DISCUSSION NOTES - MATH 32A

JOHN HOPPER

Table of Contents

Week 1

Logistics

Ta: John Hopper
Email: jshopper@math.ucla.edu, (jshopper@g.ucla.edu also works)
Office: MS 3921
OH: Thursdays 9-10am

Exercieses

Here was the bonus problem I wrote down, written out with a solution. I recommend giving it a try before reading it.

Exercise 1.1. If $J \subset \mathbb{R}$ is a closed and bounded set, and $f : \mathbb{R} \to \mathbb{R}$ is monotonically non-decreasing - that is for all $a \leq b$ that $f(a) \leq f(b)$ - then there exists $y \in J$ such that $f(y) = \sup\{f(x) : x \in J\}$.

Proof. We can recall from discussion that since f is monotonically increasing if we can find the right-most (or greatest) point in J that will be a good choice for y.

We first show that there is such a right-most point. I don't know what you covered last quarter, maybe you showed that closed and bounded subsets of $\mathbb{R}$ have a greatest point, in case you didn't we can show this with the extreme value theorem, EVT. Consider the function $g(x) = x$, which is continuous and as such we know that since J is compact (recall closed and bounded is the same as compact), then we know that g achieves supremum on J. That is there is a $y \in J$ where $g(y) \geq g(x)$ for all other $x \in J$. But recall, $g(y) = y$ and $g(x) = x$ so this is simply

Date: March 10, 2025.

a point $y \in J$ with $y \geq x$ for all $x \in J$ which is the 'right-most point' we were asking for.

We can now notice that for all $x \in J$ we know $x \leq y$ and thus by monotonicity of f that $f(x) \leq f(y)$ which shows that $f(y)$ is an upper bound for $\{f(x) : x \in J\}$, since it is in this set it must actually be the least upper bound. Thus, $f(y) = \sup\{f(x) : x \in J\}$ as we desired to show. □

I realize the last question I mentioned in discussion was a bit longer than first realized (and I rushed through it in class, which doesn't help). I have written it more carefully here. Below it is also a slightly harder, but very similar problem for you to try on your own if you want to test your understanding of this proof.

Exercise 1.2. Suppose $J = [0, 1]$ and P_n is the partition given by

$$\left\{ \left[\frac{i}{n}, \frac{i+1}{n}\right] : 0 \leq i \leq n-1, \text{ and } i \in \mathbb{Z} \right\}.$$

Suppose further that $f : \mathbb{R} \to \mathbb{R}$ is 1-Lipchitz continuous, that is f is continuous and $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Show that $\lim_{n\to+\infty} L(f, P_n) - U(f, P_n) \to 0$

Proof. First, lets write out what it is we are taking a limit of and try to unpack the notation a bit:

$$L(f, P_n) - U(f, P_n) = \sum_{J\in P_n} m_J \text{length}(J) - \sum_{J\in P_n} M_J \text{length}(J).$$

Next, we can note that the summations (the big Sigmas) are over the same set and so we can put them together,

$$\begin{aligned} \sum_{J\in P_n} m_J \cdot \text{length}(J) - \sum_{J\in P_n} M_J \cdot \text{length}(J) &= \sum_{J\in P_n} m_J \cdot \text{length}(J) - M_J \cdot \text{length}(J) \\ &= \sum_{J\in P_n} (m_J - M_J) \cdot \text{length}(J). \end{aligned}$$

Lets look more closely at each term in the sum individually. We can note that $\text{length}(J) = \frac{1}{n}$ since each interval in the partition is of the form $[\frac{i}{n}, \frac{i+1}{n}]$. Now lets turn our attention to $m_J - M_J$, we are going to try a squeeze theorem, so we need upper and lower bounds. We know that $m_J \leq M_J$ so we have $m_J - M_J \leq 0$. We also know that since f is continuous $m_J = f(x_1)$ and $M_J = f(x_2)$ where $x_1, x_2 \in J$, but J is an interval of length $1/n$ so $|x_1 - x_2| \leq 1/n$ so we can the 1-Lipchitz property to see that

$$|m_J - M_J| = |f(x_1) - f(x_2)| \leq |x_1 - x_2| \leq 1/n.$$

We also know the sign, (it is negative) so we have that

$$\frac{-1}{n} \leq m_J - M_J \leq 0.$$

We now have bounds on each part, time to put them all together:

$$\begin{aligned} L(f,P_n) - U(f,P_n) &= \sum_{J \in P_n} (m_J - M_J) \cdot \text{length}(J) \\ &= \sum_{J \in P_n} (m_J - M_J) \cdot \frac{1}{n} \\ &\leq \sum_{J \in P_n} 0 \cdot \frac{1}{n} \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} L(f,P_n) - U(f,P_n) &= \sum_{J \in P_n} (m_J - M_J) \cdot \text{length}(J) \\ &= \sum_{J \in P_n} (m_J - M_J) \cdot \frac{1}{n} \\ &\geq \sum_{J \in P_n} \frac{-1}{n} \cdot \frac{1}{n} \\ &= \sum_{J \in P_n} \frac{-1}{n^2} \\ &= \frac{-1}{n} \end{aligned}$$

the last equality comes from the fact there are n sub-intervals of P_n. We have thus shown that $\frac{-1}{n} \leq L(f,P_n) - U(f,P_n) \leq 0$, and we know that $\lim_{n\to+\infty} 0 = \lim_{n\to+\infty} \frac{-1}{n} = 0$, thus by squeeze theorem we know that $\lim_{n\to+\infty} L(f,P_n) - U(f,P_n) = 0$. □

