DISCUSSION NOTES - MATH 32A

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Week 1

Logistics

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General proof writing

Exercise 1.1. Write out truth tables for P, Q, $P \vee Q$ (or), $P \Rightarrow Q$, $Q \Rightarrow P$, not $P \rightarrow$ not Q, and not $Q \Rightarrow$ not P. Are any of these the same?

The following is an interesting exercise into proving or statements, it may be helpful to think of the above truth table.

Exercise 1.2. Let a and b be integers, prove that if ab is even, then a or b is even.

You can define a to be even if it can be written as $a = 2k$ for some integer k and odd if it can be written as $2k + 1$ for k an integer. You may assume that a number either even or odd but not both (in fact try one proof with this fact and try a proof without it).

Solution: The idea is if I want to prove an "or" statement there are generally two good ways.

(1) is by contrapositive that is in this case show if neither a nor b is even (that is they are both odd), then ab is not even (i.e. odd). This works well when the negative statements are well characterized as in this case 'not even' is 'odd' and still something useful to assume in the proof.

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(2) is by cases that is in case 1 assume a is even, in this case we are automatically done. In case 2 assume a is odd, and now show that if a is odd and ab is even then b is even. (this second method is annoying to do rigorously in this case, but there are other 'or' proofs where this method is the easier of the two). Another way to think of this is to note that a or b on a truth table is the same as not a implies b.

(3) As with most math there are other options, these are two of the most common ones.

The following is some practice on for all and there exists statements

Exercise 1.3. Determine which of the following is true, for those that are false what is the negation of the statement?:

1. For all $n, m \in \mathbb{N}$ there exists an $p \in \mathbb{N}$ where $p > m \cdot n$.
2. $\forall n \in \mathbb{N}$, $\forall m \in \mathbb{N}$, $\exists p \in \mathbb{N}$ such that $p > m/n$.
3. $\forall n \in \mathbb{N}$ $\exists p \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, $p > n \cdot m$.
4. $\exists p \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N}$ $p > n \cdot m$.

Induction

The following is in my mind the most classic type of induction problem, it is 1.1 from the book, after it I have a similar one you can do on your own.

Exercise 1.4. Show that for all $n \in \mathbb{N}$ that $\frac{1}{6}n(n+1)(2n+1) = 1 + \cdots + n^2 = \sum_{i=1}^{n} i^2$. (Bonus: do higher sums you can find the formulas here: http://www.math.com/tables/expansion/power.htm)

Proof. We only need one base case when $n = 1$ and we can see that $\frac{1}{6}(1)(1+1)(2 \cdot 1 + 1) = \frac{1 \cdot 2 \cdot 3}{6} = 1$ which is indeed equal to $\sum_{i=1}^{1} i^2 = 1^2 = 1$.

For the Inductive step we have the following inductive hypothesis, (IH): for some $n \in \mathbb{N}$ that $\frac{1}{6}n(n+1)(2n+1) = \sum_{i=1}^{n} i^2$.

We want to show that assuming the IH that the statement is true for $n+1$, that is $\frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1) = \sum_{i=1}^{n+1} i^2$.

To show this we start with $\frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1)$ and simplify and make it look like something plus $\frac{1}{6}n(n+1)(2n+1)$ to use the IH:

$$\begin{aligned}\frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1) &= \frac{1}{6}(n+1)(n+2)(2n+3)\\ &= \frac{1}{6}(n+1)(n)(2n+3) + \frac{2}{6}(n+1)(2n+3)\\ &= \frac{1}{6}(n+1)(n)(2n+1) + \frac{2}{6}\Big[(n+1)(2n+3) + (n+1)(n)\Big].\end{aligned}$$

We can now apply the IH to see that

$$\frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1) = \sum_{i=1}^{n} i^2 + \frac{2}{6}\Big[(n+1)(2n+3) + (n+1)(n)\Big].$$

If we simply we can see that $(n+1)(2n+3)+(n+1)(n) = (n+1)(3n+3) = 3(n+1)^2$, and thus we have that

$$\frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1) = \sum_{i=1}^{n} i^2 + \frac{2}{6}\Big[(n+1)(2n+3)+(n+1)(n)\Big] = \sum_{i=1}^{n} i^2 + (n+1)^2 = \sum_{i=1}^{n+1} i^2.$$

We have thus shown the inductive step and the base case and thus by the principal of induction we have shown that all $n \in \mathbb{N}$ that $\frac{1}{6}n(n+1)(2n+1) = 1 + \cdots + n^2 = \sum_{i=1}^{n} i^2$. $\square$

I like 1.11 from the book

Exercise 1.5. For each $n \in \mathbb{N}$ let P_n be the statement that $n^2 + 5n + 1$ is even:
1. Show that $P_n \Rightarrow P_{n+1}$
2. Is it true that P_n is true for all $\mathbb{N}$, why?

