Fredholm Operators and the Family Index

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Chapter 1

Introduction

Many problems in mathematics arise from the desire to understand the behavior of linear operators. The simplest example comes from elementary linear algebra, where systems of linear equations are represented by an $m \times n$ matrix; such a matrix is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. More generally, suppose that V and W are vector spaces and that $T : V \to W$ is a linear transformation. For a fixed $w \in W$, we are interested in solutions to the equation $T(v) = w$. Equipping V and W with varying amount of structure, say a metric or inner product, introduces complications and makes the study of such equations more interesting and potentially more difficult.

Partial differential equations provide a particularly motivating source of examples. For instance, Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{1.1}$$

can be written as $\Delta(f) = 0$, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a linear differential operator between appropriate vector spaces of functions. Integral equations are also of interest. Suppose that $K(x, y)$ is a continuous equation of two variables, and $g(x)$ is a fixed function. The equation

$$\int_a^b K(x,y) f(y)\, dy = g(x) \tag{1.2}$$

can be written as $K(f) = g$, where $K(f)(x) := \int_a^b K(x,y) f(y)\, dy$ is, again, a linear operator between function spaces.

In all of these examples, the solution space of the equation $T(v) = w$ can be better understood by asking the following questions: does a solution exist, and if so, is it unique? In other words, wish to measure the *surjectivity* and *injectivity* of the operator T. The *index* of an operator, defined as

$$\operatorname{ind}(T) := \dim \ker T - \dim \operatorname{coker} T \tag{1.3}$$

is one such measurement. Despite being defined algebraically, the index of an operator T is an interesting topological invariant. For example, the index distinguishes the connected components of the space of *Fredholm operators*, that is, operators for which the index in (1.3) is well-defined. This is the beginning of a much deeper connection between the functional analysis of Fredholm operators and topology.

In the 1960's, Michael Atiyah and Friedrich Hirzebruch developed *topological K-theory*, a powerful tool in algebraic topology [3, 4]. The starting point for this theory is the abelian group $K(X)$, constructed using isomorphism classes of complex vector bundles on a compact space X. Unsurprisingly, associated to any family of Fredholm operators parametrized by X is a natural element of $K(X)$, given by a generalization of the Fredholm index. That this *family index* is actually an isomorphism between homotopy classes of families of Fredholm operators and $K(X)$ is the subject of the Atiyah-Jänich theorem [5]. This theorem contains the basic topological theory of the Fredholm operators as a special case, and is a powerful connection between functional analysis and algebraic topology. The main goal of this thesis is to prove this theorem.

The structure of this text is as follows. In Chapter 2, we review the functional analytic and topological tools that will be employed in later chapters. In Chapter 3, we cover the theory of Fredholm operators and the classical index. We conclude with the fact that the index induces a bijection between the connected components of the space of Fredholm operators and the integers. As preparation for the generalization of this result, in Chapter 4 we establish the basic theory of vector bundles and the group $K(X)$, and in Chapter 5 we prove the Atiyah-Jänich theorem, which, said differently from above, states that the space of Fredholm operators is a classifying space for K-theory. Finally, in Chapter 6, we use the theory developed in the previous chapters to study a particular class of operators, the Toeplitz operators.

Were more time available, this chapter would conclude with a proof of Bott periodicity. Unfortunately, we direct the reader to Atiyah's expository papers [2] and [1] and the references therein for further details.

The material in this text is not original, but rather a personal synthesis and exposition of the work of great mathematicians such as M. Atiyah, F. Hirzebruch, R. Bott, and K. Jänich. My thanks also goes out to my advisor, Ezra Getzler, for his guidance and insight throughout the writing of this document. Without him, none of this would have been possible.

Chapter 2

Preliminaries

The goal of this exposition is to study the foundations of topological K-theory. Concretely, this entails topological analysis of the space of Fredholm operators on a Hilbert space. This requires a synthesis of ideas from functional analysis and topology, and we assume the reader is familiar with both subjects. For convenience, we use this chapter to collect some standard results. Proofs will often be omitted, and standard references will be given.

2.1 Functional Analysis

The underlying arena for the mathematics that follows is infinite dimensional Hilbert space. This section will review the basic definitions and facts of the theory. Details on any of the following results can be found in any text on functional analysis, such as [10], [11], and [19].

Definition 2.1. A **Hilbert space** $\mathcal{H}$ is a vector space with an inner product $\langle \cdot, \cdot \rangle$ which is a complete metric space in the norm induced by the inner product. A Hilbert space is **separable** if it has a countably dense subset.

Recall that a **Banach space** is a normed vector space which is complete under the metric induced by the norm. Every Hilbert space is a Banach space, but the converse is not true — not every norm comes from an inner product. Though much of what we discuss generalizes to Banach spaces, the inner product structure on a Hilbert space simplifies the theory greatly.

Proposition 2.2. *All separable Hilbert spaces admit a countable orthonormal basis. Hence, all separable infinite dimensional Hilbert spaces are isomorphic.*

Example 2.3. Let (X, μ) be a measure space. The standard example of a separable complex Hilbert space is $L^2(X, \mu)$, defined as

$$L^2(X, \mu) := \left\{ f : X \to \mathbb{C} \; : \; \int_X |f(x)|^2 \, d\mu < \infty \right\}.$$

In words, $L^2(X, \mu)$ is the space of all square-integrable functions. The inner product is given by $\langle f, g \rangle = \int_X f(x)\overline{g(x)} \, d\mu$. When $X = \mathbb{N}$, $L^2(\mathbb{N})$ is the set of square-summable sequences and is written ℓ^2.

Though inseparable spaces exist, all Hilbert spaces we consider in this text will be separable.

There is a rich theory to be developed from studying Hilbert spaces such as $L^2(X, \mu)$ in their own right, but our primary interest lies in the transformations of the space.

Definition 2.4. A linear map $T : \mathcal{H} \to \mathcal{H}$ is **bounded** if $\sup_{\|f\|=1} \|Tf\| < \infty$. The set of bounded operators on $\mathcal{H}$ is denoted $\mathfrak{L}(\mathcal{H})$. The set of invertible bounded operators is written $GL(\mathcal{H})$. When there is a distinction between the domain and codomain, we write $\mathfrak{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $GL(\mathcal{H}_1, \mathcal{H}_2)$.

Theorem 2.5 (Open Mapping Theorem)**.** *Let $T \in \mathfrak{L}(\mathcal{H})$. If T is surjective, then T is open.*

Corollary 2.6. *If $T \in \mathfrak{L}(\mathcal{H})$ is a bijection, then T is a homeomorphism.*

It is a basic fact that an operator T is bounded if and only if it is continuous. Moreover, $\mathfrak{L}(\mathcal{H})$ becomes a Banach space under the norm $\|T\| := \sup_{\|f\|=1} \|Tf\|$.

An important tool in the study and classification of Hilbert space operators is the adjoint operator.

Definition 2.7. Let $T \in \mathfrak{L}(\mathcal{H})$. The **adjoint** of T, denoted T^*, is the unique operator on $\mathcal{H}$ satisfying $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f, g \in \mathcal{H}$.

Several classes of important operators are defined in terms of the adjoint.

Definition 2.8. Let $T \in \mathfrak{L}(\mathcal{H})$.

(a) T is **normal** if $T^*T = TT^*$;

(b) T is **self-adjoint / Hermitian** if $T = T^*$;

(c) T is **unitary** if $T^* = T^{-1}$.

Note that self-adjoint and unitary operators are both normal.

The space $\mathfrak{L}(\mathcal{H})$ is the most important example of a *Banach algebra*, and in particular, a *C^*-algebra*.

Definition 2.9. A **Banach algebra** $\mathcal{B}$ is a complex algebra with identity equipped with a norm $\|\cdot\|$ such that:

(a) $\mathcal{B}$ is a Banach space under $\|\cdot\|$;

(b) $\|1\| = 1$;

(c) $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in \mathcal{B}$.

Proposition 2.10. *Let $\mathcal{B}$ be a Banach algebra. If $x \in \mathcal{B}$ with $\|1 - x\| < 1$, then x is invertible.*

The following fact is frequently used.

Proposition 2.11. *The set of invertible elements $\mathcal{B}^\times$ in a Banach algebra is open. In particular, if x is invertible, then $\mathcal{B}^\times$ contains an open ball around x of radius $1/\left\|x^{-1}\right\|$.*

Proof. Let $x \in \mathcal{B}$ be invertible. If $y \in \mathcal{B}$ with $\|x - y\| < 1/\left\|x^{-1}\right\|$, then

$$1 > \left\|x^{-1}\right\| \|x - y\| \geq \left\|1 - x^{-1}y\right\| .$$

Hence, $x^{-1}y$ is invertible, and therefore $x(x^{-1}y) = y$ is invertible. It follows that the set of invertible elements is open. ☐

Next, some basic spectral theory.

Definition 2.12. Let $\mathcal{B}$ be a Banach algebra. The **spectrum** of $x \in \mathcal{B}$ is the set

$$\sigma(x) := \{ \lambda \in \mathbb{C} \ : \ x - \lambda \text{ is not invertible in } \mathcal{B} \} .$$

The **resolvent set** of x is the complement of the spectrum:

$$\rho(x) = \mathbb{C} \setminus \sigma(x).$$

The **spectral radius** of x is $r(x) := \sup_{\lambda \in \sigma(x)} |\lambda|$.

Proposition 2.13. *Let $\mathcal{B}$ be a Banach algebra. If $x \in \mathcal{B}$, then $\sigma(x)$ is nonempty, compact, and $r(x) \leq \|x\|$.*

In fact, we can say even more about the spectral radius:

Theorem 2.14. *Let $\mathcal{B}$ be a Banach algebra. If $x \in \mathcal{B}$, then $r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$.*

With the motivating example of the Hilbert space adjoint in mind, we define an abstract involution on a Banach algebra, giving rise to the notion of a C^*-algebra.

Definition 2.15. Let $\mathcal{B}$ be a Banach algebra. An **involution** on $\mathcal{B}$ is a a mapping $x \mapsto x^*$ such that

(a) $x^{**} = x$ for all $x \in \mathcal{B}$;

(b) $(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*$ for $x, y \in \mathcal{B}$ and $\alpha, \beta \in \mathbb{C}$;

(c) $(xy)^* = y^* x^*$ for all $x, y \in \mathcal{B}$.

If $\|x^* x\| = \|x\|^2$ for all $x \in \mathcal{B}$, then $\mathcal{B}$ is a C^***-algebra**.

As $\mathfrak{L}(\mathcal{H})$ is a C^*-algebra, every operator in $\mathfrak{L}(\mathcal{H})$ has a spectrum. When the dimension of $\mathcal{H}$ is finite, this coincides with the set of eigenvalues. In infinite dimensions, the spectrum of an operator is in general more complicated. For example, an operator may have a nonempty spectrum with no eigenvalues. Of self-adjoint and unitary operators, we can say the following.

Proposition 2.16. *A normal operator $T \in \mathfrak{L}(\mathcal{H})$ is self-adjoint if and only if $\sigma(T) \subseteq \mathbb{R}$. Similarly, T is unitary if and only if $\sigma(T) \subseteq S^1 := \{ z \in \mathbb{C} \ : \ |z| = 1 \}$.*

Since the discrete spectrum of a finite dimensional operator is desirable, we introduce another important class of operators.

Definition 2.17. An operator $T \in \mathfrak{L}(\mathcal{H})$ is **compact** if $T(B_1)$ is compact, where B_1 is the closed unit ball in $\mathcal{H}$.

We denote the set of compact operators on $\mathcal{H}$ by $\mathfrak{K}(\mathcal{H})$. Note that if the image of T is finite dimensional (i.e., T is a **finite rank** operator), then T is compact. In fact, the space of compact operators is the norm-closure of the space of finite rank operators.

Proposition 2.18. *The space $\mathfrak{K}(\mathcal{H})$ is the minimal nontrivial closed ideal in $\mathfrak{L}(\mathcal{H})$, and is the closure of the ideal of finite rank operators.*

As noted above, the spectrum of an operator may not contain any eigenvalues at all. For compact operators, this is not the case. For example, when T is compact and self-adjoint, we have the following.

Theorem 2.19. *Let $T : \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator. There is an orthonormal basis $\{e_n\}$ for $\mathcal{H}$ of eigenvectors of T, with corresponding real eigenvalues $\{\lambda_n\}$ such that $\lambda_n \to 0$. Furthermore, for all $f \in \mathcal{H}$,*

$$Tf = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle e_n.$$

This result is the *spectral theorem for compact self-adjoint operators,* and is generalizes the diagonalizability of a Hermitian matrix from finite dimensional linear algebra. With more sophisticated tools, one can generalize the idea of diagonalization to normal operators.

Finally, we discuss the tensor product of Hilbert spaces. Information on tensor products can be found in [19], so we only recall the basics. The tensor product of two vector spaces V and W, denoted $V \otimes W$, satisfies the following universal property. Let $f : V \times W \to X$ be a bilinear map. There is a unique linear map $\tilde{f} : V \otimes W \to X$ making the diagram

$$\begin{array}{ccc} V \times W & \xrightarrow{\phi} & V \otimes W \\ & \searrow^{f} & \downarrow \tilde{f} \\ & & X \end{array}$$

commute, where $\phi : V \times W \to V \otimes W$ is given by $\phi(v, w) = v \otimes w$. Such a space is unique up to isomorphism. Elements of $V \otimes W$ are written as formal sums of elements $v \otimes w$ satisfying $v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w$, $v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$, and $\lambda(v \otimes w) = \lambda v \otimes w = v \otimes \lambda w$. If V and W are inner product spaces with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, then $V \otimes W$ becomes an inner product space under

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{\otimes} := \langle v_1, v_2 \rangle_V \langle w_1, w_2 \rangle_W .$$

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, then we denote by $\mathcal{H}_1 \otimes \mathcal{H}_2$ the metric space completion under $\langle \cdot, \cdot \rangle_{\otimes}$ of the vector space tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$.

Given any two linear operators $T : V \to V$ and $S : W \to W$, it follows from the universal property described above that there is a unique linear operator $T \otimes S : V \otimes W \to V \otimes W$ given by

$$(T \otimes S)(v \otimes w) = (Tv) \otimes (Sw).$$

Finally, we record some important isomorphisms.

Proposition 2.20. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$, and $\mathbb{C}^n \otimes \mathcal{H} \cong \mathcal{H}$.*

2.2 Topology

We assume familiarity with basic topological language and results. Standard references for the following material are [17] and [14].

Theorem 2.21 (Tietze Extension Theorem)**.** *Let X be a compact Hausdorff topological space and let $A \subseteq X$ be closed. Suppose that $f : A \to V$ is a continuous function into a vector space V. Then there exists a continuous extension $\tilde{f} : X \to V$ such that $\tilde{f}\mid_A = f$.*

Theorem 2.22. *Let X be compact Hausdorff, and let $\{U_i\}$ be an finite open cover. There exist continuous functions $f_i : X \to \mathbb{R}$ such that $0 \leq f_i \leq 1$, the support of each f_i is contained in U_i, and $\sum_i f_i(x) = 1$ for each $x \in X$. The collection of functions $\{f_i\}$ is called a* ***partition of unity****.*

The existence of partitions of unity is frequently used in Chapters 4 and 5.

Lemma 2.23 (Tube Lemma)**.** *Let X, Y be topological spaces with X compact. Endow $X \times Y$ with the product topology. Suppose that $y \in Y$ and $U \subseteq X \times Y$ is an open set containing $X \times \{y\}$. Then there exists an open subset $V \subseteq Y$ such that $y \in V$ and $X \times V \subseteq U$.*

We finish with some comments on elementary homotopy theory.

Definition 2.24. Let X, Y be topological spaces and let $f, g : X \to Y$ be continuous maps. A **homotopy** between f_0 and f_1 is a continuous family of functions

$$f_t : X \times I \to Y$$

such that $f_0(x) = f(x)$ and $f_1(x) = g(x)$ for all $x \in X$. If such a homotopy exists between f and g, then f and g are **homotopic**.

Informally, two functions are homotopic when they can be continuously deformed into one another. It is clear that homotopy defines an equivalence class on the set of functions $X \to Y$. We will use the notation $[X, Y]$ to denote the homotopy equivalence class of functions from $X \to Y$.

Definition 2.25. Two spaces X and Y are **homotopy equivalent** if there exist functions (called **homotopy equivalences** $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. A space is **contractible** if it is homotopy equivalent to a point.

Homotopy groups play a central role in topology. We briefly comment on their definition, and refer to [14] for details. Let S^1 be the unit circle, and let $p \in S^1$. Let X be a topological space with base point x_0. Consider the set of homotopy classes of basepoint-preserving maps $(S^1, p) \to (X, x_0)$. There is a natural notion of concatenation on this set which gives it a group structure. This group is $\pi_1(X, x_0)$, the **fundamental group** of X at x_0. If X is connected, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any $x_0, x_1 \in X$, so we can write $\pi_1(X)$ without confusion. When S^1 is replaced by S^n, the resulting group is written $\pi_n(X)$.

Chapter 3

Fredholm Operators

We begin studying a special class of Hilbert space operators, namely, the Fredholm operators. These are "almost invertible" operators, by which we mean invertible up to a compact operator. The space of Fredholm operators on a Hilbert space, written $\mathfrak{F}(\mathcal{H})$, has an interesting topology, and is the main focus of this chapter. In particular, we will construct an "index" map from $\mathfrak{F}(\mathcal{H})$ to $\mathbb{Z}$, and show that any two Fredholm operators with equal index can be connected by a path. Hence, the index map gives a bijection between the set of connected components of $\mathfrak{F}(\mathcal{H})$ and $\mathbb{Z}$. This fact will be our first exposure to the subject of K-theory, and is a special case of the main theorem of Chapter 5.

The contents of this chapter are well-known; for reference, see [11] and [7]. As a reminder, $\mathcal{H}$ will be a separable, complex Hilbert space. We denote the set of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ by $\mathfrak{L}(\mathcal{H})$, and the ideal of compact operators $K : \mathcal{H} \to \mathcal{H}$ by $\mathfrak{K}(\mathcal{H})$. Much of the following theory generalizes to Banach spaces, but emphasis is placed on Hilbert space for the sake of simplicity.

3.1 Definitions and Examples

Fredholm operators were originally studied at the turn of the 20th century in the context of solutions to certain integral equations [13]. The classical definition is as follows:

Definition 3.1. An operator $T \in \mathfrak{L}(\mathcal{H})$ is a **Fredholm** operator if $\ker T$ and $\operatorname{coker} T := \mathcal{H}/\operatorname{im} T$ are both finite dimensional.

This definition is often presented with a third condition; namely, that the image of T be closed. In fact, this follows from the finite-dimensionality of the cokernel of T.

