SECTION 2.1 was largely rewritten so that its focus is on motivating the need for limits via the concepts of velocity and the tangent line. The content on rate of change that did not treat velocity was moved elsewhere.

SECTION 4.1 was rewritten and reorganized to clarify the relationship between the different types of linear approximation. In particular, we wanted to reinforce the understanding that the various types of linear approximation are all based on the idea that the tangent line approximates the curve close to the point of tangency.

Chapter 1: Precalculus Review

1.1 Real Numbers, Functions, and Graphs
1.2 Linear and Quadratic Functions
1.3 The Basic Classes of Functions
1.4 Trigonometric Functions
1.5 Technology: Calculators and Computers

Chapter Review Exercises

Chapter 2: Limits

2.1 The Limit Idea: Instantaneous Velocity and Tangent Lines
2.2 Investigating Limits
2.3 Basic Limit Laws
2.4 Limits and Continuity
2.5 Indeterminate Forms
2.6 The Squeeze Theorem and Trigonometric Limits
2.7 Limits at Infinity
2.8 The Intermediate Value Theorem
2.9 The Formal Definition of a Limit

Chapter Review Exercises

Chapter 3: Differentiation

3.1 Definition of the Derivative
3.2 The Derivative as a Function

Chapter Review Exercises

Chapter 4: Applications of the Derivative

4.1 Linear Approximation and Applications
4.2 Extreme Values
4.3 The Mean Value Theorem and Monotonicity
4.4 The Second Derivative and Concavity
4.5 Analyzing and Sketching Graphs of Functions
4.6 Applied Optimization
4.7 Newton's Method

Chapter Review Exercises

Chapter 5: Integration

5.1 Approximating and Computing Area
5.2 The Definite Integral
5.3 The Indefinite Integral
5.4 The Fundamental Theorem of Calculus, Part I

Chapter Review Exercises

Chapter 6: Applications of the Integral

6.1 Area Between Two Curves
6.2 Setting Up Integrals: Volume, Density, Average Value
6.3 Volumes of Revolution: Disks and Washers
6.4 Volumes of Revolution: Cylindrical Shells
6.5 Work and Energy

Chapter Review Exercises

Chapter 7: Exponential and Logarithmic Functions

7.1 The Derivative of $f(x) = b^x$ and the Number e
7.2 Inverse Functions
7.3 Logarithmic Functions and Their Derivatives
7.4 Applications of Exponential and Logarithmic Functions
7.5 l'Hôpital's Rule
7.6 Inverse Trigonometric Functions
7.7 Hyperbolic Functions

Chapter Review Exercises

SECTION 7.4 is new and contains some applications from the former Sections 7.4 and 7.5, along with applications that are new to this edition. Some material from the former Sections 7.4 and 7.5 has been moved elsewhere. For example, the material on differential equations and on exponential growth and decay was moved to the chapter on differential equations.

Scanned with CamScanner
SINGLE VARIABLE CALCULUS

Contents

Chapter 8: Techniques of Integration 407
8.1 Integration by Parts 407
8.2 Trigonometric Integrals 413
8.3 Trigonometric Substitution 421
8.4 Integrals Involving Hyperbolic and Inverse Hyperbolic Functions 427
8.5 The Method of Partial Fractions 432
8.6 Strategies for Integration 441
8.7 Improper Integrals 448
8.8 Numerical Integration 459
Chapter Review Exercises 469

Chapter 9: Further Applications of the Integral 473
9.1 Probability and Integration 473
9.2 Arc Length and Surface Area 479
9.3 Fluid Pressure and Force 486
9.4 Center of Mass 492
Chapter Review Exercises 502

Chapter 10: Introduction to Differential Equations 505
10.1 Solving Differential Equations 505
10.2 Models Involving \( y' = k(y - b) \) 515
10.3 Graphical and Numerical Methods 522
10.4 The Logistic Equation 529
10.5 First-Order Linear Equations 534
Chapter Review Exercises 540

Chapter 11: Infinite Series 543
11.1 Sequences 543
11.2 Summing an Infinite Series 554
11.3 Convergence of Series with Positive Terms 566
11.4 Absolute and Conditional Convergence 575
11.5 The Ratio and Root Tests and Strategies for Choosing Tests 581
11.6 Power Series 586
11.7 Taylor Polynomials 598
11.8 Taylor Series 609
Chapter Review Exercises 621

Appendices A1
A. The Language of Mathematics A1
B. Properties of Real Numbers A7
C. Induction and the Binomial Theorem A12
D. Additional Proofs A16

Answers to Odd-Numbered Exercises A1
References R1
Index 11

Additional content can be accessed online at www.macmillanlearning.com/calculus4e:

Additional Proofs:
L'Hopital's Rule
Error Bounds for Numerical Integration
Comparison Test for Improper Integrals

Additional Content:
Second-Order Differential Equations
Complex Numbers

Section 8.1 The section on probability was moved from Section 8.8 to 8.1 so that it appears in the chapter on applications of the integral rather than the chapter on techniques of integration.

Section 10.1 was rewritten to provide a more straightforward introduction to differential equations and methods of solving them. Furthermore we wrote a few new examples that replaced a rather technical derivation to provide for a wider variety of simpler, more accessible application examples.

Section 11.1 We have chosen a somewhat traditional location for the section on Taylor polynomials, placing it directly before the section on Taylor series in Chapter 11. We feel that this placement is an improvement over the previous edition where the section was isolated in a chapter that primarily was about applications of the integral. The subject matter in the Taylor polynomials section works well as an initial step toward the important topic of Taylor series representations of specific functions. The Taylor polynomials section can serve as a follow-up to linear approximation in Section 4.1. Consequently, Taylor polynomials (except for Taylor's Theorem at the end of the section, which involves integration) can be covered at any point after Section 4.1.
INTRODUCTION TO CALCULUS

We begin with a brief introduction to some key ideas in calculus. It is not an exaggeration to say that calculus is one of the great intellectual achievements of humankind. Sending spacecraft to other planets, building computer systems for forecasting the weather, explaining the interactions between plants, insects, and animals, and understanding the structure of atoms are some of the countless scientific and technological advances that could not have been achieved without calculus. Moreover, calculus is a foundational part of the mathematical theory of analysis, a field that is under continuous development.

The primary formulation of calculus dates back to independent theories of Sir Isaac Newton and Gottfried Wilhelm Leibniz in the 1600s. However, their work only remotely resembles the topics presented in this book. Through a few centuries of development and expansion, calculus has grown into the theory we present here. Newton and Leibniz would likely be quite impressed that their calculus has evolved into a theory that many thousands of students around the world study each year.

There are two central concepts in calculus: the derivative and the integral. We introduce them next.

**The Derivative** The derivative of a function is simply the slope of its graph; it represents the rate of change of the function. For a linear function $y = 2.3x - 8.1$, the slope 2.3 indicates that $y$ changes by 2.3 for each one-unit change in $x$. How do we find the slope of a graph of a function that is not linear, such as the one in Figure 1?

Imagine that this function represents the amount $A$ of a drug in the bloodstream as a function of time $t$. Clearly, this situation is more complex than the linear case. The slope varies as we move along the curve. Initially positive because the amount of the drug in the bloodstream is increasing, the slope becomes negative as the drug is absorbed. Having an expression for the slope would enable us to know the time when the amount of the drug is a maximum (when the slope turns from positive to negative) or the time when the drug is leaving the bloodstream the fastest (a time to administer another dose).

To define the slope for a function that is not linear, we adapt the notion of slope for linear relationships. Specifically, to estimate the slope at point $P$ in Figure 2, we select a point $Q$ on the curve and draw a line between $P$ and $Q$. We can use the slope of this line to approximate the slope at $P$. To improve this approximation, we move $Q$ closer to $P$ and calculate the slope of the new line. As $Q$ moves closer to $P$, this approximation gets more precise. Although we cannot allow $P$ and $Q$ to be the same point (because we could no longer compute a slope), we instead “take the limit” of these slopes. We develop the concept of the limit in Chapter 2. Then in Chapter 3, we show that the limiting value may be defined as the exact slope at $P$.

**The Definite Integral** The definite integral, another key calculus topic, can be thought of as adding up infinitely many infinitesimally small pieces of a whole. It too is obtained through a limiting process. More precisely, it is a limit of sums over a domain that is divided into progressively more and more pieces. To explore this idea, consider a solid
ball of volume 2 cm$^3$ whose density (mass per unit volume) throughout is 1.5 g/cm$^3$. The mass of this ball is the product of density and volume, $(1.5)(2) = 3$ grams.

If the density is not the same throughout the ball (Figure 3), we can approximate its mass as follows:

- Chop the ball into a number of pieces,
- Assume the density is uniform on each piece and approximate the mass of each piece by multiplying density by volume,
- Add the approximate masses of the pieces to estimate the total mass of the ball.

We continually improve this approximation by chopping the ball into ever smaller pieces (Figure 4). Ultimately, an exact value is obtained by taking a limit of the approximate masses.

The irregular density of the moon presented a navigational challenge for spacecraft orbiting it. The first group of spacecraft (unmanned!) that circled the moon exhibited unexpected orbits. Space scientists realized that the density of the moon varied considerably and that the gravitational attraction of concentrations of mass (referred to as mascons) deflected the path of the spacecraft from the planned trajectory.

In Chapter 5, we define the definite integral in exactly this way; it is a limit of sums over an interval that is divided into progressively smaller subintervals.

**The Fundamental Theorem of Calculus** Although the derivative and the definite integral are very different concepts, it turns out they are related through an important theorem called the Fundamental Theorem of Calculus presented in Chapter 5. This theorem demonstrates that the derivative and the definite integral are, to some extent, inverses of each other, a relationship that we will find beneficial in many ways.
1 PRECALCULUS REVIEW

Calculus builds on the foundation of algebra, analytic geometry, and trigonometry. In this chapter, therefore, we review some concepts, facts, and formulas from precalculus that are used throughout the text. In the last section, we discuss ways in which technology can be used to enhance your visual understanding of functions and their properties.

1.1 Real Numbers, Functions, and Graphs

We begin with a short discussion of real numbers. This gives us the opportunity to recall some basic properties and standard notation.

A real number is a number represented by a decimal or “decimal expansion.” There are three types of decimal expansions: finite, infinite repeating, and infinite but non-repeating. For example,

\[
\frac{3}{8} = 0.375, \quad \frac{1}{7} = 0.142857142857 \ldots = 0.\overline{142857} \quad \pi = 3.141592653589793 \ldots
\]

The number \( \frac{3}{8} \) is represented by a finite decimal, whereas \( \frac{1}{7} \) is represented by an infinite repeating decimal. The bar over 142857 indicates that this sequence repeats indefinitely. The decimal expansion of \( \pi \) is infinite but nonrepeating.

