DISCUSSION NOTES - MATH 32A

JOHN HOPPER

Table of Contents

Week 1

Logistics

Ta: John Hopper
Email: jshopper@math.ucla.edu, (jshopper@g.ucla.edu also works)
Office: MS 3921
OH: TBD

Introductions

Name, subject of interest, favorite shape

Review of derivatives

You are going to need to be able to take various derivatives in this course here are a few for practice.

- $e^x + 4x^3 + \sin(x)$
- $\ln(\ln(\ln(x)))$
- $\frac{x^2+1}{2x+1}$

Answers:

Date: January 13, 2025.

- $e^x + 12x^2 + \cos(x)$
- $\frac{1}{\ln(\ln(x))\cdot\ln(x)\cdot x}$
- $\frac{2x(2x+1)-(x^2+1)(2)}{(2x+1)^2} = \frac{2x^2 2x-2}{(2x+1)^2}$

Review of integrals

You will need some integrals in this course, but not too much (32B needs more) likely you will just need u-sub and be able to apply trig formulas but will probably not need trig-sub or integration by parts, but that depends on the professor. Here are some examples of things you should be able to solve.

- $\int x^2 + \cos(x) + e^{3x}dx$
- $\int_0^{\sqrt{\pi}} x\sin(x^2)dx$
- $\int \sin(x)\cos(x)dx$
- $\int \sin^2(x)dx$

Answers:

- $\frac{x^3}{3} + \sin(x) + \frac{1}{3}e^x + C$ - Don't forget the $+C$
- 0 - Hint: u-sub with $u = x^2$
- $\frac{-1}{2}\cos^2(x) + C$ or $\frac{1}{2}\sin^2(x) + C$ depending on choice of $u = \cos(x)$ or $u = \sin(x)$ respectively, also note the negative in the first one.
- $\frac{x}{2} - \frac{1}{4}\sin(2x)$ - answer is not super important what is really important is you know that this is done with the double angle formula (something that is good to put on exam note cards) that is $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$.

Review of some limits

There will be a part of this class were you will need to compute some limits. See if you can see what the following limits are, note they are infinite, or say that they don't exist if they don't.

- $\lim_{x\to 0} \frac{x}{x+1}$
- $\lim_{x\to 0} \frac{x^2}{\sin(x)}$
- $\lim_{x\to -2} \frac{x^2-1x-6}{x+2}$
- $\lim_{x\to 0} \frac{|x|}{x}$
- $\lim_{x\to 0} \frac{1}{x}$
- $\lim_{x\to 0} \frac{1}{x^2}$

Answers:

- 0 - you can just plug in $x = 0$ since we aren't dividing by zero and everything is continuous.
- 0 - do L'hopitals rule
- -5 - factor the numerator and cancel common factor of $x+2$, then plug in $x = -2$.
- This does not exist. The limit from the left is -1 and 1 from the right, these do not agree so the limit does not exist.
- This does not exist from the left it is $-\infty$ and from the right it is $+\infty$ which are not the same.
- $+\infty$, we generally say this does exist since they agree, allowing limits to be infinity is okay but not they should have the same sign unlike the previous limit.

What is a vector?

If I have time I may take about what a vector is. Effectively it is an arrow, it contains two pieces of information, direction and magnitude (or length). Examples of vectors are velocities, I don't just care about how fast I am going but where I am going (both magnitude and direction). We often draw them sitting in the plane but they don't have to have any particular location, since there location is not encapsulated in length or direction. E.g. if I tell you that I am running north at 7 miles per hour, I have not told you where in LA I am, so the velocity vector could be sitting anywhere in space (with any base) and still represent the same velocity.

When we add vectors we put the base of one at the tip of the first and look at the vector from the base of the first to the tip of the second. Sometimes this can feel confusing because the location doesn't matter but to add the location of the second sort of depends on the first. But in reality we could work backwards and say we put the tip of the first and the base of the second, and now it is more clear that the location of the second doesn't matter. Neither has a fixed location, but their relative location (ie the tip of one at the base of the other) is fixed. When we add them they become a whole unit that moves simultaneously.