Proposition 3.2. *If $T \in \mathfrak{L}(\mathcal{H})$ is a Fredholm operator, then $\operatorname{im} T$ is closed.*

Proof. Suppose that T is Fredholm. Note that the restriction $T\,|_{(\ker T)^\perp} \colon (\ker T)^\perp \to \operatorname{im} T$ is bijective. Let $n = \dim \operatorname{coker} T < \infty$. Extend $T\,|_{(\ker T)^\perp}$ to a map

$$\tilde{T} : (\ker T)^\perp \oplus \mathbb{C}^n \to \operatorname{im} T \oplus \operatorname{coker} T$$

by sending a basis of $\mathbb{C}^n$ to a basis of $\operatorname{coker} T$. Then this map is a continuous bijection, and thus, by the open mapping theorem, a homeomorphism. Since $\ker T$ and thus $(\ker T)^\perp$ is closed, $\operatorname{im} T = \tilde{T}\left((\ker T)^\perp\right)$ is closed. □

A useful characterization quickly follows:

Corollary 3.3. *An operator $T \in \mathfrak{L}(\mathcal{H})$ is a Fredholm operator if and only if $\ker T$ and $\ker T^*$ are both finite dimensional. In particular, T is Fredholm if and only if T^* is Fredholm.*

Proof. Suppose that T is Fredholm. Recall that $\ker T^* = (\operatorname{im} T)^\perp$. Since $\operatorname{im} T$ is closed and $\mathcal{H} = \operatorname{im} T \oplus (\operatorname{im} T)^\perp$, it follows that $\ker T^* = (\operatorname{im} T)^\perp \cong \mathcal{H}/\operatorname{im} T = \operatorname{coker} T$. Hence, T is Fredholm if and only if T^* is Fredholm. □

An alternative approach to the theory of Fredholm operators is to consider the space of bounded operators modulo the ideal of compact operators. The resulting quotient Banach algebra $\mathfrak{L}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ (see [11]) is called the *Calkin algebra*, and will be denoted $\mathfrak{C}(\mathcal{H})$. Let $\pi : \mathfrak{L}(\mathcal{H}) \to \mathfrak{L}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ be the natural projection map. The following theorem, originally due to Atkinson [6], asserts that an operator is Fredholm exactly when it is image under the projection is invertible in $\mathfrak{C}(\mathcal{H})$.

Theorem 3.4. *An operator $T \in \mathfrak{L}(\mathcal{H})$ is **Fredholm** if and only if $\pi(T)$ is invertible in the Calkin algebra. In particular, an operator T is Fredholm if and only if there exists an $S \in \mathfrak{L}(\mathcal{H})$ such that*

$$ST = I + K_1 \qquad \text{and} \qquad TS = I + K_2$$

for some $K_1, K_2 \in \mathfrak{K}(\mathcal{H})$.

Proof. Let $T \in \mathfrak{L}(\mathcal{H})$.

First, suppose that $\ker T$ and $\operatorname{coker} T$ are finite dimensional. Let $P \in \mathfrak{L}(\mathcal{H})$ be the orthogonal projection onto $\ker T$, and let $Q \in \mathfrak{L}(\mathcal{H})$ be the orthogonal projection onto $(\operatorname{im} T)^\perp$. The following restriction of T is invertible:

$$T : (\ker T)^\perp \to \operatorname{im} T.$$

Let

$$S : \operatorname{im} T \to (\ker T)^\perp$$

be the inverse of this restricted operator. We can extend the domain of S to all of $\mathcal{H}$ by defining $S\left((\operatorname{im} T)^\perp\right) = 0$. With this extension, it follows that $ST + P$ and $TS + Q$ are the identity operators on $\mathcal{H}$. Since $\ker T$ and $(\operatorname{im} T)^\perp$ are finite dimensional, P and Q are compact operators. Hence, $ST = I - P$ and $TS = I - Q$, and so S is the inverse of T modulo compact operators. Thus, $\pi(T)$ is invertible in $\mathfrak{C}(\mathcal{H})$.

Next, suppose that there exists an $S \in \mathfrak{L}(\mathcal{H})$ such that $ST = I + K_1$ and $TS = I + K_2$ for some $K_1, K_2 \in \mathfrak{K}(\mathcal{H})$. Then $I - ST$ is compact. Let $f \in \ker T$. Then

$$f = (I - ST)f + STf = (I - ST)f.$$

Hence, $f \in \operatorname{im} I - ST$, which means that $\ker T \subseteq \operatorname{im} I - ST$. In particular, the closed unit ball in $\ker T$ is contained in the closed unit ball of $\operatorname{im} I - ST$. Since $I - ST$ is compact, the closed unit ball of $\ker T$ is compact. Thus, $\ker T$ is finite dimensional. Next, the equation $TS = I + K_2$ implies that $I - TS$ is compact. Taking the adjoint of this equation gives $(I - TS)^* = I - S^*T^*$, and repeating the previous argument shows that $\ker T^*$ is finite dimensional. Hence, T is Fredholm. □

3.2 The Fredholm Index

The finiteness of $\dim \ker T$ and $\dim \operatorname{coker} T$ for a Fredholm operator T allows for the definition of a particular map $\mathfrak{F}(\mathcal{H}) \to \mathbb{Z}$ which attaches an integer to each Fredholm operator, the *Fredholm index* of T.

Definition 3.5. Let $T \in \mathfrak{F}(\mathcal{H})$. The **Fredholm index** of T is

$$\begin{aligned} \operatorname{ind}(T) &:= \dim \ker T - \dim \operatorname{coker} T \\ &= \dim \ker T - \dim \ker T^*. \end{aligned}$$

The Fredholm index is a weak measure of how distant an operator is from being invertible. Indeed, if T is invertible, then T is Fredholm with index 0.

Before moving on, we consider a few examples.

Example 3.6. Suppose that $\mathcal{H}$ is finite dimensional. Then any operator $T : \mathcal{H} \to \mathcal{H}$ is Fredholm, and

$$\begin{aligned} \operatorname{ind}(T) &= \dim\ker T - \dim\operatorname{coker} T = \dim\ker T - \dim\mathcal{H}/\operatorname{im} T \\ &= \dim\ker T - \dim\mathcal{H} + \dim\operatorname{im} T. \end{aligned}$$

By the Rank-Nullity Theorem, it follows that $\operatorname{ind}(T) = 0$. In general, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional Hilbert spaces of possibly different dimension, then any linear $T : \mathcal{H}_1 \to \mathcal{H}_2$ is Fredholm (extending our definition of Fredholm to account for the different domain and range spaces in the natural way) and a similar computation gives $\operatorname{ind}(T) = \dim\mathcal{H}_1 - \dim\mathcal{H}_2$.

Example 3.7. Let $S_{-1} : \ell_2 \to \ell_2$ be the *unilateral shift* operator given by

$$S_{-1}(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots).$$

Clearly, S_{-1} is injective, so $\dim\ker S_{-1} = 0$. Furthermore, $\operatorname{coker} S_{-1} = \ell_2/S_{-1}(\ell_2) = \text{Span}(e_1)$, and so $\dim\operatorname{coker} S_{-1} = 1 < \infty$. Hence, S_{-1} is a Fredholm operator, and

$$\operatorname{ind}(S_{-1}) = \dim\ker S_{-1} - \dim\operatorname{coker} S_{-1} = 0 - 1 = -1.$$

It is a straightforward calculation to verify that $S_1 := S_{-1}^*$ is the backwards unilateral shift operator given by

$$S_1(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots).$$

In this case, $\ker S_1 = \text{Span}(e_1)$ and $\operatorname{coker} S_1 = \{0\}$. Hence, S_1 is also Fredholm and

$$\operatorname{ind}(S_1) = \dim\ker S_1 - \dim\operatorname{coker} S_1 = 1 - 0 = 1.$$

It is clear that $S_{-n} := (S_{-1})^n$ and $S_n := (S_1)^n$ are both Fredholm with index $-n$ and n, respectively. Hence, for any $k \in \mathbb{Z}$, there are Fredholm operators on ℓ_2 with index k.

Example 3.8. As a consequence of Theorem 3.4, if $K : \mathcal{H} \to \mathcal{H}$ is compact, then $I + K$ is Fredholm.

We collect some basic properties of Fredholm operators and the Fredholm index.

Proposition 3.9.

(a) *The map* $\operatorname{ind} : \mathfrak{F}(\mathcal{H}) \to \mathbb{Z}$ *is surjective;*

(b) *If* $T \in \mathfrak{F}(\mathcal{H})$*, then* $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$*;*

(c) $\mathfrak{F}(\mathcal{H})$ *is an open subset of* $\mathfrak{L}(\mathcal{H})$*.*

Proof.

(a) Every Hilbert space is isomorphic to ℓ_2, and S_n as defined in Example 3.7 is Fredholm with index n.

(b) A calculation gives:

$$\operatorname{ind}(T^*) = \dim\ker T^* - \dim\ker T = -(\dim\ker T - \dim\ker T^*) = -\operatorname{ind}(T).$$

(c) By Proposition 2.11, the set of invertible elements in the Calkin algebra $\mathfrak{C}(\mathcal{H})$ is open. Since the natural projection $\pi : \mathfrak{L}(\mathcal{H}) \to \mathfrak{C}(\mathcal{H})$ is continuous, it follows that the space of Fredholm operators (the preimage under π of the invertible elements in $\mathfrak{C}(\mathcal{H})$) is open.

□

3.3 Continuity of the Fredholm Index

The most important property of the Fredholm index is that it is a continuous mapping from $\mathfrak{F}(\mathcal{H})$ onto $\mathbb{Z}$, hence locally constant. In contrast to the topological proofs of Chapter 5, we present an algebraic proof. We begin by showing that the Fredholm index is additive, i.e., if $T, S \in \mathfrak{F}(\mathcal{H})$, then $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$.

Definition 3.10. A sequence

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_n$$

of vector spaces V_i and linear transformations f_i is **exact** if $\operatorname{im} f_{i-1} = \ker f_i$.

Proposition 3.11. *If $V \to E \to W$ is an exact sequence and V and W are finite dimensional, then E is finite dimensional.*

Proof. Suppose that $V \xrightarrow{f} E \xrightarrow{g} W$ is exact, with V and W finite dimensional. By the Rank Nullity Theorem,

$$\begin{aligned} \dim E &= \dim \operatorname{im} g + \dim \ker g \\ &= \dim \operatorname{im} g + \dim \operatorname{im} f. \end{aligned}$$

The second equality follows from exactness. Since W is finite dimensional, $\dim \operatorname{im} g < \infty$. Since V is finite dimensional, $\dim \operatorname{im} f < \infty$. Hence, $\dim E < \infty$. $\square$

Proposition 3.12. *If $0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$ is an exact sequence of finite dimensional vector spaces, then*

$$\sum_{i=1}^{n} (-1)^i \dim V_i = 0. \tag{3.1}$$

Proof. We proceed by induction on n.

If $n = 1$, then the exactness of $0 \to V_1 \to 0$ implies that $V = 0$. Perhaps more interesting is the base case $n = 2$. The exactness of the sequence

$$0 \to V_1 \xrightarrow{f} V_2 \to 0$$

implies that f is bijective. Hene, $\dim V_1 = \dim V_2$, and equation (3.1) holds.

Now, suppose that (3.1) holds for some $n > 2$. Suppose that

$$0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} V_n \xrightarrow{f_n} V_{n+1} \xrightarrow{f_{n+1}} 0$$

is exact. Note that this implies that f_n is surjective. Next, consider the reduced sequence

$$0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-2}} V_{n-1} \xrightarrow{f_{n-1}} \ker f_n \to 0.$$

By the inductive hypothesis,

$$\left(\sum_{i=1}^{n-1} (-1)^i \dim V_i \right) + (-1)^n \dim \ker f_n = 0. \tag{3.2}$$

Since f_n is surjective, by the Rank Nullity Theorem, we have

$$\dim V_n = \dim \ker f_n + \dim V_{n+1}. \tag{3.3}$$

Combining equations (3.2) and (3.3) gives

$$\left(\sum_{i=1}^{n-1}(-1)^i \dim V_i\right) + (-1)^n \left(\dim V_n - \dim V_{n+1}\right) = 0$$

and hence

$$\sum_{i=1}^{n+1}(-1)^i \dim V_i = 0.$$

□

With these two properties of exact sequences of vector spaces, we prove additivity of the index.

Proposition 3.13. *If $T, S \in \mathfrak{F}(\mathcal{H})$, then $ST \in \mathfrak{F}(\mathcal{H})$ and* $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$.

Proof. Consider the following sequence:

$$0 \to \ker T \xrightarrow{i} \ker ST \xrightarrow{T} \ker S \xrightarrow{f} \operatorname{coker} T \xrightarrow{S} \operatorname{coker} ST \xrightarrow{p} \operatorname{coker} S \to 0$$

where $i : \ker T \to \ker ST$ is the inclusion map, $f : \ker S \to \mathcal{H}/\operatorname{im} T$ is the projection map, and $p : \mathcal{H}/\operatorname{im} ST \to \mathcal{H}/\operatorname{im} T$ takes equivalence classes mod $\operatorname{im} ST$ to equivalence classes mod $\operatorname{im} T$. It is straightforward to verify that this sequence is exact. Since $\ker T$, $\ker S$, $\operatorname{coker} T$, ansd $\operatorname{coker} S$ are all finite dimensional, by Proposition 3.11, $\ker ST$ and $\operatorname{coker} ST$ are finite dimensional. Thus, ST is Fredholm. By Proposition 3.12, we have:

$$\begin{aligned} 0 &= -\dim\ker T + \dim\ker ST - \dim\ker S \\ &\quad + \dim\operatorname{coker} T - \dim\operatorname{coker} ST + \dim\operatorname{coker} S \\ &= \operatorname{ind}(ST) - \operatorname{ind}(T) - \operatorname{ind}(S). \end{aligned}$$

Thus, $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$.

□

Theorem 3.14. *The map ind : $\mathfrak{F}(\mathcal{H}) \to \mathbb{Z}$ is continuous.*

Proof. Let $T \in \mathfrak{F}(\mathcal{H})$. Let $J : (\ker T)^\perp \to \mathcal{H}$ be the inclusion of $(\ker T)^\perp$ into $\mathcal{H}$, and let $Q : \mathcal{H} \to \operatorname{im} T$ be the orthogonal projection of $\mathcal{H}$ onto $\operatorname{im} T$. Since $\ker J = 0$ and $\operatorname{coker} J = \mathcal{H}/(\ker T)^\perp = \ker T$, J is Fredholm with index

$$\operatorname{ind}(J) = \dim\ker J - \dim\operatorname{coker} J = -\dim\ker T.$$

Similarly, since $\ker Q = (\operatorname{im} T)^\perp = \operatorname{coker} T$ and $\operatorname{coker} Q = \operatorname{im} T/\operatorname{im} T = 0$, Q is Fredholm with index

$$\operatorname{ind}(Q) = \dim\ker Q - \dim\operatorname{coker} Q = \dim\operatorname{coker} T.$$

Hence,

$$\operatorname{ind}(T) + \operatorname{ind}(J) + \operatorname{ind}(Q) = 0. \tag{3.4}$$

Note that $QTJ : (\ker T)^\perp \to \operatorname{im} T$ is invertible. Fix $\epsilon = 1/\left\|(QTJ)^{-1}\right\| > 0$, and pick $T' \in \mathfrak{F}(\mathcal{H})$ such that $\|T - T'\| < \frac{\epsilon}{\|Q\|\|J\|}$. Then

$$\begin{aligned} \left\|QTJ - QT'J\right\| &= \left\|Q(T - T')J\right\| \\ &\le \|Q\| \left\|T - T'\right\| \|J\| \\ &< \epsilon. \end{aligned}$$

By Proposition 2.11, it follows that $QT'J$ is invertible. Hence, $\operatorname{ind}(QT'J) = 0$. Applying Proposition 3.13 gives us

$$\operatorname{ind}(Q) + \operatorname{ind}\left(T'\right) + \operatorname{ind}(J) = 0. \tag{3.5}$$

From equations (3.4) and (3.5) it follows that $\operatorname{ind}(T) = \operatorname{ind}(T')$. Hence, the index map is locally constant, and therefore continuous. □

One immediate application of the continuity of the Fredholm index is the following proposition, which shows that the index is invariant under perturbation by compact operators.

Corollary 3.15. *Let $T \in \mathfrak{F}(\mathcal{H})$, and let $K \in \mathfrak{K}(\mathcal{H})$. Then $T + K \in \mathfrak{F}(\mathcal{H})$, and*

$$\operatorname{ind}(T + K) = \operatorname{ind}(T).$$

Proof. Since $T + K$ and T have the same image in the Calkin algebra under the natural projection map, it is clear that $T + K$ is Fredholm if T is.

Next, consider the path $[0, 1] \to \mathfrak{F}(\mathcal{H})$ given by $t \mapsto T + tK$. This is a continuous path of Fredholm operators, and since the index is locally constant, it follows that $\operatorname{ind}(T + tK) = \operatorname{ind}(T)$ for all $t \in [0, 1]$. □

3.4 The Connected Components of $\mathfrak{F}(\mathcal{H})$.

We have demonstrated the continuity and surjectivity of the index. Next, we prove that any two Fredholm operators of equal index can be connected by a path, and hence the connected components are in bijection with $\mathbb{Z}$.

To do this we need an important fact: that the space $GL(\mathcal{H})$ of invertible operators on a Hilbert space is path-connected. This fact is typically proven using the spectral theorem for unitary operators. The proof we present here is purely elementary, and has a natural generalization that we will employ in Chapter 5 to show that $GL(\mathcal{H})$ satisfies even stronger connectivity conditions.

We begin with a general lemma.

Lemma 3.16. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space with orthogonal decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}_1$ where both $\mathcal{H}_1$ and $\mathcal{H}'$ are infinite dimensional. Then the subspace*

$$\{\, T \in GL(\mathcal{H}) \;:\; T\,|_{\mathcal{H}'} = I \;\; \text{and} \;\; T\,|_{\mathcal{H}_1} \in GL(\mathcal{H}_1\,\})$$

is contractible to $I \in GL(\mathcal{H})$.

Proof. Let $X = \{\, T \in GL(\mathcal{H}) \;:\; T\,|_{\mathcal{H}'} = I \;\text{ and }\; T\,|_{\mathcal{H}_1} \in GL(\mathcal{H}_1\,\})$. It suffices to construct a homotopy from the identity map $\mathrm{id}_X : X \to X$ to the constant map whose value is the identity operator.