The set of all real numbers is denoted by a boldface \( \mathbb{R} \). When there is no risk of confusion, we refer to a real number simply as a number. We also use the standard symbol \( \in \) for the phrase “belongs to.” Thus,

\[ a \in \mathbb{R} \quad \text{reads} \quad “a \text{ belongs to } \mathbb{R}” \]

The set of integers is commonly denoted by the letter \( \mathbb{Z} \) (this choice comes from the German word Zahl, meaning “number”). Thus, \( \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \). A whole number is a nonnegative integer—that is, one of the numbers 0, 1, 2, \ldots.

A real number is called rational if it can be represented by a fraction \( \frac{p}{q} \), where \( p \) and \( q \) are integers with \( q \neq 0 \). The set of rational numbers is denoted \( \mathbb{Q} \) (for “quotient”). Numbers that are not rational, such as \( \pi \) and \( \sqrt{2} \), are called irrational.

We can tell whether a number is rational from its decimal expansion: Rational numbers have finite or infinite repeating decimal expansions, and irrational numbers have infinite, nonrepeating decimal expansions. Furthermore, the decimal expansion of a number is unique, apart from the following exception: Every finite decimal is equal to an infinite decimal in which the digit 9 repeats. For example, \( 1/5 = 0.5 = 0.499999 \ldots \)

Two algebraic properties of the real numbers are the commutative property of addition, \( a + b = b + a \), and the distributive property of multiplication over addition, \( a(b + c) = ab + ac \). A list of further properties can be found in Appendix B. Next, we present some properties of exponents that are used regularly when we work with exponential expressions and functions.

<table>
<thead>
<tr>
<th>Exponent zero</th>
<th>Rule</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Products</td>
<td>( b^0 = 1 )</td>
<td>( 5^0 = 1 )</td>
</tr>
<tr>
<td>Quotients</td>
<td>( \frac{b^x}{b^y} = b^{x-y} )</td>
<td>( \frac{4^7}{4^5} = 4^{7-5} = 4^2 )</td>
</tr>
<tr>
<td>Negative exponents</td>
<td>( b^{-x} = \frac{1}{b^x} )</td>
<td>( 3^{-4} = \frac{1}{3^4} )</td>
</tr>
<tr>
<td>Power to a power</td>
<td>( (b^x)^y = b^{xy} )</td>
<td>( (3^2)^4 = 3^{2(4)} = 3^8 )</td>
</tr>
<tr>
<td>Roots</td>
<td>( \sqrt[b]{a} = a^{\frac{1}{b}} )</td>
<td>( 5^{1/2} = \sqrt{5} )</td>
</tr>
</tbody>
</table>
EXAMPLE 1  Rewrite as a whole number or fraction:

(a) $16^{-1/2}$  (b) $27^{2/3}$  (c) $4^{16} \cdot 4^{-18}$  (d) $9^{3/3}$

Solution

(a) $16^{-1/2} = \frac{1}{\sqrt{16}} = \frac{1}{4}$  
(b) $27^{2/3} = (27^{1/3})^2 = 3^2 = 9$

(c) $4^{16} \cdot 4^{-18} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$  
(d) $9^{3/3} = \frac{(3^2)^3}{3^3} = \frac{3^6}{3^3} = 3^3 = 27 = 3^{-1} = \frac{1}{3}$

Another important algebraic relationship is the binomial expansion of $(a + b)^n$. It is proved in Appendix C and is needed in the proof of the power law for derivatives in Section 3.2.

Expanding $(a + b)^n$ for $n = 2, 3, 4$, we obtain

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Notice there are some patterns emerging here. In each case, the first and second terms are $a^n$ and $na^{n-1}b$, while the last two terms are $nab^{n-1}$ and $b^n$. There is a general formula for the expansion, called the binomial expansion formula. It is expressed using summation notation as

$$(a + b)^n = \sum_{p=0}^{n} \binom{n}{p} a^{n-p} b^p$$

We introduce summation notation in Section 5.1. For now, you can understand the formula as saying that $(a + b)^n$ is a sum of terms $\frac{n!}{(n-p)!p!} a^{n-p} b^p$, with a term for each $p$ going from 0 to $n$. So, for example, in $(a + b)^3$, the first four terms are: $\frac{3!}{0!3!} a^3 = a^3$, $\frac{3!}{1!2!} a^2b = 3a^2b$, $\frac{3!}{2!1!} a^2b^2 = 3a^2b^2$, and $\frac{3!}{3!0!} a^0b^3 = 3a^0b^3$. Working out the rest of the terms, we find that:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

We visualize real numbers as points on a line (Figure 1), and we refer to that line as the real line. For this reason, real numbers are often called points. The point corresponding to 0 is called the origin.

The real numbers are ordered, and we can view that ordering in terms of position on the real line: $p$ is greater than $q$, written $p > q$, if $p$ is to the right of $q$ on the number line. $p$ is less than $q$, written $p < q$, if $p$ is to the left of $q$ on the number line.

A real number $x$ is said to be positive if $x > 0$, negative if $x < 0$, nonpositive if $x \leq 0$, and nonnegative if $x \geq 0$.

Two other important terms we use, related to position on the real line, are “large” and “small.” We say that $p$ is large if $p$ is distant from the origin, and $p$ is small if $p$ is close to the origin. While these definitions are somewhat vague, the meaning should be clear in the contexts in which they are used.

The absolute value of a real number $a$, denoted $|a|$, is defined by (Figure 2):

$$|a| = \begin{cases} 
    a & \text{if } a \geq 0 \\
    -a & \text{if } a < 0 
\end{cases}$$

![Figure 1](image1)

**Figure 1** The set of real numbers represented as a line.

In some texts, “larger than” is used synonymously with “greater than.” We will avoid that usage in this text.

![Figure 2](image2)

**Figure 2** $|a|$ is the distance from $a$ to the origin.
For example, \(|1.2| = 1.2\) and \(|-8.35| = -(-8.35) = 8.35\). The absolute value satisfies:

\[
|a| = |-a|, \quad |ab| = |a||b|
\]

The distance between two real numbers \(a\) and \(b\) is \(|b - a|\), which is the length of the line segment joining \(a\) and \(b\) (Figure 3).

Two real numbers \(a\) and \(b\) are close to each other if \(|b - a|\) is small, and this is the case if their decimal expansions agree to many places. More precisely, if the decimal expansions of \(a\) and \(b\) agree to \(k\) places (to the right of the decimal point), then the distance \(|b - a|\) is at most \(10^{-k}\). Thus, the distance between \(a = 3.1415\) and \(b = 3.1478\) is at most \(10^{-3}\) because \(a\) and \(b\) agree to two places. In fact, the distance is exactly \(|3.1478 - 3.1415| = 0.0063\).

Beware that \(|a + b|\) is not equal to \(|a| + |b|\) unless \(a\) and \(b\) have the same sign or at least one of \(a\) and \(b\) is zero. If they have opposite signs, cancellation occurs in the sum \(a + b\), and \(|a + b| < |a| + |b|\). For example, \(|2 + 5| = |2| + |5|\) but \(|-2 + 5| = 3\), which is less than \(|-2| + |5| = 7\). In any case, \(|a + b|\) is never greater than \(|a| + |b|\), and this gives us the simple but important triangle inequality:

\[
|a + b| \leq |a| + |b|
\]

Figure 4 shows the four intervals with endpoints \(a\) and \(b\). They all have length \(b - a\) but differ according to which endpoints are included.

The closed interval \([a, b]\) is the set of all real numbers \(x\) such that \(a \leq x \leq b\):

\[ [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \]

We usually write this more simply as \(\{x : a \leq x \leq b\}\), it being understood that \(x\) belongs to \(\mathbb{R}\). The open and half-open intervals are the sets:

\[
(a, b) = \{ x : a < x < b \} \quad , \quad [a, b) = \{ x : a \leq x < b \} \quad , \quad (a, b] = \{ x : a < x \leq b \}
\]

The infinite interval \((-\infty, \infty)\) is the entire real line \(\mathbb{R}\). A half-infinite interval is closed if it contains its finite endpoint and is open otherwise (Figure 5):

\[
[a, \infty) = \{ x : x \geq a \} \quad , \quad (-\infty, b] = \{ x : x \leq b \}
\]

Figure 6 shows the interval \((-r, r) = \{ x : |x| < r \}\).
The symbol $\leftrightarrow$ is read as "is equivalent to," and the symbol $\Rightarrow$, that we will also use, is read as "implies."

More generally, for an interval symmetric about the value $c$ (Figure 7):

$$|x - c| < r \iff c - r < x < c + r \iff x \in (c - r, c + r)$$

Closed intervals can be represented similarly, with $<$ replaced by $\leq$. We refer to $r$ as the radius and $c$ as the midpoint or center of the intervals $(c - r, c + r)$ and $[c - r, c + r]$. The intervals $(a, b)$ and $[a, b]$ have midpoint $c = \frac{1}{2}(a + b)$ and radius $r = \frac{1}{2}(b - a)$ (Figure 7).

**EXAMPLE 2** Describe $[7, 13]$ using an absolute-value inequality.

**Solution** The midpoint of the interval $[7, 13]$ is $c = \frac{1}{2}(7 + 13) = 10$, and its radius is $r = \frac{1}{2}(13 - 7) = 3$ (Figure 8). Therefore,

$$[7, 13] = \{x \in \mathbb{R} : |x - 10| \leq 3\}$$

**EXAMPLE 3** Describe the set $S = \{x : \frac{1}{2}x - 3 > 4\}$ in terms of intervals.

**Solution** It is easier to consider the opposite inequality $\frac{1}{2}x - 3 \leq 4$ first. By (2):

$$\frac{1}{2}x - 3 \leq 4 \iff -4 \leq \frac{1}{2}x - 3 \leq 4$$

$$-1 \leq \frac{1}{2}x \leq 7 \quad \text{(add 3)}$$

$$-2 \leq x \leq 14 \quad \text{(multiply by 2)}$$

Thus, $\frac{1}{2}x - 3 \leq 4$ is satisfied when $x$ belongs to $[-2, 14]$. The set $S$ is the complement, consisting of all numbers $x$ not in $[-2, 14]$. We can describe $S$ as the union of two intervals: $S = (-\infty, -2) \cup (14, \infty)$ (Figure 9).