Some fun Fibonacci sequence proofs:

Exercise 1.6. Let F_n be the nth term of the Fibonacci sequence i.e. $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$. Show that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ (how many base cases do you need?)

Exercise 1.7. Show that $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$.

Exercise 1.8. Show that if $n \equiv 1, 2 \mod 3$ (that is n is not divisible by 3) then F_n is odd and if $n = 3k$ for some $k \in \mathbb{N}$, then F_n is even by strong induction.

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Week 2

Warm up:

1. Determine if the following sets are bounded from above or not, if so state their supremum:

(1) $\{1\} \cup \{4\} \cup \{6\}$
(2) $\{x^2 : x \geq 0\}$
(3) $\{-x^4 + 4 : x \in \mathbb{R}\}$
(4) $\{\frac{n-1}{n} : n \in \mathbb{N}\}$

2. Determine if the following sequences converge if so state their limit:

(1) $\frac{1}{n}\sin(n^3)$
(2) $n^3 - n$
(3) $(\frac{1-2n}{n})^n$
(4) $\frac{2^n}{n!}$

Exercise 2.1. Suppose that $S_1, S_2 \subset \mathbb{R}$ both are bounded above and have the property that they contain their upper bound, that is $\sup S_1 \in S_1$ and $\sup S_2 \in S_2$. Show that $S_1 \cup S_2$ is bounded above and contains its own upper bound, $\sup(S_1 \cup S_2) \in S_1 \cup S_2$.

Exercise 2.2. Is this true if we change union with intersection? That is it true that $S_1 \cap S_2$ is bounded above and contains its own upper bound, $\sup(S_1 \cap S_2) \in S_1 \cap S_2$. Prove or find a counter example.

Exercise 2.3. Suppose that S is a set that is bounded below. Let L be the greatest lower bound i.e. $L = \inf\{x : x \in S\}$. Show that for all positive numbers $\epsilon > 0$ that there is an $x \in S$ where $L \leq x \leq L + \epsilon$.

Hint: What does it mean for $L + \epsilon$ not to be a lower bound.

Proof. We can argue that for all $\epsilon > 0$ that $L + \epsilon$ is not a lower bound, and as such we know that the following statement is not true: "every $x \in S$, $x \geq L+$". The negation of that statement is exactly "there exists $x \in S$ where $x < L + \epsilon$". We can note that since $x \in S$ and L is a lower bound we know that $x \geq L$, thus we have shown that there is an $x \in S$ where $L \leq x < L + \epsilon$, so certainly $L \leq x \leq L + \epsilon$. $\square$

The following are two follow up exercise on the same idea, the first is a partial converse, the second is a connection with sequences.

Exercise 2.4. Suppose that S is a set that is bounded below. Let L be a lower bound for S. Suppose further that for all $\epsilon > 0$ that there is an $x \in S$ such that $L \leq x \leq L + \epsilon$. Show that $L = \inf\{x : x \in S\}$.

We can note that there are analogous statements that U is the sup of S if for all $\epsilon > 0$ that there is an $x \in S$ where $U - \epsilon \leq x \leq U$. Use this and your knowledge of limits to solve the following:

Exercise 2.5. Suppose that a_n is an increasing sequence, that is $a_n \geq a_m$ for $n \geq m$. Suppose further that there is some R where $a_n < R$ for all n, that is $\{a_n\}_{n\in\mathbb{N}}$ is bounded above. Show that $\lim_{n\to+\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

Here is some more limit practice:

Exercise 2.6. Suppose that $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence with limit a, and $r \in \mathbb{R}$, $r > 0$ show that $\{r \cdot a_n\}_{n\in\mathbb{N}}$ is also a convergent sequence with limit $r \cdot a$.

Proof. Fix $\epsilon > 0$. We want to show that there is an $N \in \mathbb{N}$ such that for all $n > N$ that $|r \cdot a_n - r \cdot a| < \epsilon$.