Note that if we arbitrarily decided to put the base of every vector at the origin the vectors would be in one to one correspondence with points in the space, that is

$$\langle x, y, z\rangle \leftrightarrow (x, y, z).$$

This fact is a bit subtle since the location doesn't matter, we can choose the location and so now it matters somewhat.

vector addition and scalar multiplication

I should also mention the basics of vector algebra. You add two vectors by making the vector which goes from the base of one to the tip of the other if the second is placed so that its base is at the tip of the first. This is somewhat complicated, but is very simple in formula:

$$\langle 1, 2, 3\rangle + \langle 4, 5, 6\rangle = \langle 1 + 4, 2 + 5, 3 + 6\rangle = \langle 5, 7, 9\rangle.$$

Addition is just regular addition, but do it three times over, once for each component. We can then intuit what scalar multiplication (note scalar is a fancy word for number, just a way differentiating them from vectors). Recall back when you learned multiplication it was probably explained like $3*5 = 5+5+5 = 3+3+3+3+3$. We can do the same with vectors

$$2\langle 4, 5, 6\rangle = \langle 4, 5, 6\rangle + \langle 4, 5, 6\rangle = \langle 4 + 4, 5 + 5, 6 + 6\rangle = \langle 2 * 4, 2 * 5, 2 * 6\rangle = \langle 8, 10, 6\rangle.$$

We can thus also make sense of scalar multiples with non-whole numbers by saying $\lambda\langle x, y, z\rangle = \langle \lambda x, \lambda y, \lambda z\rangle$ or for a more concrete example $\sqrt{5}\langle 1, \sqrt{10}, \pi\rangle = \langle \sqrt{5}, 5\sqrt{2}, \pi\sqrt{5}\rangle$ Note that we generally do not multiply component by component, you will learn this week and next about ways you can multiply vectors which have physical meaning and applications.

magnitude and unit vectors

We say that a vector is two things, length and direction. If you want the length then by the Pythagorean theorem you can see that it is the square root of the squares of the components e.g.

$$||\langle 1, 2, 3\rangle|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

We sometimes refer to direction vectors or unit vectors which are vectors with length 1. Since we restrict their length they really only provide one piece of data, direction. Note that we can obtain a unit vector in the direction of any vector by dividing by its magnitude. For example for the vector $\langle 1,2,3\rangle$ the unit vector pointing in the same direction is

$$\frac{1}{\sqrt{14}}\langle 1,2,3\rangle = \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\rangle$$

Parallelogram law

This is one of those things which sometimes terrifies students and there is that one question on the midterm a bunch of people miss because of it. This is effectively the Pythagorean theorem a bunch of times (Wikipedia has a good picture to see this). It is sort of one of those formulas you should have on your formula sheet if you get one or should have memorized if not. Generally, if you see a bunch of lengths of vectors squared and then summed you should check and see if the formula applies and would be helpful. There is a better version when you learn what a dot-product is but for now if $\vec{v}$ and $\vec{w}$ are two vectors then

$$2||\vec{v}||^2 + 2||\vec{w}||^2 = ||\vec{v}+\vec{w}||^2 + ||\vec{v}-\vec{w}||^2.$$

Back to top

Week 2

Warm-up

Question 1: You and your friends work on a problem and are asked to find a parametric equation of a line that passes through the points $P = (1,3,2)$ and $Q = (3,1,6)$, your answer was $r(t) = \langle 1,3,2\rangle + t\langle 2,-2,4\rangle$, but your friend got $s(t) = \langle 3,1,6\rangle + t\langle -1,1,-2\rangle$, who is right?

Question 2: True or false: If two lines never intersect in 3D, then they are parallel.

Question 3: True or false, given two vectors $\vec{x}$ and $\vec{y}$ is the following identity true?

$$||\vec{x}+\vec{y}||^2 + ||\vec{x}-\vec{y}||^2 = 2||\vec{x}||^2 + 2||\vec{y}||^2$$

if so how do you prove it?

Question 4: Are $\langle 1,2,3\rangle$ and $\langle -4,1,1\rangle$ perpendicular? How do you know?

Question 5: True or False?

a: $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$

b: $\vec{x}\times\vec{y} = \vec{y}\times\vec{x}$

Answers: Q1: Both of you are right, the direction vectors are parallel, and both of you chose a point on the line.

Q2: False, there are skew lines ex $r(t) = t\langle 1,0,0\rangle$ and $s(t) = \langle 0,0,1\rangle + t\langle 0,1,0\rangle$

Q3: True this is the parallelogram identity, to prove it recall that $||\vec{v}||^2 = v\cdot v$ and then distribute and cancel where appropriate.

Q4: No, they are not. Take a dot product and see it is $-4+2+3 = 1 \neq 0$ and recall perpendicular vectors have zero dot product.

Q5: a is true, b is false the correct statement is $\vec{x}\times\vec{y} = -\vec{y}\times\vec{x}$.

How to get from vectors to vectors and numbers

For the most part, you can sort of guess the correct steps in this class by looking at what information you have and what you need. This is not perfect, there are some exceptions but should be a useful guide to this class.