**Graphing**

Graphing is a basic tool in calculus, as it is in algebra and trigonometry. Recall that rectangular (or Cartesian) coordinates in the plane are defined by choosing two perpendicular axes, the $x$-axis and the $y$-axis. To a pair of numbers $(a, b)$ we associate the point $P$ located at the intersection of the line perpendicular to the $x$-axis at $a$ and the line perpendicular to the $y$-axis at $b$ (Figure 10(A)). The numbers $a$ and $b$ are the $x$- and $y$-coordinates of $P$. The $x$-coordinate is sometimes called the abscissa and the $y$-coordinate the ordinate. The origin is the point with coordinates $(0, 0)$.

**Figure 10**: The rectangular coordinate system (A) and the four quadrants (B).
The axes divide the plane into four quadrants labeled I–IV, determined by the signs of the coordinates [Figure 10(B)]. For example, quadrant III consists of points \((x, y)\) such that \(x < 0\) and \(y < 0\).

The distance \(d\) between two points \(P_1 = (x_1, y_1)\) and \(P_2 = (x_2, y_2)\) is computed using the Pythagorean Theorem. In Figure 11, we see that \(P_1 P_2\) is the hypotenuse of a right triangle with sides \(a = |x_2 - x_1|\) and \(b = |y_2 - y_1|\). Therefore,

\[
d^2 = a^2 + b^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2
\]

We obtain the distance formula by taking square roots.

**Distance Formula** The distance between \(P_1 = (x_1, y_1)\) and \(P_2 = (x_2, y_2)\) is

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

Once we have the distance formula, we can derive the equation of a circle of radius \(r\) and center \((a, b)\) (Figure 12). A point \((x, y)\) lies on this circle if the distance from \((x, y)\) to \((a, b)\) is \(r\):

\[
\sqrt{(x - a)^2 + (y - b)^2} = r
\]

Squaring both sides, we obtain the standard equation of the circle of radius \(r\) centered at \((a, b)\):

\[
(x - a)^2 + (y - b)^2 = r^2
\]

We now review some definitions and notation concerning functions.

**Definition** A function \(f\) from a set \(D\) to a set \(Y\) is a rule that assigns, to each element \(x\) in \(D\), a unique element \(y = f(x)\) in \(Y\). We write

\[
f : D \rightarrow Y
\]

The set \(D\), called the domain of \(f\), is the set of “allowable inputs.” For \(x \in D\), \(f(x)\) is called the value of \(f\) at \(x\) (Figure 13). The range \(R\) of \(f\) is the subset of \(Y\) consisting of all values \(f(x)\):

\[
R = \{y \in Y : f(x) = y \text{ for some } x \in D\}
\]

Informally, we think of \(f\) as a “machine” that produces an output \(y\) for every input \(x\) in the domain \(D\) (Figure 14).

**Figure 13** A function assigns an element \(y = f(x)\) to each \(x \in D\).

**Figure 14** Think of \(f\) as a “machine” that takes the input \(x\) and produces the output \(f(x)\).

Writing \(y = f(x)\) for a function \(f\), we refer to \(x\) as the independent variable and \(y\) as the dependent variable (because its value depends on the choice of \(x\)).
The first part of this text deals with functions $f$, where both the domain and the range are sets of real numbers. When $f$ is defined by a formula, its natural domain is the set of real numbers $x$ for which the formula is meaningful. For example, the function $f(x) = \sqrt{9 - x}$ has domain $D = \{x : x \leq 9\}$ because $\sqrt{9 - x}$ is defined if $9 - x \geq 0$. Here are some other examples of domains and ranges:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>Domain $D$</th>
<th>Range $R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2$</td>
<td>$\mathbb{R}$</td>
<td>${y : y \geq 0}$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\mathbb{R}$</td>
<td>${y : -1 \leq y \leq 1}$</td>
</tr>
<tr>
<td>$\frac{1}{x+1}$</td>
<td>${x : x \neq -1}$</td>
<td>${y : y \neq 0}$</td>
</tr>
</tbody>
</table>

The graph of a function $y = f(x)$ is obtained by plotting the points $(a, f(a))$ for $a$ in the domain $D$ (Figure 15). If you start at $x = a$ on the $x$-axis, and move up to the graph and then over to the $y$-axis, you arrive at the value $f(a)$.

A zero or root of a function $f$ is a number $c$ such that $f(c) = 0$. The zeros are the values of $x$ where the graph intersects the $x$-axis.

In Chapter 4, we will use calculus to sketch and analyze graphs. At this stage, to sketch a graph by hand, we can make a table of function values, plot the corresponding points (including any zeros), and connect them by a smooth curve.

**EXAMPLE 4** Find the roots and sketch the graph of $f(x) = x^3 - 2x$.

**Solution** First, we solve

$$x^3 - 2x = x(x^2 - 2) = 0$$

The roots of $f$ are $x = 0$ and $x = \pm \sqrt{2}$. To sketch the graph, we plot the roots and a few values listed in Table 1 and join them by a curve (Figure 16).

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^3 - 2x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Functions arising in applications are not always given by formulas. Data collected from observation or experiment define functions for which there may be no exact formula. Such functions can be displayed either graphically or by a table of values. For example, consider the mass of the Greenland ice sheet (Figure 17) that covers most of the island of Greenland. Data in Table 2 and Figure 18 collected by NASA’s GRACE (Global Recovery and Climate Experiment) satellite show the change in the mass of the ice, $C$, as a function of time, $t$, since the beginning of 2012. (Note, for example, $t = 1.46$ means 0.46 years into 2013.) To plot this function, we plot the data points in the table and connect the points with a smooth curve. We will see that many of the tools of calculus can be applied to functions constructed from data in this way.
## TABLE 2

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2.79</td>
<td>−794.51</td>
</tr>
<tr>
<td>0.21</td>
<td>138.53</td>
<td>3.12</td>
<td>−623.4</td>
</tr>
<tr>
<td>0.54</td>
<td>−139.14</td>
<td>3.32</td>
<td>−624.41</td>
</tr>
<tr>
<td>0.89</td>
<td>−487.05</td>
<td>3.70</td>
<td>−960.08</td>
</tr>
<tr>
<td>1.12</td>
<td>−386.78</td>
<td>4.12</td>
<td>−899.86</td>
</tr>
<tr>
<td>1.46</td>
<td>−355.26</td>
<td>4.46</td>
<td>−869.46</td>
</tr>
<tr>
<td>1.87</td>
<td>−518.52</td>
<td>4.91</td>
<td>−1153.08</td>
</tr>
<tr>
<td>2.21</td>
<td>−475.14</td>
<td>5.25</td>
<td>−1110.29</td>
</tr>
<tr>
<td>2.45</td>
<td>−474.96</td>
<td>5.44</td>
<td>−1115.94</td>
</tr>
</tbody>
</table>

We can graph not just functions but, more generally, any equation relating $y$ and $x$. Figure 19 shows the graph of the equation $4y^2 - x^3 = 3$; it consists of all pairs $(x, y)$ satisfying the equation. This curve is not the graph of a function of $x$ because some $x$-values are associated with two $y$-values. For example, $x = 1$ is associated with both $y = 1$ and $y = -1$. A curve is the graph of a function of $x$ if and only if it passes the Vertical Line Test; that is, every vertical line $x = a$ intersects the curve in at most one point.

We are often interested in whether a function is increasing or decreasing. Roughly speaking, a function $f$ is increasing if its graph goes up as we move to the right and is decreasing if its graph goes down [Figures 20(A) and (B)]. More precisely, we define the notion of increase/decrease on an open interval.

A function $f$ is
- **Increasing** on $(a, b)$ if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$
- **Decreasing** on $(a, b)$ if $f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$

We say that $f$ is **monotonic** if it is either increasing or decreasing. In Figure 20(C), the function is not monotonic because, while it is increasing for some intervals of $x$ and decreasing for others, it is neither increasing nor decreasing for all $x$.

A function $f$ is called **nondecreasing** if $f(x_1) \leq f(x_2)$ for $x_1 < x_2$ (defined by $\leq$ rather than a strict inequality $<$). **Nonincreasing** functions are defined similarly. Function (D) in Figure 20 is nondecreasing, but it is not increasing on the intervals where the graph is horizontal. Function (E) is increasing everywhere, even though it levels off momentarily.
Another important property of functions is **parity**, which refers to whether a function is even or odd:

- **Even** if \( f(-x) = f(x) \).
- **Odd** if \( f(-x) = -f(x) \).

The graphs of functions with even or odd parity have a special symmetry:

- **Even function**: The graph is symmetric about the y-axis. This means that if \( P = (a, b) \) lies on the graph, then so does \( Q = (-a, b) \) [Figure 21(A)].
- **Odd function**: The graph is symmetric with respect to the origin. This means that if \( P = (a, b) \) lies on the graph, then so does \( Q = (-a, -b) \) [Figure 21(B)].

Many functions are neither even nor odd [Figure 21(C)].

\[
\begin{align*}
(A) & \quad \text{Even function: } f(-x) = f(x) \\
& \quad \text{Graph is symmetric about the y-axis.} \\
(B) & \quad \text{Odd function: } f(-x) = -f(x) \\
& \quad \text{Graph is symmetric about the origin.} \\
(C) & \quad \text{Neither even nor odd}
\end{align*}
\]

**EXAMPLE 5** Determine whether the function is even, odd, or neither.

(a) \( f(x) = x^4 \)  
(b) \( g(x) = x^{-1} \)  
(c) \( h(x) = x^2 + x \)

**Solution**

(a) \( f(-x) = (-x)^4 = x^4 \). Thus, \( f(x) = f(-x) \), and \( f \) is even.

(b) \( g(-x) = (-x)^{-1} = -x^{-1} \). Thus, \( g(x) = -g(x) \), and \( g \) is odd.

(c) \( h(-x) = (-x)^2 + (-x) = x^2 - x \). We see that \( h(-x) \) is not equal to \( h(x) \) or to \(-h(x) = -x^2 - x \). Therefore, \( h \) is neither even nor odd.

**EXAMPLE 6 Using Symmetry** Sketch the graph of \( f(x) = \frac{1}{x^2 + 1} \).

**Solution** The function \( f \) is positive [\( f(x) > 0 \)] and even [\( f(-x) = f(x) \)]. Therefore, the graph lies above the x-axis and is symmetric with respect to the y-axis.