If you want to practice, below is a very similar problem to 1.2. I would suggest taking a break then trying to solve 1.3 without looking at the solution to 1.2. If you get stuck, try look at it. I do recommend taking a (1 hour or more) break because it is often easy to miss the hard part of proofs if you just read them. The goal here is to remember some of the proof, but not all of it so it more closely resembles your working on homework/midterm problems.

Question 1.3. *Suppose $J = [0,1]$ and P_n is the partition given by*

$$\left\{ \left[\frac{i}{n}, \frac{i+1}{n} \right] : 0 \leq i \leq n-1, \text{ and } i \in \mathbb{Z} \right\}.$$

Suppose further there is a $\lambda > 0$ and $0 < \alpha < 1$ where for all $x, y \in \mathbb{R}$ we know $|f(x) - f(y)| \leq \lambda |x-y|^\alpha$. Show that $\lim_{n\to+\infty} L(f,P_n) - U(f,P_n) \to 0$.

Back to top

Week 2

Warm-up

Are the following statements true or false?

(1) For any continuous and bounded function f and box J, $M_J(f) > m_J(f)$. (F)
(2) If f is a continuous function and J is a bounded box, then f is bounded on J. (T)
(3) The area of the box $[0,1] \times [2,3] \times [0, \frac{1}{2}]$ is 2.5. (F)
(4) If P' is a refinement of the partition P, then we know that $U(f,P) \geq U(f,P')$. (T)
(5) For some particularly nasty bounded functions, f, there are partitions P and Q where $U(f,P) \leq L(f,Q)$. (F)
(6) If f is a bounded function and B a bounded box, then f is integrable. (F)
(7) If f is bounded and integrable and P is a partition of B, then $\int_B f \leq L(f,P)$. (F)
(8) If f is a bounded function and B a bounded box and for every $\epsilon > 0$ we know that there is a partition P_ϵ where $U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$ then f is integrable. (T)

Question 1

a)If f is a bounded function and B a bounded box and for every $\epsilon > 0$ we know that there is a partition P_ϵ where $U(f,P_\epsilon) - L(f,P_\epsilon) \leq \epsilon$ then f is integrable. (Caution! This is not the same as (8) above as we have a less than or equal to).

b) Suppose that f is a bounded function, and B is a bounded box. We know that for every n there is a partition P_n where $U(f,P_n) - L(f,P_n) \leq \frac{1}{n}$, then f is integrable.

Answer:

Proof of a). Fix $\epsilon > 0$ then we can note that for $\epsilon' = \epsilon/2 < \epsilon$. We can apply the assumption there is a partition P such that $U(f,P) - L(f,P) \leq \epsilon' < \epsilon$. □

proof of b). Fix $\epsilon > 0$, we wish to show that there is a partition, P, such that $U(f,P) - L(f,P) < \epsilon$.

Note that $1/\epsilon > 0$ let n be the smallest integer greater than or equal to $(1/\epsilon)+1$. In particular we have that $n > 1/\epsilon$ and thus $1/n < \epsilon$. Let $P = P_n$ so we have that $U(f,P) - L(f,P) \leq 1/n < \epsilon$. □

Question 2

Exercise 2.1. Show that if B is a box and f is integrable on B then $|f|$ is integrable on B.

Hint: fist show lemma 2.3.

Proof. Fix $\epsilon > 0$, since f is integrable we know that there is a partition P where $U(f, p) - L(f, P) < \epsilon$. We can then use the lemma to see that

$$\begin{aligned} U(|f|, P) - L|f|, P) &= \sum_{J \in P} (M_J(|f|) - m_J(|f|))\text{vol}(J) \\ &\leq \sum_{J \in P} (M_J(f) - m_J(f))\text{vol}(J) \\ &= U(f, P) - L(f, P) < \epsilon \end{aligned}$$

. Thus, we have shown for all $\epsilon > 0$ there is a partition P such that $U(|f|, P) - L|f|, P) < \epsilon$, so f is integrable.

□

Question 2.2. *Is the converse true? If $|f|$ is integrable on B then is f integrable. Either prove or find a counterexample (and prove the counterexample works, that is $|f|$ is integrable but f is not you may use the any homework problems as if you solved them).*

Answer: No, consider the function f which is 1 on the irrational numbers between 0 and 1 and -1 on the rational numbers, f is not integrable, but $|f| = 1$ is integrable.

Lemma 2.3. *Show that for a box J that $M_J(f) - m_J(f) \geq M_J(|f|) - m_J(|f|) \geq 0$*

Proof. One of the most straight forward proof is to break it into cases where 1) $M_J(f) \geq m_J(f) \geq 0$, 2) $M_J(f) \geq 0 > m_J(f)$, and $0 > M_J(f) \geq m_J(f)$.