What you have	what you want	how to get there
1 vector	scalar	take the norm
1 vector, 1 number	vector	scalar multiplication
2 vectors	scalar	dot product**
2 vectors	vector	cross product

Table 1. How do you get there?

There is one notable exception to this line you will learn, which is the area of a parallelogram, during which you take the norm of the cross product (so doing the forth line then the first line)

Equations of spheres and cylinders

This section can occur at many places depending on professor and some may not even talk about it, but if you want to take 32B it will be useful. Lets begin with the euqaion of the sphere, hopefully you agree that a sphere is all the points of distance r from a center point (a, b, c). We can write this as vectors

$$||\langle x, y, z\rangle - \langle a, b, c\rangle|| = r$$

it is easier if we square both sides this is the same as

$$||\langle x - a, y - b, z - c\rangle||^2 = r^2$$

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

which should be the familiar formula, note that $(x-a)^2 = (-1)^2(a-x)^2$ so we can also write it like

$$(a - x)^2 + (b - y)^2 + (c - z)^2 = r^2.$$

For an cylinder we can note that this is the set of points that are the same distance form a central axis (for here the central axis will be one of the coordinate axes). For example if it was the z-axis then it will look like

$$x^2 + y^2 = r^2.$$

If we want to put it through the axis (parallel to the x-axis) the line $y = b$ and $z = c$ then we can write it like

$$(y - b)^2 + (z - c)^2 = r^2.$$

It is sometimes a little confusing since if the central axis is parallel to x then there are no x's in the formula. You can think of this because the the cylinder looks the same if you move up or down in the x-axis which can help you remember that this coordinate doesn't matter and so it shouldn't appear in the formula.

Here is an example: find the equation of the sphere with diameter $\overline{PQ}$ where $P = (1, 2, 3)$ and $Q = (4, 5, 6)$.

Answer: To find the equation of the sphere we need two pieces of information, center and radius. Lets start with the center which we may be able to remember that the center is in the middle of the two the endpoints of the diameter. Recall the midpoint of two points is just the average of the sum of the points as vectors,

$$\frac{\langle 1,2,3\rangle + \langle 4,5,6\rangle}{2} = \langle \frac{5}{2}, \frac{7}{2}, \frac{9}{2}\rangle.$$

Next, we need the radius is half the length of the diameter so we can find $\overline{PQ} = \langle 3,3,3\rangle$ which has length $\sqrt{3^2+3^2+3^2} = \sqrt{27}$. We can then find the formula

$$(x-\frac{5}{2})^2 + (y-\frac{7}{2})^2 + (z-\frac{9}{2})^2 = (\sqrt{27})^2 = 27.$$

Parametric equations of lines

As was noted in the notes on week 1, we know that the location of a vector does not matter (we can move it around the plane/space and it is the same vector) as such we can choose to have it based at the origin, and in this case vectors are in one-to-one correspondence with points in the plane. Only we know how to add, subtract, multiply by scalars, take dot product and cross products of vectors so this identification lets us use what we have learned.

The first example of this is the parametric equation of a line. Recall the standard definition of a line $y = mx + b$ we can rewrite this as the vectors

$$\langle x, y\rangle = \langle x, mx+b\rangle = x\langle 1, m\rangle + \langle 0, b\rangle.$$

Note that x can be any real number where m and b are given by the specifics of the line, so we can write this as a function $r(x) = x\langle 1, m\rangle + \langle 0, b\rangle$. Note that there is one exception to $y = mx + b$ which are vertical lines where $x = c$ which can be written like $\langle c, y\rangle$ where y is any real number and so they are the image of the function

$$r(y) = y\langle 0, 1\rangle + \langle c, 0\rangle.$$

We can then try to generalize this to higher dimensions and we get that the equation of a line in 3d is given by

$$r(t) = t\vec{a} + \vec{b}$$

where $\vec{a}$ is like the slope and is called the direction vector and $\vec{b}$ is like the intercept, except it can be any point on the line (so it is easier to find in general).

Important note: as you can see from the warm-up these are not unique and you can write the same line multiple ways, and pair of parallel lines work equally well as direction vectors, and the $\vec{b}$ can be any point on the line.

Notice that the identification with vectors is useful here because I know what $r(3) = 3\langle 1,2,3\rangle + \langle 1,1,1\rangle$ is but I don't really know what $3(1,2,3)+(1,1,1)$ means. Okay, enough talk lets do some practice:

Q1: Find an equation of a line passing through $\langle 1,-2,3\rangle$ and $\langle 17,21,12\rangle$.

