DISCUSSION NOTES - MATH 31A

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Week 0

Logistics

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Introductions

Name, subject of interest, favorite shape

Review - Graphs you should know

There are a few types of graphs it will be helpful for you to know how to draw. I recommend playing around with desmos (I like the slider tool a lot) or another graphing calculator to get comfortable with the general shape of certain graphs.

Graph the following equations and answer the questions about the graphs:

(1) $y = 2x + 3$ - now what happens if we make the 2 into a 1 or a -2?
(2) $y = x^2$ - how does this compare to $y = 2x^2 - 1$?
(3) $y = (x - 2)(x + 1)$ - where does this intersect the x-axis, roughly at what value of x does it 'bottom out'?

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(4) $y = \sin(x)$ - how does this compare to $y = \cos(x)$
(5) $y = 2\sin(3x)$
(6) $y = \frac{1}{x}$ - how does this change for $y = \frac{-1}{2+x}$?
(7) $y = \frac{1}{(x-1)^2}$ - what if it where $(x-1)^4$, does the general shape change?
(8) $y = \begin{cases} x & x < -1 \\ 2 & x = -1 \\ 0 & -1 < x < 1 \\ x^2 - 1 & 1 \leq x \end{cases}$

Answers: most of these just graph but I will answer some of the open-ended questions. For (1) changing the 2 to a 1 reduces the slope and -2 makes it go 'downhill' instead of 'uphill'. For (2) it is steeper and shifted down 1. (3) it inersects at -1 and 2 and bottoms out around 0.5. For (6) note that it shifts two to the left and instead of going down then up it goes up then down. For (7) all even powers look roughly the same but get 'steeper' with higher powers (all odd powers also look the same).

Trig you should know

I recommend knowing the unit circle, i.e. what is sin, cos and tan of 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ and what do with then angles larger that $\pi/2$. You should also be able to convert radians to degrees $\pi = 180^o$. The only trig formula you should know is that $\sin(\theta)^2 + \cos(\theta)^2 = 1$ no matter what θ is (this is the pythagorean theorem if you drawn the triangle on the unit circle).

Factoring

It will be useful in this course to be able to do a bit of factoring. Luckily most of the factoring you will do will be after you find a zero of a polynomial (the way it goes is you plug in a number if it is non-zero you are good if it is zero then you may need to factor). Most of the time it will be with quadratics in which case if you are factoring $x^2 + bx + c$ it will be like $(x - A)(x - B)$ where $AB = c$ and $A + B = -b$ luckily if you know that A and B are zeros of the polynomial so if you have that 3 is a zero let $A = 3$ and then $B = \frac{c}{3} = -b + 3$, it is good to check you work and make sure $\frac{c}{3} = b - 3$, but if you are rushed you can always find whichever is easier.

Sometimes you will need something a bit more complicated, for example $ax^2 + bx + c$ which you can write as $(x - A)(Bx - C)$ where $B = a$, $-B - AC = b$ and $BC = c$. You may also have something like $x^3 + ax^2 + bx + c$ but you will likely only have to write this as something like $(x-A)(x^2+Bx+C)$ and you can find the relations as before by multiplying out the polynomials. The general trick is that A is the zero, then solve for the other capital letters. Here is some practice

(1) $x^2 - x - 6$ and $x = -2$ is a zero
(2) $x^2 - 7x + 12$ and $x = 3$ is a zero
(3) $x^2 + 6x - 9$ and $x = 3$ is a zero
(4) $2x^2 - 7x + 3$ and $x = 3$ is a zero

Answers:

(1) $(x + 2)(x - 3)$
(2) $(x - 3)(x - 4)$
(3) $(x - 3)^2$

(4) $(x-3)(2x-1)$

Week 1

Warm-up

(1) If a car travels 10 miles at 10mph and then another 10 miles at 20 mph, what is its average speed?
a) 15mph
b) >15mph
c)<15mph

(2) If a care travels for an hour at 10 mph, then another hour at 20 mph, what is its average speed?
a) 15mph
b) >15mph
c)<15mph

(3) If I know what $\lim_{x\to 17} f(x)$ and $\lim_{x\to 17} g(x)$ are can I find out what $\lim_{x\to} \left[[f(x)]^3 + f(x)g(x) + 3\right]^{1/3}$ is?
(a) yes
(b) no

(4) (True/False) The following in the correct statement of the quotient limit law: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$.