The first case $|f| = f$ so $M_J(f) - m_J(f) = M_J(|f|) - m_J(|f|)$.

The third case $|f| = -f$ so $M_J(|f|) = -m_J(f)$ and $m_J(|f|) = -M_J(f)$ and you can check that $M_J(f) - m_J(f) = (-m_J(f)) - (-M_J(f)) = M_J(|f|) - m_J(|f|)$.

The second case we can note that $M_J(|f|) = \max(M_J(f), -m_J(f)) \leq M_J(f) - m_J(f)$ and that $m_J(f) \geq 0$ since 0 is a lower bound for $|f|$ (note finding $m_J(|f|)$ exactly is not feasible without more information about f). Thus we can see that $M_J(|f|) - m_J(|f|) \leq M_J(f) - m_J(f) - 0$. □

Question 3

Exercise 2.4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded and integrable on the interval $[a, b]$. Show that $g : \mathbb{R}^2 \to \mathbb{R}$ where $g(x, y) = f(x)$ is integrable on the box $[a, b] \times [c, d]$ for any $-\infty < c < d < +\infty$.

Back to top

Question 4

Exercise 2.5. Show that $\log_{10}(x)$ is integrable on $[1, 10]$.

Hint: Notice $\log_{10}$ is increasing so $M_J f - m_J f$ where $J = [t_0, t_1]$ is $f(t_1) - f(t_0)$ (to prove this use 1.1). Can you find a partition P_n where $f(t_1) = 0.1$, $f(t_2) = 0.2$ ... what about for 0.01 and 0.02 or for $t_1 = 1/n$ and $t_2 = 2/n$, use Question 1 part b.

Week 3

Warm-up

Exercise 3.1. Show that a single point $\{x_0\} = S$ has volume zero.

Exercise 3.2. Show that $f(x) = x$ on $\mathbb{R}$ is uniformly continuous.

Uniform Continuity

Exercise 3.3. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is called α-Hölder continuous if there is a $\lambda > 0$ such that $\|f(x) - f(y)\| < \lambda \|x - y\|^\alpha$ for all $x, y \in \mathbb{R}^n$. Show that for any $0 < \alpha < 1$, if f is α-Hölder continuous then f is uniformly continuous.

Exercise 3.4. Show that $f(x) = x^2$ on all of $\mathbb{R}$ is not uniformly continuous.

Volume Zero

Exercise 3.5 (Exercise 6.5.7 in the book)**.** Prove that if S_1 and S_2 have volume zero, then so does $S_1 \cup S_2$. (Hint: $\chi_{S_1 \cup S_2} \leq \chi_{S_1} + \chi_{S_2}$.)

Corollary 3.6. *Show that for all $n \in \mathbb{N}$, if $S_1, S_2, \ldots, S_n$ have volume zero then so does $\cup_{i=1}^{n} S_i$.*

Hint: Use proof by induction

Question 3.7. *If $S_1, \cdots \subset [0,1]$ have volume zero is it true that $\cup_{i=1}^{+\infty} S_i$ has volume zero. Prove or provide a counterexample.*

Exercise 3.8. If N has volume zero and S is has volume $v \in [0, +\infty)$, show that $S \setminus N$ has volume v as well.

Back to top

Week 4

Below are some classic computations that you would probably see if you took 32B and require some tricks that it would be good to know. These assume you are familiar with polar, spherical, and cylindrical coordinates.

Exercise 4.1. Find the area of the ellipse $\frac{x^2}{a} + \frac{y^2}{b} \leq 1$.

Hint: Try a change of variables to make it a circle.

Exercise 4.2. Find the volume of the ice-cream cone: $x^2 + y^2 + z^2 \leq 4$ and $\sqrt{x^2 + y^2} \leq z$.