Q2: Find the equation of the line parallel to $s(t) = t\langle 4,5,6\rangle + \langle 1,3,6\rangle$ and passing through the origin.

Q3: Do the following lines intersect and if so where? $r(t) = t\langle 1,2,0\rangle + \langle 1,1,0\rangle$, $s(t) = t\langle 2,1,1\rangle + \langle 0,1,0\rangle$, and $q(t) = t\langle 2,3,1\rangle + \langle 6,8,3\rangle$.

A1: four common answers are $r_1(t) = t\langle 16, 23, 9\rangle + \langle 1, -2, 3\rangle$, $r_2(t) = t\langle 16, 23, 9\rangle + \langle 17, 21, 12\rangle$, $r_3 = t\langle -16, -23, -9\rangle + \langle 1, -2, 3\rangle$, $r_3 = t\langle -16, -23, -9\rangle + \langle 17, 21, 12\rangle$.

A2:Parallel means the direction vectors are parallel so we can say the direction is $\langle 4, 5, 6\rangle$ and we need one point, we are told the origin which is $\langle 0, 0, 0\rangle$, so a good answer is $r(t) = t\langle 4, 5, 6\rangle + \langle 0, 0, 0\rangle = t\langle 4, 5, 6\rangle$.

A3: r and s do note intersect, because for the third coordinate to match we need $s(0)$ and there is not t where $r(t) = s(0)$ (that is if we make the first and last coordinates match it must be $r(1)$ and $s(0)$ but the second coordinates don't match so we can't make all three match simultaneously). r and q do intersect, we can see this by noting that $r(a) = \langle a + 1, 2a + 1, 0\rangle$ and $q(b) = \langle 6 + 2b, 8 + 3b, 3 + b\rangle$ is we want the first two to match we get $a + 1 = 6 + 2b$ and $2a + 1 = 8 + 3b$ substituting $a = 5 + 2b$ we get $10 + 4b + 1 = 8 + 3b$ or that $b = -3$ and so $a = -1$, we see that $r(-1) = \langle 0, -1, 0\rangle = q(-3)$ and so we can see that these lines intersect at $(0, -1, 0)$. Similar work will show that q and s intersect at $q(-1) = s(1) = \langle 4, 3, 2\rangle$.

DOT PRODUCTS

Dot products are useful when computing projections. $\vec{v} \cdot \vec{w}$ is a weighted answer to the question how much is v along the direction w where the weights are larger the larger v and w are.

First, some practice, compute the following:

1. $\langle 1, 2, 3\rangle \cdot \langle \pi, 17, 20\rangle$
2. $\langle 20, 10\rangle \cdot \langle -1, 2\rangle$
3. $\langle 1, 1, 0\rangle \cdot \langle 0, 0, 5\rangle$
4. $\langle 1, 24\rangle \cdot \langle 0, -3, 1\rangle$

Answers:

1.$\pi + 2 \cdot 17 + 3 \cdot 20 = \pi + 94$
2. $-20 + 20 = 0$
3. 0
4. $0 - 6 + 4 = -2$

Note: It should be clear that $\vec{v} \cdot \vec{v}$ is the sum of the square of the components which is $||\vec{v}||^2$, this is something that is used often in this class and the following class, 32B as well as 33A.

PROJECTIONS

We beyond simple computations we can use dot products to find projections. This is a concept we will use multiple times in this class. If $\vec{u}$ is a unit vector, then $\vec{v} \cdot \vec{u}$ says how much v is along u. We can then decompose $\vec{v}$ to be the parts in the direction of u and a piece that is perpendicular to u. Where

$$\vec{v} = (\vec{u} \cdot \vec{v})\vec{u} + [\vec{v} - (\vec{u} \cdot \vec{v})\vec{u}]$$

this equality is trivially true we just added and subtracted $(\vec{u} \cdot \vec{v})\vec{u}$ but note that $(\vec{u} \cdot \vec{v})\vec{u}$ is parallel to u (clearly since it is a scalar (the dot product) times u). The second piece is perpendicular to u since

$$[\vec{v} - (\vec{u} \cdot \vec{v})\vec{u}] \cdot \vec{u} = \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{u}||\vec{u}||^2 = 0$$

you can see here that it is important that we chose u to be a unit vector.

We can now try a practice problem: Find the projection of $v = \langle 2, 3, -5 \rangle$ along $w = \langle 1, 0, -1 \rangle$.

