(2-r)) \quad (3.1.0.2)$$

where $\mathrm{pr}_{m\star}$ is the composition of $\mathrm{pr}_{e_i\star}$, if we write the prime factorisation of m as $\prod_i e_i$.

We use the Künneth decomposition of $H^3_{\text{ét}}$:

$$H^3_{\text{ét}}(Y(1) \times Y(m)^2_{\bar{\mathbb{Q}}}, \mathscr{L}_{[\mathbf{r}]}) = \bigoplus_{a+b+c=3} H^a_{\text{ét}}(Y(1)_{\bar{\mathbb{Q}}}, \mathscr{L}_{r_1}) \otimes H^b_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \mathscr{L}_{r_2}) \otimes H^c_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \mathscr{L}_{r_3}), \quad (3.1.0.3)$$

cf. [Mil80, Chap VI, Thm 8.5] (note that we drop $(1, Nm)$ in the notation of $\mathscr{L}_{r_i}$).

Project the class $\kappa^2_{m,\mathbf{r}}$ to the $H^1_{\text{ét}} \otimes H^1_{\text{ét}} \otimes H^1_{\text{ét}}$ component and obtain:

$$\kappa^3_{m,\mathbf{r}} \in H^1(\mathbb{Q}, H^1_{\text{ét}}(Y(1)_{\bar{\mathbb{Q}}}, \mathscr{L}_{r_1}(1)) \otimes H^1_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \mathscr{L}_{r_2}(1)) \otimes H^1_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \mathscr{L}_{r_3}(1))(-1-r)) \quad (3.1.0.4)$$

Set-up. Here are some notations and assumptions for this subsection:

1. $f \in S_2(\Gamma_0(N_f))$ is a newform.

2. K is an imaginary quadratic field.

3. ψ_1, ψ_2 are two Hecke characters over K, both of infinity type $(-1, 0)$ with conductors $\mathfrak{f}_1$, $\mathfrak{f}_2$ respectively. As recalled in Section 2.4.1, there are associated theta series $\theta_{\psi_1} \in S_2(N_{\psi_1}, \chi_{\psi_1})$ and $\theta_{\psi_2} \in S_2(N_{\psi_2}, \chi_{\psi_2})$.

4. Assume that $\chi_{\psi_1}\chi_{\psi_2} = 1$.

5. Let $p \geq 5$ be a prime that splits in K and such that $(p, \mathfrak{f}_1\mathfrak{f}_2) = 1$, as in Section 2.4. Take L/K to be a finite extension, assumed to be large enough so that its ring of integers contains the Fourier coefficients of $f, \theta_{\psi_1}, \theta_{\psi_2}$. Choose primes $\mathfrak{P}|\mathfrak{p}|p$ of $L/K/\mathbb{Q}$ respectively and let $\mathcal{O} \subset L_{\mathfrak{P}}$ be its ring of integers.

6. A newform $F \in S_k(N_F, \chi_F)$ will generate π_F, an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. For any $N_F|N$, denote $S_k(N, \chi_F)[\pi_F]$ to be the $F-$isotypic subspace of $S_k(N, \chi_F)$ attached to the automorphic representation π_F. With a basis given by $\{F(dz)\}_{d|(N/N_F)}$, this is a $\sigma_0(N/N_F)-$dimensional vector space with elements being called test vectors.

We now combine our triplet $(f, \theta_{\psi_1}, \theta_{\psi_2})$ with the constructions in Section 2.5. Since f, θ_{ψ_1}, θ_{ψ_2} all have weight 2, we will take $\boxed{\mathbf{r} = (0,0,0)}$.

Important choices. Fix a choice of test vectors:

$$\check{f} \in S_2(N)[f], \quad \check{g} \in S_2(N, \chi_{\psi_1})[\theta_{\psi_1}], \quad \check{h} \in S_2(N, \chi_{\psi_2})[\theta_{\psi_2}]$$

and a choice of maps (recall $Y(m) = Y_1(\mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})m)$):

$$\begin{aligned} H^1_{\text{ét}}(Y(1)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) &\to H^1_{\text{ét}}(Y_1(N_f)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \\ H^1_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) &\to H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \\ H^1_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) &\to H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)). \end{aligned}$$

Notation.

1. Let $\mathscr{L}_K$ be the set of split primes $\mathfrak{l}$ of K.

2. Given $\mathscr{L}$ a set of prime ideals of K, let $\mathcal{N}(\mathscr{L})$ be the set of squarefree ideals $\mathfrak{m}$ of K which is generated by prime ideals of $\mathscr{L}$ i.e. $\mathfrak{m} = \prod_i \mathfrak{l}_i$ and $\mathfrak{l}_i \in \mathscr{L}$, such that $\mathfrak{l}_i \neq \mathfrak{l}_j, \bar{\mathfrak{l}}_j$ for all $i \neq j$ (i.e. pairwise distinct and conjugatedly distinct).

Let $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_K)$ such that $m = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to p. Let $\mathfrak{l} \in \mathscr{L}_K$ be a split prime of K such that $l = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{l})$ is coprime to pm. After tensoring with

$\mathcal{O}$, we can project $\kappa^3_{m,\mathbf{r}}$ in (3.1.0.4) to:

$$\kappa^4_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1(\mathbb{Q}, T_f^\vee \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1}m)} \mathcal{O}[H^{(p)}_{\mathfrak{m}}] \otimes H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m)} \mathcal{O}[H^{(p)}_{\bar{\mathfrak{m}}}](-1)),$$

where in here we are using the $\phi_{\mathfrak{m}}$, $\phi_{\bar{\mathfrak{m}}}$ maps and construction from Proposition 2.4.1 of Section 2.4.2 for the second and third pieces.

Remark 3.1.1. Notice that the numbering here changes from m to $\mathfrak{m}$, as we will construct an 'anticyclotomic' Euler system. See more details in Section 4.3.

As $(p, \mathfrak{f}_1\mathfrak{f}_2) = 1$, condition (†) is satisfied for both ψ_1, ψ_2. We then use the isomorphisms from Propositions 2.4.2:

$$\upsilon_{\mathfrak{m}} : H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1}m)} \mathcal{O}[H^{(p)}_{\mathfrak{m}}] \xrightarrow{\sim} \operatorname{Ind}^{\mathbb{Q}}_{K^p_{\mathfrak{m}}} \mathcal{O}(\psi^{-1}_{1\mathfrak{P}})$$
$$\upsilon_{\bar{\mathfrak{m}}} : H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m)} \mathcal{O}[H^{(p)}_{\bar{\mathfrak{m}}}] \xrightarrow{\sim} \operatorname{Ind}^{\mathbb{Q}}_{K^p_{\bar{\mathfrak{m}}}} \mathcal{O}(\psi^{-1}_{2\mathfrak{P}})$$

to obtain a class:

$$\kappa^5_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1(\mathbb{Q}, T_f^\vee \otimes_{\mathcal{O}} \operatorname{Ind}^{\mathbb{Q}}_{K^p_{\mathfrak{m}}} \mathcal{O}(\psi^{-1}_{1\mathfrak{P}}) \otimes_{\mathcal{O}} \operatorname{Ind}^{\mathbb{Q}}_{K^p_{\bar{\mathfrak{m}}}} \mathcal{O}(\psi^{-1}_{2\mathfrak{P}})(-1)). \tag{3.1.0.5}$$

We write $\operatorname{Ind}^{\mathbb{Q}}_{K^p_{\mathfrak{m}}} \mathcal{O}(\psi^{-1}_{1\mathfrak{P}}) = \operatorname{Ind}^{\mathbb{Q}}_{K} \mathcal{O}_{\psi^{-1}_{1\mathfrak{P}}}[H^{(p)}_{\mathfrak{m}}]$, the notation $\mathcal{O}_\chi$ means the twisted by χ 1−dimensional Galois representation $\mathcal{O}(\chi)$, and recall that $H^{(p)}_{\mathfrak{m}} \simeq \operatorname{Gal}(K^p_{\mathfrak{m}}/K)$ is the largest $p-$quotient of the ray class group of K modulo $\mathfrak{m}$.

Since we assume that $p \geq 5$, p will be coprime to $|\mathcal{O}_K^\times|$. Given a positive integer n, the ring class field of K of conductor n is the finite abelian extension $K[n]$ of K such that $\operatorname{rec}_K : \widehat{K}^\star/K^\star\widehat{\mathbb{Q}}^\star\widehat{\mathcal{O}}^\star_n \xrightarrow{\sim} \operatorname{Gal}(K[n]/K) \xrightarrow{\sim} \operatorname{Pic}(\mathcal{O}_n)$, where $\mathcal{O}_n = \mathbb{Z} + n\mathcal{O}_K$ is the order in O_K of conductor n. Its Galois group, denoted $H[n]$, is called the ring class group of conductor n. Denote $H[n]^{(p)}$ as the maximal $p-$power quotient of $H[n]$, and $K[n]^{(p)}$ as the maximal $p-$extension inside the ring class field $K[n]$.

Furthermore, as $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_K)$, its norm satisfies $m = \mathfrak{m}\bar{\mathfrak{m}}$. Explicitly, we have the following exact sequences (we will take the $p-$part ultimately) for the ring class group:

$$\frac{\mathcal{O}_K^\times}{\mathbb{Z}^\times} \to \frac{(\mathcal{O}_K/m\mathcal{O}_K)^\times}{(\mathbb{Z}/m\mathbb{Z})^\times} \to H[m] \to H_1 \to 1,$$

and the ray class group:

$$1 \to \frac{\mathcal{O}_K^\times}{\mathcal{O}_K^\times \cap K^{\mathfrak{m},1}} \to (\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K)^\times \to H_{\mathfrak{m}} \to H_1 \to 1.$$

Assumption 3.1.1. The prime p does not divide the class number of K, i.e. $p \nmid |H_1|$.

Under this assumption, since $(\mathcal{O}_K/m\mathcal{O}_K)^\times \cong (\mathcal{O}_K/\mathfrak{m}\mathcal{O}_K)^\times \times (\mathcal{O}_K/\bar{\mathfrak{m}}\mathcal{O}_K)^\times$, and combine with the natural projections $H_m^{(p)} \twoheadrightarrow H_{\mathfrak{m}}^{(p)}$ and $H_m^{(p)} \twoheadrightarrow H_{\bar{\mathfrak{m}}}^{(p)}$, we obtain an isomorphism:

$$H_m^{(p)} \xrightarrow{\sim} H_{\mathfrak{m}}^{(p)} \times H_{\bar{\mathfrak{m}}}^{(p)}.$$

Theorem 3.1.2. *For a prime $p \geq 5$ that does not divide the class number of K, by identifying $H_{\mathfrak{m}}^{(p)} \times H_{\bar{\mathfrak{m}}}^{(p)}$ with $H_m^{(p)}$ as above, the following sequence is exact:*

$$1 \to (\mathbb{Z}/m\mathbb{Z})^{\times,(p)} \xrightarrow{\Delta} H_{\mathfrak{m}}^{(p)} \times H_{\bar{\mathfrak{m}}}^{(p)} \xrightarrow{/\Delta} H[m]^{(p)} \to 1$$
$$l \overset{\Delta}{\longmapsto} [l] \times [l]$$

for a prime l such that $(l, mp) = 1$. If $l = \mathfrak{l}\bar{\mathfrak{l}}$ splits, the image of a prime ideal $\mathfrak{l}$ in the ray class groups $H_{\mathfrak{m}}^{(p)}$ and $H_{\bar{\mathfrak{m}}}^{(p)}$ will both be denoted $[\mathfrak{l}]$. Under quotienting by the image of the diagonal Δ,

$$[\mathfrak{l}] \times [\mathfrak{l}] \overset{/\Delta}{\longmapsto} \mathrm{Frob}_{\mathfrak{l}}$$

where now $\mathrm{Frob}_{\mathfrak{l}}$ will be the Frobenius of $\mathfrak{l}$ for the ring class field.

Before applying this theorem, we make some remarks about the functorial properties of Galois and group cohomology [Mil]. If we have M, M' a $G-$ and $G'-$module respectively, together with compatible homomorphisms: $a : G' \to G$ and $b : M \to M'$ in the sense that $b(a(g') \circ m) = g' \circ b(m)$, then one can define homomorphisms of complexes of cochains and hence $H^r(G, M) \to H^r(G', M')$ for any $r \in \mathbb{Z}_{\geq 0}$. In practice, we use the following compatible pairs:

1. $H \hookrightarrow G$ a subgroup, and $\mathrm{Ind}_H^G(M) \to M$ where $\phi \mapsto \phi(1_G)$, which induces the Shapiro's lemma isomorphism:

$$H^r(G, \mathrm{Ind}_H^G(M)) \xrightarrow{\sim} H^r(H, M).$$

2. $G \xrightarrow{\mathrm{id}} G$ the identity map, and $M \to \mathrm{Ind}_H^G(M)$: $m \mapsto \phi_m$ where $\phi_m(g) = g \circ m$, which induces the restriction homomorphism:

$$\begin{array}{ccc} H^r(G, M) & \longrightarrow & H^r(G, \mathrm{Ind}_H^G(M)) \\ & \searrow^{\mathrm{res}} & \downarrow \wr \\ & & H^r(H, M) \end{array}$$

where the vertical isomorphism is the Shapiro's lemma.

3. $G \xrightarrow{\mathrm{id}} G$ the identity map, and $\mathrm{Ind}_H^G(M) \to M$: $\phi \mapsto \sum_{\sigma \in \Sigma} \sigma \circ \phi(\sigma^{-1})$ where $G = \bigcup_{\sigma \in \Sigma} \sigma H$, which induces the corestriction homomorphism:

$$\begin{array}{ccc} H^r(G, \mathrm{Ind}_H^G(M)) & \longrightarrow & H^r(G, M) \\ \wr \uparrow & \nearrow_{\mathrm{cores}} & \\ H^r(H, M) & & \end{array}$$

where the vertical isomorphism is the inverse of the Shapiro's lemma morphism.

4. Given $H \trianglelefteq G$, $G \to G/H$ is the natural projection, and $M^H \hookrightarrow M$ is the

inclusion, which induces the inflation homomorphism:

$$H^r(G/H, M^H) \xrightarrow{\text{inf}} H^r(G, M).$$

5. Given $H_2 \subset H_1$ both subgroups of G, we can define the map:

$$\text{Norm} : \text{Ind}_{H_2}^G(M) \to \text{Ind}_{H_1}^G(M)$$

where $\phi \mapsto \text{Norm}(\phi)$ with $\text{Norm}(\phi)(g) = \sum_{\sigma \in H_1/H_2} \sigma \circ \phi(\sigma^{-1} g)$. This map, induced from the corestriction homomorphism above, makes the following diagram commute:

$$\begin{array}{ccc} H^r(G, \text{Ind}_{H_2}^G(M)) & \xrightarrow{\sim} & H^r(H_2, M) \\ {\scriptstyle Norm}\downarrow & & \downarrow{\scriptstyle \text{cores}} \\ H^r(G, \text{Ind}_{H_1}^G(M)) & \xrightarrow{\sim} & H^r(H_1, M) \end{array} \tag{3.1.0.6}$$

We are now in the position to use the quotient map from Theorem 3.1.2 to define the quotient:

$$\begin{array}{ccc} \text{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \text{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{2\mathfrak{P}}^{-1}}[H_{\bar{\mathfrak{m}}}^{(p)}] & \xrightarrow{\xi} & \text{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1}}[H_{\mathfrak{m}}^{(p)} \times H_{\bar{\mathfrak{m}}}^{(p)}] \\ & \searrow^{\xi_\Delta} & \downarrow{\scriptstyle /\Delta} \\ & & \text{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1}}[H[m]^{(p)}], \end{array} \tag{3.1.0.7}$$

where the horizontal map is $f \otimes g \mapsto \xi(f \otimes g)$ with $\xi(f \otimes g)(t) = f(t) \otimes g(t)$.

Lemma 3.1.3. *The following diagram is commutative:*

$$\begin{array}{ccc} Ind_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}}[H_{\mathfrak{ml}}^{(p)}] \otimes_{\mathcal{O}} Ind_K^{\mathbb{Q}} \mathcal{O}_{\psi_{2\mathfrak{P}}^{-1}}[H_{\bar{\mathfrak{m}}\bar{\mathfrak{l}}}^{(p)}] & \longrightarrow & Ind_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1}}[H[ml]^{(p)}] \\ {\scriptstyle \text{Norm}_{\mathfrak{m}}^{\mathfrak{ml}} \otimes \text{Norm}_{\bar{\mathfrak{m}}}^{\bar{\mathfrak{m}}\bar{\mathfrak{l}}}}\downarrow & & \downarrow{\scriptstyle \text{Norm}_m^{ml}} \\ Ind_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} Ind_K^{\mathbb{Q}} \mathcal{O}_{\psi_{2\mathfrak{P}}^{-1}}[H_{\bar{\mathfrak{m}}}^{(p)}] & \longrightarrow & Ind_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1}}[H[m]^{(p)}] \end{array} \tag{3.1.0.8}$$

where each of the Norm maps is a natural one, and the two horizontal maps are ξ_Δ*, the diagonal map in equation (3.1.0.7).*

Proof. Under our assumptions, we first identify $H^{(p)}_{ml} \xrightarrow{\sim} H^{(p)}_{\mathfrak{m}\,\mathfrak{l}} \times H^{(p)}_{\bar{\mathfrak{m}}\bar{\mathfrak{l}}}$. We also identify $H^{(p)}_{ml}/H^{(p)}_{m} \xrightarrow{\sim} H^{(p)}_{\mathfrak{m}\,\mathfrak{l}}/H^{(p)}_{\mathfrak{m}} \times H^{(p)}_{\bar{\mathfrak{m}}\bar{\mathfrak{l}}}/H^{(p)}_{\bar{\mathfrak{m}}}$, and $H^{(p)}_{ml}/H^{(p)}_{m} \pmod{\Delta} \xrightarrow{\sim} H[ml]^{(p)}/H[m]^{(p)}$. Combining with the explicit natural norm map recalled in equation (3.1.0.6), one obtains the result. □

The image of the class $\kappa^5_{f,\psi_1,\psi_2,\mathfrak{m}}$ in (3.1.0.5) under the composition of (3.1.0.7) gives us a class:

$$\kappa^6_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1(\mathbb{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1}}[H[m]^{(p)}](-1))$$

and by Shapiro's lemma, we can rewrite the group cohomology:

$$\kappa^6_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1(K[m]^{(p)}, T_f^\vee(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1)).$$

In the end, the diagram (3.1.0.8) from Lemma 3.1.3 implies that we have the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathbb{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}}[H^{(p)}_{\mathfrak{m}\mathfrak{l}}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{2\mathfrak{P}}^{-1}}[H^{(p)}_{\bar{\mathfrak{m}}\bar{\mathfrak{l}}}](-1)) & \longrightarrow & H^1(K[ml]^{(p)}, T_f^\vee(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1)) \\ \downarrow {\scriptstyle 1\otimes \mathrm{Norm}^{\mathfrak{m}\mathfrak{l}}_{\mathfrak{m}} \otimes \mathrm{Norm}^{\bar{\mathfrak{m}}\bar{\mathfrak{l}}}_{\bar{\mathfrak{m}}}} & & \downarrow {\scriptstyle \mathrm{Norm}^{ml}_{m}} \\ H^1(\mathbb{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{1\mathfrak{P}}^{-1}}[H^{(p)}_{\mathfrak{m}}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbb{Q}} \mathcal{O}_{\psi_{2\mathfrak{P}}^{-1}}[H^{(p)}_{\bar{\mathfrak{m}}}](-1)) & \longrightarrow & H^1(K[m]^{(p)}, T_f^\vee(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1)) \end{array} \tag{3.1.0.9}$$

We define

$$H^1(T_f^\vee, N^{\mathfrak{m}}_{\psi_1}(m), N^{\bar{\mathfrak{m}}}_{\psi_2}(m)) := H^1(\mathbb{Q}, T_f^\vee \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1}m)} \mathcal{O}[H^{(p)}_{\mathfrak{m}}] \otimes H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m)} \mathcal{O}[H^{(p)}_{\bar{\mathfrak{m}}}](-1)).$$

Together with Proposition 2.4.2 (where $\mathrm{Norm}^{\mathfrak{m}\mathfrak{l}}_{\mathfrak{m}}$ corresponds to $\mathcal{N}^{\mathfrak{m}\mathfrak{l}}_{\mathfrak{m}}$), the diagram

(3.1.0.9) is just:

$$\begin{array}{ccc} H^1(T_f^\vee, N_{\psi_1}^{\mathfrak{m}\mathfrak{l}}(ml), N_{\psi_2}^{\bar{\mathfrak{m}}\bar{\mathfrak{l}}}(ml)) & \longrightarrow & H^1(K[ml]^{(p)}, T_f^\vee(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1)) \\ \downarrow{\scriptstyle 1\otimes\mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}}\otimes\mathcal{N}_{\bar{\mathfrak{m}}}^{\bar{\mathfrak{m}}\bar{\mathfrak{l}}}} & & \downarrow{\scriptstyle \mathrm{Norm}_m^{ml}} \\ H^1(T_f^\vee, N_{\psi_1}^{\mathfrak{m}}(m), N_{\psi_2}^{\bar{\mathfrak{m}}}(m)) & \longrightarrow & H^1(K[m]^{(p)}, T_f^\vee(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1)) \end{array} \tag{3.1.0.10}$$

Proposition 3.1.4. *Let $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_K)$ such that $m = Norm_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to p. Let $\mathfrak{l} \in \mathscr{L}_K$ be a split prime of K such that $l = Norm_{K/\mathbb{Q}}(\mathfrak{l})$ is coprime to pm. Assume further that $(ml, Np) = 1$, then we have:*

$$\begin{aligned} \mathrm{Norm}_m^{ml}(\kappa^6_{f,\psi_1,\psi_2,\mathfrak{m}\,\mathfrak{l}}) = (l-1)\Bigg(a_l(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{l}[\mathfrak{l}]\times[\mathfrak{l}] - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\bar{\mathfrak{l}})}{l}[\bar{\mathfrak{l}}]\times[\bar{\mathfrak{l}}] + \\ (1-l)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}]\times[\bar{\mathfrak{l}}]\Bigg)(\kappa^6_{f,\psi_1,\psi_2,\mathfrak{m}}) \end{aligned} \tag{3.1.0.11}$$

Proof. For simplicity, we will drop the subscripts f, ψ_1, ψ_2 and only keep track of the numbering, $\mathfrak{m}$, and $\mathfrak{l}$. Tracing back the κ^i, we calculate:

$$\begin{aligned} &(1\otimes\mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\,\mathfrak{l}}\otimes\mathcal{N}_{\bar{\mathfrak{m}}}^{\bar{\mathfrak{m}}\bar{\mathfrak{l}}})(\kappa^2_{ml}) = (1\otimes\mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\,\mathfrak{l}}\otimes\mathcal{N}_{\bar{\mathfrak{m}}}^{\bar{\mathfrak{m}}\bar{\mathfrak{l}}})(\mathrm{pr}_{ml\star},1,1)(\kappa^1_{ml}) \\ =&(\mathrm{pr}_{m\star},1,1)(\mathrm{pr}_{l\star}\otimes\mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\,\mathfrak{l}}\otimes\mathcal{N}_{\bar{\mathfrak{m}}}^{\bar{\mathfrak{m}}\bar{\mathfrak{l}}})(\kappa^1_{ml}) \\ =&(\mathrm{pr}_{m\star},1,1)\left(\mathrm{pr}_{l\star}\times(1\otimes\mathrm{pr}_{1\star} - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{l}\otimes\mathrm{pr}_{l\star})\times(1\otimes\mathrm{pr}_{1\star} - \frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{l}\otimes\mathrm{pr}_{l\star})\right)(\kappa^1_{ml}) \\ =&(\mathrm{pr}_{m\star},1,1)\Bigg((\mathrm{pr}_{l\star},\mathrm{pr}_{1\star},\mathrm{pr}_{1\star}) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{l}(\mathrm{pr}_{l\star},\mathrm{pr}_{l\star},\mathrm{pr}_{1\star}) - \frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{l}(\mathrm{pr}_{l\star},\mathrm{pr}_{1\star},\mathrm{pr}_{l\star}) \\ &+\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}]\times[\bar{\mathfrak{l}}](\mathrm{pr}_{l\star},\mathrm{pr}_{l\star},\mathrm{pr}_{l\star})\Bigg)(\kappa^1_{ml}) \\ \doteq&(l-1)(\mathrm{pr}_{m\star},1,1)\Bigg((T_l,1,1) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{l}(1,1,T_l') - (1,T_l',1)\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{l} \\ &+\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}]\times[\bar{\mathfrak{l}}](l+1)\Bigg)(\kappa^1_m) \\ =&(l-1)\Bigg((T_l,1,1) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{l}(1,1,T_l') - (1,T_l',1)\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{l} \\ &+\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}]\times[\bar{\mathfrak{l}}](l+1)\Bigg)(\kappa^2_m) \end{aligned}$$

Here we use the table in Proposition 3.1.1 for $\mathrel{\dot{\tilde{=}}}$.