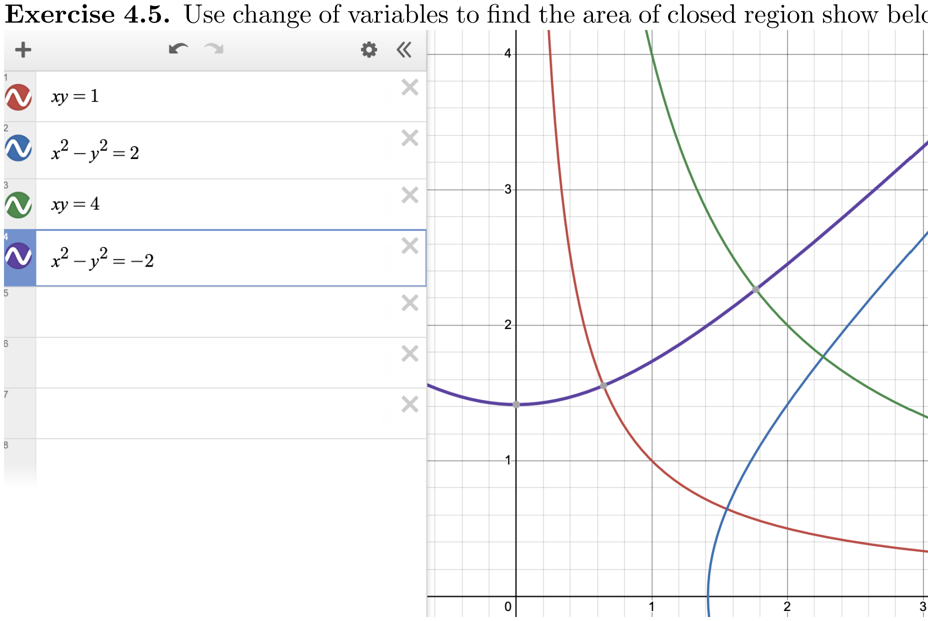
Hint: Use spherical coordinates

Exercise 4.3. Find the volume of the the frustum: $2\sqrt{x^2 + y^2} \leq \sqrt{3}z$, $1 \leq z \leq 2$.

Hint Use cylindrical coordinates, i.e. polar for x-y and leave z alone.

Exercise 4.4. Show that $\int_{\mathbb{R}} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$, by first showing that $\int_{\mathbb{R}^2} e^{-x^2-y^2} dA = \frac{\pi}{2}$ and then applying Fubini's theorem.

Exercise 4.5. Use change of variables to find the area of closed region show below:



Hint: The region looks vaguely rectangular, can you find a change of variables to make the region a rectangle?

Back to top

Week 5

Arclength

You were told in class that $\int_\Phi f = \int_D (f \circ \Phi)\text{vol}_k(D_1\Phi, \dots, D_k\Phi)$. Here we will look a little closer when $k = 1$ and $f = 1$ or in other words the arc-length of a curve.

Exercise 5.1. Show the following are equivalent for a curve $\Phi : [0, T] \to \mathbb{R}^n$,

(1) $\int_\Phi 1$
(2) $\int_0^T |\Phi'(t)|dt$

Moreover show that that the arclength of the graph $G(f)$ is given by $\int_0^T \sqrt{1 + |f'(t)|^2}dt$.

Proof. We can note that we can parameterize the 1-surface by Φ and $VolP_1(D_1\Phi) = |D_1\Phi| = |\frac{d}{dx}\Phi(x)| = |\Phi'(x)|$. Thus, our formula says that $\int_\Phi 1 = \int_0^T 1|\Phi'(t)|dt$ as desired. Now we can note that the graph $G(f)$ can be parameterized by $\Phi : (x) \mapsto (x, f_1(x), \dots, f_n(x)) = (x, f(x))$ in which case $D_1\Phi = (1, f_1'(x), \dots, f_n'(x)) = (1, f'(x))$ and $|D_1\Phi| = \sqrt{1^2 + f_1'(x)^2 + \cdots + f_n'(x)^2} = \sqrt{1 + |f'(x)|^2}$ and thus we get that the arclength is $\int_\Phi 1 = \int_0^T \sqrt{1 + |f'(x)|^2}dx$ as desired. □

Exercise 5.2. Let $n = 3$, and suppose that $|\nabla\Phi| = 1$. Show that $\frac{d}{dt}(\nabla\Phi(t))$ and $\nabla\Phi$ are perpendicular. Furthermore, argue that $\nabla\Phi(t)$, $\frac{\frac{d}{dt}\nabla\Phi(t)}{|\frac{d}{dt}\nabla\Phi(t)|}$ and their cross product form an orthonormal basis.

Surface Area

The following is a fun property about the surface area of spheres in three dimensions:

Exercise 5.3. Find the surface area of the section of the sphere $x^2 + y^2 + z^2 = 1$ where $t \leq z \leq 1$ for $t \geq 0$ by letting $\Phi : \{x^2 + y^2 \leq 1\} \to \mathbb{R}^3$ $\Phi(x, y) = (x, y, \sqrt{1 - x^2 + y^2})$. Compute the surface area where $t_1 \leq z \leq t_2$ for generic $-1 \leq t_1 \leq t_2 \leq 1$.

Proof. We will prove this with two different choices of Φ:

(1) Spherical Coordinate Φ, we can note that the surface of the sphere can be easily parameterized in spherical coordinates since $\rho \equiv 1$ and thus we can send $(\theta, \phi) \mapsto (1, \theta, \phi)$ where the latter is spherical so if we map to cartesian we can see that it is $(\theta, \phi) \mapsto (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$. We can then see that the bounds should be $[0, 2\pi] \times [0, \arccos(t)]$. We can then see that

$$D_1\Phi = (-\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0)$$

and

$$D_2\Phi = (-\cos(\phi)\cos(\theta), -\cos(\phi)\sin(\theta), \sin(\phi)).$$

We can note that these two are orthogonal (take their dot product) and so

$$vol(P_2(D_1\Phi, D_2\Phi) = |D_1\Phi| \cdot |D_2\Phi| = |\sin(\phi)| \cdot 1 = \sin(\phi).$$

Now, we can use our formula to see that the surface area is

$$\int_\Phi 1 = \int_0^{\arccos(t)} \int_0^{2\pi} \sin(\phi)d\theta d\phi = 2\pi[-\cos(\phi)]_{\phi=0}^{\phi=\arccos(t)} = 2\pi(1-t).$$