Answer: we need w to be a unit vector so first find the unit vector in the direction of w, i.e. divide by its norm which is $\sqrt{2}$ so let $u = \langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \rangle$ which the unit vector in the direction of w. We can then use the formula we have above, to see that $\vec{u} \cdot \vec{v} = \frac{2}{\sqrt{2} + \frac{5}{\sqrt{2}} = \frac{7}{\sqrt{2}}}$ and so the final answer is $(\vec{u} \cdot \vec{v})\vec{u} = \frac{7}{\sqrt{2}} \langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \rangle = \langle \frac{7}{2}, 0, -72 \rangle$.

DOT PRODUCT AND COS

There is another type of problem where we use dot products which is the following formula. We can note that $\vec{v} \cdot \vec{w} = ||v|| \cdot ||w|| \cos(\theta)$ where θ is the angle between them. Note that this makes sense with the idea a projections, if they are pointing in the same direction $\theta = 0$ and so it is just the product and if they are perpendicular $\theta = \pi/2$ and the dot product is zero.

One interesting way this comes in is questions is if you are asked about the angle between two vectors. For example, do the following vectors have an angle between them which is acute or obtuse? $u = \langle 2, 11, -2 \rangle$ and $v = \langle 1, -1, 2 \rangle$.

A: obtuse because the dot product is negative and sin of a obtuse angle is negative and $||u||$ and $||v||$ are both positive so the sign depends purely on $\sin(\theta)$.

CROSS PRODUCTS

Cross product do a couple of things, one they produce a vector which is perpendicular to two given vectors, keep this in mind since we will use this latter in the course. That is the direction but since the cross product gives a vector it has another piece of information, its magnitude which is given by $||u \times v|| = \sin(\theta)||u|| \cdot ||v||$ where θ is the angle between the vectors, note that sine is always positive for angles between 0 and π so this is always positive just like magnitudes. Also interesting this magnitude is the area of the corresponding parallelogram which is the one exception to the 'how do you get there?' table.

Tip: You can check your work when you find a cross product, by making sure it is perpendicular (i.e. has zero dot product) with the two original vectors.

Cautionary note: It is really common to drop the negative in the formula for the y component of the cross product, don't do it! Don't make me take a point off in the midterm.

Exercises: 1) Find a vector perpendicular to $\langle 1, 2, 0 \rangle$ and $\langle 0, 1, 1 \rangle$. 2) Find the cross product $\langle 1, 3, 1 \rangle \times \langle 1, 0, 1 \rangle$. 3) What is the area of the parallelogram with vertices $(0, 1, 0), (1, 1, 1), (2, 0, 2), (3, 0, 3)$.

Answers: 1) Find the cross product it is $\langle 2, -1, 1 \rangle$, technically any scalar multiple of this vector works.
2) The answer is $\langle 3, 0, -3 \rangle$.
3) Find the side lengths as vectors, $\langle 1, 0, 1 \rangle$ and $\langle 2, -1, 2 \rangle$ (go from any one to any other pair, you may get different intermediate vectors but the norm of the cross

product will be the same), find the cross product: $\langle 1, 0, -1\rangle$ whose norm is $\sqrt{2}$ which is the final answer.

Back to top

Week 3

Warm-up

Q1: Which of the following is the right amount of information to determine a plane? (more than 1 may be correct)
a) A normal vector, and a point
b) Three vectors in the plane
c) 3 points in the plane
d) Two intersecting lines in the plane
e) 2 vectors in the plane and a point in the plane

Q2: What are the following cross products? a) $\langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle$.
b) $\langle 0, 1, 0\rangle \times \langle 0, 0, 1\rangle$
c) $\langle 0, 0, 1\rangle \times \langle 1, 0, 0\rangle$

Q3: True or false the volume of a parallelepiped with side length vectors $\vec{u}, \vec{v}, \vec{w}$, is given by $\vec{v} \cdot (\vec{u} \times \vec{w})$

Q4: T or F: $(\vec{v} + \vec{w}) \times \vec{w} = \vec{v} \times \vec{w}$
A1: a, c, d, e
A2: $\langle 0, 0, 1\rangle$, $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$
A3: False, it should be the absolute value of that quantity.
A4: True, distribute this is equal to $\vec{v} \times \vec{w} + \vec{w} \times \vec{w}$ and recall that a vector crossed with itself is zero.