This implies that its image also satisfies:

$$
\begin{aligned}
&(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\,\mathfrak{l}} \otimes \mathcal{N}_{\bar{\mathfrak{m}}}^{\bar{\mathfrak{m}}\bar{\mathfrak{l}}})(\kappa^4_{\mathfrak{m}\,\mathfrak{l}}) \\
=&(l-1)\Bigg((T_l,1,1) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{l}(1,1,T_l') - (1,T_l',1)\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{l} \\
&+ \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}] \times [\bar{\mathfrak{l}}](l+1)\Bigg)(\kappa^4_{\mathfrak{m}}) \\
\mathrel{\tilde{=}}&(l-1)\Bigg(a_l(f) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{l}(\psi_2(\mathfrak{l})[\mathfrak{l}] + \psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]) - (\psi_1(\mathfrak{l})[\mathfrak{l}] + \psi_1(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}])\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{l} \\
&+ \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}] \times [\bar{\mathfrak{l}}](l+1)\Bigg)(\kappa^4_{\mathfrak{m}}) \\
=&(l-1)\Bigg(a_l(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{l}[\mathfrak{l}] \times [\mathfrak{l}] - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\bar{\mathfrak{l}})}{l}[\bar{\mathfrak{l}}] \times [\bar{\mathfrak{l}}] \\
&+ (1-l)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}] \times [\bar{\mathfrak{l}}]\Bigg)(\kappa^4_{\mathfrak{m}}),
\end{aligned}
$$

where we use for the $\mathrel{\tilde{=}}$, the fact that κ^4 lands in an isotypical piece that can be described by the map in Proposition 2.4.1. Now showing the norm relation for κ^4 is enough to conclude the proof thanks to the commutative diagram (3.1.0.10). □

Remark 3.1.2. The $(l-1)$ factor appears due to $\deg(\mu_l)T_l = \mathrm{pr}_{l\star} \circ \mathrm{pr}_1^\star$, and $\deg(\mu_l)T_l' = \mathrm{pr}_{1\star} \circ \mathrm{pr}_l^\star$, i.e. because of the μ_l degeneracy map. In the next subsection, we will get rid of this extra factor.

Remark 3.1.3. We want to emphasize that this proposition is the key result for the construction of our Euler system. Indeed, if we can get rid of $(l-1)$, the remaining factor on the RHS of Proposition 3.1.4 can be massaged to be equal to the Euler factor of the Galois representation $T_f(\psi_1\psi_2)(2)$, giving the correct norm relation which means that our class form an anticyclotomic Euler system.

3.1.1 The fix

Follow the above remark, we attempt to get rid of of $(l-1)$ using some ideas from [DR17, Sec 1.4].

Notation.

1. For this subsection, denote $Y_1(N, a) = Y(1, N(a))$.

2. For a given prime $l \neq N$ and for $i \in \{1, l\}$, define the natural degeneracy maps:

$$\begin{array}{ccc} Y_1(Nl) & & \\ \downarrow{\scriptstyle \mu_l} & \searrow^{\mathrm{pr}_i} & \\ Y_1(N,l) & \xrightarrow[\pi_i]{} & Y_1(N) \end{array}$$

   where μ_l is a cyclic Galois covering of degree $l-1$ and π_i is a non-Galois covering of degree $l+1$.

3. Denote $D_m = \{(\langle d\rangle, \langle d\rangle) : d \in (\mathbb{Z}/Nm\mathbb{Z})^\times, d \equiv 1 \pmod{N}\}$, the set of diamond operators acting diagonally on $Y_1(Nm)^2$.

4. Let $W_1(Nm) = (Y_1(Nm) \times Y_1(Nm))/D_m$ and denote by $d_m : Y_1(Nm)^2 \to W_1(Nm)$ the natural projection map, which is an étale morphism of degree $\phi(m)$.

One can obtain a class using the BSV class in (3.1.0.1):

$$\kappa_m \in H^1(\mathbb{Q}, H^3_{\text{ét}}(Y_1(N, m) \times W_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)(2)) \tag{3.1.1.1}$$

where we look at the case $\mathbf{r} = (0, 0, 0)$ such that

$$(\mu_{m\star}, d_{m\star})\kappa^1_{m,\mathbf{r}} = \phi(m)\kappa_m. \tag{3.1.1.2}$$

Then for $(m,q)=1$, as $\phi(m)(q-1)=\phi(mq)$, we have the following updated table, which gets rid of the $q-1$ factor in Proposition 3.1.1:

Proposition 3.1.5. *For a prime number q and a positive integer m such that $(m,q)=1$ and $(mq,pN)=1$,*

$$(\pi_{i\star}, pr_{j\star}, pr_{k\star})\kappa_{mq} = (\bigstar)\kappa_m$$

where

(i,j,k)	$\bigstar$	(i,j,k)	$\bigstar$
$(q,1,1)$	$(T_q,1,1)$	$(q,1,q)$	$(1,T_q',1)$
$(1,q,1)$	$(1,T_q,1)$	$(q,q,1)$	$(1,1,T_q')$
$(1,1,q)$	$(1,1,T_q)$	$(1,1,1)$	$(q+1)$
$(1,q,q)$	$(T_q',1,1)$	(q,q,q)	$(q+1)$

Now we want to proceed as above to obtain the correct norm relation (i.e., without the $q-1$ factor). This requires to be careful with the étale cohomology of $Y_1(N,m)\times W_1(Nm)$.

We begin with the Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(D_m, H^q_{\text{ét},c}(Y_1(N,m)\times Y_1(Nm)^2_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)) \Rightarrow H^{p+q}_{\text{ét},c}(Y_1(N,m)\times W_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p).$$

This leads to an exact sequence:

$$E \to H^3_{\text{ét},c}(Y_1(N,m)\times W_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \xrightarrow{(1,d_m^\star)} E_2^{0,3} \xrightarrow{d_2^{0,3}} E_2^{2,2}$$

where E is a canonical subquotient of $E_2^{1,2}\oplus E_2^{2,1}$. From this, we see that the difference between the two middle pieces are classes coming from $H^q_{\text{ét},c}(Y_1(N,m)\times Y_1(Nm)^2_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)$ with $q\leq 2$. From the Künneth decomposition here (see (3.1.0.3)), because of the condition $q\leq 2$ either $H^0_{\text{ét},c}(Y_1(N,m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)$ or $H^0_{\text{ét},c}(Y_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)$ appears as one of the factors. What we will do later is localizing at a non-Eisenstein maximal prime ideal $\mathcal{I}$ of $\mathbb{T}'_N$, which will kill these factors, hence obtain an integral isomorphism of

$$H^3_{\text{ét},c}(Y_1(N,m) \times W_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \xrightarrow{(1,d^\star_m)} E_2^{0,3} = H^3_{\text{ét},c}(Y_1(N,m) \times Y_1(Nm)^2_{\bar{\mathbb{Q}}}, \mathbb{Z}_p))^{D_m}. \tag{3.1.1.3}$$

By Poincaré duality we also obtain a map:

$$H^3_{\text{ét}}(Y_1(N,m) \times W_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \xleftarrow{(1,d_{m\star})} H^3_{\text{ét}}(Y_1(N,m) \times Y_1(Nm)^2_{\bar{\mathbb{Q}}}, \mathbb{Z}_p))_{D_m}, \tag{3.1.1.4}$$

whose kernel and cokernel will also be annihilated by localization at (the dual of) the ideal $\mathcal{I}$.

The following lemma essentially tells us that by localizing at a non-Eisenstein maximal ideal, there will be no difference between $X_1(N)$ and $Y_1(N)$, between H^1 and H^1_c.

Lemma 3.1.6. *For $\mathcal{I}$ a non-Eisenstein maximal ideal of $\mathbb{T}'_N$*

$$H^1_c(Y_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathcal{I}} \xrightarrow{\sim} H^1(X_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathcal{I}} \xrightarrow{\sim} H^1(Y_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)_{\mathcal{I}} \tag{3.1.1.5}$$

Proof. The Manin-Drinfeld theorem tells us the existence of many primes l such that $(1+l-T_l)$ kills $H^1(\partial X_1(N))$. The non-Eisenstein property tells us that we can choose l such that $(1+l-T_l) \notin \mathcal{I}$, which will be invertible after localizing at $\mathcal{I}$. □

We now recall more details from [LLZ15] besides those already recalled in Section 2.4.2. Firstly, for $p \nmid m = \mathfrak{m}\bar{\mathfrak{m}}$, we have the composition map

$$\mathbb{T}'_{Nm} \xrightarrow{\phi_{\mathfrak{m}}} \mathcal{O}_L[H_{\mathfrak{m}}] \xrightarrow{\text{aug}} \mathcal{O}_L \to \mathcal{O}_L/\mathfrak{P}$$

and define $\mathcal{I}_{\mathfrak{m}}$ to be its kernel. By [LLZ15, Prop 5.1.2], $\mathcal{I}_{\mathfrak{m}}$ can be checked to be non-Eisenstein (equivalently the associated residual representation is irreducible), ordinary and $p-$distinguished. At a later step, we will look at the following module

$$\mathcal{O}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathbb{T}'(Nm)\otimes\mathbb{Z}_p,\phi_{\mathfrak{m}}} H^1_{\text{ét}}(Y_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)),$$

and it is clear that the map from $H^1_{\text{ét}}(Y_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))$ to this module factors through completion at $\mathcal{I}_{\mathfrak{m}}$. One can choose an auxiliary prime $l \nmid Nmp$ such that $1 + l - a_l(F) \in \mathbb{Z}_p^{\times}$ and $\frac{1+l-T_l}{1+l-a_l(F)} \notin \mathcal{I}_{\mathfrak{m}}$ which annihilates $H^0_{\text{ét}}(Y_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))$ and also $H^2_{\text{ét}}(Y_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))$ which fixes the $F-$isotypical piece that we are interested in. These invertible elements after localization at $\mathcal{I}_{\mathfrak{m}}$ will annihilate $H^q_{\text{ét}}(Y_1(N,m) \times Y_1(Nm)^2_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)$ for $q \leq 2$. We then use Lemma 3.1.6 to see that after localization at non-Eisenstein maximal prime ideals, we acquire a map:

$$\begin{aligned} H^3_{\text{ét}}(Y_1(N,m) \times W_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) &\xrightarrow{(1,d_{m\star}^{-1})} H^1_{\text{ét}}(Y_1(N,m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)\otimes \\ &H^1_{\text{ét}}(Y_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \otimes_{D_m} H^1_{\text{ét}}(Y_1(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)). \end{aligned}$$

Define: $\kappa'_m = (\pi_{m\star}, 1, 1)\kappa_m$. One can adapt the notation, and mimic the construction of $\kappa^i_{\mathfrak{m},\mathbf{r}}$ for our modified κ'_m, beginning with equation (3.1.0.4) to Proposition 3.1.4. The key difference here is that the tensoring in equation (3.1.0.5) is over $\mathcal{O}$, while our class will land in:

$$H^1(\mathbb{Q}, T_f^{\vee} \otimes_{\mathcal{O}} \operatorname{Ind}^{\mathbb{Q}}_{K^p_{\mathfrak{m}}} \mathcal{O}(\psi^{-1}_{1\mathfrak{P}}) \otimes_{\mathcal{O}[D_m]} \operatorname{Ind}^{\mathbb{Q}}_{K^p_{\bar{\mathfrak{m}}}} \mathcal{O}(\psi^{-1}_{2\mathfrak{P}})(-1)). \tag{3.1.1.6}$$

Now, taking the D_m-coinvariant is compatible with the ξ_Δ map because D_m lands in its kernel. Indeed, for $(\langle d\rangle, \langle d\rangle) \in D_m$, we have $\phi_{\mathfrak{m}}(\langle d\rangle') \times \phi_{\bar{\mathfrak{m}}}(\langle d\rangle') = [d] \times [d] \in \Delta$ (by Theorem 2.4.1 and 3.1.2). In the end, we arrive at a class:

$$\kappa'_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1(K[m]^{(p)}, T_f^{\vee}(\psi^{-1}_{1\mathfrak{P}}\psi^{-1}_{2\mathfrak{P}})(-1)).$$

Proposition 3.1.7. *Let $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_K)$ such that $m = Norm_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to p. Let $\mathfrak{l} \in \mathscr{L}_K$ be a split prime of K such that $l = Norm_{K/\mathbb{Q}}(\mathfrak{l})$ is coprime to pm. Assume further that $(ml, Np) = 1$, then one has:*

$$\mathrm{Norm}_m^{ml}(\kappa'_{f,\psi_1,\psi_2,\mathfrak{m}\,\mathfrak{l}}) = \Bigg(a_l(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{l}[\mathfrak{l}] \times [\mathfrak{l}] - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\bar{\mathfrak{l}})}{l}[\bar{\mathfrak{l}}] \times [\bar{\mathfrak{l}}] + (1-l)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}] \times [\bar{\mathfrak{l}}]\Bigg)(\kappa'_{f,\psi_1,\psi_2,\mathfrak{m}}) \quad (3.1.1.7)$$

Proof. Same as Proposition 3.1.4, but instead of Proposition 3.1.1 we use Proposition 3.1.5. □

Let $P_{\mathfrak{l}}(X) = P_{\mathfrak{l}}(1 - X \cdot \mathrm{Frob}_{\mathfrak{l}} | T_f(\psi_1\psi_2)(2))$. Then we have the following congruence of endomorphisms of $H^1(K[m]^{(p)}, T_f^\vee(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1))$:

$$\begin{aligned} P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}) =& 1 - a_l(f)\frac{\psi_1\psi_2(\mathfrak{l})}{l}\mathrm{Frob}_{\mathfrak{l}} + \left(\frac{\psi_1\psi_2(\mathfrak{l})}{l}\mathrm{Frob}_{\mathfrak{l}}\right)^2 \\ \check{\equiv}& \Bigg(- a_l(f)\frac{\psi_1\psi_2(\mathfrak{l})}{l}\mathrm{Frob}_{\mathfrak{l}} + \left(\frac{\psi_1\psi_2(\mathfrak{l})}{l}\mathrm{Frob}_{\mathfrak{l}}\right)^2 + \frac{\psi_1(l)\psi_2(l)}{l^2} + \\ &(l-1)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{l^2}[\mathfrak{l}] \times [\bar{\mathfrak{l}}] \cdot \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{l}[\mathfrak{l}] \times [\mathfrak{l}]\Bigg) \\ \tilde{\equiv}& \Bigg(- \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{l}[\mathfrak{l}] \times [\mathfrak{l}]\Bigg)Q_{\mathfrak{l}} \pmod{l-1} \end{aligned}$$

where $Q_{\mathfrak{l}}$ is the factor in the RHS of equation (3.1.1.7). The congruence $\check{\equiv}$ is due to $\psi_i(l) = l\chi_{\psi_i}(l)$ and $\chi_{\psi_1}\chi_{\psi_2} = 1$. The congruence $\tilde{\equiv}$ is because of Theorem 3.1.2, where one has $\mathrm{Frob}_{\mathfrak{l}} = [\mathfrak{l}] \times [\mathfrak{l}]$ and $\mathrm{Frob}_{\mathfrak{l}}\mathrm{Frob}_{\bar{\mathfrak{l}}} = 1$ as an endomorphism of $H^1(K[m]^{(p)}, -)$.

Combine this congruence with Lemma 9.6.1 from [Rub00] (which is about the way to modify the Euler factor of a constructed cohomology class in order to obtain an Euler system, given some congruent conditions), one obtains the following theorem:

Theorem 3.1.8. *Let $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_K)$ such that its norm $m = Norm_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to $N_f N_{\psi_1} N_{\psi_2} p$. Assume that $H^1(K[m]^{(p)}, T_f^\vee(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1))$ is torsion-free for all such*

$\mathfrak{m}$. *Then there exists a collection of classes:*

$$z_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1(K[m]^{(p)}, T_f^\vee(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1))$$

such that given $\mathfrak{l} \in \mathscr{L}_K$ *a split prime of* K *satisfying* $(l, N_f N_{\psi_1} N_{\psi_2} pm) = 1$, *where* $l = Norm_{K/\mathbb{Q}}(\mathfrak{l})$, *one has the following norm relation:*

$$\mathrm{Norm}_{K[m]^{(p)}}^{K[ml]^{(p)}}(z_{f,\psi_1,\psi_2,\mathfrak{m}\,\mathfrak{l}}) = P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}})(z_{f,\psi_1,\psi_2,\mathfrak{m}})$$

where $P_{\mathfrak{l}}(X) = P_{\mathfrak{l}}(1 - X \cdot \mathrm{Frob}_{\mathfrak{l}} | T_f(\psi_1\psi_2)(2))$.

Remark 3.1.4. We assume the torsion-freeness because we want to use Lemma 9.6.1 from Rubin, in order to get an equality for the norm relation, not just a congruence modulo $(l-1)$. It will be satisfied if we have T_f being residually irreducible. Nevertheless, in practice, we only care about the $p-$power dividing $l-1$ (for the Kolyvagin system's argument) which means that we can drop the torsion-free condition.

3.2 $\Lambda-$adic tame norm relations for weights $(k, l, 2)$

In this section, we will generalise the construction in the previous one, from three newforms to three Hida families along the anticyclotomic extension, hence obtaining $\kappa^{\infty}_{f,\psi_1,\psi_2,\mathfrak{m}}$ for more general weights. The machinery for doing this comes largely from [ACR21, Sec 5] and [BSV21, Sec 4].

3.2.1 Hida families

We recall the notion of a Hida family.

Notation.

1. Let p be an odd prime.

2. Let $\Lambda = \mathbb{Z}_p[\![1 + p\,\mathbb{Z}_p]\!]$ be the completed group ring.

3. The formal spectrum of Λ: $\mathcal{W} = \mathrm{Spf}(\Lambda)$ is known as the weight space. Explicitly, for any extension $L/\mathbb{Q}_p$, $\mathcal{W}(L) = \mathrm{Hom}_{\mathrm{ct}}(1 + p\,\mathbb{Z}_p, L^\times)$.

4. An arithmetic point of $\mathcal{W}$ is an homomorphism $\nu_{r,\epsilon}$ such that $\nu_{r,\epsilon} : z \mapsto \epsilon(z) z^r$ where $r \in \mathbb{Z}_{\geq 0}$ and ϵ is a finite order character.

5. A classical point will be an arithmetic point with a trivial character ϵ, often denoted $\nu_r = \nu_{r,1}$.

6. The weight of an arithmetic point $\nu_{r,\epsilon}$ is $k = r + 2$.

7. We can generalise these notions to Λ', a normal domain finite flat over Λ and let the weight space be $\mathcal{W}_{\Lambda'} = \mathrm{Spf}(\Lambda')$. A point $x \in \mathcal{W}_{\Lambda'}(\bar{\mathbb{Q}}_p)$ is arithmetic or classical if it lies above an arithmetic point $\nu_{r,\epsilon}$ or a classical point ν_r of $\mathcal{W}(\bar{\mathbb{Q}}_p)$, respectively. The weight of x is still $k = r + 2$.

Definition. For a positive integer N such that $(N, p) = 1$, an **ordinary Hida family** of tame level N and character $\chi : (\mathbb{Z}/Mp\,\mathbb{Z})^\times \to \bar{\mathbb{Q}}_p^\times$ is a formal $q-$expansion:

$$\mathbf{f} = \sum_{n \geq 1} a_n(\mathbf{f}) q^n \in \Lambda_{\mathbf{f}}[\![q]\!],$$

where $\Lambda_{\mathbf{f}}$ is a normal domain finite flat over Λ, such that for any arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\bar{\mathbb{Q}}_p)$ lying over $\nu_{r,\epsilon}$, the power series

$$\mathbf{f}_x = \sum_{n \geq 1} a_n(\mathbf{f})(x) q^n \in \bar{\mathbb{Q}}_p[\![q]\!],$$

called the specialization at x, is the $q-$expansion of a $p-$ordinary cuspidal eigenform in $S_k(Mp^s, \chi\epsilon\omega^{-r})$. Here $s = \max(1, \mathrm{ord}_p(\mathrm{cond}(\epsilon)))$. A Hida family $\mathbf{f}$ is primitive if the specializations at arithmetic points are $p-$stabilized newforms, and is normalized if $a_1(\mathbf{f}) = 1$.

Definition/Proposition. Let $\mathbf{f}$ be a normalized primitive Hida family of tame level N. For each arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\bar{\mathbb{Q}}_p)$, let f_x be the newform that corresponds to the specializations $\mathbf{f}_x$. There exists a locally-free rank two $\Lambda_{\mathbf{f}}$ −module $\mathbb{V}_{\mathbf{f}}$, called the **big Galois representation** attached to $\mathbf{f}$, coming with an action of $G_{\mathbb{Q}}$ such that the specialization $\mathbb{V}_{\mathbf{f}} \otimes_{\Lambda_{\mathbf{f}},x} \bar{\mathbb{Q}}_p$ recovers the $G_{\mathbb{Q}}$ representation $V_{f_x}^{\vee}$ attached to f_x. If the specialization at one (equivalently at all) arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\bar{\mathbb{Q}}_p)$ of $\mathbb{V}_{\mathbf{f}}$ is residually irreducible (i.e. $T_{f_x}^{\vee}$ is residually irreducible), then $\mathbb{V}_{\mathbf{f}}$ is a free $\Lambda_{\mathbf{f}}$ module.