(2) Thinking of the surface as a graph. This is a very common trick most k-surfaces can be thought of as graphs of functions in which case $\Phi(u) = (u, G(u))$ (maybe not in that order you can imagine that the half of the sphere where $x > 0$ would be a graph and $\Phi = (G(y,z), y, z)$ but this general form is very common). We just need to solve one variable in terms of the others the initial issue is to do this we need to square root which gives us two answers instead of 1 (plus and minus) but since $z \geq 0$ we can ignore the negative answer (in some situations you can get around this by writing a surface as the union of two disjoint surfaces each which can be viewed as a graph, though this can make the notation cumbersome). We have $z = \sqrt{1-x^2-y^2} = G(x,y)$ so $\Phi(x,y) = (x, y, \sqrt{1-x^2-y^2})$. We can find

$$D_x\Phi = (1, 0, \frac{-x}{\sqrt{1-x^2-y^2}})$$

and

$$D_x\Phi = (0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}).$$

Since we are in three dimensions we can find the volume of the parallelepiped with a cross product:

$$\begin{aligned} vol(P_2(D_1\Phi, D_2\Phi)) &= |(1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}) \times (0, 1, \frac{-y}{\sqrt{1-x^2-y^2}})| \\ &= |(\frac{y}{\sqrt{1-x^2-y^2}}, \frac{x}{\sqrt{1-x^2-y^2}}, 1)| \\ &= \sqrt{\frac{x^2+y^2+(1-x^2-y^2)}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}}. \end{aligned}$$

So we want to integrate $\int_{\{(x,y):x^2+y^2\leq 1-t^2\}} \frac{1}{\sqrt{1-x^2-y^2}}$ which is best done in polar, we could have done the initial Φ in terms of polar and get to the same point but I feel like that is slightly less obvious than viewing. Converting to polar the integral becomes

$$\int_0^{\sqrt{1-t^2}} \int_0^{2\pi} \frac{r}{\sqrt{1-r^2}} d\theta dr = 2\pi \left[-\sqrt{1-r^2}\right]_{r=0}^{r=\sqrt{1-t^2}} = 2\pi[1-t].$$

For the more general problem we can note that for $t_1 \geq t_2 \geq 0$ clearly the surface area is $2\pi(1-t_2) - 2\pi(1-t_1) = 2\pi(t_1 - t_2)$ we can then argue by symmetry that this works for more general t_1 and t_2. □

Next, we will prove a formula from Calc 2 about the surface area of revolution

Exercise 5.4. Suppose that $f : [a,b] \to \mathbb{R}_{\geq 0}$ is smooth. Compute the surface area of the volume of revolution when rotating around the x-axis and show it is equal to $\int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}dx$.

Hint: Let $\Phi : [a,b] \times [0, 2\pi]$ send (x,θ) to $(x, f(x)\cos(\theta), f(x)\sin(\theta))$.

Proof. Following the hint we can see that

$$D_1\Phi = (1, f'(x)\cos(\theta), f'(x)\sin(\theta) \qquad D_2\Phi = (0, -f(x)\sin(\theta), f(x)\cos(\theta).$$

We can note that our the volume is once again the magnitude of the cross product which is

$$\begin{aligned}|D_1\Phi \times D_2\Phi| &= |(f'(x)f(x)(\sin^2(\theta)+\cos^2(\theta)), -f(x)\sin(\theta), -f(x)\cos(\theta))| \\ &= \sqrt{f'(x)^2f(x)^2+f(x)^2} = |f(x)|\sqrt{1+f'(x)^2}.\end{aligned}$$

We can then plug this into our fomula, use Fubini and note $f(x) > 0$ so $|f(x)| = f(x)$ to see that

$$\int_\Phi 1 = \int_a^b \int_0^{2\pi} 1 \cdot f(x)\sqrt{1+f'(x)^2}dx = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}.$$

□

Surface integrals are consistent

Exercise 5.5. Suppose that $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \to \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}$ all smooth with $n \leq m$ where Φ^{-1} exists and is also smooth. You take it as a fact that $f \circ g$ is smooth if f and g are. Show the following equality:

$$\int_{\Psi\circ\Phi} f = \int_\Phi f(\Psi)|\text{det}\Phi'|^{-1}$$

Hint: If A and B are matrices then $\det(AB) = \det(A)\det(B)$, in particular $\det(A^{-1}) = \det(A)^{-1}$.

Back to top

Week 6

Warm-up

1.Let S be the top have of the sphere parameterized by $\Phi(x,y) = (x, y, \sqrt{1-x^2-y^2})$ without doing the computations (by looking at the picture) determine if the $\int_S F \cdot dS$ is positive, negative or zero. when $F(x,y,z)$ is:

(1) $F(x,y,z) = \langle 1,0,0\rangle$
(2) $F(x,y,z) = \langle 0,0,1\rangle$
(3) $F(x,y,z) = \langle y,0,0\rangle$
(4) $F(x,y,z) = \langle 0,0,x\rangle$
(5) $F(x,y,z) = \langle x,y,z\rangle$.

