DISCUSSION NOTES - MATH 131A W26

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Week 1

Logistics

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General proof writing

Exercise 1.1. Write out truth tables for P, Q, $P \vee Q$ (or), $P \Rightarrow Q$, $Q \Rightarrow P$, not $P \rightarrow$ not Q, and not $Q \Rightarrow$ not P. Are any of these the same?

The following is an interesting exercise into proving or statements, it may be helpful to think of the above truth table.

Exercise 1.2. Let a and b be integers, prove that if ab is even, then a or b is even.

You can define a to be even if it can be written as $a = 2k$ for some integer k and odd if it can be written as $2k+1$ for k an integer. You may assume that a number either even or odd but not both (in fact try one proof with this fact and try a proof without it).

Solution: The idea is if I want to prove an "or" statement there are generally two good ways.

(1) is by contrapositive that is in this case show if neither a nor b is even (that is they are both odd), then ab is not even (i.e. odd). This works well when the negative statements are well characterized as in this case 'not even' is 'odd' and still something useful to assume in the proof.

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(2) is by cases that is in case 1 assume a is even, in this case we are automatically done. In case 2 assume a is odd, and now show that if a is odd and ab is even then b is even. (this second method is annoying to do rigorously in this case, but there are other 'or' proofs where this method is the easier of the two). Another way to think of this is to note that a or b on a truth table is the same as not a implies b.

(3) As with most math there are other options, these are two of the most common ones.

Likewise there are a few common methods to prove and statements, though in general proving 'and' statements I see fewer student mistakes.

Exercise 1.3. Let $a, b \in \mathbb{N}$ both be odd, then $a \cdot b$ is odd and $a + b$ is even.

(1)Directly, we know that $a = 2k+1$ and $b = 2l+1$ so $a \cdot b = 2k+2l+2(2kl)+1 = 2(k + l + 2kl) + 1$ so it is odd and $a + b = 2k + 2l + 2 = 2(k + l + 1)$.

Exercise 1.4. Let $a, b \in \mathbb{N}$, suppose that $a \cdot b$ is odd, then a and b are odd.

(2) Contrapositive and cases, we can show that if a or b is even then $a \cdot b$ is even. Case 1 is when a is even so $a = 2k$ and so $a \cdot b = 2kb = 2(kb)$ which is even. Case 2 is when b is even so $b = 2k$ and so $a \cdot b = a2k = 2(ak)$ which is even. (Note that the cases do not always have to be mutually exclusive as long as the cover all possible scenarios technically we showed that when a and b is even then $a \cdot b$ is even twice)

(3) contradiction and cases: Suppose the statement is false then we have that $a \cdot b$ is odd and a or b is even, then there is a contradiction. Again prove via cases that if a or b is even then $a \cdot b$ is even and note this contradicts the fact that $a \cdot b$ is odd.

(4) Again, there are other methods...

The following is some practice on for all and there exists statements

Exercise 1.5. Determine which of the following is true, for those that are false what is the negation of the statement?:

(1) For all even numbers n there is an odd number m such that $m > n$.
(2) There exists and odd number m such that every even number n is such that $m > n$.

(3) For all $n, m \in \mathbb{N}$ there exists an $p \in \mathbb{N}$ where $p > m \cdot n$.
(4) $\forall n \in \mathbb{N}$, $\forall m \in \mathbb{N}$, $\exists p \in \mathbb{N}$ such that $p > m/n$.
(5) $\forall n \in \mathbb{N}$ $\exists p \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, $p > n \cdot m$.
(6) $\exists p \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}$ $p > n \cdot m$.