Answers: (1) c. Recall we need to find the secant line, the car traveled 20 miles in 1.5 hours (1 hour for the first 10 miles and 0.5 for the second), and thus the average speed was $\frac{20}{1.5} = \frac{40}{3} = 13.\overline{3} < 15$.

(2) a. Again use the second line we cover $10 + 20$ miles in the two hours and so the average speed is $30/2 = 15$mph.

(3) a, yes this follows from power, root, sum and product limit laws. Note that since we have an odd root we don't need to check if it is positive, if it were an even root this could be undefined.

(4) b, no. We did not check that $\lim_{x\to c} g(x) \neq 0$, this is super important and you should make sure to check this!

Limit laws

In short, you limits of sums, products, quotient, powers, and roots can be written as these operations of the limits. The general method for basic limit laws questions is to break a large expression into a bunch of smaller limits. Here are some examples.
(1) $\lim_{x\to 94} -21$
(2) $\lim_{x\to 4} \sqrt{x} + \frac{16}{x} + 2$
(3) If $\lim_{x\to 4} f(x) = 3$, what is $\lim_{x\to 4} \sqrt{f^2(x) + \frac{f(x)}{x-1}}$

Answers: (1) -21, this is sometimes called constant limit law. (2):

$$\begin{aligned}
&= \left(\lim_{x\to4}\sqrt{x}\right)+\left(\lim_{x\to4}\frac{16}{x}\right)+\left(\lim_{x\to4}2\right) && \text{Sum limit law (twice)}\\
&= \left(\sqrt{\lim_{x\to4}x}\right)+\left(\lim_{x\to4}\frac{16}{x}\right)+\left(\lim_{x\to4}2\right) && \text{Root limit law}\\
&= \left(\sqrt{\lim_{x\to4}x}\right)+\left(\frac{\lim_{x\to4}16}{\lim_{x\to4}x}\right)+\left(\lim_{x\to4}2\right) && \text{Quotient limit law}\\
&= \left(\sqrt{\lim_{x\to4}x}\right)+\left(\frac{16}{\lim_{x\to4}x}\right)+\left(2\right) && \text{constant limit law}\\
&= \left(\sqrt{4}\right)+\left(\frac{16}{4}\right)+\left(2\right) && \text{plug in 4}\\
&= 8 && \text{combine}
\end{aligned}$$

(3):

$$\begin{aligned}
&= \sqrt{\lim_{x\to4}f^2(x)+\frac{f(x)}{x-1}} && \text{Root limit law (check not negative))}\\
&= \sqrt{\left(\lim_{x\to4}f^2(x)\right)+\left(\lim_{x\to4}\frac{f(x)}{x-1}\right)} && \text{Sum limit law}\\
&= \sqrt{\left(\lim_{x\to4}f(x)\right)^2+\left(\lim_{x\to4}\frac{f(x)}{x-1}\right)} && \text{power limit law}\\
&= \sqrt{\left(\lim_{x\to4}f(x)\right)^2+\left(\frac{\lim_{x\to4}f(x)}{\lim_{x\to4}x-1}\right)} && \text{quotient limit law (check not dividing by zero)}\\
&= \sqrt{\left(\lim_{x\to4}f(x)\right)^2+\left(\frac{\lim_{x\to4}f(x)}{(\lim_{x\to4}x)-(\lim_{x\to4}1)}\right)} && \text{Sum Limit law}\\
&= \sqrt{\left(3\right)^2+\left(\frac{3}{(\lim_{x\to4}x)-(\lim_{x\to4}1)}\right)} && \text{We are told the limit of f}\\
&= \sqrt{\left(3\right)^2+\left(\frac{3}{(\lim_{x\to4}x)-(1)}\right)} && \text{constant limit law}\\
&= \sqrt{\left(3\right)^2+\left(\frac{3}{(4)-(1)}\right)} && \text{limit of x is known}\\
&= \sqrt{10} && \text{simplify}
\end{aligned}$$