Furthermore, \( f \) is decreasing for \( x \geq 0 \) (because a larger value of \( x \) makes the denominator larger and therefore the fraction smaller). We use this information and a short table of values (Table 3) to sketch the graph (Figure 22). Note that the graph approaches the x-axis as we move away from zero, both to the right and to the left, because \( f(x) \) gets closer to zero as \( |x| \) increases.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{1}{x^2 + 1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \pm 1 )</td>
<td>2</td>
</tr>
<tr>
<td>( \pm 2 )</td>
<td>5/5</td>
</tr>
</tbody>
</table>
Two important ways of modifying a graph are translation (or shifting) and scaling. Translation consists of moving the graph horizontally or vertically:

**DEFINITION Translation (Shifting)**

- **Vertical Translation** $y = f(x) + c$: Shifts the graph of $f$ by $|c|$ units vertically, upward if $c > 0$ and downward if $c < 0$.
- **Horizontal Translation** $y = f(x + c)$: Shifts the graph of $f$ by $|c|$ units horizontally, to the right if $c < 0$ and to the left if $c > 0$.

Figure 23 shows the effect of translating the graph of $f(x) = 1/(x^2 + 1)$ vertically and horizontally.

![Figure 23](image)

(A) $y = f(x) = \frac{1}{x^2 + 1}$
(B) $y = f(x) + 1 = \frac{1}{(x+1)^2 + 1}$
(C) $y = f(x + 1) = \frac{1}{x^2 + 1}$

**EXAMPLE 7** Figure 24(A) is the graph of $f(x) = x^2$, and Figure 24(B) is a horizontal and vertical shift of (A). What is the equation of graph (B)?

![Figure 24](image)

(A) $f(x) = x^2$
(B) $y = f(x - 1)^2 - 1$

**Solution** Graph (B) is obtained by shifting graph (A) 1 unit to the right and 1 unit down. We can see this by observing that the point $(0, 0)$ on the graph of $f$ is shifted to $(1, -1)$. Therefore, (B) is the graph of $g(x) = (x - 1)^2 - 1$.

**Scaling** (also called dilation) consists of compressing or expanding the graph in the vertical or horizontal directions:

**DEFINITION Scaling**

- **Vertical scaling** $y = kf(x)$: If $|k| > 1$, the graph of $f$ is expanded vertically by the factor $|k|$. If $0 < |k| < 1$, the graph of $f$ is compressed vertically by the factor $|k|$. If $k < 0$, then the graph is also reflected across the $x$-axis (Figure 25).
- **Horizontal scaling** $y = f(kx)$: If $|k| > 1$, the graph of $f$ is compressed horizontally by the factor $|k|$. If $0 < |k| < 1$, the graph of $f$ is expanded horizontally by the factor $|k|$. If $k < 0$, then the graph is also reflected across the $y$-axis.

![Figure 25](image)

Negative vertical scale factor $k = -2$. 
EXAMPLE 8 Sketch the graphs of \( f(x) = \sin(\pi x) \) and its dilates \( f(3x) \) and \( 3f(x) \).

Solution The graph of \( f(x) = \sin(\pi x) \) is a sine curve with period 2. It completes one cycle over every interval of length 2—see Figure 26(A).

- The graph of \( f(3x) = \sin(3\pi x) \) is a compressed version of \( y = f(x) \), completing three cycles instead of one over intervals of length 2 [Figure 26(B)].
- The graph of \( y = 3f(x) = 3\sin(\pi x) \) is obtained from \( y = f(x) \) by expanding in the vertical direction by a factor of 3 [Figure 26(C)].

![Figure 26: Horizontal and vertical scaling of \( f(x) = \sin(\pi x) \).](image)

(A) \( y = f(x) = \sin(\pi x) \)  
(B) Horizontal compression: \( y = f(3x) = \sin(3\pi x) \)  
(C) Vertical expansion: \( y = 3f(x) = 3\sin(\pi x) \)

Mathematical Models

A mathematical model is a representation of a real-world phenomenon using mathematical concepts. Functions are often used as models; they provide a simple way to express a relationship between variables associated with a real-world situation. We will introduce many mathematical models in this book. Using the tools of calculus we will study models and draw conclusions about the situations they describe.

Modeling is the process of developing a mathematical model. The process usually involves making simplifying assumptions about a system in order to develop a mathematical representation that lends itself well to analysis. When such assumptions are made, the conclusions drawn from the model only approximate the real-world system. Ideally such an approximation is accurate enough to make useful predictions.

We will address different important aspects of the modeling process at various points in the book.

1.1 SUMMARY

- Important exponent laws:
  
  (i) \( b^x b^y = b^{x+y} \)  
  (ii) \( \frac{b^x}{b^y} = b^{x-y} \)  
  (iii) \( b^{-x} = \frac{1}{b^x} \)  
  (iv) \( (b^x)^y = b^{xy} \)

- Binomial expansion formula: \((a + b)^n\) is a sum of terms \( \frac{n!}{(a-p)!p!} a^{n-p} b^p \), with a term for each \( p \) going from 0 to \( n \).

- Absolute value: \(|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \)

- Triangle inequality: \(|a + b| \leq |a| + |b|\)

- Four intervals with endpoints \( a \) and \( b \):
  
  \[(a, b), \quad [a, b), \quad [a, b), \quad (a, b] \]

- Writing open and closed intervals using absolute-value inequalities:
  
  \((a, b) = \{x : |x - c| < r\}, \quad [a, b) = \{x : |x - c| \leq r\}\)

  where \( c = \frac{1}{2}(a + b) \) is the midpoint and \( r = \frac{1}{2}(b - a) \) is the radius.
SECTION 1.1 Real Numbers, Functions, and Graphs

- Distance \( d \) between \((x_1, y_1)\) and \((x_2, y_2)\):
  \[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
- Equation of circle of radius \( r \) with center \((a, b)\):
  \[ (x - a)^2 + (y - b)^2 = r^2 \]
- A zero or root of a function \( f \) is a number \( c \) such that \( f(c) = 0 \).
- Vertical Line Test: A curve in the plane is the graph of a function of \( x \) if and only if each vertical line \( x = a \) intersects the curve in at most one point.
- Increasing: \( f(x_1) < f(x_2) \) if \( x_1 < x_2 \)
- Nondecreasing: \( f(x_1) \leq f(x_2) \) if \( x_1 \leq x_2 \)
- Decreasing: \( f(x_1) > f(x_2) \) if \( x_1 < x_2 \)
- Nonincreasing: \( f(x_1) \geq f(x_2) \) if \( x_1 \leq x_2 \)
- Even function: \( f(-x) = f(x) \) (graph is symmetric about the \( y \)-axis)
- Odd function: \( f(-x) = -f(x) \) (graph is symmetric about the origin)
- Four ways to transform the graph of \( f \):
  \begin{align*}
  f(x) + c & \quad \text{Shifts graph vertically \(|c|\) units (upward if } c > 0, \text{ downward if } c < 0) \\
  f(x + c) & \quad \text{Shifts graph horizontally \(|c|\) units (to the right if } c < 0, \text{ to the left if } c > 0) \\
  kf(x) & \quad \text{Scales graph vertically by factor } |k|, \text{ stretching if } |k| > 1, \text{ compressing if } |k| < 1; \text{ if } k < 0, \text{ graph is reflected across } x\text{-axis} \\
  f(kx) & \quad \text{Scales graph horizontally by factor } |k|, \text{ compressing if } |k| > 1, \text{ stretching if } |k| < 1; \text{ if } k < 0, \text{ graph is reflected across } y\text{-axis}
  \end{align*}

1.1 EXERCISES

Preliminary Questions

1. Give an example of numbers \( a \) and \( b \) such that \( a < b \) and \( |a| > |b| \).
2. Which numbers satisfy \(|a| = a\)? Which satisfy \(|a| = -a\)? What about \(|-a| = a|\)?
3. Give an example of numbers \( a \) and \( b \) such that \(|a + b| < |a| + |b|\).
4. Are there numbers \( a \) and \( b \) such that \(|a + b| > |a| + |b|\)?
5. What are the coordinates of the point lying at the intersection of the lines \( x = 9 \) and \( y = -4\)?

Exercises

1. Which of the following equations is incorrect?
   (a) \( 3^2 \cdot 3^3 = 3^7 \)  
   (b) \( (\sqrt{5})^{4/3} = 8^{2/3} \)  
   (c) \( 3^2 - 2^3 = 1 \)  
   (d) \( (2-2)^2 = 16 \)
2. Rewrite as a whole number (without using a calculator):
   (a) \( 7^0 \)  
   (b) \( 10^2(2^2 + 5^2) \)  
   (c) \( \frac{(4^3)^5}{(4^3)^3} \)  
   (d) \( 2^{7/3} \)  
   (e) \( 8^{1/3} \cdot 8^{1/3} \)  
   (f) \( 3 \cdot 4^{1/4} - 12 \cdot 2^{3/2} \)
3. Use the binomial expansion formula to expand \((2 + x)^7\).
4. Use the binomial expansion formula to expand \((x + 1)^9\).

6. In which quadrant do the following points lie?
   (a) \( (1, 4) \)  
   (b) \( (-3, 2) \)  
   (c) \( (4, -3) \)  
   (d) \( (-4, -1) \)
7. What is the radius of the circle with equation \((x - 7)^2 + (y - 8)^2 = 9\)?
8. The equation \( f(x) = 5 \) has a solution if (choose one):
   (a) \( 5 \) belongs to the domain of \( f \).
   (b) \( 5 \) belongs to the range of \( f \).
9. What kind of symmetry does the graph have if \( f(-x) = -f(x) \)?
10. Is there a function that is both even and odd?
5. Which of (a)–(d) are true for \( a = 4 \) and \( b = -5\)?
   (a) \( -2a < -2b \)  
   (b) \( |a| < |b| \)  
   (c) \( ab < 0 \)  
   (d) \( \frac{1}{a} < \frac{1}{b} \)
6. Which of (a)–(d) are true for \( a = -3 \) and \( b = 2\)?
   (a) \( a < b \)  
   (b) \( |a| < |b| \)  
   (c) \( ab > 0 \)  
   (d) \( 3a < 3b \)

In Exercises 7–12, express the interval in terms of an inequality involving absolute value.
7. \([ -2, 2 ]\)  
8. \(( -4, 4 )\)  
9. \(( 0, 4 )\)
10. \([ -4, 0 ]\)  
11. \([ -1, 8 ]\)  
12. \(( -2.4, 1.9 )\)
In Exercises 13–16, write the inequality in the form $a < x < b$.
13. $|x| < 8$
14. $|x - 12| < 8$
15. $(x + 1) < 5$
16. $(3x - 4) < 2$

In Exercises 17–22, express the set of numbers $x$ satisfying the given condition as an interval.
17. $|x| < 4$
18. $|x| \leq 9$
19. $|x - 4| < 2$
20. $|x + 7| < 2$
21. $|x - 1| \leq 8$
22. $|3x + 5| < 1$

In Exercises 23–26, describe the set as a union of finite or infinite intervals.
23. $|x + 4| > 2$
24. $|x + 2| > 3$
25. $|x + 2| > 2$
26. $|x| > 3$

27. Match (a)–(f) with (i)–(vi).
(a) $a > 3$
(b) $|a - 5| < \frac{1}{3}$
(c) $|a - 1| < 5$
(d) $|a| > 5$
(e) $|a - 4| < 3$
(f) $1 \leq a \leq 5$

(i) $a$ lies to the right of 3.
(ii) $a$ lies between 1 and 7.
(iii) The distance from $a$ to 5 is less than $\frac{1}{3}$.
(iv) The distance from $a$ to 3 is at most 2.
(v) $a$ is less than 5 units from $\frac{1}{3}$.
(vi) $a$ lies either to the left of $-5$ or to the right of 5.