Let $\epsilon' = \epsilon/R$, not that since $\epsilon > 0$ that $\epsilon' > 0$. We can use the fact that $\lim_{n \to +\infty} a_n = a$ so there is an $N' \in \mathbb{N}$ such that for all $n > N'$ that $|a_n - a| < \epsilon'$.

Let $N = N'$, then notice that for all $n > N$,

$$|r \cdot a_n - r \cdot a| = |r(a_n - a)| = |r| \cdot |a_n - a| = r|a_n - a| < r\epsilon' = \epsilon.$$

Where $|r| = r$ since $r > 0$, as such we have shown that for all $n > N$ that $|r \cdot a_n - r \cdot a| < \epsilon$ which is what we wanted to show. □

Exercise 2.7. Suppose that $\{a_n\}_{n \in \mathbb{N}}$ is a convergent sequence with limit a. Show that $\{a_{2n} : n \in \mathbb{N}\}$ is a convergent sequence with limit a.

Exercise 2.8. Prove that $a_n = \frac{n!}{n^n}$ is a convergent sequence with limit 0.

Hint: expand the power and the factorial as a large product and compare to a simpler sequence, say $\frac{1}{n}$.

Exercise 2.9. Prove the squeeze theorem if $a_n \geq 0$ is a sequence and $b_n \geq a_n$ is a different sequence that converges with limit 0 then a_n converges with limit 0.

Week 3

Warm up: Solve the following algebraic expressions

(1) $2x - 1 = x$
(2) $\frac{2x-1}{x} = x$
(3) $x^2 = x$

Given that following sequences s_n converge to s, determine the possible values for s.

(1) $s_{n+1} = 2s_n - 1$
(2) $s_{n+2} = \frac{2s_{n+1}-1}{s_n}$
(3) $s_{n+1} = s_n^2$

also: Determine if the sequence $s_{n+2} = s_{n+1}s_n$ converge if $s_1 = 1/2$ and $s_2 = 4$? What if $s_1 = 1/2$ and $s_2 = 0$? The professor wanted me to go over the following homework problem which is the squeeze theorem:

Exercise 3.1. Suppose that $(s_n), (a_n), (b_n)$ are all sequences where for all $n \in \mathbb{N}$ we have that $a_n \leq s_n \leq b_n$ and we know that $\lim_{n\to+\infty} a_n = s = \lim_{n\to+\infty} b_n$. Show that $\lim_{n\to+\infty} s_n = s$.

Exercise 3.2. Suppose that (t_n) and (s_n) are sequences where $|s_n| \leq t_n$ for all $n \in \mathbb{N}$, conclude that if $\lim_{n\to+\infty} t_n = 0$ then $\lim_{n\to+\infty} s_n = 0$.

The following is a problem closely related to one of your homework problems:

Exercise 3.3. If (a_n) is a sequence and $\lim_{n\to+\infty} a_n = a$ and (b_n) where $b_n = a_n$ for all but finitely many $n \in \mathbb{N}$. Show that $\lim_{n\to+\infty} b_n = a$.

The following is an exercise around recursively defined series.

Exercise 3.4. For this whole exercise we will be working with the recurrence relation: $x_{n+2} = \frac{x_{n+1}+x_n}{2}$.

(1) Show that $\frac{2x_{n+2}+x_{n+1}}{3} = \frac{2x_{n+1}+x_n}{3}$ for all $n \in \mathbb{N}$, conclude that $\frac{2x_{n+1}+x_n}{3} = \frac{2x_2+x_1}{3}$ for all $n \in \mathbb{N}$.
(2) Show that $2|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$.
(3) Argue that $2|x_{n+3} - \frac{2x_2+x_1}{3}| \leq |x_{n+2} - \frac{2x_2+x_1}{3}|$ for all $n \in \mathbb{N}$.
(4) Argue that for all $n \in \mathbb{N}$ that $|x_{n+2} - \frac{2x_2+x_1}{3}| \leq \frac{1}{2^n}|a_2 - a_1|$.
(5) Conclude that the sequence x_n converges to $\frac{2x_2+x_1}{3}$.

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Week 4

Warm-up: Determine if the following sequences are increasing, decreasing, and/or bounded.

(1) $x_n = \frac{n!}{(-n)^n}$
(2) $x_n = \log(n+1)$
(3) $x_n = \frac{\log(n)}{n}$

Monotone sequences and e

Here is a fun-ish exercise proving the existence of the number e.