Consider $\mathcal{H}'$. Because $\mathcal{H}'$ is infinite dimensional, we can decompose it into a sum of infinitely many infinite-dimensional orthogonal subspaces:

$$\mathcal{H}' = \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \cdots .$$

For example, suppose $\{e_k\}$ is an orthonormal basis for $\mathcal{H}'$. Let $\mathcal{H}_2$ be the closure of the span of $\{\, e_k \;:\; k \text{ even}\,\}$, and let $\mathcal{H}_2'$ be the orthogonal complement in $\mathcal{H}'$. Repeat this process with $\mathcal{H}_2'$ to get an infinite-dimensional subspace $\mathcal{H}_3$, etc. Hence,

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \cdots .$$

Any linear operator on $\mathcal{H}$ is determined by an "infinite matrix" in terms of this decomposition; i.e, by a collection of elements $m(i,j) \in \mathfrak{L}(\mathcal{H}_i, \mathcal{H}_j)$. Using this decomposition, any $T \in X$ can be written:

$$T = \begin{bmatrix} U & & & \\ & I & & \\ & & I & \\ & & & \ddots \end{bmatrix}$$

where $U = T\mid_{\mathcal{H}_1} \in GL(\mathcal{H}_1)$. Because each I is an identity operator on an infinite-dimensional Hilbert space $\mathcal{H}_i$, we have

$$T = \begin{bmatrix} U & & & & \\ & I & & & \\ & & I & & \\ & & & I & \\ & & & & \ddots \end{bmatrix} = \begin{bmatrix} U & & & & \\ & U^{-1}U & & & \\ & & U^{-1}U & & \\ & & & U^{-1}U & \\ & & & & \ddots \end{bmatrix}.$$

We construct a homotopy $\gamma : X \times [0,\pi] \to X$ as follows. For $t \in [0, \pi/2]$, let γ be the block matrix:

$$\gamma_t(T) = \begin{bmatrix} U & & & & \\ & M_t(T) & & & \\ & & M_t(T) & & \\ & & & M_t(T) & \\ & & & & \ddots \end{bmatrix}$$

where the $M_t(T)$ blocks are given by

$$M_t(T) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

When $t = 0$, we have

$$M_0(T) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

so $\gamma_0(T) = T$. Hence, $\gamma_0 = \mathrm{id}_X$. When $t = \pi/2$,

$$\begin{aligned} M_{\pi/2}(T) &= \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & -I \\ U & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -U^{-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix}. \end{aligned}$$

Hence,

$$\gamma_{\pi/2}(T) = \begin{bmatrix} U & & & & \\ & U^{-1} & & & \\ & & U & & \\ & & & U^{-1} & \\ & & & & \ddots \end{bmatrix}.$$

The next step of the homotopy is defined similarly, with the rotation matrices shifted upwards. For $t \in [\pi/2, \pi]$, let

$$\gamma_t(T) = \begin{bmatrix} M'_t(T) & & & \\ & M'_t(T) & & \\ & & M'_t(T) & \\ & & & \ddots \end{bmatrix}$$

where

$$M'_t(T) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}.$$

When $t = \pi/2$,

$$\begin{aligned} M'_{\pi/2}(T) &= \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & -I \\ U^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -U & 0 \end{bmatrix} \\ &= \begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix} \end{aligned}$$

so $\gamma_{\pi/2}(T)$ agrees with the previous definition. When $t = \pi$, we have:

$$M'_\pi(T) = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -U^{-1} & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} -U & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

so $\gamma_\pi(T) = I$. □

Lemma 3.16 will be the last step in the proof that $GL(\mathcal{H})$ is path-connected, and will be used again in the proof of the stronger result in Chapter 5.

Proposition 3.17. *The space of invertible operators on $\mathcal{H}$ is path-connected.*

Proof. It suffices to prove that $T \in GL(\mathcal{H})$ can be connected to the identity operator I. We will build a path connecting T and I using the interval $[0, 4]$. By the previous lemma, if we can connect T to an operator which fixes an infinite dimensional subspace of $\mathcal{H}$ and is invertible on its complement, we can contract to the identity. Our first order of business is to build such an infinite dimensional subspace.

Pick any unit vector $a_1 \in \mathcal{H}$. Let $A_1 = \text{Span}(a_1, T(a_1))$. We want a unit vector a_2 such that a_2 and $T(a_2)$ are both orthogonal to A_1. Pick $a_2 \in A_1^\perp \cap T^{-1}(A_1^\perp)$, which is possible by the infinite-dimensionality of $\mathcal{H}$. Define $A_2 = \text{Span}(a_2, T(a_2))$. Continue inductively, choosing unit vectors a_i such that

$$a_k \in \bigcap_{i=1}^{k-1} A_i^\perp \cap T^{-1}(A_i^\perp).$$

Hence, for each k, a_k and $T(a_k)$ are orthogonal to $A_1, \dots, A_{k-1}$.

Next, we define planar rotations on each A_k which rotate $T(a_k)$ in the direction of a_k. Explicitly, define $R_k(t) \in GL(A_k)$ by

$$R_k(t) := \begin{bmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{bmatrix} \text{ with respect to the basis } \{T(a_k), \|T(a_k)\|\, a_k\}$$

for $0 \le t \le 1$. Then $R_k(0) = I$, and $R_k(1)(T(a_k)) = \|T(a_k)\|\, a_k$. For each k, $R_k(t)$ is a rotation, hence continuous. Next, we define $R(t)$ as the execution of all of these rotations simultaneously:

$$R(t) = \begin{cases} R_k(t) & \text{on } A_k \\ I & \text{on } (A_1 \oplus A_2 \oplus \cdots)^\perp \end{cases}$$

for $0 \le t \le 1$. By the orthogonality of the A_k's, $R(t)$ is continuous. Explicitly, fix $\epsilon > 0$. Since each $R_k(t)$ is continuous, choose $t, t' \in [0, 1]$ so that $\|R_k(t) - R_k(t')\| < \epsilon$ for all k. If $x \in \mathcal{H}$,

$$\left\|(R(t') - R(t))x\right\|^2 = \left\|(R(t') - R(t)) \sum_{k=1}^\infty \langle x, a_k\rangle\, a_k\right\|^2 = \left\|\sum_{k=1}^\infty (R_k(t') - R_k(t))(\langle x, a_k\rangle\, a_k)\right\|^2.$$

By the Pythagorean theorem,

$$\begin{aligned}\left\|(R(t') - R(t))x\right\|^2 &= \sum_{k=1}^{\infty} \left\|(R_k(t') - R_k(t))(\langle x, a_k\rangle\, a_k)\right\|^2 \\ &\le \sum_{k=1}^{\infty} \left\|R_k(t') - R_k(t)\right\|^2 \left\|\langle x, a_k\rangle\, a_k\right\|^2 \\ &< \epsilon^2 \left\|x\right\|^2.\end{aligned}$$

Hence, $R(t)$ is continuous.

The rotation function $R(t)$ lets us define the first step of the path. Let

$$T_t = R(t)T \qquad 0 \le t \le 1.$$

Then $T_0 = R(0)T = T$, and T_1 is an invertible operator that satisfies $T_1(a_k) = \|T(a_k)\|\, a_k$ for all k.

Define $\mathcal{H}' = \text{Span}(a_1, a_2, \dots)$. The operator T_1 *almost* fixes $\mathcal{H}'$; all we need to do is scale accordingly. So define

$$T_t = \begin{cases} (2-t)T_1 + (t-1)I & \text{on } \mathcal{H}' \\ T_1 & \text{on } (\mathcal{H}')^{\perp} \end{cases} \qquad 1 \le t \le 2.$$

Then T_1 agrees with the previous definition, and T_2 is an operator which restricts to the identity on $\mathcal{H}'$.

In order to invoke Lemma 3.16, we need an operator which not only fixes $\mathcal{H}'$, but also sends $(\mathcal{H}')^{\perp}$ to $(\mathcal{H}')^{\perp}$. We can achieve this as follows: let P denote the orthogonal projection onto $(\mathcal{H}')^{\perp}$ (and so $I - P$ is the orthogonal projection onto $\mathcal{H}'$), and define:

$$T_t = (3-t)T_2 + (t-2)[(I-P) + PT_2P] \qquad 2 \le t \le 3.$$

Then T_3 restricts to I on $\mathcal{H}'$, and is invertible on its orthogonal complement.

Now we apply Lemma 3.16. Let γ be the homotopy constructed in the proof of the lemma, and define

$$T_t = \gamma_{\pi(t-3)}\,(T_3) \qquad 3 \le t \le 4.$$

Then $T_3 = \gamma_0(T_3) = T_3$, and $T_4 = \gamma_\pi(T_3) = I$. Hence, T_t is a continuous path inside $GL(\mathcal{H})$ witch connects T to I. Therefore, $GL(\mathcal{H})$ is path-connected. □

Finally, we state and prove the main theorem of this section.

Theorem 3.18. *The Fredholm index induces a bijection $\pi_0(\mathfrak{F}(\mathcal{H})) \to \mathbb{Z}$.*

Proof. Surjectivity has been established. It remains to show injectivity. For each n, set $\mathfrak{F}_n = \{\, T \in \mathfrak{F}(\mathcal{H}) \,:\, \text{ind}\,(T) = n \,\}$. Since the index is locally constant, injectivity follows after we show that each $\mathfrak{F}_n$ is path-connected.

First, consider $n = 0$. Let $T \in \mathfrak{F}_0$. By Proposition 3.17, it suffices to connect T to an invertible operator by a path. Since $T \in \mathfrak{F}_0$, $\dim\ker T = \dim\ker T^*$. Let $\{v_i\}_{i=1}^k$ and $\{w_i\}_{i=1}^k$ be orthonormal bases for $\ker T$ and $\ker T^*$ respectively. Define an operator $\varphi : \mathcal{H} \to \mathcal{H}$ as follows: for $x = x_0 + \sum \lambda_i v_i \in \mathcal{H}$ where $x_0 \in (\ker T)^{\perp}$, set

$$\varphi(x) = \sum_{i=1}^{k} \lambda_i w_i.$$

Then $\ker\varphi = (\ker T)^{\perp}$ and $\text{im}\,\varphi = \ker T^*$. Note that $T + \varphi$ is invertible, since if $x \in \ker(T+\varphi)$, then $T(x) = -\varphi(x)$, and hence $x = 0$, and $\text{im}(T+\varphi) = \mathcal{H}$. In fact, the same reasoning shows that $T + t\varphi$ is invertible for all real $t > 0$. Hence, $\gamma : [0,1] \to \mathfrak{L}(H)$ given by $\gamma(t) = T + t\varphi$ is a path contained in $\mathfrak{F}(\mathcal{H})$ that connects T with an invertible operator. Thus, $\mathfrak{F}_0$ is path-connected.

For $n > 0$, we use the unilateral shift operator S_{-n}. Let $T \in \mathfrak{F}_n$. Recall from Example 3.7 that $\operatorname{ind}(S_{-n}) = -n$. Then by Proposition 3.13, TS_{-n} is Fredholm and $\operatorname{ind}(TS_{-n}) = \operatorname{ind}(T) + \operatorname{ind}(S_{-n}) = 0$, so $TS_{-n} \in \mathfrak{F}_0$. Since $S_{-n}S_n = I$, we have $T = TS_{-n}S_n \in \mathfrak{F}_0 S_n$. Hence, $\mathfrak{F}_n \subseteq \mathfrak{F}_0 S_n$. Proposition 3.13 implies that $\mathfrak{F}_0 S_n \subseteq \mathfrak{F}_n$. Therefore, $\mathfrak{F}_n = \mathfrak{F}_0 S_n$, and it follows that $\mathfrak{F}_n$ is path-connected.

For $n < 0$, it follows from taking adjoints that $\mathfrak{F}_n = \mathfrak{F}^*_{-n}$, and so $\mathfrak{F}_n$ is path-connected for all $n \in \mathbb{Z}$. □

As mentioned in the introduction, this theorem is a simple case of a deeper connection between homotopy classes of maps into $\mathfrak{F}(\mathcal{H})$ and the abelian group $K(X)$. This relation is the focus of the rest of this thesis.

Chapter 4

Vector Bundles and K-Theory

In order to develop an appropriate generalization of the theory of Fredholm operators developed in Chapter 3, it is necessary to establish the basics of the theory of vector bundles and K-theory. Indeed, T

The classical Fredholm index gives a relation between finite dimensional vector spaces (the kernel and cokernel of a Fredholm operator) and the integers. Moving forward, we are concerned with *families* of finite dimensional vector spaces and more general abelian groups. Naturally, we need to establish the basics of the theory of vector bundles and K-theory. This chapter covers the necessary theory. We refer to sources such as [5] and [18] for more details; we follow these texts closely.

4.1 Vector Bundles: Definitions and Examples

Intuitively, a vector bundle is a collection of vector spaces parameterized by a topological space continuous manner. To make this idea precise, we begin with the definition of a general family of vector spaces.

Definition 4.1. Let X be a topological space. A **family of vector spaces** over X is a topological space E and a continuous surjective map $p : E \to X$ (called the **projection**) such that each $E_x := p^{-1}(x)$ is a finite dimensional vector space compatible with the subspace topology induced by E. A family will be denoted $p : E \to X$, $E \to X$, or just E, depending on the context.

Definition 4.2. A **section** of $p : E \to X$ is a continuous map $s : X \to E$ such that $p \circ s = \mathrm{id}_X$.

Our focus is on families of vector spaces with complex vector space structures, but working over $\mathbb{R}$ is perfectly valid.

Example 4.3. Let $E = X \times \mathbb{C}^n$. Then $p : X \times \mathbb{C}^n \to X$ given by $p(x, v) = x$ is surjective and is continuous since projections onto factors are continuous. For each $x \in X$, $p^{-1}(x) = \{x\} \times \mathbb{C}^n \cong \mathbb{C}^n$. Thus, $X \times \mathbb{C}^n$ is a family of vector spaces over X. A family $X \times V$ where V is a finite dimensional vector space is called a **product family**.

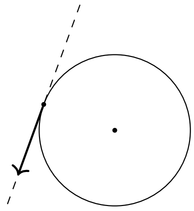
The map $s : X \to X \times \mathbb{C}^n$ given by $s(x) = (x, 0)$ is a section of this family. Any map $s(x) = (x, f(x))$ where $f : X \to \mathbb{C}^n$ is continuous is a section.

Definition 4.4. A **homomorphism** of families $p : E \to X$ and $q : F \to X$ is a continuous map $\varphi : E \to F$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

commutes, and $\varphi_x : E_x \to F_x$ is a linear transformation for every $x \in X$. If φ is a homeomorphism, then φ is an **isomorphism** of families, and we write $E \cong F$. If E is isomorphic to a product family, E is **trivial**.

Example 4.5. Let $X = S^1 = \{ (x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1 \}$. Let $TS^1 := \bigsqcup_{(x,y) \in S^1} \text{Span}\left(\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]\right)$, topologized as a subspace of $S^1 \times \mathbb{R}^2$. Then TS^1 is the collection of tangent spaces of the unit circle:



Let $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in TS^1$, and define $p : TS^1 \to S^1$ by $p(v) = \frac{1}{\|v\|}\left[\begin{smallmatrix} v_2 \\ -v_1 \end{smallmatrix}\right]$. Then p is surjective and continuous, and $p^{-1}(x, y) = \text{Span}\left(\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]\right)$. Hence, $p : TS^1 \to S^1$ is a family of vector spaces.

Let $q : S^1 \times \mathbb{R} \to S^1$ be the product family of vector spaces, and define $\varphi : S^1 \times \mathbb{R} \to TS^1$ by $\varphi((x, y), \lambda) = \lambda \left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]$. Then

$$(p \circ \varphi)((x, y), \lambda) = p(\varphi((x, y), \lambda)) = p\left(\lambda \left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]\right) = \frac{1}{\lambda}\lambda \begin{bmatrix} x \\ y \end{bmatrix} = (x, y) = q((x, y), \lambda)$$

and so φ is a homomorphism of families. Moreover, $\varphi^{-1} : TS^1 \to S^1 \times \mathbb{R}$ is given by

$$\varphi^{-1}(v) = \left(\frac{1}{\|v\|} \begin{bmatrix} v_2 \\ -v_1 \end{bmatrix}, \|v\| \right).$$

Hence, φ is an isomorphism. Thus, $TS^1 \cong S^1 \times \mathbb{R}$, and so TS^1 is a trivial family of vector spaces over S^1.

Proposition 4.6. *A family of vector spaces $E \to X$ is trivial if and only if there exist sections $s_i : X \to E$ such that $\{s_i(x)\}$ is a basis for E_x for all $x \in X$.*

Proof. Suppose $E \to X$ is trivial. Then there exists an isomorphism $\varphi : X \times V \to E$ for some n-dimensional vector space V. Let $v_1, \ldots, v_n$ be a basis for V. Define sections $\tilde{s}_i : X \to X \times V$ by $\tilde{s}_i(x) = (x, v_i)$. Set $s_i = \varphi \circ \tilde{s}_i$. By commutativity of the diagram

$$\begin{array}{ccc} X \times V & \xrightarrow{\varphi} & E \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

it follows that each s_i is a section of $E \to X$. For each $x \in X$, φ_x is injective, hence $\{s_i(x)\}$ is linearly independent. Since $\varphi_x^{-1} : E_x \to \{x\} \times V$ is injective, it follows that $\{s_i(x)\}$ is a basis for E_x for each $x \in X$.

Conversely, suppose that $s_1(x), \ldots, s_n(x)$ are sections that form a basis for E_x for each $x \in X$. Define $\varphi : E \to X \times \mathbb{C}^n$ by $\varphi(s_i(x)) = (x, e_i)$ and extend by linearity. Then φ is bijective and continuous with a continuous inverse. Hence, $E \cong X \times \mathbb{C}^n$. $\square$

Suppose that $E \to X$ is a family of vector spaces and that f is a continuous function from another topological space Y into X. We can define a family of vector spaces over Y by "pulling back" E along f:

$$\begin{array}{ccc} f^*E & & E \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Explicitly, we have the following.

Proposition 4.7. *Let $f : Y \to X$ be continuous. If $p : E \to X$ is a family of vector spaces, then the* ***induced family*** *$f^*p : f^*E \to Y$ defined by*

$$f^*E := \{ (y, e) \in Y \times E \ : \ f(y) = p(e) \} \qquad f^*p(y, e) = y$$

is a family of vector spaces over Y. Moreover, if $f : Y \to X$ and $g : Z \to Y$ are continuous, then $(f \circ g)^*E \cong g^*f^*E$.

Proof. Let $y \in Y$. Since $p : E \to X$ is surjective, there exists an $e \in E$ such that $p(e) = f(y)$. Hence, $f^*p(y, e) = y$, and so f^*p is surjective. Furthermore,

$$f^*p^{-1}(y) = \{y\} \times \{ e \in E \ : \ f(y) = p(e) \} = \{y\} \times E_{f(y)}$$

and so f^*E has the appropriate vector space structure on fibers.

Next, suppose that $g : Z \to Y$ is continuous. Then

$$\begin{aligned} g^*f^*E &= \{ (z, e') \in Z \times f^*E \ : \ g(z) = f^*p(e') \} \\ &\cong \{ (z, y, e) \in Z \times Y \times E \ : \ f(g(z)) = p(e) \} \\ &\cong \{ (z, e) \in Z \times E \ : \ (f \circ g)(z) = p(e) \} \end{aligned}$$

and it follows that $(f \circ g)^*E \cong g^*f^*E$. □

Corollary 4.8. *Let $p : E \to X$ be a family of vector spaces, and let $U \subseteq X$ be a subspace. Then the* ***restriction of** E **to** U*, defined as $E\,|_U := p^{-1}(U)$, is a family of vector spaces over U.*

Proof. Let $f : U \to X$ be the inclusion map. Then

$$f^*E = \{ (x, e) \in U \times E \ : \ p(e) = x \} \cong p^{-1}(U)$$

is a family of vector spaces over U. □

Definition 4.9. A family of vector spaces $E \to X$ is **locally trivial at** $x \in X$ if there is an open neighborhood U containing x such that $E\,|_U$ is trivial; we say $E \to X$ is **locally trivial** if E is locally trivial at every $x \in X$. A **vector bundle** over X is a locally trivial family of vector spaces over X.