3.2.2 Continuous functions and distributions

The two sets $T = \mathbb{Z}_p^{\times} \times \mathbb{Z}_p$ and $T' = p\,\mathbb{Z}_p \times \mathbb{Z}_p^{\times}$ come with a right action of

$$\Sigma_0(p) = \begin{pmatrix} \mathbb{Z}_p^{\times} & \mathbb{Z}_p \\ p\,\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \quad \text{and} \quad \Sigma_0'(p) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\,\mathbb{Z}_p & \mathbb{Z}_p^{\times} \end{pmatrix}$$

respectively on row vectors. Let $E/\,\mathbb{Q}_p$ be a finite extension, $\mathcal{O}$ its ring of integers, and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$. Let $\mathrm{Ct}(\mathbb{Z}_p, \mathcal{O})$ be the space of continuous functions from $\mathbb{Z}_p$ to $\mathcal{O}$. For a character $\nu : \mathbb{Z}_p^{\times} \to E$, one can define the following $\mathcal{O}-$modules equipped with the $\mathfrak{m}$ −adic topology:

$$\mathcal{A}_\nu = \{f : T \to \mathcal{O} \text{ s.t. } f(1,z) \in \mathrm{Ct}(\mathbb{Z}_p, \mathcal{O});\ f(a \cdot t) = \nu(a) f(t)\ \forall\, a \in \mathbb{Z}_p^{\times},\ t \in T\}$$
$$\mathcal{A}'_\nu = \{f : T' \to \mathcal{O} \text{ s.t. } f(pz,1) \in \mathrm{Ct}(\mathbb{Z}_p, \mathcal{O});\ f(a \cdot t) = \nu(a) f(t)\ \forall\, a \in \mathbb{Z}_p^{\times},\ t \in T'\}.$$

The dual $\mathcal{O}-$modules:

$$\mathcal{D}_\nu = \mathrm{Hom}_{ct}(\mathcal{A}_\nu, \mathcal{O}), \quad \text{and} \quad \mathcal{D}'_\nu = \mathrm{Hom}_{ct}(\mathcal{A}'_\nu, \mathcal{O}),$$

are equipped with the weak-$*$ topology. The right action of $\Sigma_0(p)$ on T induces a left (resp. right) $\Sigma_0(p)$ action on $\mathcal{A}_\nu$ (resp. $\mathcal{D}_\nu$). We also have similar actions of $\Sigma_0'(p)$ on

T', $\mathcal{A}'_\nu$, and $\mathcal{D}'_\nu$.

3.2.3 Group cohomology and étale cohomology

Let N, m be positive integers such that N, m and p are coprime pairwise. Let $\boxed{Y = Y(1, Nm(p))}$ and $\Gamma = \Gamma(1, Nm(p))$. Let $\mathcal{E} \to Y$ be the universal elliptic curve and denote by C_p the canonical cyclic $p-$subgroup of $\mathcal{E}$. Let $\mathscr{T}$ be the relative $p-$adic Tate module of $\mathcal{E}$ over Y. We fix a geometric point $\eta : \mathrm{Spec}(\bar{\mathbb{Q}}) \to Y$ and define an isomorphism:

$$\mathscr{T}_\eta \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$$

such that the Weil pairing on the left is identified with the determinant map on the right, and the reduction mod p of $(0, 1)$ generates $C_{p,\eta}$. Let $\mathcal{G} = \pi_1^{\text{ét}}(Y, \eta)$. The action of $\mathcal{G}$ on $\mathcal{T}$ yields an action of $\mathcal{G}$ on $\mathbb{Z}_p \oplus \mathbb{Z}_p$, i.e. a continuous representation $\rho : \mathcal{G} \to \mathrm{GL}_2(\mathbb{Z}_p)$, where $g \cdot (x, y) = (x, y)\rho(g)^{-1}$. Since the action of $\mathcal{G}$ preserves the canonical subgroup, $\rho : \mathcal{G} \to \Gamma_0(p\,\mathbb{Z}_p) \subset \mathrm{GL}_2(\mathbb{Z}_p)$. Note that the anti-involution $\gamma \to {}^\iota\gamma := \det(\gamma)\gamma^{-1}$ on $\Gamma_0(p\,\mathbb{Z}_p)$ allows us to consider the group action either as a right or a left action.

For a topological group G, define $\mathbf{M}_f(G)$ as the category of finite $G-$sets of $p-$power order. Let $\mathbf{M}_{ct}(G)$ be the category of $G-$modules which are filtered unions $\cup_i M_i$ such that $M_i \in \mathbf{M}_f(G)$. Let $\mathbf{M}(G) \subset \mathbf{M}_{ct}(G)^{\mathbb{N}}$ be the category of inverse systems of objects in $\mathbf{M}_{ct}(G)$. By taking the stalk at η, one has an equivalence of categories between $\mathbf{S}_f(Y_{\text{ét}})$, the category of locally constant constructible sheaves with finite stalk of $p-$power order at η, and $\mathbf{M}_f(\mathcal{G})$. One can define $\mathbf{S}(Y_{\text{ét}})$ similarly to $\mathbf{M}(G)$, and hence obtain an equivalence of categories between $\mathbf{M}(\mathcal{G})$ and $\mathbf{S}(Y_{\text{ét}})$. We also have a functor $\mathbf{M}(\Gamma_0(p\,\mathbb{Z}_p)) \to \mathbf{M}(\mathcal{G})$ coming from ρ. We adopt the following choice with regards to this functor: if $\mathcal{F} \in \mathbf{M}(\Gamma_0(p\,\mathbb{Z}_p))$ is given as a left (respectively right) $\Gamma_0(p\,\mathbb{Z}_p)-$module then we define the action of $\mathcal{G}$ via ρ (respectively ρ^{-1}).

For an inverse system of sheaves $\boldsymbol{\mathcal{F}} = (\boldsymbol{\mathcal{F}}_i)_{i\in\mathbb{N}} \in \mathbf{S}(Y_{\text{ét}})$, we denote by $H^j_{\text{ét}}(Y, \boldsymbol{\mathcal{F}})$ the continuous étale cohomology defined by Jannsen. We also write:

$$\mathtt{H}^j_{\text{ét}}(Y, \boldsymbol{\mathcal{F}}) = \varprojlim_i H^j_{\text{ét}}(Y, \boldsymbol{\mathcal{F}}_i).$$

Similarly we can define the compactly supported cohomology groups $H^j_{\text{ét},c}(Y, \boldsymbol{\mathcal{F}})$ and $\mathtt{H}^j_{\text{ét},c}(Y, \boldsymbol{\mathcal{F}})$. Note that there is a natural surjective morphism $H^j_{\text{ét}}(Y, \boldsymbol{\mathcal{F}}) \to \mathtt{H}^j_{\text{ét}}(Y, \boldsymbol{\mathcal{F}})$.

There is an isomorphism $\pi_1^{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \eta) \cong \hat{\Gamma}$, which induces the natural isomorphisms:

$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) \cong H^1(\hat{\Gamma}, \mathcal{F}) \cong H^1(\Gamma, \mathcal{F}) \tag{3.2.3.1}$$

where $\mathcal{F} \in \mathbf{M}_f(\mathcal{G})$ is a discrete $\mathcal{G}-$module, corresponding to $\boldsymbol{\mathcal{F}} \in \mathbf{S}_f(Y_{\text{ét}})$.

Let $\mathcal{F} \in \mathbf{M}_f(\Gamma_0(p\,\mathbb{Z}_p))$ be a left $\Gamma_0(p\,\mathbb{Z}_p)-$module and assume that the $\Gamma_0(p\,\mathbb{Z}_p)$ action extends to a left action of $\Sigma_0^{\cdot}(p)$. Let $S = \Sigma_0^{\cdot}(p) \cap \mathrm{GL}_2(\mathbb{Q})$ then the pair (Γ, S) is a Hecke pair in the sense of Ash-Stevens [AS86a, Sec 1.1]. There is also a covariant (left) action of $D(\Gamma, S)$ the Hecke algebra on $H^1(\Gamma, \mathcal{F})$ (notated $\mathscr{H}(\Gamma, S)$ in [AS86a, Sec 1.1]). Denote for each $g \in S$, $T(g) = \Gamma g \Gamma \in D(\Gamma, S)$. Define for positive integers n and a, where $(a, p) = 1$, the Hecke operators [GS93, Sec 1]:

$$T_n = T\left(\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\right), T'_n = T\left(\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\right), [a]_p = [a]'_p = T\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right).$$

For each $(b, N) = 1$, choose $\beta_a \in \Gamma_0(Npm)$ whose lower right entry is $\equiv a \pmod{N}$, and $\beta'_a \in \Gamma_0(Npm)$ whose its lower right entry is $\equiv a^{-1} \pmod{N}$. Let

$$[a]_N = T(\beta_a), \quad [a]'_N = T(\beta'_a).$$

In order to specify the maps between different levels, let $Y(m) = Y(1, Nm(p))$

and let $\Gamma(m)$ be the corresponding modular group. Fix a positive integer s and let $r = 1 + v_p(s)$. Let $\eta_s : \mathrm{Spec}(\bar{\mathbb{Q}}) \to Y(ms)$ be a geometric point lying above the point η fixed above. Choose an isomorphism $\mathscr{T}_{\eta_s} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ such that the Weil pairing on the left is identified with the natural determinant map on the right, and the reduction mod p^r of $(0,1)$ generates the canonical subgroup C_{p^r,η_s}. One can then compare the group cohomology and the étale cohomology via the following commutative diagrams:

$$\begin{array}{ccc} H^1_{\text{ét}}(Y(ms)_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) & \xrightarrow{\mathrm{pr}_{1\star}} & H^1_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) \\ \cong\downarrow & & \downarrow\cong \\ H^1(\Gamma(ms), \mathcal{F}) & \xrightarrow{\mathrm{cor}} & H^1(\Gamma(m), \mathcal{F}) \end{array} \qquad \begin{array}{ccc} H^1_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) & \xrightarrow{\mathrm{pr}_1^\star} & H^1_{\text{ét}}(Y(ms)_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) \\ \cong\downarrow & & \downarrow\cong \\ H^1(\Gamma(m), \mathcal{F}) & \xrightarrow{\mathrm{res}} & H^1(\Gamma(ms), \mathcal{F}) \end{array}$$

For $u_s = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \in \Sigma_0(p)$, we have the following commutative diagram, and we let $\mathrm{pr}_{2\star}$ be the composition of maps in its lower line:

$$\begin{array}{ccccccc} H^1_{\text{ét}}(Y(ms)_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) & \xrightarrow{\lambda_{s\star}} & H^1_{\text{ét}}(Y(ms)_{\bar{\mathbb{Q}}}, \varphi_s^\star(\boldsymbol{\mathcal{F}})) & \xrightarrow{\varphi_{s\star}} & H^1_{\text{ét}}(Y(1(s), Nm(p))_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) & \xrightarrow{\tilde{\nu}_{s\star}} & H^1_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) \\ \cong\downarrow & & \downarrow\cong & & \cong\downarrow & & \cong\downarrow \\ H^1(\Gamma(ms), \mathcal{F}) & \xrightarrow{\lambda_{s\star}} & H^1(\Gamma(ms), \varphi_s^\star(\mathcal{F})) & \xrightarrow{\varphi_{s\star}} & H^1(\Gamma(1(s), Nm(p)), \mathcal{F}) & \xrightarrow{\mathrm{cor}} & H^1(\Gamma(m), \mathcal{F}) \end{array}$$

Similarly for $l_s = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \in \Sigma_0(p)$, we have the following commutative diagram, and we let $\mathrm{pr}_2^\star$ be the composition of maps in its lower line:

$$\begin{array}{ccccccc} H^1_{\text{ét}}(Y(m)_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) & \xrightarrow{\tilde{\nu}_s^\star} & H^1_{\text{ét}}(Y(1(s), Nm(p))_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) & \xrightarrow{\varphi_s^\star} & H^1_{\text{ét}}(Y(ms)_{\bar{\mathbb{Q}}}, \varphi_s^\star(\boldsymbol{\mathcal{F}})) & \xrightarrow{\lambda_s^\star} & H^1_{\text{ét}}(Y(ms)_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) \\ \cong\downarrow & & \cong\downarrow & & \downarrow\cong & & \cong\downarrow \\ H^1(\Gamma(m), \mathcal{F}) & \xrightarrow{\mathrm{res}} & H^1(\Gamma(1(s), Nm(p)), \mathcal{F}) & \xrightarrow{\varphi_s^\star} & H^1(\Gamma(ms), \varphi_s^\star(\mathcal{F})) & \xrightarrow{\lambda_s^\star} & H^1(\Gamma(ms), \mathcal{F}) \end{array}$$

In these diagrams:

1. $\varphi_s^\star(\mathcal{F})$ is $\mathcal{F}$ with the action of $\Gamma_0(p^r\,\mathbb{Z}_p)$ conjugated by u_s.

2. $\lambda_{s\star}$ is induced by $\mathcal{F} \to \varphi_s^\star(\mathcal{F})$: $c \mapsto u_s \cdot c$.

3. $\varphi_{s\star}$ is induced from: $\Gamma(1(s), Nm(p)) \to \Gamma(1, Nm(ps)) \to \Gamma(ms)$, $\gamma \mapsto u_s^{-1}\gamma u_s$; and $\varphi_s^\star(\mathcal{F}) \to \mathcal{F}$, $c \mapsto c$.

4. $\lambda_s^\star$ is induced from $\varphi_s^\star(\mathcal{F}) \to \mathcal{F}$: $c \to l_s \cdot c$.

5. $\varphi_s^\star$ is induced by: $\Gamma(ms) \to \Gamma(1(s), Nm(p))$, $\gamma \to l_s^{-1}\gamma l_s$; $\mathcal{F} \to \varphi_s^\star(\mathcal{F})$, $c \mapsto c$.

It can be shown that $\deg(\mu_q)T_q = \mathrm{pr}_{1\star} \circ \mathrm{pr}_2^\star$ and $\deg(\mu_q)T_q' = \mathrm{pr}_{2\star} \circ \mathrm{pr}_1^\star$, i.e. under (3.2.3.1) the covariant (left) action of T_q, T_q' on the étale cohomology corresponds to the covariant action of T_q, T_q' on the group cohomology respectively. Similar correspondences hold for $\langle d\rangle$, $\langle d\rangle'$ and $[d]_N$, $[d]_N'$. Now the anti-involution ι turns a left (resp. right) action of $\Sigma_0(p)$ into a right (resp. left) action of $\Sigma_0'(p)$, i.e. for any $\mathcal{F} \in \mathbf{M}(\Gamma_0(p\,\mathbb{Z}_p))$ whose right action of $\Gamma_0(p\,\mathbb{Z}_p)$ extends to a right action of $\Sigma_0'(p)$ there is an isomorphism:

$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) \cong H^1(\Gamma, \mathcal{F}).$$

The contravariant (right) actions of $T_q, T_q', \langle d\rangle, \langle d\rangle'$ on the $H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{F}})$ correspond to the contravariant actions of $T_q, T_q', [d]_N, [d]_N'$ on the $H^1(\Gamma, \mathcal{F})$.

Denote

$$P(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \right\}, \quad P'(\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & 0 \\ pc & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \right\}.$$

As $\Gamma_0(p\,\mathbb{Z}_p)$ acts transitively on T' and $P(\mathbb{Z}_p)$ is the stabilizer of $(0,1)$, we can identify T' with $P(\mathbb{Z}_p)\backslash\Gamma_0(p\,\mathbb{Z}_p)$. Similarly we can identify T with $P'(\mathbb{Z}_p)\backslash\Gamma_0(p\,\mathbb{Z}_p)$. Let

$$\Gamma_1(p^j\,\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \text{ s.t. } c \equiv 0 \pmod{p^j}, d \equiv 1 \pmod{p^j} \right\}$$

$$\Gamma_1'(p^j\,\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \text{ s.t. } a \equiv 1 \pmod{p^j}, b \equiv 0 \pmod{p^{j-1}} \right\}$$

for any $j \in \mathbb{Z}_{\geq 1}$.

For any $i \in \mathbb{Z}_{\geq 1}$, we define the following left $\mathcal{O}[\Sigma_0^{\cdot}(p)]$-modules:

$$\mathcal{A}'_{\nu,i,j} = \left\{ f : \Gamma_1(p^j\,\mathbb{Z}_p)\backslash\Gamma_0(p\,\mathbb{Z}_p) \to \mathcal{O}/\,\mathfrak{m}^i \text{ s.t. } \begin{array}{l} f(a\cdot\gamma) = \nu(a)\cdot f(\gamma),\ \forall a \in \mathbb{Z}_p^{\times}, \\ \text{and } \gamma \in \Gamma_1(p^j\,\mathbb{Z}_p)\backslash\Gamma_0(p\,\mathbb{Z}_p) \end{array} \right\},$$

$$\mathcal{A}_{\nu,i,j} = \left\{ f : \Gamma'_1(p^j\,\mathbb{Z}_p)\backslash\Gamma_0(p\,\mathbb{Z}_p) \to \mathcal{O}/\,\mathfrak{m}^i \text{ s.t. } \begin{array}{l} f(a\cdot\gamma) = \nu(a)\cdot f(\gamma),\ \forall a \in \mathbb{Z}_p^{\times}, \\ \text{and } \gamma \in \Gamma'_1(p^j\,\mathbb{Z}_p)\backslash\Gamma_0(p\,\mathbb{Z}_p) \end{array} \right\}.$$

Let $\mathcal{A}^{\cdot}_{\nu,i} = \varinjlim_j \mathcal{A}^{\cdot}_{\nu,i,j}$ and $\mathcal{A}^{\cdot}_{\nu} = \varprojlim_i \mathcal{A}^{\cdot}_{\nu,i}$. Denote by $\boldsymbol{\mathcal{A}}^{\cdot}_{\nu} \in \mathbf{S}(Y_{\text{ét}})$ the object corresponding to $\{\mathcal{A}^{\cdot}_{\nu,i}\}_i \in \mathbf{M}(\Gamma_0(p\,\mathbb{Z}_p))$. Define the right $\mathcal{O}[\Sigma_0^{\cdot}(p)]-$modules $\mathcal{D}^{\cdot}_{\nu,i} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}^{\cdot}_{\nu,i,i}, \mathcal{O}/\,\mathfrak{m}^i)$ and $\mathcal{D}^{\cdot}_{\nu} = \varprojlim_i \mathcal{D}^{\cdot}_{\nu,i}$. Denote by $\boldsymbol{\mathcal{D}}^{\cdot}_{\nu} \in \mathbf{S}(Y_{\text{ét}})$ the object corresponding to $\{\mathcal{D}^{\cdot}_{\nu,i} \in \mathbf{M}(\Gamma_0(p\,\mathbb{Z}_p))\}_i$. There are natural morphisms of $\mathcal{O}-$modules:

$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\nu}) \to \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\nu}) \cong H^1(\Gamma, \mathcal{A}^{\cdot}_{\nu})$$
$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\nu}) \to \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\nu}) \cong H^1(\Gamma, \mathcal{D}^{\cdot}_{\nu})$$
$$H^1_{\text{ét},c}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\nu}) \cong \mathtt{H}^1_{\text{ét},c}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\nu}) \cong H^1_c(\Gamma, \mathcal{D}^{\cdot}_{\nu})$$

which are Hecke equivariant, where $H^j_c(\Gamma, -) = H^{j-1}(\Gamma, \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), -))$. These isomorphisms allow us to define continuous Galois actions on the group cohomology.

For a character $\chi : \mathbb{Z}_p^{\times} \to \mathcal{O}^{\times}$, let $\mathcal{O}(\chi)$ be the module $\mathcal{O}$ with an action $\chi \circ \det$ of $\Gamma_0(p\,\mathbb{Z}_p)$. The natural $\mathcal{G}-$equivariant map $\mathcal{A}^{\cdot}_{\nu} \otimes_{\mathcal{O}} \mathcal{D}^{\cdot}_{\nu} \to \mathcal{O}$ gives a Galois equivariant cup-product pairing:

$$H^1(\Gamma, \mathcal{A}^{\cdot}_{\nu}) \otimes_{\mathcal{O}} H^1_c(\Gamma, \mathcal{D}^{\cdot}_{\nu}) \to \mathcal{O}(-1) \tag{3.2.3.2}$$

under which the covariant Hecke action on the left is adjoint to the same operators acting contravariantly on the right. Let $\det : T' \times T \to \mathbb{Z}_p^{\times}$ where $((a,b),(c,d)) \mapsto$

$ad - bc$ and $\det_\nu = \nu \circ \det$. Evaluation at this function gives a $\mathcal{G}-$equivariant map $\mathcal{D}'_\nu \otimes \mathcal{D}_\nu \to \mathcal{O}(-\nu)$ and hence induces a Galois equivariant cup-product pairing:

$$H^1(\Gamma, \mathcal{D}'_\nu) \otimes_{\mathcal{O}} H^1_c(\Gamma, \mathcal{D}_\nu) \to \mathcal{O}(\boldsymbol{\nu})(-1), \tag{3.2.3.3}$$

where $\boldsymbol{\nu} = \nu \circ \epsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \mathcal{O}^\times$, under which the contravariant Hecke operators (e.g. T_q) on the left are adjoint to the contravariant Hecke operators (e.g. T'_q) on the right and vice versa.

3.2.4 Ordinary cohomology

For any $\mathbb{Z}_p-$algebra B, denote by $S_r(B)$ the set of homogeneous polynomials of degree r in $B[x, y]$. We let $\Sigma_{\dot{0}}(p)$ act on the left $S_r(B)$ by: $(g \cdot P)(x, y) = P((x, y) \cdot g)$. Corresponding to the $p-$adic $\Gamma_0(p\mathbb{Z}_p)-$representation $S_r = S_r(\mathbb{Z}_p)$ is a locally constant $p-$adic sheaf $\mathscr{S}_r$ on $Y_{\text{ét}}$ (see (2.1.3.1)). There is then the following Hecke equivariant isomorphism:

$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \mathscr{S}_r) \cong H^1(\Gamma, S_r)$$

with the Hecke operators acting covariantly on both sides. Via this isomorphism, we define a Galois action on the group cohomology.

Dually, let $L_r(B) = \mathrm{Hom}_B(S_r(B), B)$ and let $\Sigma_{\dot{0}}(p)$ act on the right of $L_r(B)$ by: $(H \cdot \gamma)(P(x, y)) = H(\gamma \cdot P(x, y))$ where $H \in L_r(B)$, $\gamma \in \Sigma_{\dot{0}}(p)$ and $P \in S_r(B)$. Corresponding to the $p-$adic $\Gamma_0(p\mathbb{Z}_p)-$representation $L_r = L_r(\mathbb{Z}_p)$ is a locally constant $p-$adic sheaf $\mathscr{L}_r$ on $Y_{\text{ét}}$ (see (2.1.3.1)), and there is a Hecke equivariant isomorphism:

$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \mathscr{L}_r) \cong H^1(\Gamma, L_r)$$

with the Hecke operators acting contravariantly on both sides. Using this, we can define a Galois action on the group cohomology.