Exercise 4.1. Consider the sequence $s_n = \sum_{i=1}^{n} \frac{1}{i!}$. First compare with the sequence $x_n = 1 + \sum_{i=2}^{n} \frac{1}{(i-1)^2}$ and argue that s_n is bounded. Conclude that there is sum number $e = \sum_{i=1}^{+\infty} \frac{1}{i!}$.

Exercise 4.2. Show that if $x_n > 0$ is a bounded increasing sequence if and only if is $s_n = \log(x_n)$ a bounded increasing sequence.

Exercise 4.3. Using that $x_n = (1 + \frac{1}{n})^n$ is a bounded increasing sequence. Bonus: Using the fact that $\lim_{n\to+\infty} \log(x_n) = \log(\lim_{n\to+\infty} x_n)$ to show that the limit of the above problem is e (hint: it will be useful to use the hint twice)

Infinite sequences and Cauchy sequences

Exercise 4.4. Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} x_n = +\infty$ let $s_n = \frac{1}{n}(\sum_{i=1}^{n} x_i)$ show that $\lim_{n\to+\infty} s_n = +\infty$

Hint: If you know that there is an N where $s_n > M$ for all $n > M$ is there a time where $s_n > M/2$?

Exercise 4.5. Suppose that $\lim_{n\to+\infty} s_n = -\infty$, show that there the sequence s_n is bounded from above.

Exercise 4.6. Show that if x_n is a Cauchy sequence, the show by hand (without limit theorems) that the sequence $s_n = x_{n+1} - x_n$ is Cauchy. (Bonus: what is its limit?)

Exercise 4.7. Suppose that x_n is a Cauchy sequence. Let t_n be a sequence given by the recurrence relation $t_n = t_{n-1} + x_n - x_{n-1}$ and $t_0 = 100$. Show that the sequence t_n is Cauchy.

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Week 5

Warm-up: Which of the following have an increasing subsequence? Decreasing subsequence?

(1) $a_n = (-1)^n \cdot 2$
(2) $b_n = (\frac{-1}{2})^n$
(3) $c_n = \frac{3n+2}{20n-39}$
(4) $d_n = -n^3$

Exercise 5.1. Suppose that $x_n = \begin{cases} 0 & n = 3k \\ 22 & n = 3k+1 \\ 23 & n = 3k+2 \end{cases}$ show that there is no subsequence that converges to 12.

Exercise 5.2. Show that if $\lim_{n\to+\infty} a_n = +\infty$, then every subsequence a_{n_k} we have that $\lim_{k\to+\infty} a_{n_k} = +\infty$.

Exercise 5.3. Using the previous exercise, conclude that there is no decreasing subsequence of a_n (same a_n from above).

Exercise 5.4. Suppose that b_n has no decreasing subsequence. Show that b_n is bounded from below.

Hint: A proof by contrapositive may be helpful

Exercise 5.5. Show that if every subsequence of a_n is bounded then the original subsequence is bounded.

Hint: It may be easier to do a proof by contrapositive.
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Week 6

Warm-up which of the following are continuous functions on $\mathbb{R} \setminus \{0\}$?

(1) $\frac{1}{\cos(1/x)}$
(2) $\sin(x^2)e^{1/x^2}$
(3) $\frac{1}{\sin(x)}$
(4) $\cos(1/x)x^2 - \sin(1/x)$

Exercise 6.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be α-Hölder continuous if $|f(x) - f(y)| \leq (x - y)^\alpha$. Show that for any $0 < \alpha < 1$ an α-Hölder continuous function is continuous.

Exercise 6.2. Prove that the function $\begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

Exercise 6.3. Show that if f is continuous at 0 and g is not continuous at 0, then $f + g$ is not continuous at 0.

Exercise 6.4. Find a pair of functions f and g that are not continuous at zero, but $f + g$ is continuous at zero.

Exercise 6.5. Show that $\sum_{n=1}^{+\infty} \frac{1}{17^n - 12} < +\infty$.

Exercise 6.6. Show that $\sum_{n=2}^{+\infty} \frac{\sin(n)}{n^{3/2}} < +\infty$

Exercise 6.7. Determine if $\sum_{n=1}^{+\infty} \frac{(2n)!}{(n)^{2n}}$ converges or not. Similiarly if $\sum_{n=1}^{+\infty} \frac{n!}{(n+1) \cdot 2^n}$ converges.