One immediate consequence of the local triviality of a vector bundle is that the dimension of the fibers of a vector bundle is locally constant. If X is connected, then the dimension of the fibers is constant, and we call this dimension the *rank* of E.

The definitions of homomorphisms and isomorphisms of families of vector spaces immediately extend to vector bundles. If $E \to X$ is a vector bundle, we denote its isomorphism class by $[E]$ or $[E \to X]$. The set of isomorphism classes of (complex) vector bundles on X is written $\operatorname{Vect} X$.

Example 4.10.

(i) Any product family is a vector bundle.

(ii) The family $TS^1 \to S^1$ was shown to be trivial on S^1. Hence, TS^1 is a vector bundle over S^1.

(iii) If $E \to X$ is a vector bundle and $f : Y \to X$ is continuous, then $f^*E \to Y$ is a vector bundle. Indeed, for any y, let $f(y) \in U$ be an open neighborhood such that $E\,|_U$ is trivial. Then $f^{-1}(U)$ is an open neighborhood of y such that $f^*E\,|_{f^{-1}(U)}$ is trivial.

Example 4.11. Define the *complex projective line* as $\mathbb{CP}^1 := (\mathbb{C}^2 - (0,0))/\sim$ where $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$ for all $\lambda \in \mathbb{C}$, endowed with the quotient topology. Denote the equivalence class of the point (z_1, z_2) by $[z_1; z_2]$.

Intuitively, $\mathbb{CP}^1$ is the collection of one-dimensional subspaces of $\mathbb{C}^2$. Hence, we can define a natural vector bundle over $\mathbb{CP}^1$. Let H^* denote the disjoint union of these subspaces, topologized as a subspace of $\mathbb{CP}^1 \times \mathbb{C}^2$, and let $H^* \to \mathbb{CP}^1$ be the obvious projection map. We want to show that H^* is a vector bundle over $\mathbb{CP}^1$, and thus need to demonstrate local triviality. Let

$$U_1 = \left\{ [z_1; z_2] \in \mathbb{CP}^1 \ : \ z_1 \neq 0 \right\};$$
$$U_2 = \left\{ [z_1; z_2] \in \mathbb{CP}^1 \ : \ z_2 \neq 0 \right\}.$$

Then $\mathbb{CP}^1 = U_1 \cup U_2$. Consider the function $\varphi : H^* \mid_{U_1} \to U_1 \times \mathbb{C}$ given by

$$\varphi(\text{Span}(z_1, z_2)) = ([z_1; z_2], z_2/z_1).$$

Then φ commutes with the projection maps, and its inverse is given by $([z_1; z_2], \lambda) \mapsto \lambda(z_1, z_2)$. Hence, φ is a bundle isomorphism. We can define an isomorphism over U_2 in an analogous manner. Thus, H^* is locally trivial, and is therefore a vector bundle over $\mathbb{CP}^1$.

As defined, an isomorphism of vector bundles E and F is simply a fiberwise-linear homeomorphism between E and F; as one might expect, this is equivalent to having isomorphisms of fibers.

Proposition 4.12. *A bundle homomorphism $\varphi : E \to F$ is an isomorphism if and only if $\varphi_x : E_x \to F_x$ is a vector space isomorphism for each $x \in X$.*

Proof. Suppose that $E, F \to X$ are vector bundles.

If $\varphi : E \to F$ is an isomorphism, then $\varphi_x : E_x \to F_x$ is a linear transformation for each $x \in X$, and since φ is bijective, each φ_x is necessarily bijective. Hence, φ_x is a vector space isomorphism for each $x \in X$.

Conversely, suppose that $\varphi_x : E_x \to F_x$ is a vector space isomorphism for each $x \in X$. Then φ is bijective and has a set-theoretic inverse φ^{-1}. We need to show that φ^{-1} is continuous. Let U be an open set such that $E \mid_U$ and $F \mid_U$ are trivial. Let $\gamma : E \mid_U \to U \times \mathbb{C}^n$ and $\gamma' : F \mid_U \to U \times \mathbb{C}^n$ be isomorphisms. Then the map

$$\gamma' \circ \varphi \circ \gamma^{-1} : U \times \mathbb{C}^n \to U \times \mathbb{C}^n$$

is given by $(x, v) \mapsto (x, f_x(v))$ for some transformation $f_x : \mathbb{C}^n \to \mathbb{C}^n$ where $x \mapsto f_x$ is a continuous mapping $U \to GL(n, \mathbb{C})$. Hence, $\gamma \circ \varphi^{-1} \circ (\gamma')^{-1}$ is given by $(x, v) \mapsto (x, f_x^{-1}(v))$. Since inversion is continuous is $GL(n\mathbb{C})$, φ^{-1} is continuous on $F \mid_U$. As this holds for all such neighborhoods U, φ^{-1} is continuous on F. ☐

Natural operations on vector spaces such as the direct sum can be generalized to operations on vector bundles.

Proposition 4.13. *Let E, F be vector bundles over X with projections p and q, respectively. Define $E \oplus F \to X$ by:*

$$E \oplus F := \left\{ (e, f) \in E \times F \ : \ p(e) = q(f) \right\}.$$

Then $p \oplus q : E \oplus F \to X$ given by $(p \oplus q)(e, f) := p(e) = q(f)$ is a vector bundle over X. Moreover, there are natural isomorphisms $E \oplus F \cong F \oplus E$ and $E \oplus (F \oplus G) \cong (E \oplus F) \oplus G$.

Proof. The map $p \oplus q$ as defined is clearly surjective and continuous. Furthermore, $(E \oplus F)_x \cong E_x \oplus F_x$ for each $x \in X$, and so $E \oplus F$ is a family of vector spaces. To see local triviality, fix $x \in X$ and let U_1 and U_2 be open neighborhoods of x with trivializations

$$E\mid_{U_1} \xrightarrow{\varphi_1} U_1 \times \mathbb{C}^n \qquad \text{and} \qquad F\mid_{U_2} \xrightarrow{\varphi_2} U_2 \times \mathbb{C}^m.$$

Set $U = U_1 \cap U_2$, and let $\varphi : E \oplus F\mid_U \to U \times (\mathbb{C}^n \oplus \mathbb{C}^m)$ be given by

$$\varphi(e, f) = (p(e), \varphi_{1p(e)}(e), \varphi_{2q(f)}(f)).$$

Then φ is an isomorphism of $E \oplus F\mid_U$ and $U \times (\mathbb{C}^n \oplus \mathbb{C}^m)$, and hence $E \oplus F$ is locally trivial. The vector bundle isomorphisms $E \oplus F \cong F \oplus E$ and $E \oplus (F \oplus G) \cong (E \oplus F) \oplus G$ follow from the natural isomorphisms of the corresponding vector space structures. □

We can similarly define the dual bundle E^* and quotient bundle E/F.

4.2 Vector Bundles with Compact Base Space

Vector bundles over compact base spaces posses many nice properties. We first investigate the behavior of isomorphism classes of vector bundles under homotopy, beginning with a few technical lemmas.

Lemma 4.14. *Let X be compact, and let $Y \subseteq X$ be a closed subspace. Suppose that $E, F \to X$ are vector bundles, and that $\sigma : E\mid_Y \to F\mid_Y$ is a vector bundle homomorphism. Then σ can be extended to a vector bundle homomorphism $\tilde{\sigma} : E \to F$.*

Proof. Assume without loss of generality that X is connected. Let $U_1, \ldots, U_n$ be an open cover of X so that $E\mid_{\overline{U_k}} \cong \overline{U_k} \times \mathbb{C}^N$ and $F\mid_{\overline{U_k}} \cong \overline{U_k} \times \mathbb{C}^M$. Let

$$\sigma_k : E\mid_{Y \cap \overline{U_k}} \to F\mid_{Y \cap \overline{U_k}}.$$

be the restriction of σ to $Y \cap \overline{U_k}$. Then σ_k determines a contunuous function $f_k : Y \cap \overline{U_k} \to \mathcal{M}(M, N, \mathbb{C})$ into the space of $M \times N$ matrices. By the Tietze extension theorem, f_k extends to a continuous map $\tilde{f_k} : \overline{U_k} \to \mathcal{M}(M, N, \mathbb{C})$. Each $\tilde{f_k}$ induces a bundle homomorphism $\sigma_k' : E\mid_{\overline{U_k}} \to F\mid_{\overline{U_k}}$. Let $\varphi_1, \ldots, \varphi_n$ be a partition of unity corresponding to the open cover $U_1, \ldots, U_n$. Let $p : E \to X$ be the bundle projection, and define:

$$\tilde{\sigma}_k(e) = \begin{cases} \varphi_k(p(e))\sigma'(e) & e \in p^{-1}(U_k) \\ 0 & \text{otherwise} \end{cases}$$

Then $\tilde{\sigma} = \sum_{i=1}^n \tilde{\sigma}_k$ is a bundle homomorphism which extends σ. □

Lemma 4.15. *Let X be any topological space, $E, F \to X$ vector bundles. If $\sigma : E \to F$ is a bundle homomorphism, then*

$$\mathcal{O} = \{\, x \in X \ : \ \sigma_x \text{ is an isomorphism} \,\}$$

is open in X.

Proof. Again, assume that X is connected. Suppose that $\mathcal{O}$ is nonempty. Then the rank of E and F must be equal, since their fibers are isomorphic in at least one spot. For each $x \in X$, let U_x be an open neighborhood of x so that $E\mid_{U_x} \cong U_x \times \mathbb{C}^N \cong F\mid_{U_x}$. Let σ^x denote the restriction of σ to U_x. Then, as in the proof of Lemma 4.14, σ^x induces a continuous function $f_x : U_x \to \mathcal{M}(N, \mathbb{C})$. Let $\mathcal{O}_x = f_x^{-1}(GL(N, \mathbb{C}))$. Since $GL(N, \mathbb{C})$ is open and f_x is continuous, $\mathcal{O}_x$ is open. Then $\bigcup_{x \in X} \mathcal{O}_x = \mathcal{O}$ is open. □

Lemma 4.16. *Let X be compact, and let $E \to X \times [0,1]$ be a vector bundle. If $i_t : X \to X \times [0,1]$ is the inclusion map given by $i_t(x) = (x,t)$, then $i_0^*E \cong i_1^*E$.*

Proof. Fix $\tau \in [0,1]$. Then the pullback bundle $i_\tau^*E \to X$ is isomorphic to $E\mid_{X\times\{\tau\}}$. If we define a new vector bundle $E^\tau := i_\tau^*E \times [0,1]$ over $X \times [0,1]$ in the natural way, then $E\mid_{X\times\{\tau\}} \cong E^\tau\mid_{X\times\{\tau\}}$. Choose a bundle isomorphism: $\sigma : E\mid_{X\times\{\tau\}} \to E^\tau\mid_{X\times\{\tau\}}$. Since $X \times \{\tau\} \subseteq X \times [0,1]$, Lemma 4.14 gives an extension of σ to a bundle *homo*morphism $\tilde{\sigma} : E \to E^\tau$. By Lemma 4.15,

$$\mathcal{O} = \left\{ (x,t) \in X \times [0,1] \ : \ \tilde{\sigma}_{x,t} : E_{x,t} \to E^\tau_{x,t} \text{ is an isomorphism} \right\}$$

is open in $X \times [0,1]$. Moreover, $X \times \{\tau\} \subseteq \mathcal{O}$. By the Tube Lemma, there is an open interval $I_\tau \subseteq [0,1]$ containing τ such that $X \times I_\tau \subseteq \mathcal{O}$. Since $\tilde{\sigma}_{x,t}$ is an isomorphism for every $(x,t) \in X \times I_\tau$, it follows that $E\mid_{X\times I_\tau} \cong E^\tau\mid_{X\times I_\tau}$. Thus, for any $t \in I_\tau$,

$$i_t^*E \cong E\mid_{X\times\{t\}} \cong E^\tau\mid_{X\times\{t\}} \cong i_\tau^*E.$$

So for all $t \in I_\tau$, i_t^*E are isomorphic vector bundles. Since $[0,1]$ is compact, we can find a finite open covering of such intervals and conclude that $i_0^*E \cong i_1^*E$. □

Theorem 4.17. *Let X and Y be compact, and let $E \to Y$ be a vector bundle. If $f, g : X \to Y$ are homotopic, then $f^*E \cong g^*E$.*

Proof. Since f and g are homotopic, there exists a homotopy $F : X \times [0,1] \to Y$ such that $F_0 = f$ and $F_1 = g$. If $i_t : X \to X \times [0,1]$ is as above, then $f = F \circ i_0$ and $g = F \circ i_1$. Then

$$f^*E = (F \circ i_0)^*E \cong i_0^*F^*E \cong i_1^*F^*E \cong (F \circ i_1)^*E = g^*E$$

□

Consequences of this theorem are immediate.

Corollary 4.18. *Let X and Y be compact. If $f : X \to Y$ is a homotopy equivalence, then the map f^* induces a bijection:*

$$f^* : \operatorname{Vect} Y \to \operatorname{Vect} X.$$

Proof. If $f : X \to Y$ is a homotopy equivalence, then there is a function $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identities i_Y and i_X, respectively. Hence, for any vector bundle $E \to X$,

$$f^*g^*E \cong (g \circ f)^*E \cong (i_X)^*E \cong E$$

and for any bundle $F \to Y$,

$$g^*f^*F \cong (f \circ g)^*F \cong (i_Y)^*F \cong F$$

so that f^* and g^* are inverses on the set of isomorphism classes of vector bundles. □

Corollary 4.19. *Every vector bundle over a contractible space X is trivial. Hence,* $\operatorname{Vect} X \cong \mathbb{N}$.

Proof. If X is contractible, it is homotopy equivalent to a point. Any vector bundle over a point is a single vector space, so elements of $\operatorname{Vect} X$ are distinguished by dimension. Thus, $\operatorname{Vect} X \cong \mathbb{N}$. □

Finally, we describe how to embed vector bundles into trivial bundles.

Proposition 4.20. *Let $E \to X$ be a vector bundle with X compact. Then E is isomorphic to a subbundle of a trivial vector bundle.*

Proof. Suppose that X is connected. By compactness, there exists an open cover $U_1, \ldots, U_m$ of X with isomorphisms

$$f_i : E\mid_{U_i} \to U_i \times \mathbb{C}^n$$

for each i. Let $P : X \times \mathbb{C}^n \to \mathbb{C}^n$ denote the projection onto $\mathbb{C}^n$. Let $\varphi_1, \ldots, \varphi_m$ be a partition of unity corresponding to $U_1, \ldots, U_m$. Define $\Phi : E \to \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n$ by

$$\Phi(e) = (\varphi_1(p(e))P(f_1(e)), \ldots, \varphi_m(p(e))P(f_m(e))) .$$

We are abusing notation here, since each f_i is only defined on $E\mid_{U_i}$. The partition of unity allows us to do this. Let $f(e) := (p(e), \Phi(e))$. It is clear that f commutes with the bundle projections. Since Φ is continuous and fiberwise linear, f is a bundle homomorphism from E to $X \times (\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n)$. Note that Φ is injective. Indeed, for each $x \in X$, consider Φ_x as a map from the fiber E_x. If $\Phi_x(e) = 0$ for some $e \in E_x$, then $\varphi_i(p(e))P(f_i(e)) = 0$ for all i, and so $f_i(e) = (x, 0)$ whenever this is well-defined. Since each f_i is a fiberwise isomorphism, it follows that Φ_x, and hence Φ, is injective. Using an isomorphism

$$\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n \cong \mathbb{C}^{nm}$$

and setting $N = nm$ gives the desired result.

If X is not connected, work on each connected component $X_1, \ldots, X_k$ of X and set $N = N_1 + \cdots + N_k$. □

Corollary 4.21. *Let $E \to X$ be a vector bundle over a compact space X. There exists a vector bundle $E^\perp$ such that $E \oplus E^\perp$ is trivial.*

Proof. By Proposition 4.20, E is isomorphic to a subbundle of $X \times V$ for some finite dimensional V. Set $E^\perp := (X \times V)/E$. For each $x \in X$,

$$E_x \oplus E_x^\perp = E_x \oplus ((X \times V)/E)_x \cong E_x \oplus V/E_x \cong V$$

and so $E \oplus E^\perp \cong X \times V$. □

4.3 K-Theory

To motivate the main definition of K-theory, we recall the construction of $\mathbb{Z}$ from $\mathbb{N}$. Consider the product of $\mathbb{N}$ with itself:

$$\mathbb{N} \times \mathbb{N} = \{ (m, n) \ : \ m, n \in \mathbb{N} \} .$$

Addition of elements in $\mathbb{N} \times \mathbb{N}$ is performed component wise. Place the following equivalence relation on $\mathbb{N} \times \mathbb{N}$:

$$(m_1, n_1) \sim (m_2, n_2) \iff m_1 + n_2 = m_2 + n_1.$$

Then $\mathbb{Z}$ is isomorphic to the set of equivalence classes under this relation. We are formally introducing subtraction by identifying tuples such that $m_1 - n_1 = m_2 - n_2$. An integer m is the equivalence class $[(m, 0)]$, $-m$ is the equivalence class $[(0, m)]$, etc.

We can describe the construction of $\mathbb{Z}$ from $\mathbb{N}$ in a more sophisticated manner. Under addition, $\mathbb{N}$ is an abelian monoid. The construction of $\mathbb{Z}$ is a sort of "group-completion" of $\mathbb{N}$, constructed by introducing additive inverses to $\mathbb{N}$. This is the **Grothendieck completion** of $\mathbb{N}$, and is a construction that can be applied to a general commutative monoid. The set of isomorphism classes of vector bundles over X forms a commutative monoid under the direct sum operation. The K-theory group of X, denoted $K(X)$, is defined to be the Grothendieck completion of this monoid. We now make explicit.

Theorem 4.22 (Grothendieck Completion)**.** *Let A be an abelian monoid. There exists an abelian group $\mathcal{G}(A)$ (the **Grothendieck completion of** A) with the following properties.*

(i) For any abelian group G and monoid homomorphism $\varphi : A \to G$, there is a monoid homomorphism $i : A \to \mathcal{G}(A)$ and a unique group homomorphism $\tilde{\varphi} : \mathcal{G}(A) \to G$ such that $\varphi = \tilde{\varphi} \circ i$.

(ii) The group $\mathcal{G}(A)$ possesing the properties in (i) is unique up to isomorphism.