Week 2

Warm-up

1. Determine if the following sets are bounded from above or not, if so state their supremum:

(1) $\{1\} \cup \{4\} \cup \{6\}$
(2) $\{x^2 : x \geq 0\}$
(3) $\{-x^4 + 4 : x \in \mathbb{R}\}$
(4) $\{\frac{n-1}{n} : n \in \mathbb{N}\}$

2. Determine if the following sequences converge if so state their limit:

(1) $\frac{1}{n}\sin(n^3)$
(2) $n^3 - n$
(3) $(\frac{1-2n}{n})^n$
(4) $\frac{2^n}{n!}$

Sups and Infs

Exercise 2.1. Suppose that $S_1, S_2 \subset \mathbb{R}$ both are bounded above and have the property that they contain their upper bound, that is $\sup S_1 \in S_1$ and $\sup S_2 \in S_2$, suppose that $S_1 \cap S_2 \neq \emptyset$. Show that $S_1 \cap S_2$ is bounded above, show that $\sup(S_1 \cap S_2) \leq \min\{\sup S_1, \sup S_2\}$. Is it true that $\sup(S_1 \cap S_2) = \min\{\sup S_1, \sup S_2\}$?

Exercise 2.2. Suppose that S is a set that is bounded below. Let L be the greatest lower bound i.e. $L = \inf\{x : x \in S\}$. Show that for all positive numbers $\epsilon > 0$ that there is an $x \in S$ where $L \leq x \leq L + \epsilon$.

Hint: What does it mean for $L + \epsilon$ not to be a lower bound.

Exercise 2.3. Show that every non-empty subset of $\mathbb{N}$ has a LUB in $\mathbb{N}$.

Hint You can prove this by induction, but also you can use exercise 2.2

Limits of sequences

Exercise 2.4. Suppose that $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence with limit a, and $r \in \mathbb{R}$, $r > 0$ show that $\{r \cdot a_n\}_{n\in\mathbb{N}}$ is also a convergent sequence with limit $r \cdot a$.

Exercise 2.5. Suppose that $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence with limit a. Show that $\{a_{2n} : n \in \mathbb{N}\}$ is a convergent sequence with limit a.

Exercise 2.6. Prove that $a_n = \frac{n!}{n^n}$ is a convergent sequence with limit 0.

Hint: expand the power and the factorial as a large product and compare to a simpler sequence, say $\frac{1}{n}$.

Exercise 2.7. Prove the squeeze theorem if $a_n \geq 0$ is a sequence and $b_n \geq a_n$ is a different sequence that converges with limit 0 then a_n converges with limit 0.

Week 3

Warm-up
Are the following sequences, Cauchy? Monotone? Bounded?

(1) $a_n = (n^2 + 1)/n^2$
(2) $b_n = (\frac{-1}{2})^n$
(3) $c_n = 2n^2 + \sin(\pi n)$
(4) $d_n = n\cos(\pi n)$

Are there any other combinations which do not occur in these examples?

Exercise 3.1. Suppose that S is a set that is bounded below. Let L be the greatest lower bound i.e. $L = \inf\{x : x \in S\}$. Show that for all positive numbers $\epsilon > 0$ that there is an $x \in S$ where $L \leq x \leq L + \epsilon$.

Exercise 3.2. Using 3.1 show that if $S \subset \mathbb{R}$ is complete then it has LUB and GLB properties.

Exercise 3.3. Show that for every number in $a \in \mathbb{R}$ that there is a bounded monotone sequence converging to a.

Exercise 3.4. Consider the sequence $s_n = \sum_{i=1}^{n} \frac{1}{j!}$, show that there is a number $e \in \mathbb{R}$ such that $e = \sum_{j=1}^{+\infty} \frac{1}{j!}$

Exercise 3.5. Show that every monotone non-decreasing sequence is bounded from below.

Exercise 3.6. Show that every Cauchy sequence is bounded.

Week 4

Warm-up: Which of the following have an increasing subsequence? Decreasing subsequence?