Two wrongs making a right

So fa, most of the limits we have seen are just a bunch of glorified processes of plugging in a number into an equation. There are a few interesting questions that we can think about which break this trend, for example:
(1) Find $f(x)$ and $g(x)$, such that $\lim_{x\to0}f(x)$ and $\lim_{x\to0}g(x)$ don't exist but where $\lim_{x\to0}f(x)g(x)$ does exist.
(2)Find $f(x)$ and $g(x)$, such that $\lim_{x\to0}f(x)$ and $\lim_{x\to0}g(x)$ don't exist and $g(x)\neq0$, but where $\lim_{x\to0}\frac{f(x)}{g(x)}$ does exist.

Answer, I will only solve (1), you can but once you see it something very similar works for (2). Think of the graphical examples where the limit doesn't exist. There are about 3 ways you can see this happen, one is where there is a vertical asymptote like $\frac{1}{x^2}$ (this debatably has a limit the value is just $+\infty$), another option is like $\sin(1/x)$ where you have two much oscillation (you may not have see that one yet, you will later), and finally the piece-wise definition like $f(x) = 0$ when $x \leq 0$ and $f(x) = 1$ when $x > 1$. Focus on this third option, and now consider $g(x) = 1$ when $x \leq 0$ and $g(x) = 1$ when $x > 0$, note that neither f nor g has a limit at 0 but there product is just zero everywhere which does have a limit at 1.

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Week 2

Warm-up

(1) Find $\lim_{x\to-\infty} \frac{x}{x^2-1000}$. a) ∞, b)$-\infty$, c) 0, d) None of the above.
(2) If $\sqrt{x}+14 \leq f(x) \leq x^2$ then what is $\lim_{x\to4} f(x)$? a)4, b) 14, c) It does not exist/we aren't given enough information, d) None of the above
(3) True/False: If f is continuous and $f(-100) = -1$ and $f(100) = 2$ then there is a $-100 < x < 100$ where $f(x) = \frac{\pi}{4}$?

(4) Follow, up. Is there only one such x or can there be more? a) There is only one x. b) There could be others.

Answers

(1) c)0, multiply top and bottom by $1/x$ and see that this is equal to the limit $\lim_{x\to-\infty} \frac{1}{x-1000/x} = \frac{1}{-\infty-0} = 0$.
(2) d) None of the above. The answer is 16, note that f is bounded above and below by functions whose $\lim_{x\to4}$ is 16 and thus by the squeeze theorem f has the same limit.
(3) True, this is IVT.
(4) b) IVT just says at least 1 exists there could only be 1 or be more.

Limits at infinity

In general this section involved plugging in $x = \infty$ into fractions and solving. There are a few cases we know $\lim_{x\to\pm\infty} \frac{1}{x} = 0$ or $1/(\pm\infty) = 0$ and since any number (not infinity) times zero is zero this means $4/\infty = 0$ as well (you can replace 4 with any number and this stays true. In general, these problems start that if you plug in $x = \infty$ (or $-\infty$) then you either get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ in which case you need to multiply top and bottom by (normally, a power of x) to make of a form we can get an answer (note the answer of $\infty/1$ is ∞ and is a fine answer). Here is a couple of examples:

1) $\lim_{x\to+\infty} \frac{x^2+x}{-x+17}$
2) $\lim_{x\to+\infty} \frac{\sqrt{16x^6+x}}{x^3}$.

Answers:

1) If we try to plug in right away we get $\frac{\infty+\infty}{-\infty-17}$ which is an $\frac{\infty}{\infty}$ which means we need to multiply by a power of x to get rid of some of these infinities. In general look either for the highest effective power of x in the numerator/denominator and multiply by x to the negative of this power. If you have a choice I recommend the smaller of these, and this problem will show you why.