2. Let S be the smooth boundary of a solid region in $\mathbb{R}^3$. Which, if any, of the following are the same as $\frac{1}{3}\int_S \langle x,y,z\rangle \cdot dS$?

(1) $\int_S \langle x,0,0\rangle \cdot dS$
(2) $\frac{1}{2}\int_S \langle y,x,0\rangle \cdot dS$
(3) $\int_S \langle y,x,z\rangle \cdot dS$

Green's, Stokes', divergence theorems

Exercise 6.1. Using change of variables show that if $V(x,y,z) = \nabla F(x,y,z)$ that and $\gamma : [0,T] \to \mathbb{R}^3$ a smooth curve that $\int_\gamma V(x,y,z)\cdot d\gamma = F(\gamma(T)) - F(\gamma(0))$. Conclude that if a if a vector field $V(x,y,z) : D \to \mathbb{R}^3$ is the gradient of a function F then V has no curl.

Exercise 6.2. Consider the vector-field $V(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$ defined on all of $\mathbb{R}^2 \setminus (0,0)$. Show that the vector-field has no curl. Even so, prove that there is no function $f : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}$ where $V(x,y) = \nabla f(x,y)$.

Exercise 6.3. For both of the following let S be the top half of the sphere parameterized $\Phi(x,y) = (x, y, \sqrt{1-x^2-y^2})$, compute the following quantities in a clever way using theorems from class.

$$\int_S \langle x^2 + \sin(z), \cos(y)z^2, zx + z^3 + \sqrt{x^2+y^2}\rangle \cdot dS$$

$$\int_S \langle -2z, 0, 3x^2 + 3y^2\rangle \cdot dS$$

Hint: The second vector field looks like a curl of $\langle -y^3, x^3+z^2, 0\rangle$ (answers should be $\frac{2\pi}{3}$ and $\frac{3\pi}{2}$).

1-forms

Exercise 6.4. Compute the following line integrals let $\gamma(t) = \langle t, t^2, t^3\rangle$ let $\omega_1 = xdy$, let $\omega_2 = (x^2+z)dz$ and let $\omega_3 = ydx$ find $\int_0^1 \gamma(t) d\omega$.

$(\frac{2}{3}, \frac{11}{10}, \frac{1}{3})$

Exercise 6.5. Let ω_i be as in 6.4. Are any of these closed or exact? Is there a combination of these that is? Using this can you verify what $\int_0^1 \gamma d\omega_1 + \int_0^1 \gamma\omega_3$ is?

Back to top

Week 7

Algebra of differential forms

Exercise 7.1. Show that if ω is a $2l+1$-form on then $\omega \wedge \omega = 0$.

Exercise 7.2. Show this is not true if ω is even. Furthermore, show that there is a differential form ω on $\mathbb{R}^{2n}$ such that $\omega \wedge \cdots \wedge \omega \neq 0$ for up to n copies of ω.

Exercise 7.3. Let $\omega = x^2 y dx \wedge dw - \sin(w) dy \wedge dz$ and $\eta = 3x dx - 4y dy$. Find and simplify $\omega \wedge \eta$, $\omega \wedge \omega$, $\eta \wedge \eta$, and $\eta \wedge \omega \wedge \eta$.

Green's theorem: differential forms

Exercise 7.4. Let B be a closed connected set in $\mathbb{R}^2$ with counter-clockwise oriented boundary ∂B. Argue with Green's theorem why $\int_B \frac{df}{dy}(x,y) dy \wedge dx + \int_B \frac{dg}{dx}(x,y) dx \wedge dy = \int_{\partial B} f(x,y) dx + \int_{\partial B} g(x,y) dy$ observe this almost appears to allow fractions with "$\frac{df}{dx} dx = f$".

More Stokes', Div, and Green's practice

Exercise 7.5. Let $\Phi(x,y) = (x, y, \sqrt{1-x^2-y^2})$ for $x^2+y^2 \leq 1$ parameterize the top-half of the unit sphere. Let $\omega = z dx \wedge dy - y dx \wedge dz + x dy \wedge dz$. First compute by hand $\int_\Phi \omega$. Second notice that this is equal to $\int_\Phi V \cdot N$ where $V(x,y,z) = \langle x, y, z \rangle$ and use Divergence theorem to find the integral a different way.

Exercise 7.6. Let $V(x,y) = \langle -3x^2 y, x^3 + y \sin^2(y) \rangle$, using Green's theorem find $\int_\gamma V \cdot d\gamma$ for the path $\gamma = (-\cos(\theta), \sin(\theta))$ for $\theta \in (0, \pi)$.

Exercise 7.7. Let S be the solid between $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 4$ and $y > |x|$, compute the flux integral $\int_{\partial S} V \cdot dN$ where $V(x,y,z) = \langle x^3 + 2xy^2 + xz^2, \frac{x^2}{y}, z^3 + 2zy^2 + zx^2 \rangle$.