(iii) If $\varphi : A \to B$ is a monoid homomorphism, then there is a unique group homomorphism $\mathcal{G}(\varphi) : \mathcal{G}(A) \to \mathcal{G}(B)$ making the diagram

$$\begin{array}{ccc} \mathcal{G}(A) & \xrightarrow{\mathcal{G}(\varphi)} & \mathcal{G}(B) \\ {\scriptstyle i_A}\uparrow & & {\scriptstyle i_B}\uparrow \\ A & \xrightarrow{\quad\varphi\quad} & B \end{array}$$

commute. Moreover, if $\psi : B \to C$ is a monoid homomorphism, then $\mathcal{G}(\psi \circ \varphi) = \mathcal{G}(\psi) \circ \mathcal{G}(\varphi)$, and $\mathcal{G}(1_A) = 1_{\mathcal{G}(A)}$. Hence, $\mathcal{G}(\cdot)$ is a functor from the category of abelian monoids to the category of abelian groups.

Proof. Let $\mathcal{G}(A) := A \times A/\sim$, where $\sim$ is the equivalence relation defined by

$$(a, b) \sim (a', b') \qquad \Longleftrightarrow \qquad a + b' + k = a' + b + k \text{ for some } k \in A.$$

Let $[(a, b)]$ denote the equivalence class of (a, b). Then $\mathcal{G}(A)$ clearly becomes an abelian monoid under the addition operation given by:

$$[(a, b)] + [(a', b')] := [(a + a', b + b')].$$

The identity element is $[(0, 0)]$, since $(a, b) + (0, 0) = (a, b)$. Moreover,

$$[(a, b)] + [(b, a)] = [(a + b, a + b)] = [(0, 0)]$$

and so $\mathcal{G}(A)$ is an abelian group. We now verify (i)-(iii).

(i) The map $i : A \to \mathcal{G}(A)$ given by $i(a) = [(a, 0)]$ is a monoid homomorphism. Suppose that $\varphi : A \to G$ is any monoid homomorhpism into an abelian group G. Define

$$\tilde{\varphi}([(a, b)]) := \varphi(a) - \varphi(b).$$

Clearly, $\tilde{\varphi}$ is a well-defined group homomorphism, and $\varphi = \tilde{\varphi} \circ i$. Uniqueness of $\tilde{\varphi}$ follows from the fact that the image of A under i generates all of $\mathcal{G}(A)$, and so any group homomorphism out of $\mathcal{G}(A)$ is determined by $i(A)$.

(ii) Suppose that $\hat{\mathcal{G}}(A)$ and $i' : A \to \hat{\mathcal{G}}(A)$ satisfy the same universal properties as $\mathcal{G}(A)$ and i. Since i' is a monoid homomorphism out of A, there exists a unique $\tilde{i}' : \hat{\mathcal{G}}(A) \to \mathcal{G}(A)$ such that $i' = \tilde{i}' \circ i$. Applying the same argument to $i : A \to \mathcal{G}(A)$ gives $\tilde{i} : \mathcal{G}(A) \to \hat{\mathcal{G}}(A)$ such that $i = \tilde{i} \circ i'$. Hence,

$$i = \tilde{i} \circ i' = \tilde{i} \circ \tilde{i}' \circ i$$

and so $\tilde{i} \circ \tilde{i}'$ is the identity. Similarly, $\tilde{i}' \circ \tilde{i}$ is the identity, hence $\tilde{i}$ and $\tilde{i}'$ are inverses. Thus, $\hat{\mathcal{G}}(A) \cong \mathcal{G}(A)$.

(iii) Since $\varphi \circ i_B : A \to \mathcal{G}(B)$ is a monoid homomorphism, the existence and uniqueness of $\mathcal{G}(\varphi)$ is given by (i). Functorality of $\mathcal{G}(\cdot)$ follows from the commutativity of

$$\begin{array}{ccccc} \mathcal{G}(A) & \xrightarrow{\mathcal{G}(\varphi)} & \mathcal{G}(B) & \xrightarrow{\mathcal{G}(\psi)} & \mathcal{G}(C) \\ {\scriptstyle i_A}\big\uparrow & & {\scriptstyle i_B}\big\uparrow & & {\scriptstyle i_C}\big\uparrow \\ A & \xrightarrow{\quad\varphi\quad} & B & \xrightarrow{\quad\psi\quad} & C \end{array}$$

□

With this construction, we define K-theory and develop some of its immediate properties.

Proposition 4.23. *Let X be a compact topological space. Then* $\operatorname{Vect} X$ *is an abeleian monoid under the direct sum operation.*

Proof. This is imemdiate from Proposition 4.13. The identity element is the 0 vector bundle. □

Definition 4.24. Let X be a compact topological space. The K**-theory** (or K**-group**) of X is $K(X) := \mathcal{G}(\operatorname{Vect} X)$.

Proposition 4.25. *$K(\cdot)$ is a contravariant homotopy-invariant functor from the category of compact topological spaces to the category of abelian groups.*

Proof. Homotopy invariance follows from Corollary 4.18. Indeed, if X and Y are homotopy equivalent, then $\operatorname{Vect} X \cong \operatorname{Vect} Y$, and so $K(X) \cong K(Y)$.

If $f : X \to Y$ is continuous, then by Proposition 4.7 (see also Corollary 4.18), there is an induced monoid homomorphism $f^* : \operatorname{Vect} Y \to \operatorname{Vect} X$. By Theorem 4.22,

$$K(f) := \mathcal{G}(f^*) : K(Y) \to K(X)$$

is a group homomorphism. Since $(g \circ f)^* = f^* \circ g^*$ for a continuous map $g : Y \to Z$, we have $K(g \circ f) = K(f) \circ K(g)$. If $1_X : X \to X$ is the identity map on X, then

$$K(1_X) = \mathcal{G}(1_X^*) = \mathcal{G}(1_{\operatorname{Vect} X}) = 1_{\mathcal{G}(\operatorname{Vect} X)} = 1_{K(X)}.$$

□

We often write elements $[([E], [F])]$ of $K(X)$, which are equivalence classes of pairs of isomorphism classes of vector bundles over X, as $[E] - [F]$.

Using Corollary 4.19, we can compute the K-theory of a particular class of topological spaces.

Example 4.26. Let X be a contractible space. Then $\operatorname{Vect} X \cong \mathbb{N}$, and so $K(X) \cong \mathcal{G}(\mathbb{N}) \cong \mathbb{Z}$. In particular, $K(x_0) \cong \mathbb{Z}$.

More generally, we have the following.

Proposition 4.27. *Suppose that X is a disjoint union of compact spaces $X_1, \ldots, X_n$. Then*

$$K(X) \cong K(X_1) \oplus \cdots \oplus K(X_n)$$

In particular, if X is a disjoint union of n contractible spaces, say a collection of n points, then $K(X) \cong \mathbb{Z}^n$.

Proof. If X is the disjoint union of $X_1, \ldots, X_n$, then a vector bundle over X is a choice of vector bundles over $X_1, \ldots, X_n$. Hence,

$$\text{Vect}\, X \cong \text{Vect}\, X_1 \oplus \cdots \oplus \text{Vect}\, X_n$$

from which we get

$$K(X) \cong K(X_1) \oplus \cdots \oplus K(X_n).$$

Hence, if each X_i is contractible,

$$K(X) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \cong \mathbb{Z}^n.$$

□

Finally, we make the following observation about elements of $K(X)$.

Proposition 4.28. *Every element of $K(X)$ can be written in the form $[E] - [X \times \mathbb{C}^N]$ for some N. Moreover, two elements $[E] - [X \times \mathbb{C}^N]$ and $[F] - [X \times \mathbb{C}^M]$ are equal in $K(X)$ if and only if there is an $r > 0$ such that $E \oplus (X \times \mathbb{C}^{M+r}) \cong F \oplus (X \times \mathbb{C}^{N+r})$.*

Proof. Let $[F_1] - [F_2] \in K(X)$. By Proposition 4.21, there is a vector bundle $F_2^\perp$ such that $F_2 \oplus F_2^\perp \cong X \times \mathbb{C}^N$ for some N. Then

$$[F_1] - [F_2] = ([F_1] - [F_2]) \oplus ([F_2^\perp] - [F_2^\perp]) = [F_1 \oplus F_2^\perp] - [F_2 \oplus F_2^\perp] = [F_1 \oplus F_2^\perp] - [X \times \mathbb{C}^N].$$

Setting $E = F_1 \oplus F_2^\perp$ gives the desired result.

By definition of $K(X)$ via the Grothendieck completion, $[E] - [X \times \mathbb{C}^N]$ and $[F] - [X \times \mathbb{C}^M]$ are equal if and only if there exists a vector bundle G such that

$$E \oplus (X \times \mathbb{C}^M) \oplus G \cong F \oplus (X \times \mathbb{C}^N) \oplus G.$$

Let $G^\perp$ be such that $G \oplus G^\perp \cong X \times \mathbb{C}^r$. Then

$$\begin{aligned} E \oplus (X \times \mathbb{C}^M) \oplus G \oplus G^\perp &\cong F \oplus (X \times \mathbb{C}^N) \oplus G \oplus G^\perp \\ E \oplus (X \times \mathbb{C}^M) \oplus (X \times \mathbb{C}^r) &\cong F \oplus (X \times \mathbb{C}^N) \oplus (X \times \mathbb{C}^r) \\ E \oplus (X \times \mathbb{C}^{M+r}) &\cong F \oplus (X \times \mathbb{C}^{N+r}). \end{aligned}$$

□

This is a small glimpse into K-theory, but it will suffice for the development of the Atiyah-Jänich theorem, which is the subject of the next chapter.

Chapter 5

The Atiyah-Jänich Theorem

In Chapter 3, we began studying the topology of the space of Fredholm operators. Our main result was the bijection

$$\pi_0(\mathfrak{F}(\mathcal{H})) \to \mathbb{Z} \tag{5.1}$$

induced by the classical index map $T \mapsto \operatorname{ind}(T)$. This chapter begins with the observation that $\pi_0(\mathfrak{F}(\mathcal{H})) = [x_0, \mathfrak{F}(\mathcal{H})]$, the set of homotopy classes of maps from a single point into $\mathfrak{F}(\mathcal{H})$. That is, two Fredholm operators T and S are in the same connected component exactly when the functions $x_0 \mapsto T$ and $x_0 \mapsto S$ are homotopic. In the previous chapter, we learned that $\mathbb{Z}$ is the K-theory group of a single point space. Hence, the bijection (5.1) can be written

$$[x_0, \mathfrak{F}(\mathcal{H})] \to K(x_0).$$

The goal of this chapter is to generalize this result to arbitrary compact topological spaces.

Theorem 5.1 (Atiyah-Jänich)**.** *Suppose that X is a compact topological space. Then there is a group isomorphism $[X, \mathfrak{F}(\mathcal{H})] \xrightarrow{\sim} K(X)$.*

This isomorphism will be given by a generalization of the Fredholm index. The construction of this index is our first order of business.

5.1 The Family Index

Consider a family of Fredholm operators parametrized by a topological space X, i.e., a continuous map $T : X \to \mathfrak{F}(\mathcal{H})$. We need to associate an element of $K(X)$, ideally resembling the classical index, to the family T. For every $x \in X$, $\ker T(x) =: \ker T_x$ and $\operatorname{coker} T_x$ are finite dimensional vector spaces. One might suspect that $\ker T := \bigsqcup_{x\in X} \ker T_x \to X$ and $\operatorname{coker} T \to X$, topologized as a subspace and quotient space of $X \times \mathcal{H}$ respectively, define vector bundles. If this were the case, we could define the index of the map T as

$$\operatorname{ind}(T) = [\ker T] - [\operatorname{coker} T] \in K(X).$$

If the dimension of $\ker T_x$ is constant, this is indeed possible.

Proposition 5.2. *Let $T : X \to \mathfrak{F}(\mathcal{H})$ be continuous, and suppose that $\dim \ker T_x$ is constant for all $x \in X$. Then $\ker T \to X$ is a vector bundle.*

Proof. The space $\bigsqcup_{x\in X} \ker T_x$ with projection $\ker T_x \mapsto x$ is clearly a family of vector spaces. We must show local triviality.

Let $x_0 \in X$. Let $P : \mathcal{H} \to \operatorname{im} T_{x_0}$ and $Q : \mathcal{H} \to \ker T_{x_0}$ be the orthogonal projections. For every $x \in X$, define

$$\hat{T}_x : \mathcal{H} \longrightarrow \ker T_{x_0} \oplus \operatorname{im} T_{x_0}$$

by $\hat{T}_x(f) = (Qf, PT_x f)$.

Observe that $\hat{T}_{x_0}$ is an isomorphism. Clearly, $\hat{T}_{x_0}$ is surjective. If $f \in \ker \hat{T}_{x_0}$, then $Qf = 0$ and $PT_{x_0} f = 0$. The first equality implies that $f \in (\ker T_{x_0})^{\perp}$, while the second equality implies that $T_{x_0} f \perp \operatorname{im} T_{x_0}$, which is only possible if $f \in \ker T_{x_0}$. Hence, $f = 0$, and so $\hat{T}_{x_0}$ is injective.

As infinite dimensional Hilbert spaces, $\ker T_{x_0} \oplus \operatorname{im} T_{x_0} \cong \mathcal{H}$. Since $GL(\mathcal{H})$ is open in $\mathfrak{L}(\mathcal{H})$, the set of isomorphisms in $\mathfrak{L}(\mathcal{H}, \ker T_{x_0} \oplus \operatorname{im} T_{x_0})$ is open. Hence, there is an open neighborhood of isomorphisms containing $\hat{T}_{x_0}$. Since $x \mapsto T_x$ is continuous and

$$\left\| (\hat{T}_x - \hat{T}_y) f \right\| = \| P(T_x - T_y) f \| \leq \| T_x - T_y \| \, \| f \|$$

the map $x \mapsto \hat{T}_x$ is continuous. Thus, there exists an open neighborhood U of x such that $\hat{T}_x$ is an isomorphism $\mathcal{H} \xrightarrow{\sim} \ker T_{x_0} \oplus \operatorname{im} T_{x_0}$ for all $x \in U$.

If $f \in \ker T_x$, then $\hat{T}_x(f) = (Qf, 0)$, so $\hat{T}_x(\ker T_x) \subseteq \ker T_{x_0}$. Since $\hat{T}_x$ is an isomorphism and $\dim \ker T_x = \dim \ker T_{x_0}$, $\hat{T}_x$ induces an isomorphism $\ker T_x \to \ker T_{x_0}$ on U. Hence, $\ker T$ is locally trivial, and so $\ker T \to X$ is a vector bundle. □

Similarly, when $\dim \ker T_x$ is constant, $\operatorname{coker} T \to X$ is a vector bundle. Unfortunately, as the dimensions of $\ker T_x$ and $\operatorname{coker} T_x$ can vary, $\ker T \to X$ and $\operatorname{coker} T \to X$ will in general not be vector bundles. Indeed the fact that $\hat{T}_x$ was an isomorphism of $\ker T_x$ onto $\ker T_{x_0}$ in the proof above relied on the fact that their dimensions were equal. In general, all we can say is that $\hat{T}_x(\ker T_x) \subseteq \ker T_{x_0}$. Consider the following simple example.

Example 5.3. Let $X = \mathbb{R}$ and $\mathcal{H} = \mathbb{C}$. Define $T : \mathbb{R} \to \mathfrak{F}(\mathbb{C})$ by $T_x(z) := xz$. Then

$$\dim \ker T_x = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases} .$$

Since $\mathbb{R}$ is connected, any vector bundle over $\mathbb{R}$ must have constant rank.

In order to remedy this situation, we will alter the function T so that we can extract a well-defined vector bundle. For a single operator, we do this as follows. Let $T : \mathcal{H} \to \mathcal{H}$ be Fredholm. Let $V = (\operatorname{im} T)^{\perp}$. Define $\tilde{T} : \mathcal{H} \oplus V \to \mathcal{H}$ by $\tilde{T}(f, v) = T(f) + v$. Then $\tilde{T}$ is surjective and $\ker \tilde{T} \cong \ker T$. Thus,

$$\operatorname{ind}(T) = \dim \ker \tilde{T} - d$$

where $d = \dim V = \dim \operatorname{coker} T$.

Our goal is to generalize the construction of $\tilde{T}$ to a family of Fredholm operators $T : X \to \mathfrak{F}(\mathcal{H})$ when X is compact. We begin with a lemma.

Lemma 5.4. *Let V be a finite dimensional subspace of $\mathcal{H}$. The set*

$$\mathcal{O}(V) := \{ T \in \mathfrak{F}(\mathcal{H}) \ : \ \operatorname{im} T + V = \mathcal{H} \}$$

is open in $\mathfrak{F}(\mathcal{H})$.

Proof. Let $T_0 \in \mathcal{O}(V)$. View T_0 as a map $T_0 : \ker T_0 \oplus (\ker T_0)^{\perp} \to \mathcal{H}$. Define

$$\begin{aligned} U &:= \left\{ T \in \mathfrak{F}(\mathcal{H}) \ : \ T\,|_{(\ker T_0)^{\perp}} : (\ker T_0)^{\perp} \xrightarrow{\sim} \operatorname{im} T_0 \right\} \\ &= \left\{ (T_1, T_2) \ : \ T_1 \in \mathfrak{L}(\ker T_0, \mathcal{H}), \ T_2 \in GL((\ker T_0)^{\perp}, \operatorname{im} T_0) \right\}. \end{aligned}$$

Since $GL((\ker T_0)^{\perp}, \operatorname{im} T_0)$ is open, U is an open set containing T_0. Since $\operatorname{im} T_0 + V = \mathcal{H}$, $U \subseteq \mathcal{O}(V)$. Hence, $\mathcal{O}(V) \subseteq \mathfrak{F}(\mathcal{H})$ is open. □

Let X be compact and $T : X \to \mathfrak{F}(\mathcal{H})$ continuous. Assume without loss of generality that X is connected; if not, work on each component separately. For each $x \in X$, set $V_x = (\operatorname{im} T_x)^{\perp}$. Then $\{\mathcal{O}(V_x)\}_{x \in X}$ is an open cover of $T(X)$. By compactness of $T(X)$, there is a finite subcover $\{\mathcal{O}(V_{x_i})\}_{i=1}^{n}$. Let $V = V_{x_1} + V_{x_2} + \cdots + V_{x_n}$. Then $T(X) \subseteq \mathcal{O}(V)$, since $\mathcal{O}(V_{x_i}) \subseteq \mathcal{O}(V)$.

Define $\tilde{T} : X \to \mathfrak{F}(\mathcal{H} \oplus V, \mathcal{H})$ by

$$\tilde{T}_x(f, v) = T_x(f) + v.$$

Continuity of $x \mapsto \tilde{T}_x$ follows from continuity of $x \mapsto T_x$. Since $T(X) \subseteq \mathcal{O}(V)$, $\tilde{T}_x$ is surjective for each $x \in X$. Since X is connected, $\operatorname{ind}(T_x) = \ker \tilde{T}_x$ is constant. Hence, by Proposition 5.2, $\ker \tilde{T} \to X$ is a vector bundle.