Cross products

Probably best to look at what I prepared for week 2, with also maybe doing a problem on the volume of a parallelepiped for example:

Find the volume of the parallelepiped with vertices $(1, -1, 0), (1, 1, 1), (1, 0, 0), (0, 1, 0)$ $, (1, 2, 1), (0, 3, 1), (0, 2, 0), (0, 4, 1)$
A: Find the vectors (all from one point to any three points) in this like $\langle 0, 2, 1\rangle$ (1st to 2nd), $\langle 0, 2, 0\rangle$ (1st to 3rd), $\langle -1, 2, 0\rangle$, then we plug them into the formula which is $\langle 0, 2, 1\rangle \cdot \langle 0, 2, 0\rangle \times \langle -1, 2, 0\rangle = \langle 0, 2, 1\rangle \cdot \langle 0, 0, 2\rangle = 2$. You can check that any order or other choices of vectors will also work (you can see this by the fact that the cross product of two parallel vectors is zero).

Equations of planes (parameterizations of planes)

There are many ways to write the equation of the plane, but in short it should be for a given normal vector $\vec{n}$ and a real number c (x, y, z) is in the plane if $\vec{n} \cdot \langle x, y, z\rangle = c$. So, we only need to find $\vec{n}$, c. $\vec{n}$ is orthogonal to the plane, ie orthogonal to all the vectors in the plane (that is all vectors between points in the plane or direction vectors of lines in the plane), if you have a point (a, b, c) in the plane then either you can find c by noting $c = \langle a, b, c\rangle \cdot \vec{n}$, or using this fact to rewrite the above equation (subtracting c from both sides) to get $\vec{n} \cdot \langle x - a, y - b, z - c\rangle = 0$. Pretty much every method boils down to finding (a, b, c) and $\vec{n}$, cross products are really good since they take two vectors in the plane and find a vector orthogonal to both of them (a candidate for your normal vector). Here is some practice:

Q1: Find the equation of the plane containing $(1, 2, 3), (2, 2, 2), (3, 3, 3)$.
Q2. Find the equation of the plane given the lines $r(t) = \langle 1, 1, 1\rangle + t\langle 2, 0, 0\rangle$ and $s(t) = \langle 3, 1, 1\rangle + t\langle 0, 1, 1\rangle$.
Q3: Find the equation of the plane containing the line $r(t) = \langle 1, 1, 1\rangle + t\langle 2, 0, 0\rangle$ and the point $(0, 0, 4)$.

A1: We can get two vectors by taking the difference of two vectors from a third e.g. the 1st to 2nd give $\langle 1, 0, -1\rangle$, and 1st to 3rd $\langle 2, 1, 0\rangle$. We can given two vectors find a vector orthogonal to them both by taking a cross product getting $\langle 1, -2, 1\rangle$. Now we have a normal vector and a point (I'll use $(2, 2, 2)$) so we can find the equation to be $\langle x-2, y-2, z-2\rangle \cdot \langle 1, -2, 1\rangle = 0$ which is also $x-2-2y+4+z-2 = x - 2y + z = 0$.
A2: The direction vectors gives us vectors in the plane so we can just cross product them for a normal and get $\langle 2, 0, 0\rangle \times \langle 0, 1, 1\rangle = \langle 0, -2, 2\rangle$, and use either of the points as our point, (I'll use $(3, 1, 1)$) to get $\langle x - 3, y - 1, z - 1\rangle \cdot \langle 0, -2, 2\rangle = 0$, which is the equation $2y + 2 - 2z + 2 = 0$ which simplifies to $y + z = 0$.
A3: The line give us a vector in the plane from its direction, and the point for the line $(1, 1, 1)$ and the point for the plane $(0, 0, 4)$ give us another vector by finding their difference $\langle -1, -1, 3\rangle$. So, we can then find the cross product $\langle -1, -1, 3\rangle \times \langle 2, 0, 0\rangle = \langle 0, 6, 2\rangle$, we can then use the point $(0, 0, 4)$ to get the equation of the plane which is $\langle x, y, z - 4\rangle \cdot \langle 0, 6, 2\rangle = 0$ which is $6y + 2z - 8 = 0$ which simplifies to $3y + z = 4$.

Bonus Question

Find the equation of the plane parallel to $\langle x - 3, y - 7, z + 2\rangle \cdot \langle 2, 2, -1\rangle = 0$ and goes through the origin.