We can see that the highest power of x in the numerator is 2 and in the denominator is 1. If we multiply by x^{-2} we get that $\lim_{x\to+\infty} \frac{1+1/x}{-1/x+17/x^2} = \frac{1}{0}$ which is $\pm\infty$ but we don't know which it depends on the sign of the denominator. Either you can argue that $\lim_{x\to+\infty} -1/x + 17/x^2$ is approaches 0 from the negative (which I would do by multiplying by x and seeing that it is now negative and noting x was positive).

But, what would likely have been easier is to multiply by x^{-1} on top and bottom to get $\lim_{x\to+\infty} \frac{x+1}{-1+\frac{17}{x}} = \frac{\infty+1}{-1+0} = -\infty$, since now the sign of the denominator (and the numerator) is clear.

2) Here it is clear what the highest power of the denominator is, 3, but what about the numerator. We see x^6 but this is under a square root, so effectively the powers in the square root are 1/2 of what they appear, so in this case the effective power of x in both the numerator and the denominator is 3. If we multiply top and bottom by x^{-3} we can get $\lim_{x\to+\infty} \frac{x^{-3}\sqrt{16x^6+x}}{1} = \lim_{x\to+\infty} \frac{\sqrt{16+x^{-5}}}{1} = \frac{\sqrt{16+0}}{1} = 4$.

Squeeze theorem and Trig

The most common question I get with squeeze theorem is what function do I keep and which do I bound, and also how do I know when to use it. These have mostly the same answer, in general squeeze theorem shows up if there is a function which is changing but stays bounded the classic examples are $\sin(1/x)$, $\cos(x)$ and $\frac{|x|}{x}$, but in general any time you see a sin or a cos you should ask yourself if squeeze theorem would be helpful (you should also check if you can use the limits $\lim_{x\to0} \frac{\sin(x)}{x} = 1$ or $\lim_{x\to0} \frac{\cos(x)-1}{x} = 0$. Since these both have the same basic flag (containing trig functions) I will have examples where you need to decide which is appropriate.

Try these problems:

1) $\lim_{x\to1} \frac{\sin((x-1)(x+1))}{x-1}$

2) $\lim_{x\to1}(x-1)\sin(\frac{1}{|x-1|^{1/2}})^3$

3) Tricky question: $\lim_{x\to1} \frac{\cos(x-1)\cos(\frac{1}{x-1})-\cos(\frac{1}{x-1})}{x-1}$ (Hint: you need both squeeze theorem and your trig limits)

Answers:

1) First, is it squeeze theorem or trig rules. If we try to plug in $x = 1$ we get 0/0 which is a hint that the trig rules is what we want. Next, we can try a change of variables to make it look like $\sin(x)$, let $y(x) = (x-1)(x+1)$ (whatever is in the sin). We, then multiply and divide but whatever we need to make the denominator also $y(x)$ (in this case $(x+1)$, thus we have $\lim_{x\to1} \frac{\sin(y(x)(x+1)}{y(x)} = \lim_{x\to1} \frac{\sin(y(x))}{y(x))} \lim_{x\to1}(x+1) = 1 \cdot 2 = 2$. Note we use the product limit law.

2)First, is it squeeze theorem or trig rules. If we try to plug in $x = 1$ we get 0 times something we can't understand, zero times unknown is a classic squeeze theorem set up. In this case the unknown is $\sin(\frac{1}{|x-1|^{1/2}})^3$ which is bounded (because of the sin) and so this is almost certainly a squeeze theorem question. We can recall that $|\sin(x)| \le 1$ for all x so we have $|(x-1)\sin(\frac{1}{|x-1|^{1/2}})^3| \le |x| \cdot |1|$ and so $-|x-1| \le x\sin(\frac{1}{|x-1|^{1/2}})^3 \le |x-1|$. I put absolute values around to avoid issues of negatives in my inequalities (since they flip inequalities). We can check that $\lim_{x\to1} |x-1| = 0 = \lim_{x\to1} -|x-1|$ and so by the squeeze theorem $\lim_{x\to1}(x-1)\sin(\frac{1}{|x-1|^{1/2}})^3 = 0$.