Back to top

Week 8

Exercise 8.1. If $\omega \in \Omega^\bullet(\mathbb{R}^n)$ is closed and $\eta \in \Omega^\bullet(\mathbb{R}^n)$ is exact is it necessarily true that $\omega \wedge \eta$ is closed or exact prove or find a counter example for each.

Exercise 8.2. Show that $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is closed but not exact on $\mathbb{R}^2 \setminus \{0\}$. Can you find a similar 2-form on $\mathbb{R}^3 \setminus \{0\}$ that is closed but not exact (maybe just find something that similarly is closed showing it is exact uses some more general Stokes' theorem)? What about higher dimensions?

Remark 8.3 (Just think about don't prove). For this problem I will call a k-form bounded if it is of the form $f(x_1, \ldots, x_n)dx_{i_1} \wedge \cdots dx_{i_k}$ if f is bounded (we still require f is be smooth). Show that there is no bounded $n-1$ form whose exterior derivative is $dx_1 \wedge \cdots \wedge dx_n$.

I changed the above to a remark since it is too technical to show by hand. The idea is that any $n-1$ form whose exterior derivative is $dx_1 \wedge \cdots \wedge dx_n$ must be $\sum_{i=1}^n (-1)^{i+1} a_i x^i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \cdots \wedge dx_n + \eta$ where η is exact and $\sum_{i=1}^n a_i = 1$. Proving this is the case is rather hard to do rigorously. Finally you can note that the latter is not bounded, technically speaking I still need to show that η does not have anything of the form $f(x)dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \cdots \wedge dx_n$ which 'cancels' with the previous one to keep it bounded but in order for such a cancellation to occur it must have non-zero x_i partial derivative somewhere and thus not be exact.

Exercise 8.4. If x, y and r, θ represent Cartesian and polar coordinates. Show that $dx = \cos(\theta)dr - r\sin(\theta)d\theta$ and $dy = \sin(\theta)dr + r\cos(\theta)d\theta$. Conclude that $dx \wedge dy = rdr \wedge d\theta$. Show also that $dr = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$ and $d\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ where has the latter previously shown up? Why does it makes sense that dr and $d\theta$ undefined at the origin?

Exercise 8.5. If $n = 2k+1$ can there exist a differential k-form ω on $\mathbb{R}^n$ such that $d\omega \wedge \omega = dx_1 \wedge \cdots \wedge dx_n$ (Hint: start with $k = 1$).

Proof. Such a form does exist, we can begin in $\mathbb{R}^3$ we can note that it is impossible to work for something of the form $f(x, y, z)dx$ as it and its exterior derivative have a dx and thus will wedge to zero so we will need something sort of like $f(x, y, z)dx + g(x, y, z)dy$. First, we know a dz needs to show up so maybe f should have a z and any other terms or higher powers just stick around after the derivaitve when we want a 1 in front of our wedge product so maybe $zdx + g(x, y, z)dy$ we see that in this case $\omega \wedge d\omega = g(x, y, z)dy \wedge dz \wedge dx + z\frac{\partial g}{\partial z}(x, y, z)dz \wedge dy \wedge dx$ the first term is almost there we just need g to be 1 and in this case the second term disappears and thus $zdx + dy$ should work.

So can we extend this, well how did it work we had $\omega = \eta + \lambda$ where λ was closed (just a wedge product) and η was such that its exterior derivative is the wedge product of all the things missing in λ. So we can let $\eta = x_1 dx_2 \wedge \cdots \wedge d_{x_k+1}$ and $\lambda = dx_{k+2} \wedge \cdots \wedge dx_{2k+1}$ and we can see that $d\omega \wedge \omega = dx_1 \wedge \cdots \wedge dx_n$. □

Exercise 8.6. Is there a pair differential 1-forms η and ω such that for any smooth non-vanishing function g that $g\eta \wedge d(g\omega) = g^2\eta \wedge \omega$.

Hint: Consider 8.5.

Proof. Let $\omega = \eta$ be as in 8.5 then we can note that $d(g\omega) = dg \wedge \omega + gd\omega$ and thus we can see that $g\omega \wedge d(g\omega) = g\omega \wedge dg \wedge \omega + g^2\omega \wedge d\omega$, but since ω is a 1-form we know that $\omega \wedge dg \wedge \omega = 0$. $\square$

Back to top

Week 9

I will use $\square^k$ (I^k is the more standard notation) to denote the standard k-cube $[0,1]^k$.

Exercise 9.1. Compute the pull back of $\omega = x_3 dx_1 \wedge dx_2 \wedge dx_4$ by $T(a,b,c) = (ab, c^2, b, a^2)$.

This should be $bd(ab) \wedge d(c) \wedge d(a^2) = 2a^2 bdb \wedge dc \wedge da = 2a^2 bda \wedge db \wedge dc$.

Exercise 9.2. Let $\Phi, \Psi : \square^3 \to \mathbb{R}^4$ be $\Phi(x,y,z) = (x^2, y, xz, y+z)$ and $\Psi(x,y,z) = (2x, 3y, 4x, z)$ and consider the chain $\mathcal{C} = -\Phi + 2\Psi$ and let $\omega = x_3 dx_1 \wedge dx_2 \wedge dx_4$ find $\int_{\mathcal{C}} \omega$.