The evaluation map $S_r \otimes L_r \to \mathbb{Z}_p$ is $\Gamma_0(p\mathbb{Z}_p)-$equivariant and induces a Galois equivariant pairing:

$$H^1(\Gamma, S_r) \otimes_{\mathbb{Z}_p} H^1_c(\Gamma, L_r) \to \mathbb{Z}_p(-1) \tag{3.2.4.1}$$

which becomes perfect after inverting p. Here the covariant Hecke operators on the left are adjoint to the contravariant Hecke operator on the right.

Denote by $\nu_r : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ the character where $z \mapsto z^r$. Evaluation at $(x_1y_2 - x_2y_1)^r \in S_r \otimes S_r$ defines a $\Gamma_0(p\mathbb{Z}_p)-$equivariant map $L_r \otimes L_r \to \mathbb{Z}_p(-\nu_r)$, and hence induces a Galois equivariant pairing:

$$H^1(\Gamma, L_r) \otimes H^1_c(\Gamma, L_r) \to \mathbb{Z}_p(r-1) \tag{3.2.4.2}$$

which becomes perfect after inverting p. Here the contravariant Hecke operators (e.g. T_q) on the left are adjoint to the contravariant Hecke operators (e.g. T'_q) on the right.

From (3.2.4.1) and (3.2.4.2), one can define a Galois equivariant morphism:

$$s_{r\star} : H^1(\Gamma, S_r(\mathbb{Q}_p)) \to H^1(\Gamma, L_r(\mathbb{Q}_p))(-r) \tag{3.2.4.3}$$

which intertwines the covariant Hecke operators on the left (e.g. T_q) with the contravariant Hecke operators (e.g. T'_q) on the right. Notice that one can also define $s_{r\star}$ directly via $S_r(\mathbb{Q}_p) \cong L_r(\mathbb{Q}_p)(\nu_r)$ (see equation 2.5.0.4): the denominators appeared are bounded by $r!$, i.e. $s_{r\star}(\mathrm{im}(H^1(\Gamma, S_r) \to H^1(\Gamma, S_r(\mathbb{Q}_p)))) \subset \mathrm{im}((H^1(\Gamma, L_r) \to H^1(\Gamma, L_r(\mathbb{Q}_p))))/r!$, by Remark 3.3 in [BSV21].

By viewing two variable polynomials as functions on $T^{\cdot}$ we obtain a morphism of left $\mathbb{Z}_p[\Sigma_0^{\cdot}(p)]-$modules : $S_r \to \mathcal{A}^{\cdot}_{\nu_r}$. Dually, we also have a morphism of right $\mathbb{Z}_p[\Sigma_0^{\cdot}(p)]-$modules: $\mathcal{D}^{\cdot}_{\nu_r} \to L_r$. These induce Hecke and Galois equivariant morphisms:

$$H^1(\Gamma, S_r) \to H^1(\Gamma, \mathcal{A}^{\cdot}_{\nu_r}), \quad H^1(\Gamma, \mathcal{D}^{\cdot}_{\nu_r}) \to H^1(\Gamma, L_r)$$

By applying Hida's (anti-)ordinary projector $e^{\cdot}_{\mathrm{ord}} := \lim_{n\to\infty}(T^{\cdot}_p)^{n!}$, these morphisms become isomorphisms:

$$e^{\cdot}_{\mathrm{ord}}H^1(\Gamma, S_r) \cong e^{\cdot}_{\mathrm{ord}}H^1(\Gamma, \mathcal{A}^{\cdot}_{\nu_r}), \quad e^{\cdot}_{\mathrm{ord}}H^1(\Gamma, \mathcal{D}^{\cdot}_{\nu_r}) \cong e^{\cdot}_{\mathrm{ord}}H^1(\Gamma, L_r)$$

Note that the pairings (3.2.4.1) and (3.2.4.2) correspond to the pairings (3.2.3.2) and (3.2.3.3) respectively under these isomorphisms.

3.2.5 $\Lambda-$adic Poincaré pairing

For $d \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$ such that $d \equiv a \pmod N$ and $d \equiv b \pmod{p^r}$, the diamond operator $\langle d\rangle$ will be written as $\langle a; b\rangle$. We write $\epsilon_N : G_{\mathbb{Q}} \to (\mathbb{Z}/N\mathbb{Z})^\times$ for the mod N cyclotomic character (factoring through $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$). Define for each $s \in \mathbb{Z}_{\geq 1}$:

$$G_s = 1 + p(\mathbb{Z}/p^s\mathbb{Z}) \subset \Gamma_s := (\mathbb{Z}/p^s\mathbb{Z})^\times$$

together with its $\mathbb{Z}_p-$coefficients associated group rings:

$$\Lambda_s = \mathbb{Z}_p[G_s] \hookrightarrow \tilde{\Lambda}_s = \mathbb{Z}_p[\Gamma_s], \quad \Lambda = \varprojlim_s \Lambda_s = \mathbb{Z}_p[\![1 + p\,\mathbb{Z}_p]\!] \hookrightarrow \tilde{\Lambda} = \varprojlim_s \tilde{\Lambda}_s = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!].$$

For each $i \in (\mathbb{Z}/(p-1)\mathbb{Z})$, define idemopotents:

$$e_i = \frac{1}{p-1}\sum_{\zeta\in\mu_{p-1}} \zeta^{-i}[\zeta] \in \tilde{\Lambda}.$$

Let

$$\kappa_i : \mathbb{Z}_p^\times \to \Lambda^\times, \quad \kappa_i(z) = \omega^i(z)[\langle z\rangle], \tag{3.2.5.1}$$

and $\boldsymbol{\kappa}_i = \kappa_i \circ \epsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \Lambda^\times$.

Dropping the 1 and N, we put $X(p^s m) = X(1, Np^s m)$ and let:

$$H^1_{\text{ét}}(X_\infty(m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) = \varprojlim_s H^1_{\text{ét}}(X(p^s m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p).$$

There is a natural action of $\tilde{\Lambda}_s$ and $\tilde{\Lambda}$ on the cohomology with the group element $[\alpha]$ acting as the diamond operator $\langle 1; \alpha\rangle'$.

Following [DR17, Sec 1], we fix a norm-compatible collection $\{\zeta_{p^s}\}_{s\geq 1}$ of primitive roots of unity of $p-$power order, and then similarly define Atkin-Lehner automorphisms ω_{p^s} and ω for the curve $X(p^s m)$. These actions satisfy the following relation:

$$\omega_{p^s}^\sigma = \langle 1; \epsilon_{\text{cyc}}(\sigma)\rangle \omega_{p^s}, \quad \omega^\sigma = \langle \epsilon_N(\sigma); 1\rangle \omega, \quad \text{for } \sigma \in G_{\mathbb{Q}},$$

and we let them act on cohomology via pullback.

The Galois equivariant pairing:

$$\langle ., . \rangle_{G_s} : e_i H^1_{\text{ét}}(X(p^s m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \times e_{-i} H^1_{\text{ét}}(X(p^s m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \to \Lambda_s(-1)$$
$$\langle \theta, \delta \rangle_{G_s} \mapsto \sum_{\sigma \in G_s} \langle \theta^\sigma, \delta \rangle \sigma^{-1},$$

where $\langle ., . \rangle_{G_s}$ on the left hand side is the natural Poincaré pairing, is linear in the first and anti-linear in the second argument. The modifying pairing

$$[\theta, \delta]_{G_s} = \langle \theta, \omega\omega_{p^r} \cdot (T_p')^r \cdot \delta \rangle_{G_s}$$

is Galois equivariant and Λ_s-linear in both its arguments:

$$[., .]_{G_s} : e_i H^1_{\text{ét}}(X(p^s m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \times e_i H^1_{\text{ét}}(X(p^s m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)(\langle \epsilon_N^{-1}; 1\rangle') \to \Lambda_s(\boldsymbol{\kappa}_i)(-1).$$

The pairing $[., .]_{G_s}$ are compatible with the $\text{pr}_{1\star}$ map in the sense that the following diagram commutes:

$$\begin{array}{ccc} e_iH^1_{\text{ét}}(X(p^{s+1}m)_{\bar{\mathbb{Q}}},\mathbb{Z}_p)\times e_iH^1_{\text{ét}}(X(p^{s+1}m)_{\bar{\mathbb{Q}}},\mathbb{Z}_p)(\langle\epsilon_N^{-1};1\rangle') & \xrightarrow{[,]_{G_{s+1}}} & \Lambda_{s+1}(\boldsymbol{\kappa}_i)(-1) \\ \downarrow{\scriptstyle \mathrm{pr}_{1\star}\times\mathrm{pr}_{1\star}} & & \downarrow \\ e_iH^1_{\text{ét}}(X(p^{s}m)_{\bar{\mathbb{Q}}},\mathbb{Z}_p)\times e_iH^1_{\text{ét}}(X(p^{s}m)_{\bar{\mathbb{Q}}},\mathbb{Z}_p)(\langle\epsilon_N^{-1};1\rangle') & \xrightarrow{[,]_{G_{s}}} & \Lambda_{s}(\boldsymbol{\kappa}_i)(-1) \end{array}$$

Taking the inverse limits, this yields a perfect Galois-equivariant $\Lambda-$adic pairing:

$$e_iH^1_{\text{ét}}(X_\infty(m)_{\bar{\mathbb{Q}}},\mathbb{Z}_p)^{\text{ord}}\times e_iH^1_{\text{ét}}(X_\infty(m)_{\bar{\mathbb{Q}}},\mathbb{Z}_p)^{\text{ord}}(\langle\epsilon_N^{-1};1\rangle')\to\Lambda_s(\boldsymbol{\kappa}_i)(-1) \qquad (3.2.5.2)$$

where $H^1(-)^{\text{ord}} = e'_{\text{ord}}H^1(-)$. The Hecke operators are all self-adjoint under this pairing.

3.2.6 The big Galois representation

We upgrade the constructions in the previous Sections 3.2.3 and 3.2.4, from $\mathcal{O}$-modules to $\Lambda-$modules.

Let $\mathfrak{m}_\Lambda$ be the maximal ideal of Λ. Let $\mathrm{Ct}(\mathbb{Z}_p,\Lambda)$ be the space of continuous functions from $\mathbb{Z}_p$ to Λ. Let κ be one of the $\kappa_i : \mathbb{Z}_p^\times \to \Lambda^\times$ (so $\kappa(z) = \omega^i(z)[\langle z\rangle]$). Define the following $\Lambda-$modules equipped with the $\mathfrak{m}_\Lambda-$adic topology:

$$\mathcal{A}'_\kappa = \{f : T' \to \Lambda \text{ s.t. } f(pz,1)\in\mathrm{Ct}(\mathbb{Z}_p,\Lambda);\ f(a\cdot t)=\nu(a)f(t)\ \forall\ a\in\mathbb{Z}_p^\times,\ t\in T'\}.$$

Similarly we define its $\Lambda-$dual: $\mathcal{D}'_\kappa = \mathrm{Hom}_{\mathrm{ct}}(\mathcal{A}'_\kappa,\Lambda)$. These are also $\Lambda[\Sigma_0(p)']$-modules.

For any $i,j\in\mathbb{Z}_{\geq 1}$, we define the following left $\Lambda[\Sigma'_0(p)]$-modules:

$$\mathcal{A}'_{\kappa,i,j} = \left\{ f:\Gamma_1(p^j\mathbb{Z}_p)\backslash\Gamma_0(p\mathbb{Z}_p)\to\Lambda/\mathfrak{m}_\Lambda^i \text{ s.t. } \begin{array}{l} f(a\cdot\gamma)=\kappa(a)\cdot f(\gamma),\forall\ a\in\mathbb{Z}_p^\times \\ \text{and } \gamma\in\Gamma_1(p^j\mathbb{Z}_p)\backslash\Gamma_0(p\mathbb{Z}_p) \end{array} \right\},$$

Let $\mathcal{A}'_{\kappa,i} = \varinjlim_j \mathcal{A}'_{\kappa,i,j}$ and $\mathcal{A}'_\kappa = \varprojlim_i \mathcal{A}'_{\kappa,i}$. Let $\boldsymbol{\mathcal{A}}'_\kappa \in \mathbf{S}(Y_{\text{ét}})$ be the object corresponding to $\{\mathcal{A}'_{\kappa,i}\}_i \in \mathbf{M}(\Gamma_0(p\,\mathbb{Z}_p))$. Define the right $\Lambda[\Sigma'_0(p)]$-modules $\mathcal{D}'_{\kappa,i} = \mathrm{Hom}_\Lambda(\mathcal{A}'_{\kappa,i,i}, \Lambda\,/\,\mathfrak{m}^i_\Lambda)$ and $\mathcal{D}'_\kappa = \varprojlim_i \mathcal{D}'_{\kappa,i}$. The object $\boldsymbol{\mathcal{D}}'_\kappa \in \mathbf{S}(Y_{\text{ét}})$ corresponds to $\{\mathcal{D}'_{\kappa,i} \in \mathbf{M}(\Gamma_0(p\,\mathbb{Z}_p))\}_i$.

There are natural morphisms of $\Lambda-$modules:

$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}'_\kappa) \to \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}'_\kappa) \cong H^1(\Gamma, \mathcal{A}'_\kappa),$$

$$H^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}'_\kappa) \to \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}'_\kappa) \cong H^1(\Gamma, \mathcal{D}'_\kappa),$$

$$H^1_{\text{ét},c}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}'_\kappa) \cong \mathtt{H}^1_{\text{ét},c}(Y_{\bar{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}'_\kappa) \cong H^1_c(\Gamma, \mathcal{D}'_\kappa),$$

which are Hecke equivariant. These isomorphisms allow us to define continuous Galois actions on the group cohomology. The natural $\mathcal{G}-$equivariant map $\mathcal{A}'_\kappa \otimes \mathcal{D}'_\kappa \to \Lambda$ gives a Galois equivariant cup-product pairing:

$$H^1(\Gamma, \mathcal{A}'_\kappa) \otimes_\Lambda H^1_c(\Gamma, \mathcal{D}'_\kappa) \to \Lambda(-1) \tag{3.2.6.1}$$

under which the covariant actions of Hecke operators on the left are adjoint to the same operators acting contravariantly on the right.

Recall the notation $\Gamma = \Gamma(1, Nm(p))$. Let $S = \Sigma'_0(p) \cap \mathrm{GL}_2(\mathbb{Q})$ and for $r \in \mathbb{Z}_{\geq 1}$ define:

$$\Sigma'_1(p^r) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^r\,\mathbb{Z}_p & 1 + p^r\,\mathbb{Z}_p \end{pmatrix}, \quad S_r = \Sigma'_1(p^r) \cap \mathrm{GL}_2(\mathbb{Q}), \quad \Gamma_r = \Gamma(1, Nm(p^r)).$$

We say that the Hecke pair (Γ_r, S_r) is compatible to the Hecke pair (Γ_s, S_s) [AS86a, Sec 1.1] if $(\Gamma_r, S_r) \subset (\Gamma_s, S_s)$, $S_r\,\Gamma_s = S_s$ and $\Gamma_s \cap S_r^{-1} S_r = \Gamma_r$ (note the changing left-right conventions). If Γ_r has finite index in Γ_s then for any S_r-module M we define the induction, equipped with a right action of S_s:

$$\mathrm{Ind}_{\Gamma_r}^{\Gamma_s} M = \{\phi : \Gamma_s \to M \text{ s.t. } \phi(xy) = \phi(y)x^{-1}\ \forall\, x \in \Gamma_r, y \in \Gamma_s\}$$
$$(\phi \cdot g)(x) = \sum_{\gamma \in \Gamma_r \backslash \Gamma \cap S_r x g^{-1}} \phi(\gamma)\gamma g x^{-1}, \forall\, \phi \in \mathrm{Ind}_{\Gamma_r}^{\Gamma_s} M,\ g \in S_r.$$

Note that by definition, the Hecke pair (Γ_r, S_r) is compatible to the Hecke pair (Γ_s, S_s) if $r \geq s$ and also to the Hecke pair (Γ, S). Let

$$A'_{\kappa,s} = \left\{ f : \Gamma_1(p^s\,\mathbb{Z}_p)\backslash\Gamma_0(p\,\mathbb{Z}_p) \to \Lambda_s \text{ s.t. } \begin{array}{l} f(a\cdot\gamma) = \kappa(a)\cdot f(\gamma),\ \forall a \in \mathbb{Z}_p^{\times}, \\ \text{and } \gamma \in \Gamma_1(p^s\,\mathbb{Z}_p)\backslash\Gamma_0(p\,\mathbb{Z}_p) \end{array} \right\},$$

and $D'_{\kappa,r} = \mathrm{Hom}(A'_{\kappa,r}, \Lambda_r)$. One obtains $\mathcal{D}'_{\kappa} = \varprojlim_r D'_{\kappa,r}$.

Let S_r act trivially on $\mathbb{Z}_p$, and consider the right $\mathbb{Z}_p[S_1]-$module $\mathrm{Ind}_{\Gamma_r}^{\Gamma_1}\mathbb{Z}_p$. The map

$$\mathrm{Ind}_{\Gamma_r}^{\Gamma_1}\mathbb{Z}_p \to D'_{\kappa,r} : \phi \mapsto [f \mapsto \sum_{r \in \Gamma_r \backslash \Gamma_1} \phi(r) f(r)]$$

is an isomorphism of right $\mathbb{Z}_p[S_1]-$modules, hence induces the natural isomorphisms:

$$H^1(\Gamma_1, \mathcal{D}'_{\kappa}) \cong \varprojlim_r H^1(\Gamma_1, D'_{\kappa,r}) \cong \varprojlim_r H^1(\Gamma_r, \mathbb{Z}_p),$$

which are Hecke equivariant (following [AS86a], both corestriction and the Shapiro map commute with the action of (Γ, S) via restriction of Hecke algebras).

Denote

$$H^1_{\text{ét}}(Y_\infty(m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) = \varprojlim_r H^1_{\text{ét}}(Y(1, Np^r m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p),$$

where the inverse limit is with respect to $\mathrm{pr}_{1\star}$. Then

$$H^1(\Gamma_1, \mathcal{D}'_{\kappa}) \cong H^1_{\text{ét}}(Y_\infty(m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \qquad (3.2.6.2)$$

where we use: $\varprojlim_r H^1_{\text{ét}}(Y(1, Np^r(m))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \cong \varprojlim_r H^1(\Gamma_r, \mathbb{Z}_p)$ (one needs to choose a compatible system of geometric points for $Y(1, Np^r(m))$ and suitably compatible bases for the corresponding Tate modules). Here, the contravariant Hecke operators (e.g. T'_q, $[d]'_N$, $[a]'_p$) on the left correspond to the contravariant Hecke operators (e.g. $T'_q, \langle d; 1\rangle', \langle 1; a\rangle'$) on the right (defined via the compatibility with $\mathrm{pr}_{1\star}$). Since $\kappa = \kappa_i$, the restriction maps yield a Hecke equivariant isomorphism:

$$H^1(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1(\Gamma_1, \mathcal{D}'_\kappa)$$

i.e. one obtains from (3.2.6.2):

$$H^1(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1_{\text{ét}}(Y_\infty(m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \tag{3.2.6.3}$$

and also

$$H^1_c(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1_{\text{ét},c}(Y_\infty(m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p) \tag{3.2.6.4}$$

using [AS86b, Prop 4.2].

3.2.7 Proof of the Λ −adic tame norm relations

Following [ACR21, Sec 6] we adopt the constructions in [BSV21, Sec 8], which applies to three Hida families $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$. Then we specialize to f a newform, $\mathbf{g}$ a CM Hida family, and h a CM form attached to a Hecke character of infinity type $(-1, 0)$.

We first recall the set-up:

Set-up.

1. $k, l \geq 2$ are positive even integers.

2. $f \in S_k(\Gamma_0(N_f))$ is a newform, ordinary at p.

3. K is an imaginary quadratic field.

4. ψ_1, ψ_2 are two Hecke characters over K of infinity type $(1-l, 0)$, $(-1, 0)$ with conductors $\mathfrak{f}_1$, $\mathfrak{f}_2$ respectively. As recalled in Section 2.4.1, one can associate with ψ_1 and ψ_2 two theta series $\theta_{\psi_1} \in S_l(N_{\psi_1}, \chi_{\psi_1})$ and $\theta_{\psi_2} \in S_2(N_{\psi_2}, \chi_{\psi_2})$.

5. We do not need to assume that ψ_1 and ψ_2 satisfy condtion (†) because p will be chosen to be coprime with $\mathfrak{f}_1\mathfrak{f}_2$.

6. We assume that $\chi_{\psi_1}\chi_{\psi_2} = 1$.

7. $N = \mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})$.

8. $p \geq 5$, is a prime such that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and $(p, h_K\mathfrak{f}_1\mathfrak{f}_2) = 1$, as in Section 2.4. Let L/K be a finite extension, large enough so that its ring of integers contains the Fourier coefficients of $f, \theta_{\psi_1}, \theta_{\psi_2}$. Fix primes $\mathfrak{P}|\mathfrak{p}|p$ of $L/K/\mathbb{Q}$ respectively and let $\mathcal{O} \subset L_{\mathfrak{P}}$ be its ring of integers.

9. $\Gamma_{\mathfrak{p}}$ is the unique $\mathbb{Z}_p$ extension of K unramified outside $\mathfrak{p}$. Following [BL18, Sec 3], denoted by ψ_0 the unique Hecke character of infinity type $(-1, 0)$ of conductor $\mathfrak{p}$ whose $p-$adic avatar factors through $\Gamma_{\mathfrak{p}}$. Then ψ_1 can be written uniquely as $\alpha\psi_0^{l-1}$ where α is a ray class character of conductor dividing $\mathfrak{f}_1\mathfrak{p}$ (using the fact that the quotient of these two characters has finite $p-$power order with conductor dividing $\mathfrak{p}$ and that $p \nmid h_K$).

10. $H_{\mathfrak{f}_1\mathfrak{p}^\infty}$ is the maximal pro$-p$ quotient of the ray class group of K of conductor $\mathfrak{f}_1\mathfrak{p}^\infty$. Let $[\mathfrak{a}]$ be the image of $\mathfrak{a}$ in $H_{\mathfrak{f}_1\mathfrak{p}^\infty}$ under the geometric Artin map. Note that $p \nmid h_K$ implies that $H_{\mathfrak{f}_1\mathfrak{p}^\infty} \cong H_{\mathfrak{f}_1}^{(p)} \times \Gamma_{\mathfrak{p}}$.