Definition 5.5. Let X be compact and $T : X \to \mathfrak{F}(\mathcal{H})$ continuous. The **family index** (also **generalized index** or **index bundle**) of T is

$$\operatorname{ind}(T) := [\ker \tilde{T}] - [X \times V] \in K(X)$$

where $V \subseteq \mathcal{H}$ is a finite dimensional subspace with $T(X) \subseteq \mathcal{O}(V)$ and $\tilde{T} : X \to \mathfrak{F}(\mathcal{H} \oplus V, \mathcal{H})$ is given by $\tilde{T}_x(f, v) = T_x(f) + v$.

We have associated an index in $K(X)$ to each family of Fredholm operators $T : X \to \mathfrak{F}(\mathcal{H})$. Next, we show that the family index is well-defined, homotopy invariant, and a homomorphism.

Lemma 5.6. *The family index is well-defined, i.e.,* $\operatorname{ind}(T)$ *does not depend on the choice of subspace* V.

Proof. Suppose that V' is another finite dimensional subspace such with $T(X) \subseteq \mathcal{O}(V')$. Since $V + V'$ is another such subspace, we can assume without loss of generality that $V \subseteq V'$. Moreover, it suffices to check the case when V' extends V by one dimension.

Suppose that $V' = V + \text{Span}(g)$ for some $g \notin V$. Any element of V' can be written $v + \lambda g$ for some $v \in V$ and $\lambda \in \mathbb{C}$. If $\tilde{T} : X \to \mathfrak{F}(\mathcal{H} \oplus V, \mathcal{H})$ and $\hat{T} : X \to \mathfrak{F}(\mathcal{H} \oplus V', \mathcal{H})$ are defined by

$$\tilde{T}_x(f, v) = T_x(f) + v \qquad \text{and} \qquad \hat{T}_x(f, v + \lambda g) = T_x(f) + v + \lambda g$$

then we need to show that $[\ker \tilde{T}] - [X \times V] = [\ker \hat{T}] - [X \times V']$.

Define a homotopy $\hat{T}^t$ by $\hat{T}^t_x(f, v + \lambda g) = T_x(f) + v + (1 - t)\lambda g$. Since $\hat{T}^t_x$ is surjective for all $t \in [0, 1]$, $\ker \hat{T}^t \to X \times [0, 1]$ is a vector bundle. Let $i_t : X \to X \times [0, 1]$ be the usual inclusion map. By Lemma 4.16,

$$i_0^* \left(\ker \hat{T}^t \right) \cong i_1^* \left(\ker \hat{T}^t \right). \tag{5.2}$$

Note that $i_0^* \left(\ker \hat{T}^t \right) \cong \ker \hat{T}$, and $i_1^* \left(\ker \hat{T}^t \right) \cong \ker S$, where $S : X \to \mathfrak{F}(\mathcal{H} \oplus V \oplus \mathbb{C}, \mathcal{H})$ is defined by $S_x(f, v, \lambda) = T_x(f) + v$. Since $\ker S \cong \ker \tilde{T} \oplus (X \times \mathbb{C})$, (5.2) gives $\ker \hat{T} \cong \ker \tilde{T} \oplus (X \times \mathbb{C})$. Hence,

$$\ker \hat{T} \oplus (X \times V) \cong \tilde{T} \oplus (X \times \mathbb{C}) \oplus (X \times V).$$

Since $V' \cong V \oplus \mathbb{C}$,

$$[\ker \hat{T}] - [X \times V'] = [\ker \tilde{T}] - [X \times V]$$

as desired. $\square$

Next, we show that the family index is homotopy-invariant.

Lemma 5.7. *If $S, T : X \to \mathfrak{F}(\mathcal{H})$ are homotopic, then* $\operatorname{ind}(T) = \operatorname{ind}(S)$. *Hence, the map*

$$\operatorname{ind} : [X, \mathfrak{F}(\mathcal{H})] \longrightarrow K(X)$$

is well-defined.

Proof. Since S and T are homotopic, there exists a homotopy $F : X \times [0,1] \to \mathfrak{F}(\mathcal{H})$ such that $S = F \circ i_0$ and $T = F \circ i_1$. By functorality of $K(\cdot)$, $\operatorname{ind}(S) = \operatorname{ind}(F \circ i_0) = i_0^* \operatorname{ind}(F)$. By Lemma 4.16, $i_0^* \operatorname{ind}(F) = i_1^* \operatorname{ind}(F)$. Hence,

$$\operatorname{ind}(S) = i_1^* \operatorname{ind}(F) = \operatorname{ind}(F \circ i_1) = \operatorname{ind}(T).$$

□

Finally, we consider the additive properties of the index.

Lemma 5.8. *Suppose that $T, S : X \to \mathfrak{F}(\mathcal{H})$. Define $T \oplus S : X \to \mathfrak{F}(\mathcal{H} \oplus \mathcal{H})$ by*

$$(T \oplus S)_x = \begin{bmatrix} T_x & 0 \\ 0 & S_x \end{bmatrix}$$

for $x \in X$. Then $\operatorname{ind}(T \oplus S) = \operatorname{ind}(T) \oplus \operatorname{ind}(S)$.

Proof. Let V and W be finite dimensional subspaces such that $T(X) \subseteq \mathcal{O}(V)$ and $S(X) \subseteq \mathcal{O}(W)$, and let $\tilde{T}$ and $\tilde{S}$ be defined as usual. Then $T \oplus S \subseteq \mathcal{O}(V \oplus W)$, and $\widetilde{T \oplus S} = \tilde{T} \oplus \tilde{S}$. Hence,

$$\operatorname{ind}(T \oplus S) = [\ker(\tilde{T} \oplus \tilde{S})] - [X \times (V \oplus W)].$$

There is an obvious isomorphism of vector bundles $\ker(\tilde{T} \oplus \tilde{S}) \cong \ker \tilde{T} \oplus \ker \tilde{S}$. Thus,

$$\begin{aligned} \operatorname{ind}(T \oplus S) &= [\ker(\tilde{T} \oplus \tilde{S})] - [X \times (V \oplus W)] \\ &= [\ker \tilde{T}] \oplus [\ker \tilde{S}] - ([X \times V] \oplus [X \times W]) \\ &= \Big([\ker \tilde{T}] - [X \times V]\Big) \oplus \Big([\ker \tilde{S}] - [X \times W]\Big) \\ &= \operatorname{ind}(T) \oplus \operatorname{ind}(S). \end{aligned}$$

□

Lemma 5.9. *Let $T, S : X \to \mathfrak{F}(\mathcal{H})$. Define $TS : X \to \mathfrak{F}(\mathcal{H})$ by $(TS)_x = T_x S_x$. Then* $\operatorname{ind}(TS) = \operatorname{ind}(T) \oplus \operatorname{ind}(S)$.

Proof. Let $T, S : X \to \mathfrak{F}(\mathcal{H})$. The identity map $I : X \to \mathfrak{F}(\mathcal{H})$ given by $I_x = I$ has index [0]. By the previous lemma,

$$\operatorname{ind}(TS) = \operatorname{ind}(TS) \oplus \operatorname{ind}(I) = \operatorname{ind}(TS \oplus I)$$

Consider the following homotopy for $0 \le t \le \pi/2$:

$$\begin{bmatrix} TS & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix}.$$

When $t = 0$,

$$\begin{bmatrix} TS & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} TS & 0 \\ 0 & I \end{bmatrix}.$$

When $t = \pi/2$,

$$\begin{bmatrix} TS & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} = \begin{bmatrix} 0 & -TS \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -S^{-1} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix}$$
$$= \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}.$$

Hence, $TS \oplus I$ and $T \oplus S$ are homotopic maps. Thus,

$$\operatorname{ind}(TS) = \operatorname{ind}(TS \oplus I) = \operatorname{ind}(T \oplus S) = \operatorname{ind}(T) \oplus \operatorname{ind}(S).$$

□

Pointwise composition of families of Fredholm operators turns $[X, \mathfrak{F}(\mathcal{H})]$ into an abelian monoid with identity element $x \mapsto I$. On the other hand, $K(X)$ is an ableian monoid under the direct sum. We have shown that the family index is a well-defined map from $[X, \mathfrak{F}(\mathcal{H})]$ to $K(X)$ satisfying $\operatorname{ind}(TS) = \operatorname{ind}(T) \oplus \operatorname{ind}(S)$. This proves the following theorem, which summarizes our knowledge of the family index thus far.

Theorem 5.10. *Let X be compact. The family index is a monoid homomorphism $[X, \mathfrak{F}(\mathcal{H})] \to K(X)$.*

5.2 Kuiper's Theorem

One of the significant results from Chapter 3 is that the space of invertible operators on a Hilbert space is path connected. In Section 5.3, we will complete the proof of Theorem 5.1 by proving bijectivity of the index, which relies on a generalization of the aforementioned result due to Nicolaas Kuiper.

Theorem 5.11 (Kuiper)**.** *Let X be compact, and let $GL(\mathcal{H})$ be the space of invertible bounded operators on an infinite dimensional Hilbert space. Then $[X, GL(\mathcal{H})] = 0$.*

The proof of Theorem 5.11 is quite direct in that we will consider a continuous map $f : X \to GL(\mathcal{H})$ and simply construct a homotopy from f to the identity $x \mapsto I$. The will be done in a series of steps. We follow Kuiper's approach closely [15].

Step 1: Reducing to Finite Dimensions. Let $f : X \to GL(\mathcal{H})$ be continuous. We will show that f is homotopic to a map whose image is contained in a finite dimensional vector space in $\mathfrak{F}(\mathcal{H})$. Explicitly,

Lemma 5.12. *The map f is homotopic to a map f_1 such that $f_1(X) \subseteq GL(\mathcal{H})$ is contained in a finite dimensional subspace of $\mathfrak{L}(\mathcal{H})$.*

Proof. Let $f_0 = f$. Recall that $GL(\mathcal{H})$ is open in $\mathfrak{L}(\mathcal{H})$. For each $x \in X$, place an open ball around $f_0(x)$ in $GL(\mathcal{H})$. For reasons that will soon become apparent, we require the radius ϵ of each ball to be small enough so that the ball of radius 3ϵ is still in $GL(\mathcal{H})$. This gives an open cover of $f_0(X)$. Since f_0 is continuous and X is compact, $f_0(X)$ is compact, and so there is a finite subcover of open balls $U_1, \ldots, U_N$ centered at $f_0(x_1), \ldots, f_0(x_N)$. Let $\varphi_1, \ldots, \varphi_N$ be a partition of unity corresponding to $U_1, \ldots, U_N$, and define, for each $T \in \bigcup U_i$,

$$h_t(T) = (1-t)T + t \sum_{i=1}^{N} \varphi_i(T) f_0(x_i).$$

Then h_0 is the identity on $\bigcup U_i$, and the image of h_1 is contained in a finite dimensional vector space spanned by $f_0(x_1), \ldots, f_0(x_N)$. Moreover, $h_t(T) \in GL(\mathcal{H})$ for all t. To see this, let $U_{i_1}, \ldots, U_{i_k}$ be the open balls such that $T \in U_{i_j}$. Let U' denote the ball from this set of largest radius ϵ. Then by the triangle equality, each $U_{i_j} \subseteq 3U'$, where $3U'$ is the ball of radius 3ϵ with the same center as U'. Since open balls are convex, it follows that the convex hull of $T, f_0(x_{i_1}), \ldots, f_0(x_{i_k})$ is contained in $3U'$, which is contained in $GL(\mathcal{H})$ by choice of the open cover. Hence, $h_t(T)$ is contained in $GL(\mathcal{H})$.

Set $f_t := h_t f_0$ for $0 \leq t \leq 1$. Then $f_0 = f$, and $f_1 = h_1 f_0$ is contained in a finite dimensional vector subspace of $\mathfrak{L}(\mathcal{H})$.

□

Step 2: Planar Rotations. The next step of the homotopy involves a series of rotations in finite dimensional subspaces of $\mathcal{H}$. We inductively define a sequence of unit vectors and subspaces as follows.

First, to simplify notation, let $g_1, \ldots, g_N$ be the operators $f_0(x_1), \ldots, f_0(x_n)$. Let W be the vector space spanned by $g_1, \ldots, g_N$.

Let $a_1 \in \mathcal{H}$ be any unit vector. Consider the vectors $g_1(a_1), \ldots, g_N(a_1)$; let a_1^0 be a unit vector orthogonal to $a_1, g_1(a_1), \ldots, g_N(a_1)$, and let $A_1 \subseteq \mathcal{H}$ be an $N+2$ dimensional subspace containing $a_1, g_1(a_1), \ldots, g_N(a_1), a_1^0$.

Next, suppose that a_j, a_j^0, and A_j have been defined for $j < i$. We want to define a_i so that $a_i, g_1(a_i), \ldots, g_N(a_i)$ are orthogonal to A_j for all $j < i$. Consider the space

$$\tilde{A}_i = \bigcap_{j=1}^{i-1} \left[A_j^\perp \cap \left(\bigcap_{\ell=1}^{N} g_\ell^{-1}\left(A_j^\perp\right) \right) \right].$$

We claim that $\tilde{A}_i$ is nonempty. Since each A_j is finite dimensional, $A_j^\perp$ is finite codimensional. Invertibility of the g_ℓ's implies the same for $g_\ell^{-1}(A_j^\perp)$. In general, we have $V^\perp \cap W^\perp = (V+W)^\perp$ for finite dimensional V, W. Thus, since the A_j's and $g_\ell^{-1}(A_j)$'s are finite dimensional, the intersection of their orthogonal complements is the orthogonal complement of a finite dimensional space, and is therefore nonempty. Hence, by choosing a unit vector a_i in $\tilde{A}_i$, we have $a_i \in A_j^\perp$ and $g_\ell(a_i) \in A_j^\perp$ for all j. Let a_i^0 be a unit vector orthogonal to $a_i, g_1(a_i), \ldots, g_N(a_i)$, and let A_i be an $N+2$ dimensional subspace containing these $N+2$ vectors.

The homotopy that we construct will be based on rotating the vectors in each A_i in a particular way. First, choose $C \geq 1$ such that $\|f_1(x)\| \leq C$ nd $\left\|f_1(x)^{-1}\right\| \leq C$ for all $x \in X$. Define the following space:

$$W_C := \left\{ w \in W \cap GL(\mathcal{H}) \;:\; \|w\| \leq C, \left\|w^{-1}\right\| \leq C \right\}.$$

Then $f_1(X) \subseteq W_C$, and W_C is closed under inversion. By construction, for every $w \in W_C$, $w(a_i) \in A_i$, and that $w(a_i)$ is orthogonal to a_i^0. Define the function

$$k_i : W_C \times [0,1] \to GL(A_i)$$

as follows. If $0 \leq t \leq \frac{1}{2}$, then $k_i(w,t)$ is a rotation in the plane spanned by $w(a_i)$ and a_i^0:

$$k_i(w,t) := \begin{cases} \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix} & \text{on } \operatorname{Span}\left(w(a_i), \|w(a_i)\|\, a_i^0\right) \\ I & \text{on } \operatorname{Span}\left(w(a_i), \|w(a_i)\|\, a_i^0\right)^\perp \end{cases}.$$

If $\frac{1}{2} \leq t \leq 1$, then $k_i(w,t)$ is a rotation in the plane spanned by a_i^0 and a_i:

$$k_i(w,t) := \begin{cases} \begin{bmatrix} \cos\left(\pi\left(t-\frac{1}{2}\right)\right) & -\sin\left(\pi\left(t-\frac{1}{2}\right)\right) \\ \sin\left(\pi\left(t-\frac{1}{2}\right)\right) & \cos\left(\pi\left(t-\frac{1}{2}\right)\right) \end{bmatrix} k_i\left(w, \frac{1}{2}\right) & \text{on } \operatorname{Span}\left(a_i^0, a_i\right) \\ k_i\left(w, \frac{1}{2}\right) & \text{on } \operatorname{Span}\left(a_i^0, a_i\right)^\perp \end{cases}.$$

Each of the rotation matrices above are defined with respect to the bases

$$\{w(a_i), \|w(a_i)\|\, a_i^0\} \text{ and } \{a_i^0, a_i\}$$

respectively.

Lemma 5.13. *For each i, k_i is continuous in both w and t.*

Proof. Let $w, w' \in W_C$ and let $t, t' \in [0,1]$. The triangle inequality gives

$$\begin{aligned}\left\|k_i(w',t') - k_i(w,t)\right\| &\leq \left\|k_i(w',t') - k_i(w',t)\right\| + \left\|k_i(w',t) - k_i(w,t)\right\| \\ &\leq \left\|k_i(w',t')k_i(w',t)^{-1} - I\right\| \left\|k_i(w',t)^{-1}\right\| + \\ &\qquad \left\|k_i(w',t)k_i(w,t)^{-1} - I\right\| \left\|k_i(w,t)^{-1}\right\| \\ &= \left\|k_i(w',t')k_i(w',t)^{-1} - I\right\| + \left\|k_i(w',t)k_i(w,t)^{-1} - I\right\|.\end{aligned}$$

The last equality follows from the fact that $k_i(w,t)$ is unitary.

First, consider the case where $t, t' \in [0, 1/2]$. Then $k_i(w',t')k_i(w',t)^{-1}$ is a rotation in a plane by an angle of $\theta = \pi(|t'-t|)$. Since the length of a chord of a circle between two points is less than the arc length of the portion of circle connecting those points, it follows that

$$\left\|k_i(w',t')k_i(w',t)^{-1} - I\right\| \leq 2\pi\frac{\theta}{2\pi} = \pi(|t'-t|).$$

Next, we consider $k_i(w',t)k_i(w,t)^{-1}$ for $t', t \in [0,1/2]$. Suppose that $w' \neq w$. Since $w', w \in W_C$, we have $\|w'(a_i)\|, \|w(a_i)\| \geq C^{-1}$. Let α be the angle between $w'(a_i)$ and $w(a_i)$. Since an isosceles triangle with legs of length C^{-1} and angle α has base length $2C^{-1}\sin(\alpha/2)$, it follows that

$$2C^{-1}\sin(\alpha/2) \leq \left\|w'(a_i) - w(a_i)\right\| \leq \left\|w' - w\right\|. \tag{5.3}$$

By construction, both $w'(\alpha_i)$ and $w(\alpha_i)$ are orthogonal to a_i^0. Let E be the vector space spanned by $\alpha_i^0, w(\alpha_i)$ and $w'(\alpha_i)$. Since $k_i(w',t)k_i(w,t)^{-1}$ comprises of rotations inside E, it leaves $E^\perp$ fixed. Moreover, since $k_i(w',t)k_i(w,t)^{-1}$ is the product of two rotations about an angle α, by the same argument as above, $\left\|k_i(w',t)k_i(w,t)^{-1} - I\right\| \leq 2 \cdot 2\sin(\alpha/2)$. With equation (5.3), this implies that

$$\left\|k_i(w',t)k_i(w,t)^{-1} - I\right\| \leq 2C\left\|w' - w\right\|.$$

Thus, for $t', t \in [0,1/2]$,

$$\left\|k_i(w',t') - k_i(w,t)\right\| \leq \pi(|t'-t|) + 2C\left\|w' - w\right\|$$

and so k_i is continuous in w and t for $t \in [0, 1/2]$.