3) As I said this one is a bit tricky. We can guess the use of a trig rule since there is something that looks like $\frac{\cos(x)-1}{\cos(x)}$ only instead of x it is $x-1$ and there are some other cosines in the numerator. Maybe you can see that we can plug out a factor of $\cos(\frac{1}{x-1})$ in the numerator and get $\left(\cos(\frac{1}{x-1})\right)\left(\frac{\cos(x-1)-1}{\cos(x-1)}\right)$ which is an unknown times something going to zero, so a squeeze theorem set up. We know the unknown is $\cos(\frac{1}{x-1})$ which is bounded by 1 in absolute value, $|\cos(\frac{1}{x-1})| \leq 1$. By a change of variables $y = x-1$ we get that $\lim_{x\to 1} \frac{\cos(x-1)-1}{\cos(x-1)} = 0$ and so we have the following squeeze theorem $-|\frac{\cos(x-1)-1}{\cos(x-1)}| \leq \left(\cos(\frac{1}{x-1})\right)\left(\frac{\cos(x-1)-1}{\cos(x-1)}\right) \leq |\frac{\cos(x-1)-1}{\cos(x-1)}|$, and the limit of the left most and right most terms is 0 as $x \to 1$ so by the squeeze theorem $\lim_{x\to 1} \frac{\cos(x-1)\cos(\frac{1}{x-1})-\cos(\frac{1}{x-1})}{x-1} = 0$.

IVT IVT questions come in two flavors, straightforward applications and clever applications. There is not much you can do to prepare for the latter other than note that if you are supposed to show something exists or find a point where something has a value then you are in business, there is one common trick with intersections I will show. Luckily most of the time you get the more straightforward types of questions.

In short, the IVT says if your continuous function was at some point small, then got large, then there was a time when it was medium sized. If you think of jump discontinuities as the exception they jump from small values to large values (or vice versa) without taking intermediate values, this is not a contradiction since they aren't continuous so the theorem does not apply. (So continuous means no jumps and no jumps means you had to be in the middle at some time).

The goal is to find a c where $a < c < b$ and $f(c) = y$. The answers have two steps. Step 1, show that $f(a) < y < f(b)$. Step 2, is say why f is continuous. Step 3, say by IVT there is a value c with $a < c < b$ and $f(c) = y$.

1) Show there is a $1 < c < 20$ where $c^2 - 17 = 340$.
2) Show that there is a real number x where $\sin(x) = 437/576$.
3) Show there is a x where $x^3 + 1$ and x^4 intersect.

Answers:
1) This is a classic formulation of the question in this case first find what the functions values are at the bounds they give $1^2 - 17 = -16$ and $20^2 - 17 = 400 - 17 = 383$ and $-16 < 340 < 383$ (step 1). Also, $x^2 - 17$ is continuous because it is a polynomial (step 2) so, by IVT there is a $1 < c < 20$ with $c^2 - 17 = 340$.

2)Here it is strange since we are not given a and b so we can choose them so that $f(a) < y < f(b)$ (really choose points that are easy to compute). Note $0 < \frac{437}{500} < 1$ and we can find poinys where sin is 0 and 1 namely $x = 0$ and $x = \pi/2$ and since $\sin(x)$ is continuous by IVT there is a $0 < c < \pi/2$ with $\sin(c) = \frac{437}{500}$. Note that the fact $0 < c < \pi/2$ is not super relevant since we weren't asked for the interval but a point between 0 and $\pi/2$ is still a real number so we are good. It is not uncommon to be given a larger interval than you need, if it is easier to work on a smaller interval do so.

3) This initially seems not like an IVT question we have neither a function nor an interval. THe only tip off is that we need to show that a value of x exists but we don't need to find x exactly. The trick is to note that $x^3 + 1 = x^4$ if and only if $x^4 - x^3 - 1 = 0$. Now we have a continuous function $x^4 - x^3 + 1$ and any real number (so this should feel like example 2 where the interval was not given). Note that if I plug in some easy values like say 0 we get $0^4 - 0^3 - 1 = -1$ so now we need a point

that it is positive, for example when $x = 10$ we get $10000 - 1000 - 1 = 9899 > 0$ (I like powers of ten since they are easy you can check $2^4 - 2^3 - 1 = 7$ which also works and isn't too hard.) Thus we can apply IVT to this to show that there is a real number $0 < x < 10$ with $x^4 - x^3 - 1 = 0$ and so $x^3 + 1 = x^4$.