28. Describe $\left\{ x : \frac{x}{x + 1} < 0 \right\}$ as an interval. Hint: Consider the sign of $x$ and $x + 1$ individually.
29. Describe $\left\{ x : x^2 + 2x < 3 \right\}$ as an interval. Hint: Consider the graph of $y = x^2 + 2x - 3$.
30. Describe the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ as a half-infinite interval.

31. Show that if $a > b$, and $a, b \neq 0$, then $b^{-1} > a^{-1}$, provided that $a$ and $b$ have the same sign. What happens if $a > 0$ and $b < 0$?
32. Which $x$ satisfies both $|x - 3| < 2$ and $|x - 5| < 1$?
33. Show that if $|a - 5| < \frac{1}{3}$ and $|b - 8| < \frac{1}{2}$, then we can conclude that $|a + b - 13| < 1$. Hint: Use the triangle inequality $|a + b| \leq |a| + |b|$.

34. Suppose that $|x - 4| \leq 1$.
(a) What is the maximum possible value of $|x + 4|$?
(b) Show that $|x^2 - 16| \leq 9$.
35. Suppose that $|a - 6| \leq 2$ and $|b| \leq 3$.
(a) What is the largest possible value of $|a + b|$?
(b) What is the smallest possible value of $|a + b|$?
36. Prove that $|x| - |y| \leq |x - y|$. Hint: Apply the triangle inequality to $y$ and $x - y$.

37. Express $r_1 = 0.273$ as a fraction. Hint: $100r_1 - r_1$ is an integer. Then express $r_2 = 0.266\ldots$ as a fraction.
38. Represent 1/7 and 4/27 as infinite repeating decimals.
39. Plot each pair of points and compute the distance between them:
(a) $(1, 4)$ and $(3, 2)$
(b) $(2, 1)$ and $(2, 4)$
40. Plot each pair of points and compute the distance between them:
(a) $(0, 0)$ and $(-2, 3)$
(b) $(-3, -3)$ and $(-2, 3)$

41. Find the equation of the circle with center $(2, 4)$:
(a) With radius $r = 3$
(b) That passes through $(1, -1)$
42. Find all points in the $xy$-plane with integer coordinates located at a distance 5 from the origin. Then find all points with integer coordinates located at a distance 5 from $(2, 3)$.

43. Determine the domain and range of the function $f : [a, b, c, d] \rightarrow [A, B, C, D, E]$ defined by $f(x) = A, f(x) = B, f(t) = B, f(u) = E$.
44. Give an example of a function whose domain $D$ has three elements and whose range $R$ has two elements. Does a function exist whose domain $D$ has two elements and whose range $R$ has three elements?

In Exercises 45–52, find the domain and range of the function.
45. $f(x) = -x$
46. $g(t) = t^4$
47. $f(x) = x$
48. $g(t) = \sqrt{2 - t}$
49. $f(x) = |x|$
50. $h(x) = \frac{1}{x}$
51. $f(x) = \frac{1}{x^2}$
52. $g(t) = \frac{1}{\sqrt{1-t}}$

In Exercises 53–56, determine where $f$ is increasing.
53. $f(x) = |x|$
54. $f(x) = x^3$
55. $f(x) = x^4$
56. $f(x) = \frac{1}{x^2 + x^2 - 1}$

In Exercises 57–62, find the zeros of $f$ and sketch its graph by plotting points. Use symmetry and increasing/decreasing information where appropriate.
57. $f(x) = x^2 - 4$
58. $f(x) = 2x^2 - 4$
59. $f(x) = x^3 - 4x$
60. $f(x) = x^3$
61. $f(x) = x^2 - x^3$
62. $f(x) = \frac{1}{x^2 - 1 + 1}$

63. Which of the curves in Figure 27 is the graph of a function of $x$?

![Figure 27](image)

64. Of the curves in Figure 27 that are graphs of functions, which is the graph of an odd function? Of an even function?
65. Determine whether the function is even, odd, or neither.
(a) $f(x) = x^2$
(b) $g(t) = t^3 - t^2$
(c) $F(t) = \frac{1}{t^4 + 1}$
66. Determine whether the function is even, odd, or neither.
   (a) $f(x) = 2x - x^2$
   (b) $g(x) = (1 - x)^3 + (1 + x)^3$
   (c) $f(t) = \frac{1}{t^n + t + 1} - \frac{1}{t^n - t + 1}$

67. Write $f(x) = 2x^4 - 5x^3 + 12x^2 - 3x + 4$ as the sum of an even and an odd function.

68. Assume that $p$ is a function that is defined for all $x$.
   (a) Prove that if $f$ is defined by $f(x) = p(x) + p(-x)$ then $f$ is even.
   (b) Prove that if $g$ is defined by $g(x) = p(x) - p(-x)$ then $g$ is odd.

69. Assume that $p$ is a function that is defined for $x > 0$ and satisfies $p(a/b) = p(b) - p(a)$. Prove that $f(x) = p\left(\frac{2 - x}{2 + x}\right)$ is an odd function.

70. State whether the function is increasing, decreasing, or neither.
   (a) Surface area of a sphere as a function of its radius
   (b) Temperature at a point on the equator as a function of time
   (c) Price of an airline ticket as a function of the price of oil
   (d) Pressure of the gas in a piston as a function of volume

In Exercises 71-76, let $f$ be the function shown in Figure 28.

71. Find the domain and range of $f$.

72. Sketch the graphs of $y = f(x + 2)$ and $y = f(x) + 2$.

73. Sketch the graphs of $y = f(2x)$, $y = f\left(\frac{1}{2}x\right)$, and $y = 2f(x)$.

74. Sketch the graphs of $y = f(-x)$ and $y = -f(-x)$.

75. Extend the graph of $f$ to $[-4, 4]$ so that it is an even function.

76. Extend the graph of $f$ to $[-4, 4]$ so that it is an odd function.

77. Suppose that $f$ has domain $[4, 8]$ and range $[2, 6]$. Find the domain and range of:
   (a) $y = f(x) + 3$
   (b) $y = f(x + 3)$
   (c) $y = f(3x)$
   (d) $y = 3f(x)$

78. Let $f(x) = x^2$. Sketch the graph over $[-2, 2]$ of:
   (a) $y = f(x + 1)$
   (b) $y = f(x) + 1$
   (c) $y = f(5x)$
   (d) $y = 5f(x)$

79. Suppose that the graph of $f(x) = x^4 - x^2$ is compressed horizontally by a factor of 2 and then shifted 5 units to the right.
   (a) What is the equation for the new graph?
   (b) What is the equation if you first shift by 5 and then compress by 2?
   (c) Verify your answers by plotting your equations.

80. Figure 29 shows the graph of $f(x) = |x| + 1$. Match the functions
   (a)-(e) with their graphs (i)-(v).
   (a) $y = f(x - 1)$
   (b) $y = f(-x)$
   (c) $y = -f(x) + 2$
   (d) $y = f(x - 1) - 2$
   (e) $y = f(x + 1)$

81. Sketch the graph of $y = f(2x)$ and $y = f\left(\frac{1}{2}x\right)$, where $f(x) = |x| + 1$ (Figure 29).

82. Find the function $f$ whose graph is obtained by shifting the parabola $y = x^2$ by 3 units to the right and 4 units down, as in Figure 30.

83. Define $f(x)$ to be the larger of $x$ and $2 - x$. Sketch the graph of $f$.
   What are its domain and range? Express $f(x)$ in terms of the absolute value function.

84. For each curve in Figure 31, state whether it is symmetric with respect to the $y$-axis, the origin, both, or neither.

85. Show that the sum of two even functions is even and the sum of two odd functions is odd.
86. Suppose that \( f \) and \( g \) are both odd. Which of the following functions are even? Which are odd?
(a) \( y = f(x)g(x) \)
(b) \( y = f(x)^3 \)
(c) \( y = f(x) - g(x) \)
(d) \( y = \frac{f(x)}{g(x)} \)

87. Prove that the only function whose graph is symmetric with respect to both the \( y \)-axis and the origin is the function \( f(x) = 0 \).

Further Insights and Challenges

88. Prove the triangle inequality \( |a + b| \leq |a| + |b| \) by adding the two inequalities:
\[
-|a| \leq a \leq |a|, \quad -|b| \leq b \leq |b|
\]

89. Show that a fraction \( r = a/b \) in lowest terms has a finite decimal expansion if and only if
\[
b = 2^m5^n \quad \text{for some } n, m \geq 0
\]

Hint: Observe that \( r \) has a finite decimal expansion when \( 10^m r \) is an integer for some \( N \geq 0 \) (and hence \( b \) divides \( 10^m \)).

90. Let \( p = p_1 \ldots p_k \) be an integer with digits \( p_1, \ldots, p_k \). Show that
\[
\frac{p}{10^k - 1} = 0.p_1 \ldots p_k
\]

91. A function \( f \) is symmetric with respect to the vertical line \( x = a \) if \( f(a - x) = f(a + x) \).
(a) Draw the graph of a function that is symmetric with respect to \( x = 2 \).
(b) Show that if \( f \) is symmetric with respect to \( x = a \), then \( g(x) = f(x + a) \) is even.

92. Formulate a condition for \( f \) to be symmetric with respect to the point \((a, 0)\) on the \( x \)-axis.

1.2 Linear and Quadratic Functions

Linear functions are the simplest of all functions, and their graphs (lines) are the simplest of all curves. However, linear functions and lines play an enormously important role in calculus. For this reason, you should be thoroughly familiar with the basic properties of linear functions and the different ways of writing an equation of a line.

Let's recall that a linear function is a function of the form
\[
f(x) = mx + b \quad (m \text{ and } b \text{ constants})
\]

The graph of \( f \) is a line of slope \( m \), and since \( f(0) = b \), the graph intersects the \( y \)-axis at the point \((0, b)\) (Figure 1). The number \( b \) is called the \( y \)-intercept.