Next, suppose that $t', t \in [1/2, 1]$. As before, $k_i(w',t')k_i(w',t)^{-1}$ is a planar rotation by an angle of $\theta = \pi(|t'-t|)$, and so

$$\left\|k_i(w',t')k_i(w',t)^{-1} - I\right\| \leq 2\pi\frac{\theta}{2\pi} = \pi(|t'-t|).$$

For $1/2 \leq t \leq 1$, the rotation is in the a_i^0-a_i plane, hence $k_i(w',t)k_i(w,t)^{-1}$ is the identity. Thus,

$$\left\|k_i(w',t)k_i(w,t)^{-1} - I\right\| = 0.$$

Hence, for $t', t \in [1/2, 1]$,

$$\left\|k_i(w',t') - k_i(w,t)\right\| \leq \pi(|t'-t|)$$

and so k_i is continuous in both t and w. Hence, k_i is continuous in both t and w for $t \in [0,1]$. ☐

The continuity of each rotation k_i on A_i did not depend on the choice of i. This allows us to execute all the rotations at once in a continuous map.

Lemma 5.14. *Let $k : W_C \times [0,1] \to GL(\mathcal{H})$ be defined by*

$$k(w,t) := \begin{cases} k_i(w,t) & \text{on } A_i \\ I & \text{on } (A_1 \oplus A_2 \oplus \cdots)^{\perp} \end{cases}.$$

Then k is continuous in w and t.

Proof. Let $x \in \mathcal{H}$. Let x_i be the orthogonal projection of x_i onto A_i, and let x_0 be the orthogonal projection of x onto $(A_1 \oplus A_2 \oplus \cdots)^{\perp}$. Then

$$x = x_0 + x_1 + x_2 \cdots$$

Let $w', w \in W_C$ and $t', t \in [0,1]$. Then

$$\begin{aligned} \left\|(k(w',t') - k(w,t))x\right\| &= \left\|(k(w',t') - k(w,t))\left(\sum_{i=0}^{\infty} x_i\right)\right\| \\ &= \left\|(k(w',t') - k(w,t))x_0 + \sum_{i=1}^{\infty}(k(w',t') - k(w,t))x_i\right\| \\ &= \left\|\sum_{i=1}^{\infty}(k_i(w',t') - k_i(w,t))x_i\right\|. \end{aligned}$$

Since the A_i's are orthogonal, we can apply the Pythagorean Theorem, followed by our continuity estimates from above:

$$\begin{aligned} &= \sqrt{\sum_{i=1}^{\infty} \|(k_i(w',t') - k_i(w,t))x_i\|^2} \\ &\leq \sqrt{\sum_{i=1}^{\infty} \|[\pi(|t'-t|) + 2C\|w'-w\|]\, x_i\|^2} \\ &= \sqrt{[\pi(|t'-t|) + 2C\|w'-w\|]^2 \sum_{i=1}^{\infty} \|x_i\|^2} \\ &\leq \left[\pi(|t'-t|) + 2C\left\|w'-w\right\|\right]\|x\|. \end{aligned}$$

Therefore,

$$\left\|k(w',t') - k(w,t)\right\| \leq \pi(|t'-t|) + 2C\left\|w'-w\right\|$$

so k is continuous in both w and t. ☐

With these technicalities out of the way, we can construct the next step of our homotopy. By the definition of k, for any w, $k(w,0)$ is the identity transformation (at time 0 we have not rotated anything), and $k(w,1)$ rotates $w(a_i)$ in the direction of a_i. Explicitly, $k(w,1)(w(a_i)) = \|w(a_i)\|\, a_i$. In particular, for $x \in X$, $k(f_1(x),1)(f_1(x)(a_i)) = \|f_1(x)(a_i)\|\, a_i$. This gives us the following lemma:

Lemma 5.15. *The map $f : X \to GL(\mathcal{H})$ is homotopic to a map $f_2 : X \to GL(\mathcal{H})$ such that*

$$f_2(x)(a_i) = \|f_1(x)(a_i)\|\, a_i$$

for every a_i and every $x \in X$.

Proof. For $1 \leq t \leq 2$, define $f_t(x) = k(f_1(x), t-1)f_1(x)$. This map is continuous, and when $t = 1$ we have

$$k(f_1(x), 0)f_1(x) = If_1(x) = f_1(x).$$

When $t = 2$,

$$f_2(x)(a_i) = k(f_1(x), 1)f_1(x)(a_i) = \|f_1(x)(a_i)\| \, a_i.$$

Since f is homotopic to f_1, f is homotopic to f_2. $\square$

Step 3: Scaling Down. Let $\mathcal{H}'$ denote the closure of the span of the vectors $a_1, a_2, \ldots$. Let $\mathcal{H}_1 = (\mathcal{H}')^\perp$. By the previous lemma, any transformation $f_2(x)$ acts by scaling $\mathcal{H}'$. The next step of the homotopy is to "pull back" all of the scaled a_i's so that $f_3(x)(a_i) = a_i$ for all i. We achieve this as follows. For $t \in [2, 3]$, define:

$$\begin{cases} f_t(x)(a_i) = (3-t)f_2(x)(a_i) + (t-2)a_i & i = 1, 2, \ldots \\ f_t(x)v = f_2(x)v & v \in \mathcal{H}_1 \end{cases}.$$

When $t = 2$, this definition agrees with the definition of f_2. When $t = 3$, $f_3(x)(a_i) = a_i$ for each i, and $f_3(x)v = f_2(s)v$ for $v \in \mathcal{H}_1$. In effect, we have homotoped f to the identity on $\mathcal{H}'$. All that remains is to homotope f_2 to the identity on $\mathcal{H}_1$.

Step 4: Projecting onto $\mathcal{H}_1$. Let P' denote the projection onto $\mathcal{H}'$, and let P_1 be the projection onto $\mathcal{H}_1$. Consider the following space of operators:

$$\mathcal{E} = \left\{ w \in GL(\mathcal{H}) \ : \ w = P' + wP_1 \right\}.$$

In words, $\mathcal{E}$ is the space of operators that act as the identity on $\mathcal{H}'$. Note that $f_3(x) \in \mathcal{E}$ for all $x \in X$. We define a map $\omega : \mathcal{E} \to GL(\mathcal{H})$ by

$$\omega(w) = P' + P_1 w P_1 \qquad \text{for all } w \in \mathcal{E}.$$

Then $\omega(w)$ acts as the identity on $\mathcal{H}'$, and $\omega(w)(\mathcal{H}_1) = \mathcal{H}_1$. This allows us to construct the next step of the homotopy. Define

$$f_t(x) = (4-t)f_3(x) + (t-3)\omega(f_3(x)); \qquad 3 \leq t \leq 4.$$

When $t = 3$, we have $f_3(x)$, and when $t = 4$, we have $\omega(f_3(x))$, which is the identity on $\mathcal{H}'$ and maps $\mathcal{H}_1$ to $\mathcal{H}_1$.

Step 5: Contracting onto I. Our homotopy thus far has morphed the image of X under f in $GL(\mathcal{H})$ so that it is contained in the following subspace:

$$\{ g \in GL(\mathcal{H}) \ : \ g \mid_{\mathcal{H}'} = I \ \text{ and } \ g(\mathcal{H}_1) = \mathcal{H}_1 \}.$$

In Chapter 3, we proved a general result about the contractibility of such spaces (see Lemma 3.16). This gives us the final step in our homotopy.

Proof of Theorem 5.11. Let γ be defined as in the proof of Lemma 3.16. For $4 \leq t \leq 5$, define

$$f_t(x) = \gamma_{\pi(t-4)}(f_4(x)).$$

When $t = 4$, $\gamma_0(f_4(x)) = f_4(x)$, and when $t = 5$, we have $f_5(x) = \gamma_\pi(f_4(x)) = I$. Hence $f_t(x)$ for $0 \leq t \leq 5$ is a homotopy for which $f_0 = f$ and f_5 is the constant map $X \mapsto I$. Therefore, every continuous map $f : X \to GL(\mathcal{H})$ is homotopic to the constant map, and so $[X, GL(\mathcal{H})] = 0$. $\square$

Corollary 5.16. *For $n = 0, 1, 2, \ldots$, $\pi_n(GL(\mathcal{H}) = 0$.*

Proof. For any n, S^n is compact. By Kuiper's Theorem, $[S^n, GL(\mathcal{H})] = 0$, and so $\pi_n(GL(\mathcal{H}))$. $\square$

Though we do not prove it here, it is worth mentioning that another corollary of Kuiper's Theorem is the contractibility of $GL(\mathcal{H})$. For details, see Kuiper's paper [15].

5.3 Bijectivity of the Index

Summarizing our progress, we have shown that for a compact space X, there is a well-defined *monoid* homomorphism $[X, \mathfrak{F}(\mathcal{H})] \to K(X)$ given by the family index. In this section, we demonstrate the bijectivity of the index. As a bijective homomorphism of monoids, $[X, \mathfrak{F}(\mathcal{H})]$ is a group, and the family index is actually a *group* isomorphism.

We begin with surjectivity. Recall that every element of $K(X)$ can be written in the form $[E] - [X \times \mathbb{C}^k]$. We produce a family of Fredholm operators with index $-[X \times \mathbb{C}^k]$, and then a family with index $[E]$. Surjectivity then follows from the additive property of the index.

Lemma 5.17. *Let $\mathcal{H}$ have orthonormal basis $e_1, e_2, \ldots$. For each nonnegative integer k, let S_k be the shift operator $S_k(e_j) = e_{j-k}$. Then if $S_k : X \to \mathfrak{F}(\mathcal{H})$ denotes the constant map, then* $\operatorname{ind}(S_k) = [X \times \mathbb{C}^k]$ *and* $\operatorname{ind}(S_{-k}) = -[X \times \mathbb{C}^k]$.

Proof. Let $k > 0$ be an integer. Since $S_k : \mathcal{H} \to \mathcal{H}$ is surjective, $\operatorname{ind}(S_k) = [\ker S_k]$. Since $\ker S_k = \text{Span}(e_1, \ldots, e_k) \cong \mathbb{C}^k$,

$$[\ker S_k] = \left[\bigsqcup_{x \in X} \ker S_k\right] = [X \times \mathbb{C}^k].$$

The fact that $\operatorname{ind}(S_{-k}) = -[X \times \mathbb{C}^k]$ follows from

$$[0] = \operatorname{ind}(I) = \operatorname{ind}(S_k S_{-k}) = \operatorname{ind}(S_k) \oplus \operatorname{ind}(S_{-k}).$$

□

Proposition 5.18. *The family index $[X, \mathfrak{F}(\mathcal{H})] \to K(X)$ is surjective.*

Proof. Let $E \to X$ be a vector bundle. We will construct a family T such that $\operatorname{ind}(T) = [E]$. Then $\operatorname{ind}(TS_{-k}) = \operatorname{ind}(T) \oplus \operatorname{ind}(S_{-k}) = [E] - [X \times \mathbb{C}^k]$.

By Proposition 4.20, E is isomorphic to a subbundle of a trivial bundle of rank N. Let $P_x : \mathbb{C}^N \to \mathbb{C}^N$ be the orthogonal projection onto E_x, and let $Q_x = I - P_x$. We will use these projections to build a new family of Fredholm operators on the tensor space $\mathbb{C}^N \otimes \mathcal{H}$. Let S_1 be the shift operator, and define $T : X \to \mathfrak{F}(\mathbb{C}^N \otimes \mathcal{H})$ by

$$T_x := P_x \otimes S_1 + Q_x \otimes I.$$

Then T_x is surjective for each $x \in X$, since if $v \otimes e_i \in \mathbb{C}^N \otimes \mathcal{H}$ with e_i a basis element,

$$\begin{aligned} T_x(Q_x(v) \otimes e_i + P_x(v) \otimes e_{i+1}) &= (P_x \otimes S_1 + Q_x \otimes I)(Q_x(v) \otimes e_i + P_x(v) \otimes e_{i+1}) \\ &= P_x Q_x(v) \otimes S_1(e_1) + Q_x^2(v) \otimes e_i \\ &\quad + P_x^2(v) \otimes S_1(e_{i+1}) + Q_x P_x(v) \otimes e_{i+1} \\ &= Q_x(v) \otimes e_i + P_x(v) \otimes e_i \\ &= v \otimes e_i. \end{aligned}$$

Thus, $\ker T$ is a vector bundle. Suppose that $v \otimes f \in \ker T_x$ for $v, f \neq 0$. Then

$$P_x(v) \otimes S_1(f) + Q_x(v) \otimes f = 0.$$

Since $f \neq 0$ and $P_x + Q_x = I$,

$$P_x(v) \otimes S_1(f) = 0 \quad \text{and} \quad Q_x(v) \otimes f = 0.$$

Since $f \neq 0$, $Q_x(v) \otimes f = 0$ exactly when $v \in E_x$. Since $v \in E_x$, $P_x(v) \otimes S_1(f) = 0$ exactly when $f \in \text{Span}(e_1)$. Hence, $\ker T_x = E_x \otimes \text{Span}(e_1) \cong E_x$, and $\mathbb{C}^N \otimes \mathcal{H} \cong \mathcal{H}$, and so $T : X \to \mathfrak{F}(\mathbb{C}^N \otimes \mathcal{H}) \cong \mathfrak{F}(\mathcal{H})$ is a continuous map with $\ker T_x \cong E_x$ for each $x \in X$. Therefore $\ker T \cong E$ as vector bundles. By surjectivity of each T_x, $\operatorname{ind}(T) = [E]$. □

Next, we prove injectivity. In the classical case $X = \{x_0\}$, this amounts to showing that a Fredholm operator with index 0 is in the same connected component as the invertible operators. To recall (see the proof of Theorem 3.18), suppose that $T \in \mathfrak{F}(\mathcal{H})$ with $\mathrm{ind}\,(T) = 0$. Since $\dim\ker T = \dim\operatorname{coker} T$, we can choose an isomorphism

$$\varphi : \ker T \to (\operatorname{im} T)^{\perp}$$

and define

$$\Phi = \begin{cases} \varphi & \text{on } \ker T \\ 0 & \text{on } (\ker T)^{\perp} \end{cases}.$$

Then $T + \Phi : \mathcal{H} \to \mathcal{H}$ is invertible. Since Φ is a finite rank operator, $T + t\Phi \subseteq \mathfrak{F}(\mathcal{H})$ is a path from T to an invertible operator.

The generalization of this argument to a family of Fredholm operators $T : X \to \mathfrak{F}(\mathcal{H})$ invokes Kuiper's Theorem.

Proposition 5.19. *The family index $[X, \mathfrak{F}(\mathcal{H})] \to K(X)$ is injective.*

Proof. Let $T : X \to \mathfrak{F}(\mathcal{H})$ be a family of Fredholm operators with $\mathrm{ind}\,(T) = [0]$. It suffices to show that T is homotopic to a function $X \to GL(\mathcal{H})$. Indeed, by Kuiper's Theorem, any map $X \to GL(\mathcal{H})$ is homotopic to the identity map $X \mapsto I$. Since $[X, \mathfrak{F}(\mathcal{H})]$ is a monoid with identity element $[X \to I]$ and the index is a monoid homomorphism, injectivity then follows.

Pick a subspace V with $T(X) \subseteq \mathcal{O}(V)$ and define $\tilde{T} : X \to \mathfrak{F}(\mathcal{H} \oplus V, \mathcal{H})$ as usual. Then $[0] = [\ker \tilde{T}] - [X \times V]$. Since the family index is well-defined, and by Propositon 4.28, we can pick V of large enough dimension so that

$$\ker \tilde{T} \cong X \times V.$$

Let $\varphi : \ker \tilde{T} \to X \times V$ be a bundle isomorphism. For each x, $\tilde{T}_x : \mathcal{H} \oplus V \to \mathcal{H}$ is surjective and $\varphi_x : \ker \tilde{T}_x \subseteq \mathcal{H} \oplus V \to V$ is an isomorphism. We want to construct an invertible operator on $\mathcal{H} \oplus V$. Set

$$\Phi_x = \begin{cases} \varphi_x & \text{on } \ker \tilde{T}_x \\ 0 & \text{on } (\ker \tilde{T}_x)^{\perp} \end{cases}$$

and define $\gamma_x : \mathcal{H} \oplus V \to \mathcal{H} \oplus V$ by

$$\gamma_x(f, v) := \left(\tilde{T}_x(f, v), \Phi_x(f, v)\right).$$

This is the analogue of the operator $T + \Phi$ from above. Indeed, γ_x is injective and surjective, hence $\gamma : X \to GL(\mathcal{H} \oplus V)$. We need to show that this map is homotopic to T.

By definition, $\tilde{T}_x(f, v) = T_x(f) + v$. Hence,

$$\gamma_x^t(f, v) = (T_x(f) + tv, t\Phi_x(f, v)) \qquad 0 \le t \le 1$$

is a homotopy from $\gamma^0 = T$ to an invertible family γ^1. Since Φ_x is finite rank, γ^t remains in $\mathfrak{F}(\mathcal{H} \oplus V)$ for all t.

□

Proof of Theorem 5.1. It has been established that $\mathrm{ind} : [X, \mathfrak{F}(\mathcal{H})] \to K(X)$ is a well-defined bijective monoid homomorphism. Since $K(X)$ is a group, it follows that $[X, \mathfrak{F}(\mathcal{H})]$ is a group, and thus the generalized index is a group isomorphism. □

Chapter 6

Toeplitz Operators

We close this thesis with a brief study of a particular class of operators, the *Toeplitz operators*. This chapter serves two purposes: to give an interesting realization of the abstract Fredholm theory developed in Chapter 3, and to offer a glimpse of more general theories in mathematics.

One of the primary resources for the study of Toeplitz operators is [9]. More details can be found in [11], [8], and [16].

6.1 The Hardy Space

Recall that $L^2(S^1)$ is the complex Hilbert space of square-integrable functions on the unit circle. There is a natural identification of $L^2(S^1)$ with $L^2([0, 2\pi])$ which we employ without comment. In this chapter, we normalize the inner product as follows:

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{S^1} f(z)\overline{g(z)}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)\overline{g\left(e^{i\theta}\right)}\,d\theta.$$

The functions $\{z^n = e^{in\theta}\}_{n=-\infty}^{\infty}$ form an orthonormal basis for this space. We are interested in subspace which is the closed span of $\{z^n\}_{n=0}^{\infty}$.

Definition 6.1. The **Hardy space** is defined as:

$$H^2(S^1) := \left\{\, f \in L^2(S^1) \;:\; \langle f, z^n\rangle = 0 \text{ for } n < 0 \,\right\}.$$

Observe that a function f is in $H^2(S^1)$ if it can be written in the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Throughout this chapter, we let $P : L^2(S^1) \to H^2(S^1)$ be the orthogonal projection onto $H^2(S^1)$.

The definition of the Hardy space $H^2(S^1)$ can be extended quite naturally to spaces $H^p(S^1) \subseteq L^p(S^1)$, though we restrict ourselves to $H^2(S^1)$ for the sake of simplicity.