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Week 3

Warm-up

All these problems are true/false questions

(1) If f is continuous then f is differentiable.
(2) $(7f(x) + 3g(x) - 105h(x)) = 7f'(x) + 3g'(x) - 105h'(x)$
(3) If $f(x) = x^{-3}$, then $f'(x) = -3x^{-2}$
(4) $\frac{d}{dx} f(x)g(x) = f'(x)g'(x) + f(x)g(x)$
(5) $\frac{d}{dx} \frac{g(x)}{f(x)} = \frac{f(x)g'(x) - f'(x)g(x)}{f^2(x)}$

Answers:

(1) False, the opposite is true if f is differentiable then f is continuous.
(2) True, this is linearity of the derivative.
(3) False, the $f'(x) = -3x^{-4}$ because $-3 - 1 = -4$ (this is a super common error!)
(4) False, the primes don't go together, $f'(x)g(x) + f(x)g'(x)$
(5) True, note that I swapped the role of f and g in the classical statement.

Graphs of derivatives

There are a number of classic questions that show up in this area where you are expected to either sketch derivatives, sketch original functions given the derivative, or identify which graph is the derivative of what. I find the most common mistake for these is overthinking things, here I will talk about the basic graphical detail you are expected to know about.

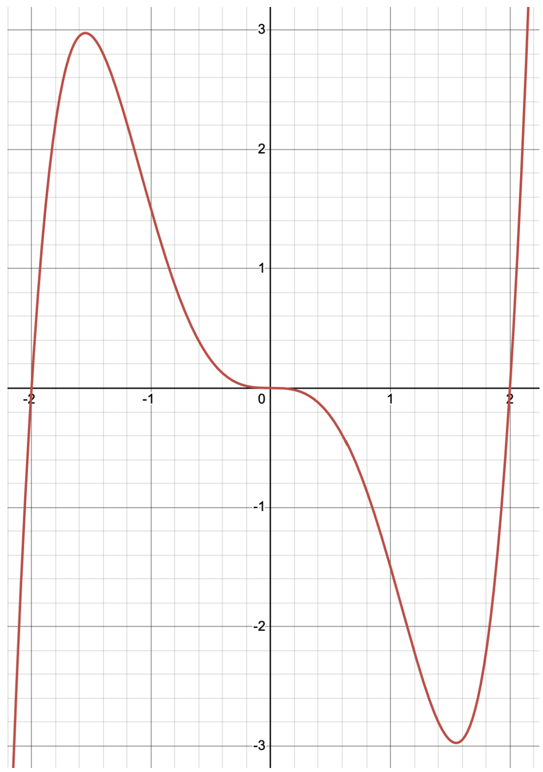

Figure 1. This graph has a peak, a valley, and a half-peak.

Peaks and valleys and half-peaks, look at figure 1, you can see that it has a clear peak at around $x = -1.5$ and a valley at around $x = 1.5$. It also has what I call a half-peak half-valley at $x = 0$. A half-peak is a point that from one direction looks like a peak and from the other looks like a valley (cover up with you hand the everything to the left of the y-axis and it will look like a peak, if you cover everything on the right of the y-axis it looks like a valley). These points are all important because they are where the tangent line has slope zero, that is the derivative is zero. The graph of the derivative must then touch the x-axis at three spots $-1.5, 0.1.5$ and since there are no other points with zero derivative these are the only places it will intersect the x-axis.