The slope-intercept form of the line with slope \( m \) and \( y \)-intercept \( b \) is given by
\[
y = mx + b
\]

The Greek letter \( \Delta \) (delta) is commonly used to denote the change in a variable or function. Thus, letting \( \Delta x \) and \( \Delta y \) denote the change in \( x \) and \( y = f(x) \) over an interval \([x_1, x_2] \), we have
\[
\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1 = f(x_2) - f(x_1)
\]

The slope \( m \) of a line (Figure 1) is equal to the ratio
\[
m = \frac{\Delta y}{\Delta x} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}
\]
This follows from the formula \( y = mx + b \):

\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m
\]

The slope \( m \) measures the \textit{rate of change} of \( y \) with respect to \( x \). In fact, by writing

\[ \Delta y = m \Delta x \]

we see that a 1-unit increase in \( x \) (i.e., \( \Delta x = 1 \)) produces an \( m \)-unit change \( \Delta y \) in \( y \). For example, if \( m = 5 \), then \( y \) increases by 5 units per unit increase in \( x \). The rate-of-change interpretation of the slope is fundamental in calculus.

Graphically, the slope \( m \) measures the steepness of the line \( y = mx + b \). Figure 2(A) shows lines through a point of varying slope \( m \). Note the following properties:

- **Steepness**: The larger the absolute value \( |m| \), the steeper the line.
- **Positive slope**: If \( m > 0 \), the line slants upward from left to right.
- **Negative slope**: If \( m < 0 \), the line slants downward from left to right.
- **\( f(x) = mx + b \) is increasing if \( m > 0 \) and decreasing if \( m < 0 \).**
- **The horizontal line \( y = b \) has slope \( m = 0 \)** [Figure 2(B)].
- **A vertical line has equation \( x = c \)**, where \( c \) is a constant. The slope of a vertical line is undefined. It is not possible to write the equation of a vertical line in slope-intercept form \( y = mx + b \). A vertical line is not the graph of a function [Figure 2(B)].

![Figure 2](image)

**FIGURE 2**

Scale is especially important in applications because the steepness of a graph depends on the choice of units for the \( x \)- and \( y \)-axes. We can create very different subjective impressions by changing the scale. Figure 3 shows the growth of company profits over a 4-year period. The two plots convey the same information, but the left-hand plot makes the growth look more dramatic.

![Figure 3](image)
Next, we recall the relation between the slopes of parallel and perpendicular lines that are not vertical (Figure 4):

- Lines of slopes $m_1$ and $m_2$ are parallel if and only if $m_1 = m_2$.
- Lines of slopes $m_1$ and $m_2$ are perpendicular if and only if $m_1 = -\frac{1}{m_2}$ (or $m_1 m_2 = -1$)

**CONCEPTUAL INSIGHT** The changes over an interval $[x_1, x_2]$

$$\Delta x = x_2 - x_1, \quad \Delta y = f(x_2) - f(x_1)$$

are defined for any function $f$ (linear or not), but the rise-over-run ratio $\Delta y/\Delta x$ may depend on the interval (Figure 5). The characteristic property of a linear function $f(x) = mx + b$ is that $\Delta y/\Delta x$ has the same value $m$ for every interval. In other words, $y$ has a constant rate of change with respect to $x$. We can use this property to test if two quantities are related by a linear equation.

**EXAMPLE 1 Testing for a Linear Relationship** Do the data in Table 1 suggest a linear relation between the pressure $P$ and temperature $T$ of a gas?

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature (°C)</td>
</tr>
<tr>
<td>40</td>
</tr>
<tr>
<td>45</td>
</tr>
<tr>
<td>55</td>
</tr>
<tr>
<td>70</td>
</tr>
<tr>
<td>80</td>
</tr>
</tbody>
</table>
Real experimental data generally do not display perfect linearity. To model a data set, the statistical tool called linear regression is used to find the linear function that best approximates the data.

**Solution** We calculate $\Delta P/\Delta T$ at successive data points and check whether this ratio is constant:

<table>
<thead>
<tr>
<th>$(T_1, P_1)$</th>
<th>$(T_2, P_2)$</th>
<th>$\Delta P$</th>
<th>$\Delta T$</th>
<th>$\frac{\Delta P}{\Delta T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(40, 1365.80)</td>
<td>(45, 1385.40)</td>
<td>1385.40 - 1365.80</td>
<td>45 - 40</td>
<td>$\frac{39}{40} = 3.92$</td>
</tr>
<tr>
<td>(45, 1385.40)</td>
<td>(55, 1424.60)</td>
<td>1424.60 - 1385.40</td>
<td>55 - 45</td>
<td>$\frac{39}{40} = 3.92$</td>
</tr>
<tr>
<td>(55, 1424.60)</td>
<td>(70, 1483.40)</td>
<td>1483.40 - 1424.60</td>
<td>70 - 55</td>
<td>$\frac{39}{40} = 3.92$</td>
</tr>
<tr>
<td>(70, 1483.40)</td>
<td>(80, 1522.60)</td>
<td>1522.60 - 1483.40</td>
<td>80 - 70</td>
<td>$\frac{39}{40} = 3.92$</td>
</tr>
</tbody>
</table>

Because $\Delta P/\Delta T$ has the constant value 3.92, the data points lie on a line with slope $m = 3.92$. This is confirmed in the plot in Figure 6.

As mentioned above, it is important to be familiar with the standard ways of writing the equation of a line. The general **linear equation** is

$$ax + by = c$$

where $a$ and $b$ are not both zero. For $b = 0$, we obtain the vertical line $ax = c$. For $a = 0$, we obtain the horizontal line $by = c$. When $b \neq 0$, we can rewrite Eq. (1) in slope-intercept form. For example, $-6x + 2y = 3$ can be rewritten as $y = 3x + \frac{3}{2}$.

Another important form for an equation of a line is the **point-slope form**. Given the slope of the line and a point on it (Figure 7), we can use this form to obtain an equation for the line.

**The point-slope form** of the line through $P = (a, b)$ with slope $m$ is

$$y - b = m(x - a)$$

**EXAMPLE 2** Line of Given Slope Through a Given Point

Find the slope-intercept equation of the line through $(9, 2)$ with slope $-\frac{2}{3}$.

**Solution** In point-slope form:

$$y - 2 = -\frac{2}{3}(x - 9)$$

In slope-intercept form: $y = -\frac{2}{3}(x - 9) + 2$ or $y = -\frac{2}{3}x + 8$. See Figure 8.

**EXAMPLE 3** Line Through Two Points

Find an equation of the line through $(2, 1)$ and $(9, 5)$.

**Solution** The line has slope

$$m = \frac{5 - 1}{9 - 2} = \frac{4}{7}$$

Because $(2, 1)$ lies on the line, its equation in point-slope form is $y - 1 = \frac{4}{7}(x - 2)$.

Recall that the slope $m$ of a line measures the rate of change of the dependent variable $y$ with respect to the independent variable $x$, that is $m = \frac{\Delta y}{\Delta x}$. In applications, the slope has units:

$$\text{units of slope} = \frac{\text{units of dependent variable}}{\text{units of independent variable}}$$

Scanned with CamScanner
For example:

- Let \( T = -0.1h + 52 \) represent the temperature (in °C) as a function of height (in m) measured by a weather balloon as it rose from the ground to 1000 meters. The slope is \(-0.1 \, \text{°C/m}\) indicating that the temperature dropped by one-tenth of a degree for every meter the balloon rose.
- For the time period 1900 to 1920, \( P = 33.3t + 57.7 \) models the population of Saskatchewan (in thousands) \( t \) years after 1900. The slope is \(33.3 \, \text{thousand people/year} \), indicating the population rose by 33.3 thousand people per year during the time period.

A quadratic function is a function defined by a quadratic polynomial:

\[
f(x) = ax^2 + bx + c \quad (a, b, c \text{ are constants with } a \neq 0)
\]

The graph of \( f \) is a parabola (Figure 10). The parabola opens upward if the leading coefficient \( a \) is positive and downward if \( a \) is negative. Ignoring air resistance, the path of a struck baseball is modeled by a downward-opening parabola (Figure 9).

The discriminant of \( f(x) = ax^2 + bx + c \) is the quantity

\[
D = b^2 - 4ac
\]

The roots of \( f \) are given by the quadratic formula (see Exercise 60):

\[
	ext{roots of } f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}
\]

The sign of \( D \) determines the nature of the roots (Figure 10). If \( D > 0 \), then \( f \) has two real roots, and if \( D = 0 \), it has one real root (a “double root”). If \( D < 0 \), then \( f \) has no roots that are real numbers but has two roots that are complex numbers. We focus primarily on real numbers and real-number roots (“real roots”) of functions in this text.

![Figure 9: A parabola models the path of the baseball.](image)

![Figure 10: Graphs of quadratic functions](image)

When \( f \) has two real roots \( r_1 \) and \( r_2 \), then \( f(x) \) factors as

\[
f(x) = a(x - r_1)(x - r_2)
\]

For example, \( f(x) = 2x^2 - 3x + 1 \) has discriminant \( D = b^2 - 4ac = 9 - 8 = 1 > 0 \), and by the quadratic formula, its roots are \((3 \pm 1)/4\), that is, 1 and \(\frac{1}{2}\). Therefore,

\[
f(x) = 2x^2 - 3x + 1 = 2(x - 1)\left(x - \frac{1}{2}\right)
\]

The technique of completing the square consists of writing a quadratic polynomial as a multiple of a square plus a constant. For \( x^2 + bx + c \), add and subtract the square of half the coefficient of the \( x \)-term so that a square term \((x + \frac{b}{2})^2\) can be made:

\[
x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c
\]

If the \( x^2 \) term has a coefficient \( a \), we factor that out first, as demonstrated in the following example.
EXAMPLE 4  Completing the Square  Complete the square for the quadratic polynomial \( f(x) = 4x^2 - 12x + 3 \).

Solution  First factor out the leading coefficient:

\[
4x^2 - 12x + 3 = 4 \left( x^2 - 3x + \frac{3}{4} \right)
\]

Then complete the square for the term \( x^2 - 3x \):

\[
x^2 - 3x = x^2 - 3x + \left( \frac{3}{2} \right)^2 - \left( \frac{3}{2} \right)^2 = \left( x - \frac{3}{2} \right)^2 - \frac{9}{4}
\]

Therefore,

\[
4x^2 - 12x + 3 = 4 \left( x - \frac{3}{2} \right)^2 - 6
\]

The method of completing the square can be used to find the minimum or maximum value of a quadratic function, as we do in the next example.