A: Recall that planes are parallel implies that they have the same normal vector (or their normal vectors are scalar multiples of each other since the scalar multiple of a normal vector is also a normal vector). So, we know that the normal vector in this case is $\langle 2, 2, -1\rangle$ and so we also need a point. We can recall that the origin is the point $(0, 0, 0)$ and so the equaiton of the plane is $0 = \langle x-0, y-0, z-0\rangle \cdot \langle 2, 2, -1\rangle = 2x + 2y - z$
Back to top

Week 4

Warm-up

Q1: Which of the following derivative laws are true?
(1) $\langle f_1(x), f_2(x)\rangle' = \langle f_1'(x), f_2(x)\rangle + \langle f_1(x), f_2'(x)\rangle$
(2) $(\vec{f}_1(x) \cdot \vec{f}_2(x))' = \vec{f}_1{}'(x) \cdot \vec{f}_2(x) + \vec{f}_1(x) \cdot \vec{f}_2{}'(x)$

(3) $(\vec{f_1}(x) \times \vec{f_2}(x))' = \vec{f_1}'(x) \times \vec{f_2}(x) + \vec{f_2}'(x) \times \vec{f_1}(x)$
(4) $(a\vec{f_1}(x) + b\vec{f_2}(x))' = a\vec{f_1}'(x) + b\vec{f_2}'(x)$
(5) $(||\vec{f}(x)||^2)' = 2\vec{f}'(x) \cdot \vec{f}(x)$
(6) $\left(\int_0^x \vec{f}(y)dy\right)' = \vec{f}(x)$
(7) $\vec{f}(g(x))' = \vec{f}'(g(x))$
(8) $\int_a^b \vec{f}'(x)dx = \vec{f}(b) - \vec{f}(a)$

Q2: If the arc-length of a curve $\vec{f}(x)$ between 0 and 1 is 4, is true that $||\vec{f}(1) - \vec{f}(0)|| = 4$?

Q3: If $\vec{r}(t)$ is a vector valued function which is parameterized by arc-length, what is the length of the curve between $t = 1$ and $t = 6$?

A1: Here are the true equalities for the false
(1) $\langle f_1(x), f_2(x)\rangle' = \langle f_1'(x), f_2'(x)\rangle$
(2) True
(3) $(\vec{f_1}(x) \times \vec{f_2}(x))' = \vec{f_1}'(x) \times \vec{f_2}(x) + \vec{f_1}(x) \times \vec{f_2}'(x)$
(4) True
(5) True, be ready also for $||\vec{f}(x)||' = \sqrt{\vec{f}(x) \cdot \vec{f}(x)} = \frac{\vec{f}'(x) \cdot \vec{f}(x)}{||\vec{f}(x)||}$
(6) True
(7) $\vec{f}(g(x))' = \vec{f}'(g(x))g'(x)$
(8) True

A2: This is false, we know that $||\vec{f}(1) - \vec{f}(0)|| \leq 4$ since you can't travel less than 4m and get 4m away, but you could go around in a circle and end up less than 4m away.
A3: The length of the curve is 5 since we can note that if we are parameterized by arc-length, the length of the curve is the difference in the times (since we curve moves at 1 distance per 1 time)

Arclength and speed

There are really two types of arc-length problems, either finding the length of a curve and/or the speed of the curve (these are closely related and live in a similar place in my mind) or taking a curve and parameterize it by arc-length. Here is one of each:

Q1: Given the curve $\vec{r}(t) = \langle \sqrt{3}t^2, 2t, t^3\rangle$ find the speed of the curve and compute the the length of the curve from $t = 1$ to $t = 4$.

Q2: Find the speed the following curve $\vec{r}(t) = \langle \frac{4}{3}t^3, -\cos(t^3), 4 - \sin(t^3)\rangle$.

A1: First we find the derivative, this is $\vec{r}'(t) = \langle 2\sqrt{3}t, 2, 3t^2\rangle$, we can recall that the speed is $s(t) = ||\vec{r}'(t)|| = \sqrt{12t^2 + 4 + 9t^4} = \sqrt{(3t^2 + 2)^2} = 3t^2 + 2$. The length of the curve is then $\int_1^5 3t^2 + 2dt = t^3 + 2t|_{t=1}^{t=5} = 125 + 10 - (1 + 2) = 132$.

A2: We start the same way by finding the speed, and so need to take a derivative and see that $\vec{r}'(t) = \langle 4t^2, 3t^2\sin(t^3), -3t^2\cos(t^3)\rangle$. We can then see that the speed is $||\vec{r}'(t)|| = \sqrt{16t^4 + 9t^4\sin^2(t) + 9t^4\cos^2(t)} = 5t^2$. We now write $s = \int_0^t 5x^2 dx = \frac{5}{3}t^3$ and so if we want to find the inverse, find t in terms of s and get that $t = \sqrt[3]{\frac{3s}{5}}$, we can then see that this

$$\vec{r}(s) = \left\langle \frac{4}{3}\left(\sqrt[3]{\frac{3s}{5}}\right)^3, -\cos\left(\left(\sqrt[3]{\frac{3s}{5}}\right)^3\right), 4-\sin\left(\left(\sqrt[3]{\frac{3s}{5}}\right)^3\right)\right\rangle = \left\langle \frac{4}{5}s, -\cos\left(\frac{3s}{5}\right), 4-\sin\left(\frac{3s}{5}\right)\right\rangle$$