Next, we care about where the function is increasing versus decreasing. To do this look at the the regions between the peaks, valleys and half-peaks (as these are where the function has derivative zero and thus can chance from positive to negative or vice versa). You can see here that $(-\infty, -1.5)$ the function is increasing, from

$(-1.5, 0)$ it is decreasing from $(0, 1.5)$ it is still decreasing, finally from $(1.5, +\infty)$ it is increasing. Thus our sketch should of the derivative should start off (from the left) positive, hit the x-axis at -1.5, then be negative, then hit the x-axis at 0 but bounce to stay negative, finally cross above the x-axis at 1.5. You can see figure 2 for the real graph of the derivative as an example of something doing this.

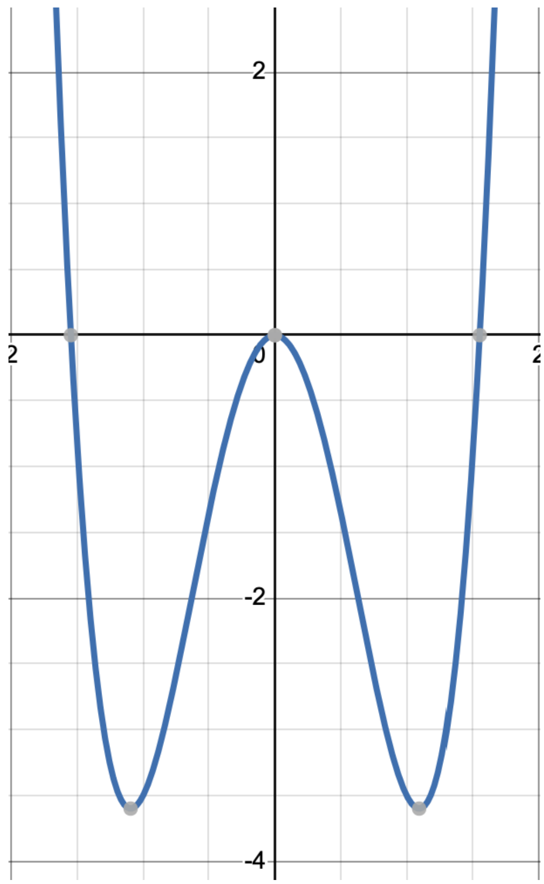

FIGURE 2. This graph is the derivative of the function in figure 1

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Midterm Recap
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WEEK 5

WARM-UP

(1) If an implicitly defined graph has a tangent line at $(1, 3)$ $y - 1 = 2(x - 3)$ which of the following are potential implicit derivatives $\frac{dy}{dx}$ (select all that apply), a) $2y$, b)$2x$, c) $2x + 2y$, d)None of these are possible.

(2) What is $\frac{dV}{dt}$ as a function of $\frac{dr}{dt}$ if $V = \frac{4}{3}\pi r^3$? a) $\frac{4}{3}\pi(\frac{dr}{dt})^3$, b) $4\pi(\frac{dr}{dt})^2$, c) $4\pi(r)^2$, d) None of the above.
(3) If $f'(3) = 10$ and $f(3) = 6$ approximate $f(2.9)$. a) 5.9, b) 6.1 c) 5, d) 6 e) none of the above,
(4) Does the method used above work well to approximate $f(100)$?

Answers:

(1) b because x=1 so $2x$ is the slope of the tangent line.
(2) d) None of the above, it should be $4r^2\frac{dr}{dt}$.
(3) c
(4) No, in general these approximations would for nearby values, unless we have very small units 100 is not very close to 3.

Related rates and implicit derivatives

I will include implicit derivatives with related rates since most related rates have a lot in common with implicit derivatives and thus we can kill to birds with one practice problem. Typically, a related rates question involves reading a word problem to set up an equation that relates two variables (often sides of a triangle, or volume of some shape, but sometimes there are more creative questions). The the step is to take an implicit derivative, typically with respect to time or some other third variable not in the original equation, and solving for the desired rate.

Example 1) John is watching a bird fly. It is approximately $20\sqrt{3}$ ft in the air and just flew over a tree that is 20 ft away from me. I notice that I am moving my binoculars at a rate of 1 radian per second, how fast is the bird flying (assuming it is moving horizontally toward me).