EXAMPLE 5  Finding the Maximum of a Quadratic Function  Complete the square and find the maximum value of \( f(x) = -x^2 + 4x + 1 \).

Solution  Since the \( x^2 \) term has a coefficient of \(-1\), we factor it out:

\[
f(x) = -(x^2 - 4x - 1) = -(x^2 - 4x + 4 - 4 - 1) = -(x - 2)^2 - 5 = -(x - 2)^2 + 5
\]

Now, \( f(2) = 5 \), and \( -(x - 2)^2 + 5 < 5 \) for all other \( x \). Thus, the maximum value of \( f \) is 5 occurring at \( x = 2 \) (Figure 11).

1.2 SUMMARY

- A linear function is a function of the form \( f(x) = mx + b \).
- The general equation of a line is \( ax + by = c \). The line \( y = c \) is horizontal, and \( x = c \) is vertical.
- Two convenient ways of writing the equation of a nonvertical line:
  - Slope-intercept form: \( y = mx + b \) (slope \( m \) and \( y \)-intercept \( b \))
  - Point-slope form: \( y - b = m(x - a) \) (slope \( m \), passes through \( (a, b) \))
- Two lines of slopes \( m_1 \) and \( m_2 \) are parallel if and only if \( m_1 = m_2 \), and they are perpendicular if and only if \( m_1 = -\frac{1}{m_2} \).
- Quadratic function: \( f(x) = ax^2 + bx + c \). The roots are \( x = \frac{-b \pm \sqrt{D}}{2a} \), where \( D = b^2 - 4ac \) is the discriminant. The roots are real and distinct if \( D > 0 \), there is a double real root if \( D = 0 \), and there are no real roots if \( D < 0 \).
- Completing the square consists of writing a quadratic function \( f(x) = ax^2 + bx + c \) as a multiple of a square plus a constant; that is, \( f(x) = a(x + p)^2 + q \).

1.2 EXERCISES

Preliminary Questions

1. What is the slope of the line \( y = -4x - 97 \)?
2. Are the lines \( y = 2x + 1 \) and \( y = -2x - 4 \) perpendicular?
3. When is the line \( ax + by = c \) parallel to the \( y \)-axis? To the \( x \)-axis?
4. Suppose \( y = 3x + 2 \). What is \( \Delta y \) if \( x \) increases by 3?
5. What is the minimum of \( f(x) = (x + 3)^2 - 4 \)?
6. What is the result of completing the square for \( f(x) = x^2 + 1 \)?
7. Describe how the parabolas \( y = ax^2 + 1 \) change as \( a \) changes from \(-\infty \) to \( \infty \).
8. Describe how the parabolas \( y = x^2 + bx \) change as \( b \) changes from \(-\infty \) to \( \infty \).
Exercises

In Exercises 1–4, find the slope, the y-intercept, and the x-intercept of the line with the given equation.

1. \( y = 3x + 12 \)
2. \( y = 4 - x \)
3. \( 4x + 9y = 3 \)
4. \( y - 3 = \frac{1}{2}(x - 6) \)

In Exercises 5–8, find the slope of the line.

5. \( y = 3x + 2 \)
6. \( y = 3(x - 9) + 2 \)
7. \( 3x + 4y = 12 \)
8. \( 3x + 4y = -8 \)

In Exercises 9–20, find the equation of the line with the given description.

9. Slope 3, y-intercept 8
10. Slope -2, y-intercept 3
11. Slope 3, passes through (7, 9)
12. Slope -5, passes through (0, 0)
13. Horizontal, passes through (0, -2)
14. Passes through (-1, 4) and (2, 7)
15. Parallel to \( y = 3x - 4 \), passes through (1, 1)
16. Passes through (1, 4) and (12, -3)
17. Perpendicular to \( 3x + 5y = 9 \), passes through (2, 3)
18. Vertical, passes through (-4, 9)
19. Horizontal, passes through (8, 4)
20. Slope 3, x-intercept 6

21. Find the equation of the perpendicular bisector of the segment joining (1, 2) and (5, 4) (Figure 12). Hint: The midpoint \( Q \) of the segment joining \((a, b)\) and \((c, d)\) is \( \left( \frac{a + c}{2}, \frac{b + d}{2} \right) \).

22. Intercept–Intercept Form  Show that if \( a, b \neq 0 \), then the line with x-intercept \( x = a \) and y-intercept \( y = b \) has equation (Figure 13)

\[
\frac{x}{a} + \frac{y}{b} = 1
\]

23. Find an equation of the line with x-intercept \( x = 4 \) and y-intercept \( y = 3 \).
24. Find y such that \( (3, y) \) lies on the line of slope \( m = 2 \) through \((1, 4)\).
25. Determine whether there exists a constant \( c \) such that the line \( cx + cy = 1 \):
   (a) Has slope 4
   (b) Passes through (3, 1)
   (c) Is horizontal
   (d) Is vertical
26. Determine whether there exists a constant \( c \) such that the line \( cx - 2y = 4 \):
   (a) Has slope 4
   (b) Passes through (1, -6)
   (c) Is horizontal
   (d) Is vertical
27. Suppose that the number of Bob’s Bites computers that can be sold when the computer’s price is \( P \) (in dollars) is given by a linear function \( N(P) \), where \( N(1000) = 10,000 \) and \( N(1500) = 7500 \).
   (a) Determine \( N(P) \).
   (b) What is the slope of the graph of \( N(P) \), including units? Describe what the slope represents.
   (c) What is the change \( \Delta N \) in the number of computers sold if the price is increased by \( \Delta P = 1000 \) dollars?
28. Suppose that the demand for Colin’s kidney pies is linear in the price \( P \). Further, assume that he can sell 100 pies when the price is \$5.00 and 40 pies when the price is \$10.00.
   (a) Determine the demand \( N \) (number of pies sold) as a function of the price \( P \) (in dollars).
   (b) What is the slope of the graph of \( N(P) \), including units? Describe what the slope represents.
   (c) Determine the revenue \( R = N \times P \) for prices \( P = 5, 6, 7, 8, 9, 10 \) and then choose a price to maximize the revenue.
29. In each case, identify the slope and give its meaning with the appropriate units.
   (a) The function \( N = -70t + 5000 \) models the enrollment at Maple Grove College during the fall of 2018, where \( N \) represents the number of students and \( t \) represents the time in weeks since the start of the semester.
   (b) The function \( C = 3.5n + 700 \) represents the cost (in dollars) to rent the Shakedown Street Dance Hall for an evening if \( n \) people attend the dance.
30. In each case, identify the slope and give its meaning with the appropriate units.
   (a) The function \( N = 3.97 - 178.8 \) models the number of times, \( N \), that a cricket chirps in a minute when the temperature is \( T \) °Celsius.
   (b) The function \( V = 47.50 \sqrt{d} \) gives the volume \( V \), in gallons, of molasses in the storage tank in relation to the depth \( d \), in feet, of the molasses.
31. Materials expand when heated. Consider a metal rod of length \( L_0 \) at temperature \( T_0 \). If the temperature is changed by an amount \( \Delta T \), then the rod’s length approximately changes by \( \Delta L = \alpha L_0 \Delta T \), where \( \alpha \) is the thermal expansion coefficient and \( \Delta T \) is not an extreme temperature change. For steel, \( \alpha = 1.24 \times 10^{-5} \, ^\circ C^{-1} \).
   (a) A steel rod has length \( L_0 = 40 \) cm at \( T_0 = 40 \) °C. Find its length at \( T = 90 \) °C.
   (b) Find its length at \( T = 50 \) °C if its length at \( T_0 = 100 \) °C is 65 cm.
   (c) Express length \( L \) as a function of \( T \) if \( L_0 = 65 \) cm at \( T_0 = 100 \) °C.
32. Do the points \((0.5, 1), (1, 1.2), (2, 2)\) lie on a line?
33. Find \( b \) such that \((2, -1), (3, 2), (b, 5)\) lie on a line.
34. Find an expression for the velocity $v$ as a linear function of $t$ that matches the following data:

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$ (m/s)</td>
<td>39.2</td>
<td>58.6</td>
<td>78</td>
<td>97.4</td>
</tr>
</tbody>
</table>

35. The period $T$ of a pendulum is measured for pendulums of different lengths $L$. Based on the following data, does $T$ appear to be a linear function of $L$?

<table>
<thead>
<tr>
<th>$L$ (cm)</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$ (s)</td>
<td>0.9</td>
<td>1.1</td>
<td>1.27</td>
<td>1.42</td>
</tr>
</tbody>
</table>

36. Show that $f$ is linear of slope $m$ if and only if

$$f(x + h) - f(x) = mh \quad \text{(for all } x \text{ and } h)$$

That is to say, prove the following two statements:

(a) $f$ is linear of slope $m$ implies $f(x + h) - f(x) = mh$ (for all $x$ and $h$).

(b) $f(x + h) - f(x) = mh$ (for all $x$ and $h$) implies $f$ is linear of slope $m$.

37. Find the roots of the quadratic polynomials:

(a) $f(x) = 4x^2 - 3x - 1$

(b) $f(x) = x^2 - 2x - 1$

In Exercises 38–45, complete the square and find the minimum or maximum value of the quadratic function.

38. $y = x^2 + 2x + 5$

39. $y = x^2 - 6x + 9$

40. $y = -9x^2 + x$

41. $y = x^2 + 6x + 2$

42. $y = 2x^2 - 4x - 7$

43. $y = -4x^2 + 6x + 8$

44. $y = 3x^2 + 12x - 5$

45. $y = 4x^2 - 12x + 2$

46. Sketch the graph of $y = x^2 - 6x + 8$ by plotting the roots and the minimum point.

47. Sketch the graph of $y = x^2 + 4x + 6$ by plotting the minimum point, the $y$-intercept, and one other point.

48. If the alleles $A$ and $B$ of the cystic fibrosis gene occur in a population with frequencies $p$ and $1 - p$ (where $p$ is between 0 and 1), then the frequency of heterozygous carriers (carriers with both alleles) is $2p(1 - p)$. Which value of $p$ gives the largest frequency of heterozygous carriers?

49. For which values of $c$ does $f(x) = x^2 + cx + 1$ have a double root? No real roots?

50. Let $f(x) = x^2 + x - 1$.

(a) Show that the lines $y = x + 3$, $y = x - 1$, and $y = x - 3$ intersect the graph of $f$ in two, one, and zero points, respectively.

(b) Sketch the graph of $f$ and the three lines from (a).

(c) Describe the relationship between the graph of $f$ and the lines $y = x + c$ as $c$ changes from $-\infty$ to $\infty$.