Back to top

Week 5

Warm-up

Q1: Which of the following curves have larger curvature?
a)Circular motion with radius 2 vs Circular motion with radius 3?
b) Circular motion with radius 2 vs moving in a helix where the radius is also 2

Q2: Which of the following are correct formulas for curvature?
a) $\kappa(s) = ||\frac{dT}{ds}||$
b) $\kappa(t) = ||r'(t)|| \cdot ||\frac{dT}{dt}||$
c) $\kappa(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^2}$

Q3: What direction is the normal vector pointing in circular motion? How about for helical motion?

A1: rad 2 circle has higher curvature in both cases, the larger the circle the longer it takes to make a loop and so the slower we change direction. For the helix, if you move in the z direction as well you circle slower (since you travel farther to make one rotation) also your vectors all have a constant z component so the normal vectors have smaller x and y components which are changing.

A2: The first is the only one that is correct, the second should have $\frac{1}{||r'(t)||}$ instead of multiplying by in and the second should have a cube in the denominator.

A3: Toward the center of the circle, or toward the line of z for the helix (the normal vector has no z component)

Finding curvature and normal vectors

Q1. Given the motion $r(s) = \frac{s+1}{\sqrt{2}}\langle\cos(\ln(\frac{s+1}{\sqrt{2}})), \sin(\ln(\frac{s+1}{\sqrt{2}}))\rangle$ is a curve parameterized by arc length, find its curvature and its normal vector.

Q2. Given the motion $r(t) = \langle t^4, t^2 + 1, t\rangle$ find its curvature.

Q3. Given the motion $r(t) = \langle\sin(t), \cos(t), t^2\rangle$ find its curvature, normal vector, and bi-normal vector.

A1: Since we know that this is parameterized by arclength we can do all these 'by hand' by findin $\frac{dT}{ds}$ for the curvature and normalizing for the normal vector.

$$r'(s) = T(s) = \frac{1}{\sqrt{2}}\langle \cos(\ln(\frac{s+1}{\sqrt{2}})), \sin(\ln(\frac{s+1}{\sqrt{2}}))\rangle + \frac{1}{\sqrt{2}}\langle -\sin(\ln(\frac{s+1}{\sqrt{2}})), \cos(\ln(\frac{s+1}{\sqrt{2}}))\rangle$$

(careful with chain rule on the natural log). We can then

$$T'(s) = \frac{-1}{\sqrt{2}(s+1)}\langle \cos(\ln(\frac{s+1}{\sqrt{2}})), \sin(\ln(\frac{s+1}{\sqrt{2}}))\rangle + \frac{1}{\sqrt{2}(s+1)}\langle -\sin(\ln(\frac{s+1}{\sqrt{2}})), \cos(\ln(\frac{s+1}{\sqrt{2}}))\rangle$$

You can find that this has norm $\frac{1}{s+1}$ and so $\kappa(s) = \frac{1}{s+1}$ and that it has a normal vector which is

$$N(s) = \frac{-1}{\sqrt{2}}\langle \cos(\ln(\frac{s+1}{\sqrt{2}})), \sin(\ln(\frac{s+1}{\sqrt{2}}))\rangle + \frac{1}{\sqrt{2}}\langle -\sin(\ln(\frac{s+1}{\sqrt{2}})), \cos(\ln(\frac{s+1}{\sqrt{2}}))\rangle$$

.

A2: This is super easy to take derivatives of and we can even see that some terms drop out after we take derivatives so we likely want to use the cross product formula. $r'(t) = \langle 4t^3, 2t, 1\rangle$, $r''(t) = \langle 12t^2, 2, 0\rangle$ so $r''(t) \times r'(t)$

Back to top

Week 6

Warm-up

(1) Which of the following are level curves for which functions $f(x, y) = xy$, $f(x, y) = x + y$, $f(x, y) = x^4 + y^4$, and $f(x, y) = x^2$ (also traces)
(2) Find the average rate of change if from P to Q if $P = (1, 2)$ and $Q = (5, -1)$ and $f(x, y) = x + y$. a) 1 b) 4 c)$\frac{1}{5}$, d)none of the above
(3) Find the average rate of change of from the contour plot where P and Q are both on the same level curve. a)1 b)0 c) not enough information given.

Back to top