6.2 Definitions and Properties

If $\varphi \in L^\infty(S^1)$, then the multiplication operator M_φ defined by $M_\varphi f(z) = \varphi(z)f(z)$ is a bounded operator on $L^2(S^1)$. In particular,

$$\begin{aligned}\|M_\varphi f\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(e^{i\theta}\right) f\left(e^{i\theta}\right)\overline{\varphi\left(e^{i\theta}\right) f\left(e^{i\theta}\right)}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\varphi\left(e^{i\theta}\right)\right|^2 \left|f\left(e^{i\theta}\right)\right|^2 d\theta \\ &\le \|\varphi\|_\infty^2 \|f\|^2.\end{aligned}$$

Hence, $\|M_\varphi\| \leq \|\varphi\|_\infty$. A *Toeplitz operator* is the "compression" of a multiplication operator to the Hardy space.

Definition 6.2. Let $\varphi \in L^\infty(S^1)$. The **Toeplitz operator with symbol** φ is the operator $T_\varphi : H^2(S^1) \to H^2(S^1)$ defined by

$$T_\varphi f := PM_\varphi f$$

for all $f \in H^2(S^1)$.

Basic properties of Toeplitz operators are immediate.

Proposition 6.3. *Let $\varphi \in L^\infty(S^1)$. Then $\|T_\varphi\| \leq \|\varphi\|_\infty$ and $T_\varphi^* = T_{\overline{\varphi}}$.*

Proof. Compute:

$$\|T_\varphi\| = \|PM_\varphi\| \leq \|M_\varphi\| \leq \|\varphi\|_\infty .$$

For $f, g \in H^2(S^1)$,

$$\begin{aligned}\left\langle T_\varphi^* f, g\right\rangle &= \langle f, T_\varphi g\rangle = \langle f, PM_\varphi g\rangle = \langle Pf, M_\varphi g\rangle = \langle f, \varphi g\rangle \\ &= \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)\overline{\varphi\left(e^{i\theta}\right) g\left(e^{i\theta}\right)}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \overline{\varphi\left(e^{i\theta}\right)} f\left(e^{i\theta}\right)\overline{g\left(e^{i\theta}\right)}\, d\theta\end{aligned}$$

and so $\left\langle T_\varphi^* f, g\right\rangle = \langle \overline{\varphi} f, g\rangle = \langle T_{\overline{\varphi}} f, g\rangle$. ◻

Toeplitz operators have nice interpretations as matrices in terms of the orthonormal basis $\{z^n\}_{n=0}^\infty$. A *Toeplitz matrix* is a (possibly infinite) matrix with constant diagonals; i.e., a matrix of the form:

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \\ a_1 & a_0 & a_{-1} & a_{-2} & \ddots \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Proposition 6.4. *If $T : H^2(S^1) \to H^2(S^1)$ is a Toeplitz operator, then its matrix with respect to the orthonormal basis $\{z^n\}_{n=0}^\infty$ is a Toeplitz matrix.*

Proof. Let $n, m \geq 0$. Suppose that T is a Toeplitz operator with symbol φ. Note that

$$\langle Tz^n, z^m\rangle = \langle PM_\varphi z^n, z^m\rangle = \langle M_\varphi z^n, Pz^m\rangle = \langle M_\varphi z^n, z^m\rangle .$$

Since

$$\begin{aligned}\langle M_\varphi z^n, z^m\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(e^{i\theta}\right) e^{i(n-m)\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(e^{i\theta}\right) e^{i((n+1)-(m+1))\theta}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(e^{i\theta}\right) e^{i(n+1)\theta} e^{-i(m+1)\theta}\, d\theta \\ &= \left\langle M_\varphi z^{n+1}, z^{m+1}\right\rangle\end{aligned}$$

we have $\langle Tz^n, z^m\rangle = \left\langle M_\varphi z^{n+1}, z^{m+1}\right\rangle = \left\langle Tz^{n+1}, z^{m+1}\right\rangle$. Hence, the diagonals of the matrix representation of T are constant. ◻

A natural question to ask is the following: when is a Toeplitz operator a Fredholm operator? For Toeplitz operators with continuous symbol, we will give a partial answer. We begin with a lemma.

On $L^2(S^1)$, the map $\varphi \to M_\varphi$ is multiplicative in that $M_{\varphi\psi} = M_{\varphi\psi}$. That this is not the case for Toeplitz operators is one of the main reasons why the theory is more difficult. For Toeplitz operators with continuous symbol, we can say the following.

Lemma 6.5. *Let $\varphi \in C(S^1)$. Then $PM_\varphi - M_\varphi P$ is a compact operator on $L^2(S^1)$.*

Proof. We consider the case when $\varphi(z) = z$, and apply the operator to each element of the standard basis for $L^2(S^1)$. If $n \geq 0$, we have

$$(PM_z - M_zP)z^n = PM_zz^n - M_zPz^n = Pz^{n+1} - z^{n+1} = 0.$$

If $n < -1$,

$$(PM_z - M_zP)z^n = Pz^{n+1} - M_z0 = 0 - 0 = 0.$$

If $n = -1$,

$$(PM_z - M_zP)z^{-1} = PM_zz^{-1} - M_zPz^{-1} = Pz^0 = z^0.$$

Hence, the image of the set of polynomials under the operator $PM_z - M_zP$ has dimension 1. By the Stone-Weierstrauss theorem, the image of $L^2(S^1)$ has dimension 1. Thus, $PM_z - M_zP$ is finite rank, hence compact.

The next observation that we make is that

$$E = \{ \varphi \in C(S^1) \ : \ PM_\varphi - M_\varphi P \text{ is compact} \}$$

is a C^*-subalgebra of $C(S^1)$. To see this, consider

$$(PM_\varphi - M_\varphi P)^* = M_\varphi^* P^* - P^* M_\varphi^* = M_{\overline{\varphi}} P - PM_{\overline{\varphi}} = -(PM_{\overline{\varphi}} - M_{\overline{\varphi}} P).$$

So compactness of $PM_\varphi - M_\varphi P$ implies compactness of $PM_{\overline{\varphi}} - M_{\overline{\varphi}} P$. Hence, E is invariant under conjugation. That E is closed under addition and scalar multiplication is clear. Since $PM_\varphi - M_\varphi P$ depends linearly on φ, the map $\varphi \mapsto PM_\varphi - M_\varphi P$ is continuous and so E is norm-closed. It remains to verify that E is closed under multiplication. Let $\psi \in E$. Then

$$\begin{aligned} PM_{\varphi\psi} - M_{\varphi\psi} P &= PM_{\varphi\psi} - M_\varphi PM_\psi + M_\varphi PM_\psi - M_{\varphi\psi} P \\ &= (PM_\varphi - M_\varphi P)M_\psi + M_\varphi(PM_\psi - M_\psi P). \end{aligned}$$

Since $PM_\varphi - M_\varphi P$ and $PM_\psi - M_\psi P$ are compact and $\mathfrak{K}(L^2(S^1))$ is an ideal, E is closed under multiplication. Hence, E is a C^*-subalgebra of $C(S^1)$ containing z. By the Stone-Weierstrauss theorem, z generates $C(S^1)$, so $E = C(S^1)$. Therefore, $PM_\varphi - M_\varphi P$ is compact for all $\varphi \in C(S^1)$. □

Proposition 6.6. *If $\varphi \in C(S^1)$ is nowhere-zero, then T_φ is Fredholm.*

Proof. Suppose that $\varphi \in C(S^1)$ is nowhere-zero. We show that T_φ is invertible up to a compact operator. Since φ is nonzero on S^1, the function $1/\varphi$ is well-defined. Then

$$\begin{aligned} T_\varphi T_{1/\varphi} &= PM_\varphi PM_{1/\varphi} \\ &= PM_\varphi M_{1/\varphi} + PM_\varphi PM_{1/\varphi} - PM_\varphi M_{1/\varphi} \\ &= P + P(M_\varphi P - PM_\varphi)M_{1/\varphi}. \end{aligned}$$

Since $P = I$ on $H^2(S^1)$, and since $M_\varphi P - PM_\varphi$ is compact by Lemma 6.5,

$$T_\varphi T_{1/\varphi} = I + K_1$$

where K_1 is compact. A similar calculation gives $T_{1/\varphi} T_\varphi = I + K_2$ for some compact operator K_2. Hence, T_φ is invertible modulo compact operators, and is therefore Fredholm. □

On $L^2(S^1)$, the map $\varphi \to M_\varphi$ is multiplicative in that $M_{\varphi\psi} = M_{\varphi\psi}$. This is not the case for Toeplitz operators. However, generalizing the calculation in the proof above gives us the following fact.

Corollary 6.7. *Suppose that $\varphi, \psi \in C(S^1)$. Then $T_{\varphi\psi} = T_\varphi T_\psi + K$ for some compact operator K.*

Proof. Compute as above with ψ in place of $1/\varphi$. □

6.3 The Toeplitz Index Theorem

In this section, we give an elegant relationship between the Fredholm index of a Toeplitz operator and the winding number of its symbol. This connection between an analytic invariant (the index) with a topological invariant (the winding number) is suggestive of the power of index theory.

One of the basic facts of algebraic topology is the isomorphism $\pi_1(S^1) \cong \mathbb{Z}$. Similarly, $\pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$. Since $\pi_1(\mathbb{C}^\times)$ consists of homotopy classes of continuous functions $\varphi : S^1 \to \mathbb{C}^\times$, every such function has an associated integer n which is invariant under homotopy. This number is what we call the winding number of φ.

Definition 6.8. Let $\varphi : S^1 \to \mathbb{C}^\times$ be continuous. The **winding number** of φ, denoted $w(\varphi)$, is the image of the homotopy class of φ under the isomorphism $\pi_1(\mathbb{C}^\times) \to \mathbb{Z}$.

There are many equivalent definitions of the winding number; for example, from complex analysis, we have

$$w(\varphi) = \frac{1}{2\pi i} \oint_{\varphi(S^1)} \frac{1}{z}\, dz.$$

The main result of this chapter equates, up to a sign difference, the Fredholm index of a Toeplitz operator to the winding number of its symbol.

Theorem 6.9 (Toeplitz Index Theorem). *If $\varphi : S^1 \to \mathbb{C}^\times$ is continuous, then* $\operatorname{ind}(T_\varphi) = -w(\varphi)$.

Proof. First, we compute the index of T_z. Recall that $\{z^k\}_{k=0}^\infty$ is an orthonormal basis for $H^2(S^1)$. For each $k = 0, 1, \ldots$, we have

$$T_z(z^k) = z^{k+1}.$$

Since T_z is injective, $\dim\ker T_z = 0$. Since $\operatorname{coker} T = \text{Span}(z^0)$, $\dim\operatorname{coker} T = 1$. Hence, $\operatorname{ind}(T_z) = -1$.

Next, by Corollary 6.7, Corollary 3.15, and Proposition 3.13, for nonvanishing $\varphi, \psi \in C(S^1)$,

$$\operatorname{ind}(T_{\varphi\psi}) = \operatorname{ind}(T_\varphi T_\psi - K) = \operatorname{ind}(T_\varphi T_\psi) = \operatorname{ind}(T_\varphi) + \operatorname{ind}(T_\psi).$$

By induction on n, for $n \geq 1$,

$$\operatorname{ind}(T_{z^n}) = -n.$$

The fact that $\operatorname{ind}(T_{z^{-n}}) = n$ follows from $T_\varphi^* = T_{\bar\varphi}$.

Next, let φ be any nonvanishing continuous function. Then by the isomorphism $\pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$, φ is homotopic to z^n for some n. Hence, $w(\varphi) = n$. This homotopy gives a continuous path in $C(S^1)$ from φ to z^n. Composing this map with the map $C(S^1) \to \mathfrak{L}(H^2(S^1))$ which sends $\psi \mapsto T_\psi$ gives a path in $\mathfrak{F}(H^2(S^1))$ from T_φ to T_{z^n}. Since the Fredholm index is locally constant and the winding number is homotopy invariant, we then have

$$\operatorname{ind}(T_\varphi) = \operatorname{ind}(T_{z^n}) = -n = -w(z^n) = -w(\varphi)$$

as desired. □

6.4 Generalizations

We finish with a brief discussion on generalizations of the theory of Toeplitz operators developed in this chapter. As per the introduction, this section contains a natural progression of the family index theory from Chapter 5, and would have more details were time available. Unfortunately, we restrict ourselves to a few comments, mostly without proof.

Extensions of C^*-algebras.

Consider the following proposition.

Proposition 6.10. *Let $\pi : H^2(S^1) \to \mathfrak{C}(H^2(S^1))$ be the projection onto the Calkin algebra. The map $\varphi \to \pi(T_\varphi)$ is an injective $*$-homomorphism from $C(S^1)$ to $\mathfrak{C}(H^2(S^1))$.*

Proof. We have $T_\varphi^* = T_{\overline{\varphi}}$, and $T_{\varphi\psi} = T_\varphi T_\psi + K$ for some compact K. Hence, $\varphi \to \pi(T_\varphi)$ is a $*$-homomorphism into $\mathfrak{C}(H^2(S^1))$. It remains to check injectivity.

Observe the following about the spectrum of a Toeplitz operator with continuous symbol φ. For $f \in H^2(S^1)$, we have

$$\begin{aligned}(T_\varphi - \lambda I)f &= T_\varphi f - \lambda f = P(\varphi f) - P(\lambda f) \\ &= P((\varphi - \lambda)f) \\ &= T_{\varphi-\lambda} f.\end{aligned}$$

So if $T_{\varphi-\lambda}$ is not invertible for some λ, then $\lambda \in \sigma(T_\varphi)$. If $\lambda \in \operatorname{im}\varphi$, then $\varphi - \lambda = 0$ at some point in S^1, and $T_{\varphi-\lambda}$ will not have a bounded inverse. Hence, $\operatorname{im}\varphi \subseteq \sigma(T_\varphi)$.

Suppose that $\pi(T_\varphi) = 0$. Then T_φ is compact. But the spectrum of a compact operator is discrete, which means $\operatorname{im}\varphi \subseteq \sigma(T_\varphi)$ is discrete. Since φ is continuous and S^1 is connected, $\varphi(z) = a$ for some constant $a \in \mathbb{C}$. But if $a \neq 0$, then T_φ is a nonzero multiple of the identity and is not compact. Therefore, $\varphi = 0$, and injectivity follows. □

We can rephrase this proposition in the following way. Let $\mathfrak{T} \subseteq \mathfrak{L}(H^2(S^1))$ be the C^*-algebra generated by the Toeplitz operators with continuous symbols and the compact operators. Then $\mathfrak{K}(H^2(S^1))$ injects into $\mathfrak{T}$. By Proposition 6.10, $\pi(\mathfrak{T})$ is isomorphic to $C(S^1)$ as a subalgebra of $\mathfrak{C}(H^2(S^1))$. Hence, there is a short exact sequence of the form

$$0 \to \mathfrak{K}(H^2(S^1)) \to \mathfrak{T} \to C(S^1) \to 0.$$

This exact sequence is called the *Toeplitz extension*. It is of interest to study more general extensions of the form

$$0 \to \mathfrak{K}(\mathcal{H}) \to \mathcal{E} \to C(X) \to 0$$

where $\mathcal{H}$ is a Hilbert space, $X \subseteq \mathbb{C}$ is compact, and $\mathcal{E}$ is some C^*-algebra. We refer to [12] for further details.

Generalizing the Toeplitz Index Theorem.

We can generalize our index computation in Theorem 6.9 in a few ways. The results in this section we state without proof, and refer to [2] and [7] for details.

Consider the following vector-valued generalization of $L^2(S^1)$:

$$L^2(S^1;\mathbb{C}^n) := \left\{ f : S^1 \to \mathbb{C}^n \ : \ \int_{S^1} \|f(z)\|_{\mathbb{C}^n}^2 \, dz < \infty \right\}$$

with inner product given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{S^1} \langle f(z), g(z)\rangle_{\mathbb{C}^n} \, dz.$$

There is an evident isomorphism $L^2(S^1;\mathbb{C}^n) \cong L^2(S^1) \otimes \mathbb{C}^n$. Hence, we define

$$H^2(S^1;\mathbb{C}^n) := H^2(S^1) \otimes \mathbb{C}^n$$

and let $P_n : L^2(S^1;\mathbb{C}^n) \cong L^2(S^1) \otimes \mathbb{C}^n \to H^2(S^1;\mathbb{C}^n)$ be the orthogonal projection.

We replace the scalar-valued maps $S^1 \to \mathbb{C}^\times$ with maps $\varphi : S^1 \to GL(n,\mathbb{C})$. If φ is a such a map, then the multiplication operator $M_\varphi : L^2(S^1;\mathbb{C}^n) \to L^2(S^1;\mathbb{C}^n)$ given by $M_\varphi f(z) = \varphi(z)f(z)$ is a bounded operator. Define the Toeplitz operator $T_\varphi : H^2(S^1;\mathbb{C}^n) \to H^2(S^1;\mathbb{C}^n)$ by $T_\varphi := P_n M_\varphi$. The proofs from the previous sections generalize naturally, and we get the following theorem.

Theorem 6.11. *Let $\varphi : S^1 \to GL(n,\mathbb{C})$ be continuous. Then T_φ is a Fredholm operator on $H^2(S^1;\mathbb{C}^n)$, and $\operatorname{ind}(T_\varphi) = -w(\det\varphi)$.*

We can generalize further by considering *families* of Toeplitz operators. For example, suppose that X is a compact space. Let $\varphi : S^1 \times X \to GL(n,\mathbb{C})$ be continuous. Define a family of Fredholm operators $X \to \mathfrak{F}(H^2(S^1;\mathbb{C}^n))$ by $x \mapsto T_{\varphi_x}$. Then this family has a well-defined index $\operatorname{ind}(T_\varphi) \in K(X)$ per the construction in Chapter 5.

Even more generally, let $E \to X$ be a complex vector bundle of rank n. Let $p : S^1 \times X \to X$ be the projection. Then $p^*E \to S^1 \times X$ is a vector bundle. If $\varphi : p^*E \to p^*E$ is a vector bundle automorphism, then for every $x \in X$, $\varphi_x : S^1 \to GL(E_x)$ is a continuous map. In this way, we get a family of Fredholm operators $X \to \mathfrak{F}(H^2(S^1;E_x))$ given by $x \mapsto T_{\varphi_x}$. Though it may seem problematic that the Hilbert space $H^2(S^1;E_x)$ varies with x, it turns out that there is no issue. Take $\bigsqcup_{x\in X} H^2(S^1;E_x) \to X$ to be a vector bundle of Hilbert spaces over X. Using Kuiper's Theorem, it is possible to show that every vector bundle of Hilbert spaces is trivial, and so $\bigsqcup_{x\in X} H^2(S^1;E_x) \cong X \times \mathcal{H}$. Hence, we can view T_φ as a family $X \to \mathfrak{F}(\mathcal{H})$, and the family index $\operatorname{ind}(T_\varphi) \in K(X)$ is then well-defined.

Continuing down this path leads to the concept of *Bott periodicity*, which is one of the building blocks of K-theory. We end our discussion here, and refer to resources such as [5], [2], and [7] for more details on families of Toeplitz operators, Bott periodicity, and the relation between these concepts.

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