The formal $q-$expansion [LLZ15, Sec 6.2]

$$\Theta = \sum_{(\mathfrak{a}, \mathfrak{f}_1\mathfrak{p}=1)} [\mathfrak{a}] q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \mathcal{O}[\![H_{\mathfrak{f}_1\mathfrak{p}^\infty}]\!][\![q]\!]$$

can be specialized to

$$\mathbf{g} = \sum_{(\mathfrak{a},\mathfrak{f}_1\mathfrak{p}=1)} \alpha(\mathfrak{a})\psi_0(\mathfrak{a})[\mathfrak{a}]q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \mathcal{O}[\![\Gamma_\mathfrak{p}]\!][\![q]\!] =: \Lambda_\mathbf{g}[\![q]\!].$$

Under the identification of $\Gamma_\mathfrak{p}$ with $\Gamma = 1 + p\mathbb{Z}_p$ (via $\Gamma \cong \mathcal{O}_{K,\mathfrak{p}}^{(1)} \to \Gamma_\mathfrak{p}$: $s \mapsto \mathrm{art}_\mathfrak{p}(s)^{-1}$), one can view $\mathbf{g}$ as a primitive Hida family passing through the ordinary $p-$stabilization of θ_{ψ_1}. Explicitly, for a general Hecke character $\psi = \alpha\psi_0^{k_0-1}$ such that $\psi((m)) = m^{k_0-1}\chi(m)$ for all integers $(m, N_{K/\mathbb{Q}}(\mathfrak{f})) = 1$, one has $\chi = \alpha\omega^{1-k_0}$ and

$$\mathbf{g}_{k_0} = \sum_{(\mathfrak{a},\mathfrak{f}_1\mathfrak{p}=1)} \alpha(\mathfrak{a})\psi_0^{k_0-1}(\mathfrak{a})q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in S_{k_0}^{\mathrm{ord}}(N_\psi p, \alpha\omega^{1-k_0}\epsilon_K).$$

Let $\chi_\mathbb{Q}$ be the adelic character attached to χ, $\chi_K = \chi_\mathbb{Q} \circ N_{K/\mathbb{Q}}$, and $\psi^\star = \chi_K^{-1}\psi$. Note that the Hida family $\mathbf{g}^\star$ attached to $\psi^\star$ (defined similar as above) is just $\mathbf{g} \otimes \chi^{-1}$.

Let $(r_1, r_2, r_3) = (k-2, l-2, 0)$. Let $\kappa = \kappa_{r_2}$ and choose the square root $\kappa^{1/2}$ of this character defined by $\kappa^{1/2}(s) = \omega(s)^{r_2/2}[\langle u\rangle^{1/2}]$. Following [BSV21, Sec 8.1], we obtain a class

$$\kappa_m^0 \in H^0_{\text{ét}}(Y(1, Nm(p)), \boldsymbol{\mathcal{A}}_{r_1} \otimes \boldsymbol{\mathcal{A}}'_\kappa \otimes \boldsymbol{\mathcal{A}}_0(-\kappa^{1/2} - \nu_{r_1/2}))$$

by specializing the Hida families $\mathbf{f}$ and $\mathbf{h}$ to f, θ_{ψ_2} respectively (note the change from working with modules of locally analytic functions in [BSV21] to working with modules of continuous function in [ACR21]). Following the notation of [BSV21], we define:

$$\kappa_m^1 = (e_{\mathrm{ord}} \otimes e'_{\mathrm{ord}} \otimes e_{\mathrm{ord}}) \circ \mathtt{K} \circ \mathtt{HS} \circ d_\star(\mathbf{Det}_m^{f\mathbf{g}h}) \in H^1(\mathbb{Q}, H^1(Y(1, Nm(p)), \mathcal{A}_{r_1})^{\mathrm{ord}} \otimes H^1(Y(1, Nm(p)), \mathcal{A}'_\kappa)^{\mathrm{ord}} \otimes H^1(Y(1, Nm(p)), \mathcal{A}_0)^{\mathrm{ord}}(\boldsymbol{\kappa}^{1/2} + 2 + r_1/2))$$

where $\boldsymbol{\kappa}^{1/2} = \kappa^{1/2} \circ \epsilon_{\mathrm{cyc}}$, and $\mathtt{K}$ comes from the Künneth decomposition (see definitions of $\mathtt{HS}$ and $d_\star$ in Section 2.5).

Proposition 3.2.1. *For a prime number q and a positive integer m if $(mq, pN) = 1$ then*

$$(pr_{i\star}, pr_{j\star}, pr_{k\star})\kappa^1_{mq} = (\bigstar)\kappa^1_m$$

where

(i,j,k)	$\bigstar$
$(q,1,1)$	$(q-1)(T'_q,1,1)$
$(1,q,1)$	$(q-1)(1,T'_q,1)$
$(1,1,q)$	$(q-1)(1,1,T'_q)$
$(1,q,q)$	$(q-1)q^{-r_1/2}\kappa^{1/2}(q)(T_q,1,1)$
$(q,1,q)$	$(q-1)\kappa^{-1/2}(q)q^{r_1/2}(1,T_q,1)$
$(q,q,1)$	$(q-1)\kappa^{1/2}(q)q^{r_1/2}(1,1,T_q)$

If we also have that $(q,m) = 1$ then

(i,j,k)	$\bigstar$
$(1,1,1)$	(q^2-1)
(q,q,q)	$(q^2-1)q^{r_1/2}\kappa^{1/2}(q)$

Proof. See equations (174) and (176) in [BSV21]. □

3.2.8 Another fix

This subsection will largely follow Section 3.1.1 in order to get rid of the unwanted factor $(q-1)$ in Proposition 3.2.1. The pairings in (3.2.6.1) and (3.2.6.4) induce a map:

$$\begin{aligned} H^1(\Gamma(1, Nm(p)), \mathcal{A}'_\kappa) &\to \mathrm{Hom}_\Lambda(H^1_c(\Gamma(1, Nmp), \mathcal{D}'_\kappa), \Lambda)(-1) \\ &\cong \mathrm{Hom}_\Lambda(e_{r_2} H^1_{\text{ét},c}(Y_\infty(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p), \Lambda)(-1). \end{aligned}$$

By localizing at the $p-$ordinary maximal ideal $\mathcal{I}_{\mathfrak{n}}$ of $\mathbb{T}'(1, Nmp^{\infty})_{\text{ord}}$ corresponding to the Hida family $\mathbf{g}^{\star}$ (using condition (†)), one can go back and forth between cohomology of the open and closed curves, and étale cohomology and étale cohomology with compact support:

$$H^1_{\text{ét},c}(Y_\infty, \mathbb{Z}_p)^{\text{ord}}_{\mathcal{I}_{\mathfrak{n}}} \cong H^1_{\text{ét},c}(X_\infty, \mathbb{Z}_p)^{\text{ord}}_{\mathcal{I}_{\mathfrak{n}}} \cong H^1_{\text{ét}}(Y_\infty, \mathbb{Z}_p)^{\text{ord}}_{\mathcal{I}_{\mathfrak{n}}} \tag{3.2.8.1}$$

(see Lemma 3.1.6). Note that this choice is compatible with taking the inverse limit of the map

$$\phi_{\bar{\mathfrak{m}},r} : \mathbb{T}(1, N_{\psi_1} mp^r)'_{\text{ord}} \to \mathcal{O}[H^{(p)}_{\bar{\mathfrak{m}}\mathfrak{p}^r}]$$

attached to $\alpha\psi_0\chi_K^{-1}$ (see Proposition 2.4.1) to get:

$$\phi_{\bar{\mathfrak{m}},\infty} : \mathbb{T}(1, N_{\psi_1} mp^{\infty})'_{\text{ord}} \to \mathcal{O}[H_{\bar{\mathfrak{m}}\mathfrak{p}^\infty}] = \mathcal{O}[H^{(p)}_{\bar{\mathfrak{m}}}] \otimes \mathcal{O}[\![\Gamma_{\mathfrak{p}}]\!].$$

Combining with the pairings (3.2.5.2) and (3.2.6.1), one obtains a morphism:

$$\mathscr{M}_{\mathbf{g}^\star} : H^1(\Gamma(1, Nm(p)), \mathcal{A}'_\kappa)^{\text{ord}} \to e_{r_2} H^1_{\text{ét}}(Y_\infty(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p)^{\text{ord}}_{\mathcal{I}_{\mathfrak{n}}}(\langle \epsilon_N^{-1}; 1\rangle')(-\boldsymbol{\kappa})$$

which is Galois equivariant, where the covariant action of T'_q, $[d]'_N$, $[a]'_p$ on the left corresponding to the contravariant action of T'_q, $\langle d; 1\rangle'$, $\langle 1; a\rangle'$ on the right.

Following Section 3.1.1, we obtain our amended class: $\phi(m)\kappa^2_m = (\mu_{m\star}, d_{m\star})\kappa^1_m$ where

$$\begin{gathered}\kappa^2_m \in H^1(\mathbb{Q}, H^1(Y(1, N(p)), \mathcal{A}_{r_1})^{\text{ord}} \otimes \\ H^1(Y(1, Nm(p)), \mathcal{A}'_\kappa)^{\text{ord}} \otimes_{\mathcal{O}[D_m]} H^1(Y(1, Nm(p)), \mathcal{A}_0)^{\text{ord}}(\boldsymbol{\kappa}^{1/2} + 2 + r_1/2)).\end{gathered} \tag{3.2.8.2}$$

We also have a parallel lemma with Lemma 3.2.1, but getting rid of the $(q-1)$ factor as expected, similar to Lemma 3.1.5.

Important choices. Fix a choice of level N test vector $\breve{\mathbf{g}}$ for $\mathbf{g}$ and let :

$$\breve{f} \in S_k(N, \chi_f)[f], \quad \breve{\mathbf{g}}^\star = \breve{\mathbf{g}} \otimes \chi^{-1}, \quad \breve{h} \in S_2(N, \chi_{\psi_2})[\theta_{\psi_2}].$$

Fix also choices of maps (recall $Y(m) = Y(1, \mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})m)$:

$$\begin{aligned} &H^1_{\text{ét}}(Y(1, N(p))_{\bar{\mathbb{Q}}}, \mathcal{L}_{r_1}(1)) \xrightarrow{\mu_p^\star} H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbb{Q}}}, \mathcal{L}_{r_1}(1)) \to H^1_{\text{ét}}(Y_1(N_f)_{\bar{\mathbb{Q}}}, \mathcal{L}_{r_1}(1)), \\ &H^1_{\text{ét}}(Y_\infty(Nm)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))^{\text{ord}} \to H^1_{\text{ét}}(Y_\infty(N_{\psi_1}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))^{\text{ord}}, \\ &H^1_{\text{ét}}(Y(1, Nm(p))_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\mu_p^\star} H^1_{\text{ét}}(Y(1, Nmp)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \to H^1_{\text{ét}}(Y(1, N_{\psi_2}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)), \end{aligned}$$

which are compatible with $\breve{f}$, $\breve{\mathbf{g}}^\star$, and $\breve{h}$.

Let $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_K)$ such that $m = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to p. Let $\mathfrak{l} \in \mathscr{L}_K$ be a split prime of K such that $l = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{l})$ is coprime to pm. Assume further that $(ml, Np) = 1$. After tensoring with $\mathcal{O}$, we can project (3.2.8) to:

$$\begin{aligned} \kappa^3_{\mathfrak{m}} \in H^1(\mathbb{Q}, T_f^\vee \otimes H^1_{\text{ét}}(Y_\infty(N_{\psi_1}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))(\langle \epsilon_N^{-1}; 1\rangle')(-\boldsymbol{\kappa}^{-1/2}) \otimes_{\phi_{\bar{\mathfrak{m}},\infty}} \mathcal{O}[H^{(p)}_{\bar{\mathfrak{m}}}] \otimes_{\mathcal{O}} \mathcal{O}[\![\Gamma_{\mathfrak{p}}]\!] \\ \otimes_{\mathcal{O}[D_m]} H^1_{\text{ét}}(Y(1, N_{\psi_2}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(1, N_{\psi_2}m)} \mathcal{O}[H^{(p)}_{\mathfrak{m}}](-k/2)). \quad (3.2.8.3) \end{aligned}$$

Using the geometric normalised Artin map, we identify $\Gamma^- = \mathrm{Gal}(K^-_\infty/K)$ with the anti-diagonal in $(1 + p\,\mathbb{Z}_p) \times (1 + p\,\mathbb{Z}_p) \simeq \mathcal{O}^{(1)}_{K,\mathfrak{p}} \times \mathcal{O}^{(1)}_{K,\bar{\mathfrak{p}}}$ and define:

$$\begin{aligned} \kappa_{ac} : \Gamma^- \to \mathbb{Z}_p^\times \quad &\text{where} \quad ((1+p)^{-1/2}, (1+p)^{1/2}) \mapsto (1+p) \\ \boldsymbol{\kappa}_{ac} : \Gamma^- \to \Lambda^\times \quad &\text{where} \quad ((1+p)^{-1/2}, (1+p)^{1/2}) \mapsto [(1+p)]. \end{aligned}$$

We then obtain a Galois equivariant isomorphism of $\Lambda_{\mathcal{O}}[H^{(p)}_{\bar{\mathfrak{m}}}]-$modules:

$$\begin{aligned} H^1_{\text{ét}}(Y_\infty(N_{\psi_1}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))(\langle\epsilon_N^{-1};1\rangle')(-\boldsymbol{\kappa}^{-1/2}) \otimes_{\phi_{\bar{\mathfrak{m}},\infty}} \mathcal{O}[H^{(p)}_{\bar{\mathfrak{m}}}] \otimes_{\mathcal{O}} \mathcal{O}[\![\Gamma_{\mathfrak{p}}]\!] \\ \cong \quad \mathrm{Ind}^{\mathbb{Q}}_{K^p_{\bar{\mathfrak{m}}}} \Lambda_{\mathcal{O}}(\psi^{-1}_{1\mathfrak{P}}\kappa_{ac}^{r_2/2}\boldsymbol{\kappa}_{ac}^{-1/2})(-r_2/2), \end{aligned}$$

see [ACR21, Eq 6.4]. Combining with the map:

$$v_{\mathfrak{m}} : H^1_{\text{ét}}(Y(1, N_{\psi_2}m)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathbb{T}'(1,N_{\psi_2}m)} \mathcal{O}[H^{(p)}_{\mathfrak{m}}] \xrightarrow{\sim} \mathrm{Ind}^{\mathbb{Q}}_{K^p_{\mathfrak{m}}} \mathcal{O}(\psi^{-1}_{2\mathfrak{P}})$$

and using the ξ_Δ map from equation (3.1.0.7) as in Section 3.1, we arrive at a class:

$$\kappa^4_{\mathfrak{m}} \in H^1(\mathbb{Q}, T_f^\vee(-k/2) \otimes \mathrm{Ind}^{\mathbb{Q}}_{K[m]^{(p)}} \Lambda_{\mathcal{O}}(\psi^{-1}_{1\mathfrak{P}}\psi^{-1}_{2\mathfrak{P}}\kappa_{ac}^{r_2/2}\boldsymbol{\kappa}_{ac}^{-1/2})(-r_2/2)).$$

Using Shapiro's lemma, we rewrite the cohomology group as:

$$\kappa^4_{\mathfrak{m}} \in H^1(K[m]^{(p)}, T_f^\vee(-k/2) \otimes \Lambda_{\mathcal{O}}(\psi^{-1}_{1\mathfrak{P}}\psi^{-1}_{2\mathfrak{P}}\kappa_{ac}^{r_2/2}\boldsymbol{\kappa}_{ac}^{-1/2})(-r_2/2)). \tag{3.2.8.4}$$

Definition. For any $L-$valued G_K representation V with a Galois stable $\mathcal{O}-$lattice T, the Iwasawa cohomology is defined as

$$H^1_{\mathrm{Iw}}(K[mp^\infty], T) = \varprojlim_r H^1(K[mp^r]^{(p)}, T)$$

that lies in

$$H^1_{\mathrm{Iw}}(K[mp^\infty], V) = H^1_{\mathrm{Iw}}(K[mp^\infty], T) \otimes L,$$

where the inverse limit is taken with respect to the corestriction maps.

Under this notation,

$$\kappa^4_{\mathfrak{m}} \in H^1_{\mathrm{Iw}}(K[mp^\infty], T_f^\vee(1-k/2)\chi_{12\mathfrak{P}}\kappa_{ac}^{r_2/2})$$

where $\chi_{12} = \psi_1^{-1}\psi_2^{-1}\mathbf{N}^{-r_2/2-1}$ is an anticyclotomic Hecke character of infinity type $(l/2, -l/2)$. We are now in the position to state the 'Λ −adic tame norm' relations:

Theorem 3.2.2. *Let $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_K)$ such that its norm $m = Norm_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to $N_f N_{\psi_1} N_{\psi_2} p$. Assume that $H^1(K[mp^r], T_f^\vee(1-k/2)\chi_{12\mathfrak{P}}\kappa_{ac}^{r_2/2})$ is torsion-free for all such $\mathfrak{m}$ and $r \geq 0$. Then there exists a collection of classes:*

$$\kappa_{f,\psi_1,\psi_2,\mathfrak{m}}^{\infty} \in H^1_{Iw}(K[mp^\infty], T_f^\vee(1-k/2)\chi_{12\mathfrak{P}})$$

such that given $\mathfrak{l} \in \mathscr{L}_K$ a split prime of K satisfying $(l, N_f N_{\psi_1} N_{\psi_2} pm) = 1$, where $l = Norm_{K/\mathbb{Q}}(\mathfrak{l})$, one has the following norm relation:

$$\mathrm{Norm}_{K[m]^{(p)}}^{K[ml]^{(p)}}(\kappa_{f,\psi_1,\psi_2,\mathfrak{m}\,\mathfrak{l}}^{\infty}) = P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}})(\kappa_{f,\psi_1,\psi_2,\mathfrak{m}}^{\infty})$$

where $P_{\mathfrak{l}}(X) = P_{\mathfrak{l}}(1 - X \cdot \mathrm{Frob}_{\mathfrak{l}} | T_f(\psi_1\psi_2)((k+l)/2))$.

Proof. Similar to what we did in the proof of Theorem 3.1.8, using the fact that the morphism $\mathscr{M}_{\mathbf{g}^\star}$ interchanges the degeneracy maps $\mathrm{pr}_{1\star}$ and $\mathrm{pr}_{l\star}$, one obtains the result but for $T_f^\vee(1-k/2)\chi_{12\mathfrak{P}}\kappa_{ac}^{r_2/2}$. Now we use the twisting result of Rubin (see [Rub00, Thm 6.3.5]) to finish the proof. □

Chapter 4

Triple product $p-$adic $L-$functions and Selmer groups

4.1 Triple product $p-$adic $L-$functions

We start with three primitive Hida families $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ of tame level $N_{\mathbf{f}}$, $N_{\mathbf{g}}$, $N_{\mathbf{h}}$ with character $\chi_{\mathbf{f}}$, $\chi_{\mathbf{g}}$, $\chi_{\mathbf{h}}$ and coefficients in $\Lambda_{\mathbf{f}}$, $\Lambda_{\mathbf{g}}$, $\Lambda_{\mathbf{h}}$ respectively, such that $\chi_{\mathbf{f}}\chi_{\mathbf{g}}\chi_{\mathbf{h}} = \omega^a$ for some even integer a. For our application we want $\mathbf{g}$ and $\mathbf{h}$ to pass through the ordinary $p-$stabilizations of θ_{ψ_1} and θ_{ψ_2} at some arithmetic points y_0 and z_0 respectively.

We may assume that $\Lambda_{\mathbf{f}}$ is a finite flat extension of $\Lambda_{\mathcal{O}}$ and consider only the arithmetic points of $\mathcal{W}_{\Lambda_{\mathbf{f}}}(\bar{\mathbb{Q}}_p)$ lying in $\mathrm{Hom}_{\mathrm{ct},\mathcal{O}}(\Lambda_{\mathbf{f}}, \bar{\mathbb{Q}}_p)$. Moreover, despite the fact that $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ might not be regular, we can still consider the $\Lambda-$adic families $\mathbf{g}$ and $\mathbf{h}$ coming from embedding $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ into the rings of functions of suitable wide open connected subsets $U_{\mathbf{g}}$ and $U_{\mathbf{h}}$ of $\mathcal{W}(\bar{\mathbb{Q}}_p)$ defined over some finite extension L (with $\mathcal{O}$ its ring of integers) of $\mathbb{Q}_p$ containing y_0 and z_0 respectively. We denote these rings also by $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$. They are non-canonically isomorphic to $\mathcal{O}[\![T]\!]$ (so regular). The prime ideal generated by $\mathbf{l} - l$ in $\Lambda_{\mathbf{g}}$ corresponding to the point y_0 and similarly for

$(\mathbf{m}-m)$ in $\Lambda_{\mathbf{h}}$ with the point z_0.

Define $\kappa_{\mathbf{f}}$ and $\kappa_{\mathbf{f}}^{1/2}$ to be the composition $\mathbb{Z}_p^\times \xrightarrow{\kappa_{r_1}} \Lambda^\times \hookrightarrow \Lambda_{\mathbf{f}}^\times$ and the fixed choice of square root $\kappa_{r_1}^{1/2}$, respectively, as in equation (3.2.5.1). Define $\kappa_{\mathbf{g}\,\mathbf{h}}(u) = \omega(u)^{l+m-4}\langle u\rangle^{\mathbf{l}+\mathbf{m}-4}$ as a character $\mathbb{Z}_p^\times \to (\Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}})^\times$ (recall $u = \omega(u)\langle u\rangle$) and also choose its square root $\kappa_{\mathbf{g}\,\mathbf{h}}^{1/2}$. Denote $\Lambda_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}} = \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$ and consider the self-dual Galois $\Lambda_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}$ $-$module:

$$\mathbb{V}^\dagger_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}} := \mathbb{V}_{\mathbf{f}} \hat{\otimes} \mathbb{V}_{\mathbf{g}} \hat{\otimes} \mathbb{V}_{\mathbf{h}}(\Xi_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}) \text{ where } \Xi_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}} = \epsilon_{\text{cyc}}^{-1}\kappa_{\mathbf{f}}^{-1/2}\kappa_{\mathbf{g}\,\mathbf{h}}^{-1/2}$$

where $\mathbb{V}_{\boldsymbol{\phi}}$ is the big Galois representation attached to $\boldsymbol{\phi}$ for $\boldsymbol{\phi} \in \{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$. Its 'specialization' as a $\Lambda_{\mathbf{f}}$ $-$module:

$$\mathbb{V}^\dagger_{\mathbf{f}\,gh} := \mathbb{V}_{\mathbf{f}} \otimes T_g \otimes T_h(\Xi_{\mathbf{f}\,gh}) \text{ where } \Xi_{\mathbf{f}\,gh} = \epsilon_{\text{cyc}}^{(2-l-m)/2}\kappa_{\mathbf{f}}^{-1/2}$$

is also Galois self-dual.

Definition.

1. Given a triple of integers (a, b, c),

$$(a,b,c) \text{ is called } \begin{cases} \text{balanced} & \text{if } a+b>c,\, b+c>a,\, c+a>b, \\ f\text{-unbalanced} & \text{if } a \geq b+c, \\ g\text{-unbalanced} & \text{if } b \geq a+c, \\ h\text{-unbalanced} & \text{if } c \geq a+b. \end{cases}$$

   Moreover, ϕ-unbalanced for $\phi \in \{f, g, h\}$ is also called unbalanced.

2. Given a set $\Sigma \subset \mathbb{Z}^3$, Σ is called balanced if all of its elements are balanced. Similarly, we have the same definition for unbalanced, f-, g-, and h-unbalanced set.