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Week 7

Exercise 7.1. Prove that for any constant $r \in \mathbb{R}$ there is an angle $\theta_r \in [0, \frac{\pi}{2}]$ where $\sin(\theta_r) = \cos(r\theta_r)$.

Proof. Consider the function $f(x) = \sin(x) - \cos(rx)$ note that $f(0) = -1$ and $f(\pi/2) \geq 0$.

Case 1: If it is equal to zero we are done as $\sin(\pi/2) = \cos(r\pi/2)$.

Case 2: If $f(\pi/2) > 0$ by IVT we know that there is a $x \in [0, \pi]$ where $f(x) = 0$ and this x solves the problem. $\square$

Exercise 7.2. Given that $\sqrt{x+y} - \sqrt{x} \leq \sqrt{y}$ for $x, y \geq 0$ show that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, +\infty)$.

Exercise 7.3. Let $g(x)$ be a continuous function on $[0, +\infty)$ with the property that for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $x > N$ $x \in \mathbb{R}$ that $|g(x)| < \epsilon$. Show that $g(x)$ is uniformly continuous.

Proof. Fix $\epsilon > 0$.

There exists and N such that if $x > N$ then $g(x) < \frac{\epsilon}{2}$.

We can note that g is continuous on the closed interval $[0, N+1]$ and thus uniformly continuous, that is there is a $\delta' >$ such that if $|x - y| < \delta'$ with $x, y \in [0, N+1]$ then $|f(x) - f(y)| < \epsilon$.

Let $\delta = \min\{\delta', 1\}$. If $|x - y| < \delta$ then either $x, y > N$ or $x, y \leq N + 1$. In the first case we have that $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$ by the triangle inequality. In the second case we have that $x, y \in [0, N+1]$ and $|x - y| < \delta'$ so $|f(x) - f(y)| < \epsilon$. $\square$

Exercise 7.4. Suppose that $f(x) : [0, +\infty) \to \mathbb{R}$ is a continuous function that is uniformly continuous on $[1,000, +\infty)$ show that f in uniformly continuous on $[0, +\infty)$.

Proof. Fix $\epsilon > 0$.

Note that f is continuous on the closed and bounded interval $[0, 1, 002]$ and thus uniformly continuous on this interval. Thus, there exists a $\delta_1 > 0$ such that if $|x - y| < \delta_1$ and $x, y \in [0, 1002]$ then $|f(x) - f(y)| < \epsilon$.

We are given that f is uniformly continuous from $[1000, +\infty)$ so there exists $\delta_2 > 0$ such that if $|x - y| < \delta_2$ and $x, y \geq 1000$ then $|f(x) - f(y)| < \epsilon$.

Define $\delta = \min\{\delta_1, \delta_2, 2\}$. If $|x - y| < \delta \leq 2$ and $x, y \in [0, +\infty)$ then we can note that either $0 \leq x, y \leq 1002$ or $x, y \geq 1000$ (this is because otherwise if $x \leq 1000$ and $y > 1002$ then $|x - y| > 2$ contradiction that $|x - y| < \delta$).

In the first case we have that $x, y \in [0, 1002]$ and $|x - y| < \delta \leq \delta_1$ so we know that $|f(x) - f(y)| < \epsilon$.

In the second case we have that $x, y \in [1000, +\infty)$ and that $|x - y| < \delta \leq \delta_2$ so we know that $|f(x) - f(y)| < \epsilon$.

Thus, in either possible case we have that $|f(x) - f(y)| < \epsilon$, so we have shown that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$ proving the statement. $\square$

Exercise 7.5. Show that the sequence $a_n = e^{1/2} \cdot e^{1/4} \dots e^{1/2^n} = \Pi_{j=1}^{n} e^{1/2^j}$ is a Cauchy sequence.

Exercise 7.6. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous (but not necessarily uniformly continuous on all of $\mathbb{R}$) and a_n is a Cauchy sequence, then $f(a_n)$ is a Cauchy sequence.

Remark 7.7. The above exercise 7.6 does not show that general continuous functions send Cauchy sequences to Cauchy sequences. For example $f(x) = \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$ but it sends the Cauchy sequence $1/n$ to the sequence n which is not Cauchy, so the assumption that f was continuous on all of $\mathbb{R}$ was important for the exercise (as failing to be continuous at a single point in $\mathbb{R}$ allows for counter examples).

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