Hint: First compute the pull back

Exercise 9.3. Let $\Phi(u,v) = (u\cos(2\pi c), u\sin(2\pi v), \sqrt{1-u^2})$ and $\Psi(u,v) = (u\cos(2\pi c), u\sin(2\pi v), -\sqrt{1-u^2})$.

(1) Find $\partial(\Phi + \Psi)$ and $\partial(\Phi - \Psi)$.
(2) Given a two form $\eta = x_3 dx_1 \wedge dx_2 - x_2 dx_1 \wedge dx_3 + x_1 dx_2 \wedge dx_3$ is its integral over ∂B_1 positive, negative, or zero?
(3) Compute $\int_{\Phi \pm \Psi} \eta$.
(4) Which of the two is the "correct" way to write ∂B_1 as a chain?

Exercise 9.4. Consider $\Phi(u,v) = (\sin(\pi u)\cos(2\pi v), \sin(\pi u)\sin(2\pi v), \cos(2\pi v))$ show that $\partial\Phi = 0$.

Exercise 9.5. If $\omega = df$ is an exact 1-form and $\Phi : \square^1 \to \mathbb{R}^n$ is a 1-box. Show that $\int_\Phi \omega = f(\Phi(1)) - f(\Phi(0))$, how does this related to $\int_{\partial\Phi} f$ (integral of a zero form)? How does this change is we instead consider the chain $\mathcal{C} = a\Phi$?

Exercise 9.6. Suppose that $\Phi, \Psi : \square^1 \to \mathbb{R}^n$, are 1-boxes let $\mathcal{C}_1 = \Phi - \Psi$ and $\mathcal{C}_2 = 2\Phi - 2\Psi$. Show that if $\Phi(1) = \Psi(0)$ then there is a different 1-box $\Gamma_1 : \square^1 \to \mathbb{R}^n$ such that $\partial\Gamma = \partial\mathcal{C}_1$. Under what circumstances is there $\Gamma_2 : \square^1 \to \mathbb{R}^n$ such that $\partial\Gamma = \partial\mathcal{C}_2$?

Back to top

Week 10

Exercise 10.1 (9.14.3 from the book)**.** Let $\mathcal{C}$ be a k-chain, $f : \mathbb{R}^n \to \mathbb{R}$ be a function and ω be a $(k-1)$-form show the following 'integration by parts' and explain when it gives the standard integration by parts formula:

$$\int_{\mathcal{C}} f d\omega = \int_{\partial C} f\omega - \int_{\mathcal{C}} df \wedge \omega.$$

Exercise 10.2. Let $\Phi(a,b) = \langle a^2, ab, b^2 \rangle$ and $\Psi(a,b) = \langle a, a, b \rangle$ and $\mathcal{C} = 2\Phi + \Psi$ and let $\omega = zdx \wedge dy$, using generalized Stokes' theorem, find

$$\int_{\mathcal{C}} \omega.$$

Answer: $\frac{11}{6}$ most of the differential forms should pull back to zero.

Exercise 10.3. Let $\Phi(\theta, \phi) = \langle 2\cos(2\pi\theta)\sin(\pi\phi), 2\sin(2\pi\theta)\sin(\pi\phi), \cos(\pi\phi) \rangle$, using Stokes' theorem argue why

$$\int_{\Phi} d\omega = 0$$

for any 1-form ω.

Exercise 10.4. Use the generalized Stokes' theorem to prove the fundamental theorem of calculus that $\int_a^b f'(x)dx = f(b) - f(a)$.

Exercise 10.5. Let $f : [0,1] \to (0, +\infty)$ be a smooth function where $f(0) = f(1)$, let $\gamma(t) = \langle \cos(2\pi t)f(t), \sin(2\pi t)f(t) \rangle$, and $v(x,y) = \langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \rangle$. Using generalized Stokes' prove that $\int \gamma' \cdot v dt = 2\pi$.

Hint: Note that we can write $\gamma' \cdot v$ as $\omega(\gamma')$ for $\omega = \frac{xdy - ydx}{x^2+y^2}$ find $d\omega$ and a 2-chain $\mathcal{C}$ where $\partial\mathcal{C} = \gamma - \eta$ where η is something easy to integrate, say counterclockwise oriented circle or some large/small radius.

Exercise 10.6. Let

(1) $f_1(s,t) = \langle 0, st, 0, s \rangle$
(2) $f_2(s,t) = \langle s, st, t, s \rangle$
(3) $f_3(s,t) = \langle 0, 0, st, 0 \rangle$
(4) $f_4(s,t) = \langle s, t, st, 1 \rangle$
(5) $f_5(s,t) = \langle st, 0, 0, t \rangle$
(6) $f_6(s,t) = \langle st, t, s, t \rangle$

and let $\mathcal{C} = -f_1 + f_2 + f_3 - f_4 - f_5 + f_6$ and let $\omega = xdy \wedge dz + wzdx \wedge dy$, find

$$\int_{\mathcal{C}} \omega$$

Hint: Consider $\Phi(a,b,c) = \langle ab, bc, ca, b \rangle$

Back to top