For each choice of a triple of test vectors $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ for $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of level N, Harris-Tilouine [HT01] (for N=1) and Darmon-Rotger [DR17] construct a triple product $p-$adic $L-$function:

$$\mathcal{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) \in \mathrm{Frac}(\Lambda_{\mathbf{f}}) \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$$

that interpolates the **square root** of the central critical values

$$L(f_k \otimes g_l \otimes h_m, c) = L(\mathbb{V}^{\dagger,\vee}_{f_k g_l h_m}(1), 0),$$

where $c = (k+l+m-2)/2$ given that $k \geq l+m$ (i.e. f-unbalanced) (see the explicit definition of the LHS in [DR14, Sec 4.1]).

Subsequently, Hsieh [Hsi21] constructed an explicit choice of test vector for which

$$\mathcal{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) \in \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$$

and satisfies a precise and simpler interpolation formula in the same range $k \geq l+m$ (i.e. f-unbalanced).

Assumption 4.1.1. (‡)

1. $\gcd(N_{\mathbf{f}}, N_{\mathbf{g}}, N_{\mathbf{h}})$ is squarefree, (imposed for the local Rankin-Selberg calculation).

2. There is a triple of arithmetic points with weights (k, l, m) such that the local sign $\epsilon_q(\mathbb{V}^{\dagger}_{\mathbf{f}_k\, \mathbf{g}_l\, \mathbf{h}_m}) = 1$ for all primes $q|N$. Because the local sign at infinity depends on whether the weights are balanced or unbalanced, this condition implies that $\epsilon(\mathbb{V}^{\dagger}_{\mathbf{f}_k\, \mathbf{g}_l\, \mathbf{h}_m})$ equals to 1 in the unbalanced range and -1 in the balanced range.

3. There is a classical point k such that $V_{\mathbf{f}_k}$ is residually absolutely irreducible and $p-$distinguished. This implies the Gorenstein property of the local ring

$\Lambda_{\mathbf{f}}$ by Mazur-Wiles [MW86, Sec 9, Prop 2] and Wiles [Wil95]. Hida [Hid88] then proved that the congruence module of $\mathbf{f}$ is isomorphic to $\Lambda_{\mathbf{f}}/(\zeta)$ for some nonzero $\zeta \in \Lambda_{\mathbf{f}}$, (ζ) is normally called the congruence ideal of $\mathbf{f}$.

The following theorem is Theorem A in [Hsi21]. The $p-$adic $L-$function constructed there is unique up to a choice of generator ζ of the congruence ideal of $\mathbf{f}$. Nevertheless, the ratio by ζ is a genuine $p-$adic $L-$function.

Theorem 4.1.1. *Under the assumption (‡), there exists a choice of a triple of test vector* $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ *for* $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ *of level* N *and an element* $\mathcal{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) \in \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$ *such that for all triples of* $f-$*unbalanced arithmetic points* (k, l, m), $\mathcal{L}_p^{f,\zeta}$ *satisfies the following interpolation property:*

$$\boxed{\mathcal{L}_p^{f,\zeta}(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})^2(k, l, m) = \mathcal{C}_{k,l,m} \frac{L(\mathbb{V}^{\dagger}_{\mathbf{f}_k \mathbf{g}_l \mathbf{h}_m}, 0))}{\pi^{2k-4}(\mathbf{f}_k^{\#}, \mathbf{f}_k^{\#})_{N_{\mathbf{f}}}}}$$

where $\mathcal{C}_{k,l,m}$ *is an explicit nonzero constant depending on* $\{p, \mathbf{f}_k,\ \mathbf{g}_l,\ \mathbf{h}_m\}$, $\mathbf{f}_k^{\#}$ *is the newform associated to the* $p-$*stabilized form* $\mathbf{f}_k$, *and* $(,)_{N_{\mathbf{f}}}$ *is the Peterson inner product.*

Recall that we will choose $\mathbf{g}$ and $\mathbf{h}$ to pass through the ordinary $p-$stabilizations of θ_{ψ_1} and θ_{ψ_2} at some arithmetic points y_0 and z_0 respectively. We can specialize Theorem 4.1.1 to obtain the existence of test vectors $(\breve{\mathbf{f}}, \breve{g}, \breve{h})$ for $(\mathbf{f}, g, h)$ of level N and an element

$$\mathcal{L}_p^{f,\zeta}(\breve{\mathbf{f}}, \breve{g}, \breve{h}) \in \Lambda_{\mathbf{f}}$$

such that its square $L_p^f(\mathbf{f}, g, h) = \mathcal{L}_p^{f,\zeta}(\breve{\mathbf{f}}, \breve{g}, \breve{h})^2$ interpolates $L(\mathbb{V}^{\dagger}_{\mathbf{f}_k gh}, 0)$. Even though $L_p^{f,\zeta}$ depends on the choice of ζ, the principal ideal it generates in $\Lambda_{\mathbf{f}}$ will be independent of that choice.

4.2 The reciprocity law

For $\boldsymbol{\phi} \in \{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$, assume that the Galois representation attached to $\boldsymbol{\phi}$ is residually absolutely irreducible and $p-$distinguished (for the definition, see [Hsi21, p. 415]). Restricting to $G_{\mathbb{Q}_p}$, $\mathbb{V}_{\boldsymbol{\phi}}$ admits a filtration:

$$0 \to \mathbb{V}_{\boldsymbol{\phi}}^{+} \to \mathbb{V}_{\boldsymbol{\phi}} \to \mathbb{V}_{\boldsymbol{\phi}}^{-} \to 0$$

where $\mathbb{V}_{\boldsymbol{\phi}}^{\pm}$ is free of rank one over $\Lambda_{\boldsymbol{\phi}}$. The Frobenius acts on $\mathbb{V}_{\boldsymbol{\phi}}^{-}$ as multiplication by $a_p(\boldsymbol{\phi})$. One obtains the following filtration of $G_{\mathbb{Q}_p}-$stable of $\Lambda_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}-$modules:

$$0 \subset \mathscr{F}^3\, \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger} \subset \mathscr{F}^2\, \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger} \subset \mathscr{F}^1\, \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger} \subset \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger},$$

where

$$\mathscr{F}^3\, \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger} = \mathbb{V}_{\mathbf{f}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{g}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{h}}^{+}(\Xi_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}})$$

$$\mathscr{F}^2\, \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger} = (\mathbb{V}_{\mathbf{f}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{g}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{h}} + \mathbb{V}_{\mathbf{f}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{g}} \hat{\otimes} \mathbb{V}_{\mathbf{h}}^{+} + \mathbb{V}_{\mathbf{f}} \hat{\otimes} \mathbb{V}_{\mathbf{g}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{h}}^{+})(\Xi_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}})$$

$$\mathscr{F}^1\, \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger} = (\mathbb{V}_{\mathbf{f}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{g}} \hat{\otimes} \mathbb{V}_{\mathbf{h}} + \mathbb{V}_{\mathbf{f}} \hat{\otimes} \mathbb{V}_{\mathbf{g}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{h}} + \mathbb{V}_{\mathbf{f}} \hat{\otimes} \mathbb{V}_{\mathbf{g}} \hat{\otimes} \mathbb{V}_{\mathbf{h}}^{+})(\Xi_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}).$$

If we also let

$$\mathbb{V}_{\mathbf{f}}^{\mathbf{g}\,\mathbf{h}} := \mathbb{V}_{\mathbf{f}}^{-} \hat{\otimes} \mathbb{V}_{\mathbf{g}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{h}}^{+},$$

$$\mathbb{V}_{\mathbf{g}}^{\mathbf{f}\,\mathbf{h}} := \mathbb{V}_{\mathbf{f}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{g}}^{-} \hat{\otimes} \mathbb{V}_{\mathbf{h}}^{+},$$

$$\mathbb{V}_{\mathbf{h}}^{\mathbf{f}\,\mathbf{g}} := \mathbb{V}_{\mathbf{f}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{g}}^{+} \hat{\otimes} \mathbb{V}_{\mathbf{h}}^{-},$$

then

$$\mathscr{F}^2\, \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger} \,/\, \mathscr{F}^3\, \mathbb{V}_{\mathbf{f}\,\mathbf{g}\,\mathbf{h}}^{\dagger} \cong \mathbb{V}_{\mathbf{f}}^{\mathbf{g}\,\mathbf{h}} \oplus \mathbb{V}_{\mathbf{g}}^{\mathbf{f}\,\mathbf{h}} \oplus \mathbb{V}_{\mathbf{h}}^{\mathbf{f}\,\mathbf{g}}. \tag{4.2.0.1}$$

A similar notation will be used when we specialize to (f, g, h).

Important choices. The class κ_1^3 in (3.2.8.3) that corresponds to the choice of level N test vectors given by Hsieh's construction in Theorem 4.1.1, gives a class $\kappa(\mathbf{f}, g, h) \in H^1(\mathbb{Q}, \mathbb{V}^{\dagger}_{\mathbf{f}gh})$ by the augmentation map $\mathcal{O}[H_1^{(p)} \times H_1^{(p)}] \to \mathcal{O}$ (but as we assume $(p, h_K) = 1$, this map is constant). Corollary 8.2 in [BSV21] tells us that

$$\mathrm{res}_p(\kappa(\mathbf{f}, g, h)) \in H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{f}gh}) := \mathrm{im}(H^1(\mathbb{Q}_p, \mathscr{F}^2\, \mathbb{V}^{\dagger}_{\mathbf{f}gh}) \xrightarrow{i} H^1(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{f}gh})).$$

As i is an injection, we may view $\mathrm{res}_p(\kappa(\mathbf{f}, g, h)) \in H^1(\mathbb{Q}_p, \mathscr{F}^2\, \mathbb{V}^{\dagger}_{\mathbf{f}gh})$.

Then the projection onto the first direct summand of (4.2.0.1) gives a a map:

$$\mathrm{proj}_{\mathbf{f}} : H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{f}gh}) \to H^1(\mathbb{Q}_p, \mathbb{V}^{gh}_{\mathbf{f}}). \tag{4.2.0.2}$$

Building on the work of Kings-Loeffler-Zerbes [KLZ17], Bertolini-Seveso-Venerucci constructed a three-variable Perrin-Riou regulator map [BSV21, Sec 7.1] (see [ACR21, Prop 7.3] for details)

$$\mathfrak{Log}^{\zeta} : H^1(\mathbb{Q}_p, \mathbb{V}^{gh}_{\mathbf{f}}) \to \Lambda_{\mathbf{f}}$$

which is an injective $\Lambda_{\mathbf{f}}$ −module homomorphisms with pseudo-null cokernel such that:

Theorem 4.2.1. ***(Explicit Reciprocity Law)*** *One has the reciprocity law:*

$$\boxed{\mathfrak{Log}^{\zeta}(\mathrm{proj}_{\mathbf{f}}(\mathrm{res}_p(\kappa(\mathbf{f}, g, h)))) = \mathcal{L}^f_p(\breve{\mathbf{f}}, \breve{g}, \breve{h})} \tag{4.2.0.3}$$

Proof. This is the reciprocity law attached to the triple $(\mathbf{f}, g, h)$, which is Theorem A in [BSV21]. □

4.3 Anticyclotomic Euler systems

We discuss the theory of anticyclotomic Euler systems together with applications constructed by Jetchev-Nekovář-Skinner [JNS].

Set-up.

1. Fix an odd prime p.

2. Let K be an imaginary quadratic field. For an integral prime ideal $\mathfrak{m}$ of K, denote by $K(\mathfrak{m})$ the maximal $p-$subextension of the ray class field of conductor $\mathfrak{m}$. For a positive integer m, denote by $K[m]$ the maximal $p-$subextension of the ring class field of conductor m. Denote by K_∞^- the anticyclotomic $\mathbb{Z}_p$ $-$extension of K.

3. Assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K.

4. Let $\Phi/\mathbb{Q}_p$ be a finite extension and $\mathcal{O}$ be its ring of integers. Let $\varpi \in \mathcal{O}$ be a uniformizer and denote by $\mathbb{F} = \mathcal{O}/\varpi\mathcal{O}$ the residue field.

5. Let V be a finite dimensional representation of G_K over Φ, unramified outside a finite set of primes Σ (in practice we want V to be geometric), and let $T \subset V$ be a Galois stable $\mathcal{O}-$lattice. Let $A = V/T$.

Assumption 4.3.1.

1. There exists a non-degenerate symmetric $\mathcal{O}-$bilinear pairing: $\langle , \rangle : T \times T \to \mathcal{O}(1)$ such that $\langle x^\sigma, y^{c\sigma c^{-1}} \rangle = \langle x, y \rangle^\sigma$ for all $x, y \in T$, $\sigma \in G_K$, where $c \in G_\mathbb{Q}$ is a complex conjugation. This implies that $V^c \cong V^\vee(1)$, where V^c is the vector space V but with $g \in G_K$ acting as cgc^{-1}.

   (a) Note that if the above pairing is perfect then we also have $T^c \cong T^\vee(1)$.

(b) For a finite extension M/K where M is $c-$stable with w a finite place of M, denote $\bar{w} = w^c$. Then we have local pairings (induced from $\langle , \rangle$):

$$H^1(M_w, V) \times H^1(M_{\bar{w}}, V) \to \Phi, \text{ and } H^1(M_w, T) \times H^1(M_{\bar{w}}, T) \to \mathcal{O}.$$

The isomorphism $H^1(M_{\bar{w}}, V) \cong H^1(M_w, V^c)$ (where $G_{M_{\bar{w}}} \to G_{M_w}\colon \sigma \mapsto c\sigma c$ and $V \to V^c\colon x \mapsto x$) combined with $V^c \cong V^\vee(1)$ implies that

$$H^1(M_w, V^\vee(1)) \cong H^1(M_{\bar{w}}, V),$$

and the above local pairing is just the natural cup-product pairing.

2. **(abs-irr)**: V is an absolutely irreducible G_K-representation.

3. **(res-irr)**: $\bar{T} = T/\varpi T$ is an absolutely irreducible G_K-representation.

4. **(per)**: The pairing $\langle , \rangle : T \times T \to \mathcal{O}(1)$ is perfect.

5. Hyp($\boldsymbol{\sigma}$): There exists $\sigma \in G_K$ such that:

   (a) σ fixes $K[1](\mu_{p^\infty})$

   (b) $\dim_\Phi V/(\sigma - 1)V = 1$

   Note that this hypothesis can be deduced from the existence of an element $\sigma \in G_K$ acting nontrivially and unipotently on V. In particular, verifying this hypothesis relies on some 'big image results', which often hold in practice. Hyp($\boldsymbol{\sigma}$) will then be used to show some finite Galois modules are free of rank 1 in the 'Kolyvagin system argument' (see more in [Rub00, Chap 5, Sec 2]).

6. Hyp($\boldsymbol{\gamma}$): There exists $\gamma \in G_K$ such that:

   (a) γ fixes $K[1](\mu_{p^\infty}, \mathcal{O}_K^{\times, 1/p^\infty})$

(b) $V = (\gamma - 1)V$ (i.e. 1 is not an eigenvalue γ)

This hypothesis ensures the 'rigidity' of an anticyclotomic Euler system (see [Rub00, Sec 9.1]). Specifically, the standard assumption that there exists a $\mathbb{Z}_p$ −extension of K in which no finite prime splits completely cannot be satisfied for the anticyclotomic extension K_∞^- (inert primes split completely in K_∞^-).

7. Hyp($\boldsymbol{\zeta}$): There exists $\zeta \in G_K$ such that ζ acts on $\bar{T} = T/\varpi T$ as multiplication by some scalar $1 \neq a_\zeta \in \mathbb{F}^\times$.

Definition. We assume that Hyp($\boldsymbol{\sigma}$) holds.

1. For each positive integer n, the set of split-σ Kolyvagin primes level n, denoted $\mathscr{L}_n^\sigma$, is a collection of primes $l \in \mathbb{Q}$ such that:

   (a) $l \nmid 2p$, and l splits in K such that $l = \mathfrak{l}\bar{\mathfrak{l}}$.

   (b) V is unramified at $\mathfrak{l}$ and $\bar{\mathfrak{l}}$.

   (c) Frob_l lies in the G_K conjugacy class of σ in $\mathrm{Gal}(\Omega_n/K)$, where $T_n = T/\varpi^n T$, $\Omega_n = K[1]K(\mu_{p^n})K(T_n)$, and $K(T_n)$ denotes the smallest extension of K such that $G_{K(T_n)}$ acts trivially on T_n.

2. Denote $\mathscr{L}_n^{K,\sigma} = \{\text{primes } \mathfrak{l} \text{ of } K \text{ such that } \mathfrak{l}|l \text{ for some } l \in \mathscr{L}_n^\sigma\}$.

3. For $\mathscr{L}$ a set of primes of K, we denote $\mathcal{N}(\mathscr{L}) = \{\mathfrak{a} = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_r^{a_r} \subset \mathcal{O}_K$, where $\mathfrak{p}_i \in \mathscr{L}, a_i = 1$ if $\mathfrak{p}_i \nmid p$, and $\mathfrak{p}_i \neq \mathfrak{p}_j, \bar{\mathfrak{p}}_j\}$.

For each prime $v \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ lying above p of K, we choose a G_{K_v}−stable $\mathcal{O}$-submodule $F_v^+(T)$ of T and denote $F_v^-(T) = T/F_v^+(T)$. Define: $F_v^+(V) = F_v^+(T) \otimes_{\mathcal{O}} L_\mathfrak{p} \subset V$ and $F_v^-(V) = V/F_v^+(V)$. Let M/K be a c−stable finite extension. For each place w of M, define the Greenberg Selmer group:

$$H^1_{Gr}(M, V) = \ker\left(H^1(M, V) \to \prod_w \frac{H^1(M_w, V)}{H^1_{Gr}(M_w, V)}\right)$$

where the local condition is defined as follow:

$$H^1_{Gr}(M_w, V) = \begin{cases} \ker(H^1(M_w, V) \to H^1(M_w^{ur}, V)) & \text{if } w \nmid p \\ \ker(H^1(M_w, V) \to H^1(M_w, F^-_{\mathfrak{p}}(V))) & \text{if } w \mid \mathfrak{p} \\ \ker(H^1(M_w, V) \to H^1(M_w, F^-_{\bar{\mathfrak{p}}}(V))) & \text{if } w \mid \bar{\mathfrak{p}}. \end{cases}$$

In addition, instead of $H^1_{Gr}(M_w, V)$, we may choose the following local conditions for $w|p$:

$$H^1_g(M_w, V) = \ker(H^1(M_w, V) \to H^1(M_w, V \otimes \mathbf{B}_{dR})) \tag{4.3.0.1}$$

$$H^1_f(M_w, V) = \ker(H^1(M_w, V) \to H^1(M_w, V \otimes \mathbf{B}_{cris})) \tag{4.3.0.2}$$

and define the Bloch-Kato Selmer groups $H^1_g(M, V)$ and $H^1_f(M, V)$ with these new conditions.

We can also define the Greenberg Selmer group for T and $A = V/T$ by propagating the local conditions as follows:

1. $H^1_{Gr}(M_w, T)$ is the preimage of $H^1_{Gr}(M_w, V)$ from the map $H^1(M_w, T) \to H^1(M_w, V)$.

2. $H^1_{Gr}(M_w, A)$ is the image of $H^1_{Gr}(M_w, V)$ from the map $H^1(M_w, V) \to H^1(M_w, A)$.

Assumption 4.3.2. **(orth)** For all squarefree integer m which is divisible by only primes $l \in \mathscr{L}^{\sigma}_n$ and all places w of $K[m]$ above p, the local conditions $H^1_{Gr}(K[m]_w, V)$ and $H^1_{Gr}(K[m]_{\bar{w}}, V)$ are orthogonal complements under the local pairing

$$H^1(K[m]_w, V) \times H^1(K[m]_{\bar{w}}, V) \to \Phi.$$

Note that this holds for places away from p by [Rub00, Prop 1.4.2]. If **(orth)** holds, $H^1_{Gr}(K[m]_w, T)$ and $H^1_{Gr}(K[m]_{\bar{w}}, T)$ are also orthogonal complements (see [Rub00, Prop B.2.4]).

A special case (of interests) of the Greenberg Selmer group local condition at $v \in \{\mathfrak{p}, \bar{\mathfrak{p}}\}$ is when

1. $F_v^+(V) = 0$, we call it the **strict** condition

2. $F_v^+(V) = V$, we call it the **relaxed** condition.

If the Greenberg Selmer group is defined by the relaxed and strict conditions at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ respectively, i.e. $F_{\mathfrak{p}}^+(V) = V$ and $F_{\bar{\mathfrak{p}}}^+(V) = 0$, then (**orth**) is automatic. We call it the 'relaxed-strict' Greenberg Selmer groups.

4.3.1 The 'relaxed-strict' Greenberg Selmer groups

In this subsection, we show that the classes constructed in Section 3 do land in the 'relaxed-strict' Greenberg Selmer groups. We assume that Hyp($\boldsymbol{\sigma}$) holds and we are in the Set-up of Section 3.2.7.

As recalled in Section 2.3.2, since f is $p-$ordinary, we have an exact sequence for $V_f^\vee$ restricted to $G_{\mathbb{Q}_p}$:

$$0 \to V_f^{\vee,+} \to V_f^\vee \to V_f^{\vee,-} \to 0 \tag{4.3.1.1}$$

with $\dim(V_f^{\vee,\pm}) = 1$ and the sub-representation $V_f^{\vee,-}$ is unramified. Since our convention is that $\mathbb{Q}_p(1)$ has HT weight -1, then the HT weight of $V_f^{\vee,-}$ is 0 and the HT weight of $V_f^{\vee,+}$ is $1-k$. If χ is an algebraic Hecke character over K of infinity type (a,b) then the HT weight of its $p-$adic avatar $\chi_{\mathfrak{P}}$ (see our convention in Section 2.4.1) is $-a$ at $\mathfrak{p}$, and $-b$ at $\bar{\mathfrak{p}}$.

Denote by $V = V_f^\vee(1-k/2)(\chi_{12\mathfrak{P}})$, where $\chi_{12} = \psi_1^{-1}\psi_2^{-1}\mathbf{N}^{-l/2}$ is an anticyclotomic Hecke character of infinity type $(l/2, -l/2)$, $V^+ = V_f^{\vee,+}(1-k/2)(\chi_{12\mathfrak{P}})$ and $V^- = V_f^{\vee,-}(1-k/2)(\chi_{12\mathfrak{P}})$. Beside the relaxed and the strict conditions, we can also define the **ordinary** condition as follows:

$$F_v^+(V) = V^+ \text{ for } v|p.$$

Notation. Denote by $H^1_{\alpha,\beta}$ the subgroup of $H^1(K,V)$ where classes are unramified at all primes $v \nmid p$ and satisfy the conditions α, β at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ respectively, where $\alpha, \beta \in \{\mathrm{rel}, \mathrm{str}, \mathrm{ord}\}$, and these conditions correspond to the relaxed, strict, and ordinary condition respectively.