(1) $a_n = (-1)^n \cdot 2$
(2) $b_n = (\frac{-1}{2})^n$
(3) $c_n = \frac{3n+2}{20n-39}$
(4) $d_n = -n^3$

Exercise 4.1. Suppose that $x_n = \begin{cases} 0 & n = 3k \\ 22 & n = 3k+1 \\ 23 & n = 3k+2 \end{cases}$ Prove the set of subsequential limits is $\{0, 22, 23\}$.

Exercise 4.2. Show that if $\lim_{n\to+\infty} a_n = +\infty$, then every subsequence a_{n_k} we have that $\lim_{k\to+\infty} a_{n_k} = +\infty$.

Exercise 4.3. Using the previous exercise, conclude that there is no decreasing subsequence of a_n (same a_n from above).

Exercise 4.4. Suppose that b_n has no decreasing subsequence. Show that b_n is bounded from below.

Hint: A proof by contrapositive may be helpful

Exercise 4.5. Let $a_0 = x$ and $a_1 = y$ where let $a_{n+2} = \frac{a_n + 2a_{n+1} + 1}{4}$. Show that a_n is a convergent sequence, then conclude that $\lim_{n\to+\infty} a_n = 1$.

How would 4.5 change if $a_0 = a_1 = 10$? Would the sequence still converge? Would the limit change?

Week 5

Warm-up: Are the following sets, open, closed or neither (as subsets of $\mathbb{R}$?

(1) $\emptyset$
(1) $(0, 1) \cup [3, 4]$
(2) $\cup_{n\in\mathbb{N}}[n, n + \frac{1}{n}]$
(3) $\{x : x^2 < 3\}$
(4) $\{\sum_{i=1}^{n} \frac{1}{i^2} : n \in \mathbb{N}\}$

Exercise 5.1. Suppose that $A \subset \mathbb{R}^n$ has the property that for every $a \in A$ there is an open set U_a such that $a \in U_a \subset A$. Show that A is open.

Exercise 5.2. Suppose that $X \subset \mathbb{R}^n$ is a dense set, and U is an open set containing X must $U = \mathbb{R}^n$?

Exercise 5.3. Suppose $A \subset \mathbb{R}^n$ is a closed set and $x \notin A$, show that there are open sets U_x, U_A with $x \in U_x$ and $A \subset U_A$ such that $U_x \cap U_A = \emptyset$. This property is sometimes known as regular Hausdorff or T_3

Exercise 5.4. Suppose that $A, B \subset \mathbb{R}^n$ are closed such that $A \cap B \neq \emptyset$ then show that there are open sets $U_A, U_B \subset \mathbb{R}^n$ such that $A \subset U_A$ and $B \subset U_B$ but $U_A \cap U_B = \emptyset$. This property is known as normal Hausdorff or T_4. *Hint:* First show 5.3

Exercise 5.5. Given a set $A \subset \mathbb{R}^n$ we can define $A_\delta := \cup_{a \in A}\{x : |x - a| < \delta\}$. Show that A_δ is open. Show that there closed sets $A, B \subset \mathbb{R}^n$ with $A \cap B = \emptyset$ such that for all $\delta_1, \delta_2 > 0$ $A_{\delta_1} \cap A_{\delta_2} \neq \emptyset$, why does this not contradict 5.4?

Exercise 5.6. Show that if $A \subset \mathbb{R}$ is a closed set and $x \in \mathbb{R}$ then there is a $\alpha \in A$ with $|\alpha - x| = \inf_{a \in A} |a - x|$. *Hint:* Can you construct a sequence of a_n which is bounded such that $\lim_{n \to +\infty} |a_n - x| = \inf_{a \in A} |a - x|$

Exercise 5.7. If $A, B \subset \mathbb{R}$ are closed sets is there an $\alpha \in A$ and $\beta \in B$ such that $|\alpha - \beta| = \inf_{a \in A, b \in B} |a - b|$? Think about 5.5.

Exercise 5.8. Show that a set that is compact is closed. If this is part of your definition show that if C has the property that every open cover has a finite subcover then C is closed.