Answer: If we draw the triangle we see that it is a right triangle with height $h = 20\sqrt{3}ft$ and length $l = 20ft$. Some trig, should tell us that this means the angle from vertical is $60^o = \pi/3$, angle is changing by 1 radian a second (and getting smaller). I know that the variables I know about are l, h, and the angle θ so I want an equation involving these and the most notable is $\tan(\theta) = \frac{l}{h}$ and then we can take a time derivative, since the bird is moving horizontally we know that h is a constant and so we get $\sec(\theta)^2\frac{d\theta}{dt} = \frac{1}{h}\frac{dl}{dt}$. We can then plug in the numbers we know: $\frac{4}{3} = \sec(\theta)^2$ and $\frac{d\theta}{dt} = -1$ and $h = 20\sqrt{3}$ so we have that $\frac{-4}{3} \cdot 20\sqrt{3} = \frac{dl}{dt}$. The original question was how fast it was flying and so this should be positive so the answer is $\frac{80}{\sqrt{3}}$ after simplifying.

Linear approximation

The idea of this section is the tangent line is a good approximation of a function at nearby values. So if I want to find $f(x + h)$ where h is small positive or small negative then I can use instead $f(x)+hf'(x)$ as a good approximation, (if you solve for $f(x+h)$ in the limit definition of the derivative (and drop the limit) this is what you get).

Examples:

1) Approximate: $\sqrt{101}$ 2) Approximate $\sin(0.01)$
3) Approximate $(999)^2$ with linearization around 1000.

Answers: We need a value of square root that we know that is close to 101, the natural choice is 100. So let $x = 100$ $h = 1$ and $f = \sqrt{x}$ so $f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{20}$ we can then see that this should be about $10 + 1(0.05) = 10.05$.

Again, what is a value of sin we know that is near 0.1, zero. So, $x = 0$ $h = 0.01$ and $f'(x) = \cos(x) = 1$, thus we can see that $f(0.01)$ is approximately $0+0.01(1) = 0.01$ this is also known as the small angle approximation of sin.

This one we are told what we are to approximate around. Note that we could multiply this out but this will be quicker (if you are practiced with this material). We know $1000^2 = 1,000,000$ and $\frac{d}{dx}x^2 = 2x$ and $h = -1$ so we should have this is approximately $1,000,000 - 1(2 \cdot 1,000) = 998,000$.

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Week 6

Warm-up

(1) When trying to maximize a function $f(x)$ on $[a, b]$ what points should be check 1) a, 2)b, 3) Where $f'(x) = 0$, 4) where f' does not exist, 5) where $f''(x) = 0$.
(2) If we f' is negative before x and positive after x is x a local min or a local max?
(3) If f is continuous and differentiable and $f(1) = 2$ and $f(3) = 6$ what must be the value of f' for some c between $1 < c < 3$? a) 1 b) 2 c)3 d)6 e) none of the above
(4) draw a picture where is f concave, convex, does it have inflection points?

Answers 1. is is 1, 2, 3, and 4
2. local max
3. c because $\frac{6-2}{3-1} = 2$

Extreme points

Find the largest and smallest value f takes on the interval and idetify any local minima and maxima: $f(x) = x^4 - 2x^2 - 2$ for $-2 \leq x \leq 1$.

To do this we will look at the derivative $f'(x) = 4x^3 - 4x$ we can see that $f'(x) = 0$ is when $4x^3 - 4x = 0$ note that $4x^3 - 4x = (4x)(x^2 - 1) = (4x)(x-1)(x+1)$ so this is zero when $x = -1, 0, 1$. We can then see if these are local minima or maxima, for this I recommend the second derivative test $f''(x) = 12x^2 - 4$, we can see that $f''(-1) = 8$, $f''(0) = -4$, and $f''(1) = 8$ so -1 and 1 are local minima since the second derivative is positive and 0 is a local max. For the largest and smallest values we plug these into f along with the endpoints note that one of the critical points is an endpoint, that is okay (if it were outside the interval we would ignore it) $f(-2) = 6$ $f(-1) = -3$ $f(0) = -2$, and $f(1) = -3$ so the largest value f takes is 6 and the smallest is -3.

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