Lemma 4.3.1. *The Bloch-Kato Selmer group satisfies*

$$H^1_f(K,V) = \begin{cases} H^1_{\mathrm{rel,str}}(K,V) & \text{if} \quad l \geq k, \\ H^1_{\mathrm{ord,ord}}(K,V) & \text{if} \quad k \geq l+2. \end{cases} \tag{4.3.1.2}$$

Proof. By using the following table of Hodge-Tate weights:

	V^+	V^-
HT weight at $\mathfrak{p}$	$\frac{-k-l}{2}$	$\frac{k-2-l}{2}$
HT weight at $\bar{\mathfrak{p}}$	$\frac{l-k}{2}$	$\frac{k+l-2}{2}$

and looking at the Panchiskin condition ([Gre94], [BK07, Thm 4.1(ii)]), one obtains the result (e.g. if $l \geq k$, the HT weights of both $V^{\pm}$ at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are < 0 and ≥ 0 respectively). □

Proposition 4.3.2. *Let $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_1^{K,\sigma})$ such that $m = Norm_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to $N_f N_{\psi_1} N_{\psi_2} p$, then the class constructed in Theorem 3.1.8 satisfies:*

$$z_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1_{Gr}(K[m], T_f^{\vee}(\psi_{1\mathfrak{P}}^{-1}\psi_{2\mathfrak{P}}^{-1})(-1)),$$

where the Greenberg Selmer group is defined by the relaxed and strict conditions at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ respectively.

Proof. If $w \nmid p$, because V is conjugate self-dual, and pure of weight (-1) [Nek93, Sec 8.3], we have $H^0(K[m]_w, V) = 0 = H^2(K[m]_w, V^{\vee}(1))$ i.e. $H^2(K[m]_w, V) = 0$. Hence

$H^1(K[m]_w, V) = 0$ by the local Euler characteristic formula.

If $w|p$, by [BSV21, Prop 3.2], the class κ^1 is geometric at p, thus its restriction at w lands in H^1_g. The geometricity is preserved after taking the (direct sum) quotient ξ_Δ. Moreover, H^1_g equals to the Bloch-Kato subspace H^1_f ([Nek93, Prop 1.24(2)]). Because we are in the case $k = l = 2$, using Lemma 4.3.1, we have $H^1_f = H^1_{Gr}$. □

Definition. The Iwasawa Greenberg Selmer group is defined as:

$$H^1_{Gr}(K[mp^\infty], T) = \varprojlim_r H^1_{Gr}(K[mp^r], T)$$

that lies inside

$$H^1_{Gr}(K[mp^\infty], V) = H^1_{Gr}(K[mp^\infty], T) \otimes_{\mathcal{O}} L.$$

Proposition 4.3.3. *Let $\mathfrak{m} \in \mathcal{N}(\mathscr{L}_1^{K,\sigma})$ such that $m = Norm_{K/\mathbb{Q}}(\mathfrak{m})$ is coprime to $N_f N_{\psi_1} N_{\psi_2} p$, the class constructed in Theorem 3.2.2 satisfies:*

$$\kappa^\infty_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1_{Gr}(K[mp^\infty], T_f^\vee(1-k/2)(\chi_{12\mathfrak{P}}))$$

where it is propagated from the relaxed condition at $\mathfrak{p}$ and the strict condtion at $\bar{\mathfrak{p}}$.

Proof. Similar to Proposition 4.3.2, one can show that the class vanishes at all prime $v \nmid p$ by showing that each layer $H^1(K[mp^r]_v, V)$ vanishes. For $v|p$, by using that κ^1_m lands in the balanced Selmer group, our classes will land in the image of $H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}^\dagger_{f\theta_{\psi_1}\theta_{\psi_2}})$ under the (direct sum) quotient ξ_Δ. Furthermore, the reducibility of the restriction of $V^\vee_{\theta_\psi}$ to the decomposition group $G_{\mathbb{Q}_p}$ allows us to write $V^{\vee,+}_{\theta_\psi} = \mathcal{O}(\psi^{-1})$ and $V^{\vee,-}_{\theta_\psi} = \mathcal{O}(\psi^{-c})$ for $\psi \in \{\psi_1, \psi_2\}$. Finally, we interpret the ξ map as quotienting by the direct summand $(V_f^\vee \otimes \mathcal{O}(\psi_1^{-1}) \otimes \mathcal{O}(\psi_2^{-c}) \oplus V_f^\vee \otimes \mathcal{O}(\psi_1^{-c}) \otimes \mathcal{O}(\psi_2^{-1}))(1-(k+l)/2)$, and we can show that the local condition at $\mathfrak{p}$ is $V_f^\vee(1-k/2)(\chi_{12\mathfrak{P}})$ (i.e. relaxed) and the local condition at $\bar{\mathfrak{p}}$ is 0 (i.e. strict). □

4.3.2 About split-σ Kolyvagin primes and the anticyclotomic Euler system

For this subsection, in order to define an anticyclotomic Euler system, we will assume that Hyp($\boldsymbol{\sigma}$) holds.

Notation. Given $\mathscr{L}_i$ a set consisting of primes of K for $i \in \{1,2\}$, we write $\mathscr{L}_1 \dot{\supset} \mathscr{L}_2$ if the natural density of $(\mathscr{L}_2 \setminus (\mathscr{L}_2 \cap \mathscr{L}_1))$ is 0.

Definition. (**Euler system**) Fix a choice of the Greenberg Selmer group. Let $\mathscr{L}$ be a set consisting of primes of K such that $\mathscr{L} \dot{\supset} \mathscr{L}_n^{K,\sigma}$ for some $n \geq 1$. A (split-σ) anticyclotomic Euler system for $(T, \mathscr{L})$ (in the sense of Jetchev-Nekovář-Skinner) [JNS] is a collection of cohomology classes $\mathbf{c} = \{c_{\mathfrak{m}},$ where $\mathfrak{m} \in \mathcal{N}(\mathscr{L})\}$ such that:

1. $c_{\mathfrak{m}} \in H^1_{Gr}(K[m], T)$, where $m = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{m})$
2. For $\mathfrak{m}\mathfrak{l} \in \mathcal{N}(\mathscr{L})$, where $\mathfrak{l}$ is a prime of K with $l = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{l})$, we have the following norm relation:

$$\mathrm{cores}_{K[ml]/K[m]}(c_{\mathfrak{m}\,\mathfrak{l}}) = P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}^{-1}) c_{\mathfrak{m}} \tag{4.3.2.1}$$

where $P_{\mathfrak{l}}(X) = \det(1 - \mathrm{Frob}_{\mathfrak{l}}^{-1} X | T^{\vee}(1))$.

Remark 4.3.1. The asymmetry comes from $\mathscr{L}_n^{\sigma}$. In particular, for each $l \in \mathscr{L}_n^{\sigma}$, there is only one prime $\lambda | l$ of $\mathscr{L}_n^{K,\sigma}$ such that Frob_{λ} lies in the conjugacy class of σ in $\mathrm{Gal}(\Omega_n/K)$.

Definition. Fix a choice of the Greenberg Selmer group and given a set $\mathscr{L}$ consisting of primes of K such that $\mathscr{L} \dot{\supset} \mathscr{L}_n^{K,\sigma}$ for some $n \geq 1$. A (split-σ) Λ_K^-−adic anticyclotomic Euler system for $(T, \mathscr{L})$ is a collection of cohomology classes $\mathbf{c}_{\infty} = \{c_{\mathfrak{m},\infty} \in H^1_{Gr}(K[mp^{\infty}], T)$, where $\mathfrak{m} \in \mathcal{N}(\mathscr{L})$ and $m = \mathrm{Norm}_{K/\mathbb{Q}}(\mathfrak{m})\}$ that satisfies the same norm relation (4.3.2.1). In this case, we have the following map (where γ is a topological generator of Λ_K^- that maps to $(1+T)$ under the isomorphism $\Lambda_K^- \xrightarrow{\sim} \mathbb{Z}_p[\![T]\!]$) :

$$H^1_{Gr}(K[mp^\infty], T) = H^1_{Gr}(K[m], T \hat{\otimes} \Lambda_K^-) \xrightarrow{/(\gamma-1)} H^1_{Gr}(K[m], T)$$

where the first identification is by Shapiro's Lemma, and we denote its composition as $\mathrm{proj}_{K[m]}$. Then $c_{\mathfrak{m}} := \mathrm{proj}_{K[m]}(c_{\mathfrak{m},\infty}) \in H^1_{Gr}(K[m], T)$ forms a (split-σ) anticyclotomic Euler system, and we say that $\mathbf{c} = \{c_{\mathfrak{m}}, \mathfrak{m} \in \mathcal{N}(\mathscr{L})\}$ extends along the anticyclotomic $\mathbb{Z}_p$ −extension.

The main contribution of this thesis is the following theorem:

Theorem 4.3.4. *The classes constructed in Theorem 3.2.2*

$$\kappa^\infty_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1_{Gr}(K[mp^\infty], T_f^\vee(1-k/2)\chi_{12\mathfrak{P}})$$

form a Λ_K^- *−adic anticyclotomic Euler system for* $(T_f^\vee(1-k/2)\chi_{12}, \mathscr{L}_1^{K,\sigma})$*, where* $\chi_{12} = \psi_1^{-1}\psi_2^{-1}\mathbf{N}^{-l/2}$ *is an anticyclotomic of infinity type* $(l/2, -l/2)$.

Proof. By combining Theorem 3.2.2 and Proposition 4.3.3, we obtain the result. □

Before finishing this section, we record two applications of [JNS].

Theorem 4.3.5. *[JNS] Assume that p splits in K, that **(abs-irr)**, Hyp($\boldsymbol{\sigma}$),Hyp($\boldsymbol{\gamma}$), and **(orth)** hold. Let* $\mathbf{c}$ *be a* Λ *−adic anticyclotomic Euler system for* $(T, \mathscr{L})$ *that extends along the anticyclotomic* $\mathbb{Z}_p$ *−extension. If* $\mathrm{cores}_{K[1]/K} c_1 \neq 0$ *then* $H^1_{Gr}(K, T)$ *has* $\mathcal{O}$*−rank one.*

Notation. For a topological $\mathbb{Z}_p$ −module M we denote its Pontrjagin dual

$$M^\star = \mathrm{Hom}_{\mathrm{ct}}(M, \mathbb{Q}_p / \mathbb{Z}_p)$$

Under a stronger assumption, the machinery in [JNS] also gives us one divisibility of the Iwasawa Main Conjecture without $L-$function (see more in [Ski, Sec 4.3]).

Theorem 4.3.6. *[JNS] Assume that p splits in K, that* ***(res-irr)***, ***(per)***, ***(orth)*** *and Hyp($\boldsymbol{\zeta}$) hold. Let $\mathbf{c}$ be a Λ_K^-−adic anticyclotomic Euler system for $(T, \mathscr{L})$ that extends along the anticyclotomic $\mathbb{Z}_p$−extension. If $c_\infty = \mathrm{cores}_{K[1]/K} c_{1,\infty} \in H^1_{Gr}(K_\infty^-, T)$ is not Λ_K^-−torsion then one has*

$$\mathrm{rank}_{\Lambda_K^-}(X^\star) = \mathrm{rank}_{\Lambda_K^-}(H^1_{Gr}(K_\infty^-, T)) = 1, \ \textit{where } X = H^1_{Gr}(K_\infty^-, T \otimes {\Lambda_K^-}^\star),$$

and the following divisibility of characteristic ideals:

$$\mathrm{char}_{\Lambda_K^-}(X^\star_{tors}) \quad | \quad \mathrm{char}_\Lambda \left(\frac{H^1_{Gr}(K_\infty^-, T)}{\Lambda_K^- \, c_\infty} \right)^2,$$

where $X^\star_{tors}$ is the torsion part of $X^\star$ as an Λ_K^- module.

Chapter 5

Applications

5.1 Main applications

In this section, we present the proof of the main arithmetic applications of the Euler system constructed. First we make precise the setup of our main applications.

Set-up.

1. Let K be an imaginary quadratic field.

2. Let $p \geq 5$ be a prime.

3. Let $k, l \geq 2$ be two positive even integers.

4. Let $f \in S_k(\Gamma_0(N_f))$ be a newform, ordinary at p.

5. Let ψ_1, ψ_2 be two Hecke characters over K of infinity type $(1-l, 0)$, $(-1, 0)$ with conductor $\mathfrak{f}_1$, $\mathfrak{f}_2$ respectively. As recalled in Section 2.4.1, there are two theta series $\theta_{\psi_1} \in S_l(N_{\psi_1}, \chi_{\psi_1})$ and $\theta_{\psi_2} \in S_2(N_{\psi_2}, \chi_{\psi_2})$ associated with ψ_1 and ψ_2. Assume that $\chi_{\psi_1}\chi_{\psi_2} = 1$.

6. Let $N = \mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})$.

7. We assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K, and $(p, h_K\mathfrak{f}_1\mathfrak{f}_2) = 1$ as in Section 2.4. Take L/K to be a finite extension, and assume it to be large enough so that its ring of integers contains the Fourier coefficients of $f, \theta_{\psi_1}, \theta_{\psi_2}$. Choose primes $\mathfrak{P}|\mathfrak{p}|p$ of $L/K/\mathbb{Q}$ respectively and let $\mathcal{O} \subset L_{\mathfrak{P}} = \Phi$ be its ring of integers.

8. Let $\chi_{12} = \psi_1^{-1}\psi_2^{-1}\mathbf{N}^{-l/2}$. This is an anticyclotomic Hecke character of infinity type $(l/2, -l/2)$ of conductor dividing $\mathfrak{f}_1\mathfrak{f}_2$.

9. Note that since $(p, h_K) = 1$, we have $K[1] = K$.

Theorem 5.1.1. *(**Rank 1** result) Let $\kappa_{f,\chi_{12}} = \mathrm{proj}_K(\kappa^{\infty}_{f,\psi_1,\psi_2,1})$ be the base class of the Euler system in Theorem 4.3.4. Then we have*

$$\boxed{\kappa_{f,\chi_{12}} \neq 0 \quad \Longrightarrow \quad \dim_{\Phi} H^1_{Gr}(K, V_f^{\vee}(1-k/2)(\chi_{12})) = 1}$$

where we also assume that f is not of CM type. The Greenberg Selmer group local condition will be the 'relaxed-strict' condition i.e. $F^+_{\mathfrak{p}}(V) = V$ and $F^+_{\bar{\mathfrak{p}}}(V) = 0$.

Proof. The assumption that f is not CM implies the big image result ([Mom81], [Rib85]), in other words the image of G_K (open in $G_{\mathbb{Q}}$) inside $\mathrm{Aut}(V_f^{\vee})(1-k/2) \cong \mathrm{GL}_2(L_{\mathfrak{P}})$ contains an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. This induces the irreducibility (hence absolute irreducibility) over G_K. Moreover, the image of $\mathrm{Gal}(\bar{K}/K^{\mathrm{ab}})$ is open inside $\mathrm{SL}_2(\mathbb{Z}_p)$ (as the derived subgroup), fixing χ_{12} hence containing a nontrivial element $\sigma = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. This implies that Hyp$(\boldsymbol{\sigma})$ holds true (see more in [LLZ15, Sec 7.1]). The condition **(orth)** is automatic for the relaxed-strict condition. Now we are in a position to apply the Jetchev-Nekovář-Skinner machinery (Theorem 4.3.5) to our anticyclotomic Euler system in Theorem 4.3.4, which finishes the proof. □

Corollary 5.1.2. *Assume the same conditions as in Theorem 5.1.1 and assume also that* $\boxed{l \geq k}$. *Then we have:*

$$\boxed{\kappa_{f,\chi_{12}} \neq 0 \quad \Longrightarrow \quad \dim_{\Phi} H^1_f(K, V_f^{\vee}(1-k/2)(\chi_{12})) = 1}$$

Proof. By using Lemma 4.3.1 and Theorem 5.1.1, we obtain the result. □

5.1.1 The case $k \geq l+2$

Now we focus on the case when $\boxed{k \geq l+2}$. Assume that f is not of CM type (i.e. we can apply Theorem 5.1.1) and $\kappa_{f,\chi_{12}} \neq 0$. By Poitou-Tate global duality [Rub00, Thm 1.7.3], one has the following exact hexagon:

$$\begin{array}{ccccccc}
 & & H^1_{\mathrm{ord}}(K_{\bar{\mathfrak{p}}}, V) & \xrightarrow{\mathrm{res}_{\bar{\mathfrak{p}}}^{\vee}} & H^1_{\mathrm{str,rel}}(K, V^{\vee}(1))^{\vee} & & \\
 & \overset{\mathrm{res}_{\bar{\mathfrak{p}}}}{\nearrow} & & & & \searrow & \\
H^1_{\mathrm{rel,ord}}(K, V) & & & & & & H^1_{\mathrm{str,ord}}(K, V^{\vee}(1))^{\vee} \\
 & \underset{\mathrm{res}_{\mathfrak{p}}}{\searrow} & & & & \nearrow & \\
 & & \frac{H^1(K_{\mathfrak{p}}, V)}{H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, V)} & \xrightarrow{\mathrm{res}_{\mathfrak{p}}^{\vee}} & H^1_{\mathrm{ord,ord}}(K, V^{\vee}(1))^{\vee} & &
\end{array} \tag{5.1.1.1}$$

where $V = V_f^{\vee}(1-k/2)(\chi_{12})$. Here we use notations in Section 4.3.1. We also have the following observations.

1. By Lemma 4.3.1, $H^1_{\mathrm{ord,ord}}(K, V)$ will be the Bloch-Kato Selmer group $H^1_f(K, V)$ while $H^1_{\mathrm{rel,str}}(K, V)$ will be our Greenberg Selmer group $H^1_{Gr}(K, V)$ that $\kappa_{f,\chi_{12}}$ lands in. Note that $H^1_{\mathrm{ord}}(K_{\bar{\mathfrak{p}}}, V)$, $\frac{H^1(K_{\mathfrak{p}}, V)}{H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, V)}$, and $H^1_{\mathrm{rel,str}}(K, V)$ (by Theorem 5.1.1) are all one-dimensional. Furthermore, as V is conjugate self-dual, we have $H^1_{\mathrm{str,rel}}(K, V^{\vee}(1)) \cong H^1_{\mathrm{str,rel}}(K, V^c) = H^1_{\mathrm{rel,str}}(K, V)$, i.e. $H^1_{\mathrm{str,rel}}(K, V^{\vee}(1))^{\vee}$ is also one-dimensional.

2. Recall that we are assuming $\kappa_{f,\chi_{12}} \neq 0$. We have:

$$\kappa_{f,\chi_{12}} \in H^1_{\text{rel,str}}(K, V) \hookrightarrow H^1_{\text{rel,ord}}(K, V).$$

Firstly, we use the explicit reciprocity law to obtain the surjectivity of $\text{res}_{\mathfrak{p}}$ (as its range is one-dimensional, and also the projection of the reciprocity law $\text{proj}_{\mathbf{f}}$ in Theorem 4.1.1 maps to $V_f^{\vee,-}$). This implies that $\text{res}_{\bar{\mathfrak{p}}}^{\vee}$ is injective, i.e. an isomorphism (as both of its range and domain are one-dimensional). Hence we have $H^1_{\text{str,ord}}(K, V^{\vee}(1))^{\vee} = 0$ by the exactness of the top sides of the hexagon (5.1.1.1). Moreover, as $\text{res}_{\mathfrak{p}}^{\vee} = 0$ (by the exactness at H^1/H^1_{ord}), we obtain

$$H^1_{\text{ord,ord}}(K, V^{\vee}(1))^{\vee} = H^1_{\text{str,ord}}(K, V^{\vee}(1))^{\vee} = 0.$$

This shows that $H^1_f(K, V) = H^1_{\text{ord,ord}}(K, V) = 0$. In the end, we arrive at:

$$\kappa_{f,\chi_{12}} \neq 0 \quad \Longrightarrow \quad H^1_f(K, V_f^{\vee}(1 - k/2)(\chi_{12})) = 0. \tag{5.1.1.2}$$

Note that in order to be able to apply the reciprocity law as well as the interpolation property, we need the following conditions to hold. They are assumption 4.1.1, which come from the Hsieh triple product $p-$adic $L-$function contruction that we recalled in Section 4.1:

Assumption 5.1.1. (‡)

1. $\gcd(N_{\mathbf{f}}, N_{\mathbf{g}}, N_{\mathbf{h}})$ is squarefree. (for the local Rankin-Selberg calculation)

2. There is a triple of arithmetic point (k, l, m) such that the local root numbers $\epsilon_q(\mathbb{V}^{\dagger}_{\mathbf{f}_k\,\mathbf{g}_l\,\mathbf{h}_m}) = 1$ for all primes $q|N$.

3. The Galois representation attached to f is residually absolutely irreducible and $p-$distinguished.

Using the observations above, we obtain the following case of the Bloch-Kato conjecture:

Theorem 5.1.3. *(**Rank 0** Bloch-Kato) Assume (‡). assume further that:*

- *(**a**) (**non-Heeg**): For $N_f = N_f^+ N_f^-$ where N_f^+ and N_f^- are the products of split and inert primes in K, respectively, N_f^- is a squarefree product of an odd number of (inert) primes.*
- *(**b**) f is a newform of weight $k \geq 4$*
- *(**c**) (**p large**): $p \geq k+2$*
- *(**d**) $(N_f, D_K) = 1$ and $p \nmid N_f D_K$*

Then for all anticyclotomic Hecke characters χ_{12} of infinity type $(l/2, -l/2)$ such that

$$(pN_f D_K, cond(\chi_{12})) = 1, \tag{5.1.1.3}$$

we have

$$\boxed{L(V_f^\vee(1-k/2)(\chi_{12}), 0) \neq 0 \quad \Longrightarrow \quad H_f^1(K, V_f^\vee(1-k/2)(\chi_{12})) = 0.} \tag{5.1.1.4}$$

Remark 5.1.1. Note that for conjugate self-dual representation V i.e. $V^\vee(1) \cong V^c$, one has $L(V,0) = L(V^c,0) = L(V^\vee(1),0)$ so we can write the Bloch-Kato conjecture in this way. Also (**non-Heeg**) combining with $L-$value nonvanishing implies that $k \geq l+2$ and $\epsilon(f, \chi_{12}) = 1$.

Proof. The Hypothesis (**non-Heeg**) implies that f is not a CM form i.e. we can apply our main theorem about **Rank 1** (Theorem 5.1.1).

For χ_{12} an anticyclotomic character of infinity type $(l/2, -l/2)$ with conductors satisfying (5.1.1.3), we want firstly the implication:

$$L(V_f^\vee(1-k/2)(\chi_{12}),0)\neq 0 \quad\Longrightarrow\quad L(V^\dagger_{f\theta_{\psi_1}\theta_{\psi_2}},0)\neq 0 \tag{5.1.1.5}$$

Note that changing the pair $(\psi_1,\psi_2)\mapsto(\psi_1\chi,\psi_2\chi^{-1})$ where χ is a ring class character does not change χ_{12}. Also the decomposition of the Galois representation $V^\dagger_{f\theta_{\psi_1}\theta_{\psi_2}}$ gives:

$$L(V^\dagger_{f\theta_{\psi_1}\theta_{\psi_2}},0)=L(V_f^\vee(1-k/2)(\chi_{12}),0)L(V_f^\vee(1-k/2)(\rho_{12}),0)$$

where $\rho_{12}=\psi_1^{-1}\psi_2^{-c}\mathbf{N}^{-l/2}$ an anticyclotomic Hecke character of infinity type $(l/2-1,1-l/2)$. Changing the pair $(\psi_1,\psi_2)\mapsto(\psi_1\chi,\psi_2\chi^{-1})$ replaces ρ_{12} with $\rho_{12}\cdot\chi^{-2}$. We will show that for any given ψ_1, ψ_2 there exists a ring class character χ such that

$$L(V_f^\vee(1-k/2)(\rho_{12}\chi^{-2}),0)\neq 0. \tag{5.1.1.6}$$

Indeed, such a χ exists (and will have q-power conductor where q is a prime not dividing pN_fD_K) by Lemma 5.1.4 below and [CH18b, Thm 5.9] under the assumption that:

1. $(D_K,N_f^-)=1$ (which holds by (***d***)).
2. $p\nmid N_fD_K$, $p\geq k-2$ (holds by (***d***) and (***p* large**)) and the $p-$adic Galois representation attached to f is absolutely residually irreducible (holds by ‡(3)).

Choosing such a χ, i.e. (5.1.1.5) holds, and noting that $(k,l,2)$ lies in the f-unbalanced range so we can use not only the reciprocity law (Theorem 4.2.1) but also the interpolation property of the triple product $p-$adic $L-$function (Theorem 4.1.1). Furthermore, we would want the implication:

$$L(V^\dagger_{f\theta_{\psi_1}\theta_{\psi_2}},0)\neq 0 \quad\Longrightarrow\quad \kappa_{\mathbf{f},g,h}\neq 0 \quad =\kappa_{f,\rho_{12}}\oplus\kappa_{f,\chi_{12}} \quad\overset{?}{\Longrightarrow}\quad \kappa_{f,\chi_{12}}\neq 0 \tag{5.1.1.7}$$

Here, $\rho_{12} = \psi_1^{-1}\psi_2^{-c}\mathbf{N}^{-l/2}$, which is a Hecke character of infinity type $(l/2-1, 1-l/2)$. The Selmer group $H^1_{Gr}(K, V^\dagger_{f\theta_{\psi_1}\theta_{\psi_2}})$ decomposes under $V^\dagger_{f\theta_{\psi_1}\theta_{\psi_2}} = V_f^\vee(1-k/2)(\rho_{12}) \oplus V_f^\vee(1-k/2)(\chi_{12})$ into $H^1_f(V_f^\vee(1-k/2)(\rho_{12})) \oplus H^1_{Gr}(V_f^\vee(1-k/2)(\chi_{12}))$ (check that the local conditions at both $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ for the first direct summand are $V_f^{\vee,+}(1-k/2)(\rho_{12\mathfrak{P}})$), and the classes $\kappa_{f,\rho_{12}}, \kappa_{f,\chi_{12}}$ are projections to the correspondingly summand. For the $\check{\Longrightarrow}$ one would need $\kappa_{f,\rho_{12}} = 0$, which would follow from

$$H^1_f(K, V_f^\vee(1-k/2)(\rho_{12})) = 0,$$

i.e. the Bloch-Kato conjecture for $V_f^\vee(1-k/2)(\rho_{12})$ would need to hold.

However, because the projection $\mathrm{proj}_{\mathbf{f}}$ in equation (4.2.0.2) maps to $V_f^{\vee,-}(\psi_1^{-1}\psi_2^{-1})$, the image of $\kappa_{\mathbf{f},g,h}$ under $\mathrm{proj}_{\mathbf{f}}$ will not see $\kappa_{f,\rho_{12}}$, i.e. we actually obtain the following implication:

$$L(V^\dagger_{f\theta_{\psi_1}\theta_{\psi_2}}, 0) \neq 0 \quad \Longrightarrow \quad \mathrm{proj}_{\mathbf{f}}(\kappa_{\mathbf{f},g,h}) \neq 0 \quad \Longrightarrow \quad \kappa_{f,\chi_{12}} \neq 0. \qquad (5.1.1.8)$$

We finish the proof by using the observation (5.1.1.2) to obtain that (5.1.1.4) is true for χ_{12} of infinity type $(l/2, -l/2)$. □

Lemma 5.1.4. *Given ρ_1 an anticyclotomic character over K of infinity type $(a, -a)$ and conductor $\mathfrak{f}$. There exists a prime q such that $(q, p\mathfrak{f}) = 1$, and a finite order G_K character χ such that*

$$\rho_1\chi^{-2} = \rho\nu,$$

where ρ is anticyclotomic character over K of infinity type $(a, -a)$ with conductor q, and ν is a finite order anticyclotomic character of $q-$power conductor.

Proof. We first choose a prime q such that $(q, p\mathfrak{f}) = 1$, $q = \mathfrak{q}\bar{\mathfrak{q}}$ splits in K, and $q \equiv 1 \pmod{2|\mathcal{O}_K^\times|}$. As there exists ψ_0 a Hecke character of infinity type $(1, 0)$ and conductor $\mathfrak{q}$, if we denote $\rho = (\psi_0\psi_0^{-c})^a$ then ρ is an anticyclotomic character of infinity type $(a, -a)$ and conductor q. Let $\psi = \rho_1\rho^{-1}$. Then ψ is a finite order anticyclotomic character of conductor $\mathfrak{f}q$. Now as K is an imaginary quadratic field (a finite order character of $\mathbb{A}_K^\times$ will be trivial on $K_\infty^\times$), ψ will be a character of:

$$\frac{\prod_{\lambda \notin P} \mathcal{O}_\lambda^\times \times \prod_{\lambda \in P} K_\lambda^\times}{\mathcal{O}_P^\times} \xrightarrow{\sim} \frac{\mathbb{A}_{K,f}^\times}{K^\times},$$

where P is a finite set consisting of primes $\lambda \nmid q\mathfrak{f}$ that generates the class group of K, and $\mathcal{O}_P^\times$ is the set of $P-$units. Using this identification, we can construct a global character χ such that it is anticyclotomic, its conductor divides $\mathfrak{f}q$, and $\chi_\lambda^2 = \psi_\lambda$ for all $\lambda | \mathfrak{f}$ (where we use $2|\mathcal{O}_K^\times|$ divides $q-1$ for the inclusions $\mathcal{O}_K^\times \hookrightarrow \mathcal{O}_\mathfrak{q}^\times, \mathcal{O}_{\bar{\mathfrak{q}}}^\times$ in order to kill the image of $\mathcal{O}_P^\times$). In the end, if we denote by $\chi^{-2}\rho_1\rho^{-1} = \nu$ then ν is a finite order anticyclotomic character of $q-$power conductor. □

Remark 5.1.2. The case $k = 2$ and $l = 0$ was worked out by Bertolini-Darmon in [BD05] and generalized by Longo-Vigni in [LV10], using the Euler system for CM points on Shimura curves. Hence the restriction to $k \geq 4$ is on the theorem.

Remark 5.1.3. The case $k \geq 4$ and $l = 0$ was worked out by Chida in [Chi17]. The proof in [Chi17] uses the same methods as in [BD05] for the ordinary case, and CM cycles on Kuga-Sato varieties for the non-ordinary case combined with level raising results. In order to obtain such a level raising result, Chida assumed the following hypothesis on the residual Galois representation attached to f, (which will hold for all but finitely many primes if f is not of CM type):

Hypothesis (**level raising**):

1. $p \geq k+2$ and $\#(\mathbb{F}_p^\times)^{k-1} > 5$
2. The residual Galois representation attached to f, denoted $\bar{\rho}_f$, is absolutely irreducible when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{l^\star}))$, where $l^\star = (-1)^{\frac{l-1}{2}} l$.
3. $\bar{\rho}_f$ is ramified at q if either:

(a) $q | N_f^-$ and $q^2 \equiv 1 \pmod p$

(b) $q || N_f^+$ and $q \equiv 1 \pmod p$

4. If $q^2 | N$ and $q \equiv -1 \pmod p$ then the restriction of $\bar{\rho}_f$ to the inertia group at q is irreducible.

This is Hypothesis (CR^+) in [Chi17].

Remark 5.1.4. Kings-Loeffler-Zerbes [KLZ17] also obtained similar results in the case of the Rankin-Selberg product of two modular forms f and g. Nevertheless, they require $\chi_f \chi_g \neq 1$ for their 'big image' hypothesis [KLZ17, Rem 11.1.3], meanwhile our result concerns about $\chi_f = \chi_g = 1$. Moreover, our methods can be generalized to totally real fields, which is presumably not true for the Euler system of Rankin-Eisenstein classes used in [KLZ17].

5.2 The work of Castella-Hsieh (and Magrone)

We recall the results and methods that were used in [CH18a], where they obtained the analytic rank zero case of Bloch-Kato conjecture but with the Heegner Hypothesis **(Heeg)**, which contrasts with our assumption (**non-Heeg**) in Theorem 5.1.3.

Set-up.

1. f is a newform of $S_k(\Gamma_0(N))$, where k is even.

2. K is an imaginary quadratic of odd discriminant $-D_K < -3$.

3. χ is an anticyclotomic Hecke character of infinity type $(l/2, -l/2)$, where l is an even integer, such that its conductor is prime to N.

Assumption 5.2.1.

1. **(Heeg)**: N is a product of primes that split in K.

2. $p \nmid 2(k-1)!N\varphi(N)$ is a prime that splits in K.

3. f is $p-$ordinary.

The following theorem is [CH18a, Thm A].

Theorem 5.2.1. *[CH18a](**Rank 0** Bloch-Kato) Under Assumption 5.2.1, the following implication holds:*

$$L(f,\chi,k/2) \neq 0 \quad \Longrightarrow \quad H^1_f(K, V_f^\vee(1-k/2)(\chi)) = 0$$

Remark 5.2.1. For this implication to be non-vacuous, the condition $\boxed{|l| \geq k}$ is necessary. Indeed, $L(f,\chi,k/2) \neq 0$ implies that $\epsilon(f,\chi) = 1$. Combining that with **(Heeg)**, we obtain $|l| \geq k$.

When $\boxed{|l| < k}$, so $\epsilon(f,\chi) = -1$, the $\chi-$component of z_f (some $p-$adic Abel-Jacobi image of generalized Heegner cycles), form an anticyclotomic Euler system. In this case, one has the following theorem [CH18a, Thm B].

Theorem 5.2.2. *[CH18a](**Rank 1** result)*

$$z_{f,\chi} \neq 0 \quad \Longrightarrow \quad H^1_f(K, V_f^\vee(1-k/2)(\chi)) = \Phi \cdot z_{f,\chi} \tag{5.2.0.1}$$

Remark 5.2.2. Under the generalized **(Heeg)**, i.e. N^- is a squarefree product of even number of inert primes, Magrone [Mag] in her thesis obtains a more general result using the generalized Kuga-Sato varieties instead of generalized Heegner cycles, and Brooks' generalization of BDP in a quaternionic setting [Bro14].

5.3 Updated picture

As a consequence of the results in this Section we can update the picture Ⓡ described in the Introduction.

Under the following setup:

1. f is a newform of $S_k(\Gamma_0(N))$ with k even,

2. K is an imaginary quadratic of odd discriminant $-D_K < -3$,

3. χ is an anticyclotomic Hecke character of infinity type $(l/2, -l/2)$, where l is an even integer, such that its conductor is prime to N,

4. $N = N^+N^-$, where N^+ and N^- are the product of split and inert primes in K respectively, and N^- is squarefree,

we obtain the following updated table

	(Heeg) [CH18a],[Mag] (number of inert primes $\mid N^-$ is even)	(**non-Heeg**) This thesis (number of inert primes $\mid N^-$ is odd)
$l < k$	1st quadrant $\epsilon(f, \chi) = -1$ ES of generalized Heegner cycles	2nd quadrant $\epsilon(f, \chi_{12}) = 1$ Bloch-Kato conjecture for rank 0
$l \geq k$	3rd quadrant $\epsilon(f, \chi) = 1$ Bloch-Kato conjecture for rank 0	4th quadrant $\epsilon(f, \chi_{12}) = -1$ My ES from diagonal cycles

We finish by noting that the modularity theorem associates each rational elliptic curve a newform f of weight 2, hence our anticyclotomic Euler system fits right in the 4th quadrant. But the Bloch-Kato result for analytic rank 0 that fits in the 2nd quadrant was taken from [BD05] and [LV10].

Remark 5.3.1. Notice that we do not use any Hypothesis **(Heeg)** or (**non-Heeg**) in our construction. With a slight modification in the construction, we expect a new anticyclotomic Euler system in the 1^{st} quadrant. Using a different reciprocity law in Theorem 4.2.1 ($\text{proj}_{\mathbf{g}}$ that maps cohomology classes to $\mathcal{L}_p^g$), we get classes in $H^1_f(K, V_f^\vee(1-k/2)(\chi_{12}))$ and $H^1_{Gr}(K, V_f^\vee(1-k/2)(\rho_{12}))$. By specialising to $|l| > k$, we hope to recover the Bloch-Kato conjecture for rank 0 results of [CH18a] and [Mag] in the 3^{rd} quadrant using the non vanishing result of Hsieh [Hsi14, Thm C]. In particular, we expect the Euler system constructed in this thesis to control the arithmetic results of the whole table!

Bibliography

[ACR21] Raúl Alonso, Francesc Castella, and Óscar Rivero. Iwasawa theory for $\mathrm{GL}_2 \times \mathrm{GL}_2$ and diagonal cycles, 2021. Available at https://arxiv.org/abs/2106.05322.

[AS86a] Avner Ash and Glenn Stevens. Cohomology of arithmetic groups and congruences between systems of hecke eigenvalues. *Journal für die reine und angewandte Mathematik*, 365:192–220, 1986.

[AS86b] Avner Ash and Glenn Stevens. Modular forms in characteristic l and special values of their $L-$functions. *Duke Mathematical Journal*, 53(3):849 – 868, 1986.

[BBJ20] Réda Boumasmoud, Ernest Hunter Brooks, and Dimitar P. Jetchev. Vertical distribution relations for special cycles on unitary Shimura varieties. *Int. Math. Res. Not. IMRN*, (13):3902–3926, 2020.

[BD05] Massimo Bertolini and Henri Darmon. Iwasawa's main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p-$extensions. *Annals of Mathematics*, 162(1):1–64, 2005.

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna. Generalized Heegner cycles and p-adic Rankin L-series. *Duke Mathematical Journal*, 162(6):1033 – 1148, 2013.

[Bel] Joël Bellaïche. An introduction to the conjecture of bloch and kato. Available at https://www.claymath.org/sites/default/files/bellaiche.pdf.

[BK07] Spencer Bloch and Kazuya Kato. $L-$functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift: A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck*, pages 333–400. Birkhäuser Boston, Boston, MA, 2007.

[BL18] Kâzım Büyükboduk and Antonio Lei. Anticyclotomic $p-$ordinary iwasawa theory of elliptic modular forms. *Forum Mathematicum*, 30(4):887–913, 2018.

[Bro14] Ernest Hunter Brooks. Shimura Curves and Special Values of $p-$adic $L-$functions. *International Mathematics Research Notices*, 2015(12):4177–4241, 2014.

[BSV21] Massimo Bertolini, Marco Seveso, and Rodolfo Venerucci. Reciprocity laws for balanced diagonal cycles. *Astérisque*, 2021. To appear.

[CH18a] Francesc Castella and Ming-Lun Hsieh. Heegner cycles and p-adic L-functions. *Mathematische Annalen*, 370(1):567–628, 2018.

[CH18b] Masataka Chida and Ming-Lun Hsieh. Special values of anticyclotomic L-functions for modular forms. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2018(741):87–131, 2018.

[Chi17] Masataka Chida. Selmer groups and central values of L-functions for modular forms. *Annales de l'Institut Fourier*, 67(3):1231–1276, 2017.

[DDT97] Henri Darmon, Fred Diamond, and Richard Taylor. Fermat's last theorem. In *Elliptic curves, modular forms and Fermat's last theorem*, pages 2–140. International Press, Cambridge, MA, 1997.

[Del71] Pierre Deligne. Formes modulaires et représentations l-adiques. In *Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363*, volume 175 of *Lecture Notes in Math.*, pages Exp. No. 355, 139–172. Springer, Berlin, 1971.

[Dis17] Daniel Disegni. The p-adic Gross-Zagier formula on Shimura curves. *Compos. Math.*, 153(10):1987–2074, 2017.

[DR14] Henri Darmon and Victor Rotger. Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(4):779–832, 2014.

[DR17] Henri Darmon and Victor Rotger. Diagonal cycles and Euler systems II: The Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-functions. *J. Amer. Math. Soc.*, 30(3):601–672, 2017.

[Eic54] Martin Eichler. Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion. *Arch. Math.*, 5:355–366, 1954.

[FK88] Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture*, volume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné.

[GK92] Benedict H. Gross and Stephen S. Kudla. Heights and the central critical values of triple product L-functions. *Compositio Math.*, 81(2):143–209, 1992.

[Gre94] Ralph Greenberg. Iwasawa theory and p-adic deformations of motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 193–223. Amer. Math. Soc., Providence, RI, 1994.

[Gro91] Benedict H. Gross. Kolyvagin's work on modular elliptic curves. In *L-functions and arithmetic (Durham, 1989)*, volume 153 of *London Math. Soc. Lecture Note Ser.*, pages 235–256. Cambridge Univ. Press, Cambridge, 1991.

[GS93] Ralph Greenberg and Glenn Stevens. p-adic L-functions and p-adic periods of modular forms. *Invent. Math.*, 111(2):407–447, 1993.

[GS95] Benedict H. Gross and Chad Schoen. The modified diagonal cycle on the triple product of a pointed curve. *Ann. Inst. Fourier (Grenoble)*, 45(3):649–679, 1995.

[GZ86] Benedict H. Gross and Don B. Zagier. Heegner points and derivatives of L-series. *Invent. Math.*, 84(2):225–320, 1986.

[Hid88] Haruzo Hida. Modules of congruence of Hecke algebras and L-functions associated with cusp forms. *Amer. J. Math.*, 110(2):323–382, 1988.

[Hsi14] Ming-Lun Hsieh. Special values of anticyclotomic Rankin-Selberg L-functions. *Doc. Math.*, 19:709–767, 2014.

[Hsi21] Ming-Lun Hsieh. Hida families and p-adic triple product L-functions. *Amer. J. Math.*, 143(2):411–532, 2021.

[HT01] Michael Harris and Jacques Tilouine. p-adic measures and square roots of special values of triple product L-functions. *Math. Ann.*, 320(1):127–147, 2001.

[Iha75] Yasutaka Ihara. On modular curves over finite fields. In *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, pages 161–202. Oxford Univ. Press, 1975.

[Jet] Dimitar P. Jetchev. Hecke and galois properties of special cycles on unitary shimura varieties. Available at https://arxiv.org/abs/1410.6692.

[JNS] Dimitar P. Jetchev, Jan Nekovář, and Christopher Skinner. preprint.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. In *Cohomologies p-adiques et applications arithmétiques. III*, number 295, pages ix, 117–290. Astérisque, 2004.

[KLZ17] Guido Kings, David Loeffler, and Sarah Livia Zerbes. Rankin-Eisenstein classes and explicit reciprocity laws. *Camb. J. Math.*, 5(1):1–122, 2017.

[Kol90] V. A. Kolyvagin. Euler systems. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 435–483. Birkhäuser Boston, Boston, MA, 1990.

[LLZ15] Antonio Lei, David Loeffler, and Sarah Livia Zerbes. Euler systems for modular forms over imaginary quadratic fields. *Compos. Math.*, 151(9):1585–1625, 2015.

[LV10] Matteo Longo and Stefano Vigni. On the vanishing of Selmer groups for elliptic curves over ring class fields. *J. Number Theory*, 130(1):128–163, 2010.

[Mag] Paola Magrone. Generalized heegner cycles and $p-$adic $l-$functions in a quaternionic setting. Available at https://arxiv.org/abs/2008.13500.

[Mil] James S. Milne. Class field theory. available at https://www.jmilne.org/math/CourseNotes/cft.html.

[Mil80] James S. Milne. *Étale cohomology*. Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, N.J., 1980.

[Miy89] Toshitsune Miyake. *Modular forms.* Springer-Verlag, Berlin, 1989. Translated from the Japanese by Yoshitaka Maeda.

[MN19] Ahmed Matar and Jan Nekovář. Kolyvagin's result on the vanishing of $Ш(E/K)[p^\infty]$ and its consequences for anticyclotomic Iwasawa theory. *J. Théor. Nombres Bordeaux*, 31(2):455–501, 2019.

[Mom81] Fumiyuki Momose. On the l-adic representations attached to modular forms. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(1):89–109, 1981.

[MW84] B. Mazur and A. Wiles. Class fields of abelian extensions of $\mathbf{Q}$. *Invent. Math.*, 76(2):179–330, 1984.

[MW86] B. Mazur and A. Wiles. On p-adic analytic families of Galois representations. *Compositio Math.*, 59(2):231–264, 1986.

[Nek93] Jan Nekovář. On p-adic height pairings. In *Séminaire de Théorie des Nombres, Paris, 1990–91*, volume 108 of *Progr. Math.*, pages 127–202. Birkhäuser Boston, Boston, MA, 1993.

[NN16] Jan Nekovář and Wiesł awa Nizioł. Syntomic cohomology and p-adic regulators for varieties over p-adic fields. *Algebra Number Theory*, 10(8):1695–1790, 2016. With appendices by Laurent Berger and Frédéric Déglise.

[PR87] Bernadette Perrin-Riou. Fonctions L p-adiques, théorie d'Iwasawa et points de Heegner. *Bull. Soc. Math. France*, 115(4):399–456, 1987.

[Rib77] Kenneth A. Ribet. Galois representations attached to eigenforms with Nebentypus. In *Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 17–51. Lecture Notes in Math., Vol. 601, 1977.

[Rib84] Kenneth A. Ribet. Congruence relations between modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 503–514. PWN, Warsaw, 1984.

[Rib85] Kenneth A. Ribet. On l-adic representations attached to modular forms. II. *Glasgow Math. J.*, 27:185–194, 1985.

[Rub00] Karl Rubin. *Euler systems*, volume 147 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.

[Sch90] A. J. Scholl. Motives for modular forms. *Invent. Math.*, 100(2):419–430, 1990.

[Shi58] Goro Shimura. Correspondances modulaires et les fonctions ζ de courbes algébriques. *J. Math. Soc. Japan*, 10:1–28, 1958.

[Shi71] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*. Kanô Memorial Lectures, No. 1. Iwanami Shoten Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. Publications of the Mathematical Society of Japan, No. 11.

[Ski] Christopher Skinner. Lectures on the iwasawa theory of elliptic curves. Available at https://swc-math.github.io/aws/2018/2018SkinnerNotes.pdf.

[SU14] Christopher Skinner and Eric Urban. The Iwasawa main conjectures for GL_2. *Invent. Math.*, 195(1):1–277, 2014.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

[YZZ] Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang. Triple product $L-$series and gross-kudla-schoen cycles. Available at https://math.mit.edu/~wz2113/math/online/triple.pdf.