Further Insights and Challenges

57. Show that if $f$ and $g$ are linear, then so is $f + g$. Is the same true of $fg$?

58. Show that if $f$ and $g$ are linear functions such that $f(0) = g(0)$ and $f(1) = g(1)$, then $f = g$.

59. Show that $\Delta y/\Delta x$ for the function $f(x) = x^2$ over the interval $[x_1, x_2]$ is not a constant, but depends on the interval. Determine the exact dependence of $\Delta y/\Delta x$ on $x_1$ and $x_2$.

60. Complete the square and use the result to derive the quadratic formula for the roots of $ax^2 + bx + c = 0$.

61. Let $a, c \neq 0$. Show that the roots of

$$ax^2 + bx + c = 0$$

are reciprocals of each other.

62. Let $a, b > 0$. Show that the geometric mean $\sqrt{ab}$ is not larger than the arithmetic mean $(a + b)/2$. Hint: Consider $(a^{1/2} - b^{1/2})^2$.
62. Show, by completing the square, that the parabola
\[ y = ax^2 + bx + c \]
can be obtained from \( y = ax^2 \) by a vertical and horizontal translation.

63. Prove Viète's Formulas: The quadratic polynomial with \( a \) and \( b \) as roots is \( x^2 + bx + c \), where \( b = -a - b \) and \( c = ab \).

1.3 The Basic Classes of Functions

The primary condition on a function \( f \) is that it assigns to each element \( x \) of its domain a unique element \( f(x) \) in its range. There are no other restrictions on how that relation is defined. Usually we describe the relationship by a formula for the function, sometimes by a table or a graph, but the association could be quite complicated, not lending itself to any simple description. The possibilities for functions are endless. In calculus we make no attempt to deal with all possible functions. The techniques of calculus, powerful and general as they are, apply only to functions that are sufficiently "well behaved" (we will see what well behaved means when we study the derivative in Chapter 3). Fortunately, such functions are adequate for a vast range of applications.

Most of the functions considered in this text are constructed from the following familiar classes of well-behaved functions:

- polynomials
- rational functions
- algebraic functions
- exponential functions
- trigonometric functions
- logarithmic functions
- inverse trigonometric functions

We shall refer to these as the basic functions.

- **Polynomials**: For any real number \( m \), \( f(x) = x^m \) is called the power function with exponent \( m \). Power functions include \( f(x) = x^3 \), \( f(x) = x^{-7} \), and \( f(x) = x^2 \). The base is the variable, and the exponent is a constant. For now, we are interested in power functions with exponents that are positive integers. A polynomial is a sum of multiples of power functions with exponents that are positive integers or zero (Figure 1):
  \[ f(x) = x^5 - 5x^3 + 4x, \quad g(t) = 7t^6 + t^3 - 3t - 1, \quad h(x) = x^9 \]

  Thus, the function \( f(x) = x + x^{-1} \) is not a polynomial because it includes a term \( x^{-1} \) with a negative exponent. The general polynomial \( P \) in the variable \( x \) may be written
  \[ P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \]
  - The numbers \( a_0, a_1, \ldots, a_n \) are called coefficients.
  - The degree of \( P \) is \( n \) (assuming that \( a_n \neq 0 \)).
  - The coefficient \( a_n \) is called the leading coefficient.
  - The domain of \( P \) is \( \mathbb{R} \).

- **A rational function** is a quotient of two polynomials (Figure 2):
  \[ f(x) = \frac{P(x)}{Q(x)} \quad [P(x) \text{ and } Q(x) \text{ polynomials}] \]

  The domain of \( f \) is the set of numbers \( x \) such that \( Q(x) \neq 0 \). For example,
  \[ f(x) = \frac{1}{x^2} \quad \text{domain } \{ x : x \neq 0 \} \]
  \[ h(t) = \frac{7t^6 + t^3 - 3t - 1}{t^2 - 1} \quad \text{domain } \{ t : t \neq \pm 1 \} \]

  Every polynomial is also a rational function [with \( Q(x) = 1 \)].
An algebraic function is produced by taking sums, products, and quotients of roots of polynomials and rational functions (Figure 3):

\[ f(x) = \sqrt{1 + 3x^2 - x^4}, \quad g(t) = (\sqrt{t} - 2)^2, \quad h(z) = \frac{z + z^{-5/3}}{5z^3 - \sqrt{z}} \]

A number \( x \) belongs to the domain of \( f \) if each term in the formula is defined and the result does not involve division by zero. For example, \( g(t) \) is defined if \( t \geq 0 \) and \( \sqrt{t} \neq 2 \), so the domain of \( g \) is \( D = \{ t : t \geq 0 \text{ and } t \neq 4 \} \).

- **Exponential functions**: The function \( f(x) = b^x \), where \( b > 0 \) and \( b \neq 1 \), is called the exponential function with base \( b \). Some examples are
  \[ f(x) = 2^x, \quad g(t) = 10^t, \quad h(x) = \left( \frac{1}{2} \right)^x, \quad p(t) = (\sqrt{5})^t \]

Exponential functions and their inverses, the logarithmic functions, are treated in greater detail in Chapter 7.

- **Trigonometric functions** are functions built from \( \sin x \) and \( \cos x \). These functions are discussed in the next section.

### Constructing New Functions

Given functions \( f \) and \( g \), we can construct new functions by forming the sum, difference, product, and quotient functions:

\[ (f + g)(x) = f(x) + g(x), \quad (f - g)(x) = f(x) - g(x) \]
\[ (f \cdot g)(x) = f(x) \cdot g(x), \quad \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \quad \text{(where} \ g(x) \neq 0) \]

For example, if \( f(x) = x^2 \) and \( g(x) = \sin x \), then

\[ (f + g)(x) = x^2 + \sin x, \quad (f - g)(x) = x^2 - \sin x \]
\[ (f \cdot g)(x) = x^2 \sin x, \quad \left( \frac{f}{g} \right)(x) = \frac{x^2}{\sin x} \]

We can also multiply functions by constants. A function of the form

\[ h(x) = c_1 f(x) + c_2 g(x) \quad (c_1, c_2 \text{ constants}) \]

is called a linear combination of \( f \) and \( g \).

Composition is another important way of constructing new functions. The composition of \( f \) and \( g \) is the function \( f \circ g \) defined by \( (f \circ g)(x) = f(g(x)) \). The domain of \( f \circ g \) is the set of values of \( x \) in the domain of \( g \) such that \( g(x) \) lies in the domain of \( f \).

**EXAMPLE 1** Compute the composite functions \( f \circ g \) and \( g \circ f \) and discuss their domains, where

\[ f(x) = \sqrt{x}, \quad g(x) = 1 - x \]

**Solution** We have

\[ (f \circ g)(x) = f(g(x)) = f(1 - x) = \sqrt{1 - x} \]

The square root \( \sqrt{1 - x} \) is defined if \( 1 - x \geq 0 \), that is, for \( x \leq 1 \). Therefore, the domain of \( f \circ g \) is \( \{ x : x \leq 1 \} \). On the other hand,

\[ (g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = 1 - \sqrt{x} \]

The domain of \( g \circ f \) is \( \{ x : x \geq 0 \} \).

**EXAMPLE 2. Surface Area and Volume** Express the surface area \( S \) of a cube as a function of its volume \( V \).
Solution  We will derive a relationship via a composition of functions.

The volume \( V \) and the length \( L \) of a side of a cube are related by \( V = L^3 \) (Figure 4). Therefore, \( L = V^{1/3} \). Thus, \( L(V) = V^{1/3} \) expresses the side length as a function of the volume.

The surface area \( S \) is a function of the side length defined by \( S(L) = 6L^2 \) (the cube has six sides, each with area \( L^2 \)). Thus, \( S \) depends on \( V \) by the composition

\[
S \circ L(V) = S(L(V)) = S(V^{1/3}) = 6(V^{1/3})^2 = 6V^{2/3}
\]

It follows that we can express surface area as a function of volume by \( S(V) = 6V^{2/3} \).

The simple geometric relationship derived in the previous example is the basis for a variety of theoretical power laws in biology and ecology in which an attribute proportional to an animal’s surface area is related to an attribute proportional to its volume. For example, in a particular species, the mass \( M \) of an individual is proportional to its volume, and the mass \( F \) of its fur might be proportional to its surface area. Thus, the relationship between fur mass and animal mass could be modeled by a power law \( F = kM^{2/3} \). Typically, scientists collect data to check the proposed relationship that either confirms this model or suggests adjustments or other factors that must be considered.

### Elementary Functions

As noted above, we can produce new functions by applying the operations of addition, subtraction, multiplication, division, and composition. It is convenient to refer to a function constructed in this way from the basic functions listed above as an elementary function. The following functions are elementary:

\[
f(x) = \sqrt{2x} + \sin x, \quad f(x) = 10\sqrt{x}, \quad f(x) = \frac{1 + x^{-1}}{1 + \cos x}
\]

### Piecewise-Defined Functions

We can also create new functions by piecing together functions defined over limited domains, obtaining piecewise-defined functions. One example we have already seen is the absolute value function defined by

\[
|x| = \begin{cases} 
-x & \text{when } x < 0 \\
x & \text{when } x \geq 0
\end{cases}
\]

**Example 3** Given the function \( f \), determine its domain, range, and intervals where it is increasing or decreasing.

\[
f(x) = \begin{cases} 
1 & \text{when } x < 0 \\
x + 1 & \text{when } x \geq 0
\end{cases}
\]

**Solution** The graph of \( f \) appears in Figure 5. The function is defined for all values of \( x \), so the domain is all real numbers. Now, for all \( x < 0 \), the output of \( f \) is just the single value \( 1 \), and for \( x \geq 0 \), the output covers all values greater than or equal to \( 1 \). Hence, the range of the function is \( \{ y : y \geq 1 \} \). The function is neither increasing nor decreasing for \( x < 0 \); however, the function is increasing for \( x \geq 0 \).

### 1.3 SUMMARY

- For \( m \) a real number, \( f(x) = x^m \) is called the power function with exponent \( m \).
- A polynomial \( P \) is a sum of multiples of \( x^m \), where \( m \) is a whole number:
  \[
P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0
\]
This polynomial has degree \( n \) (assuming that \( a_n 
eq 0 \)), and \( a_n \) is called the leading coefficient.

- A rational function is a quotient \( P/Q \) of two polynomials [defined when \( Q(x) 
eq 0 \)].
- An algebraic function is produced by taking sums, products, and quotients of roots of polynomials and rational functions.
- Exponential function: \( f(x) = b^x \), where \( b > 0 \) and \( b 
eq 1 \) (\( b \) is called the base).
- The composite function \( f \circ g \) is defined by \( (f \circ g)(x) = f(g(x)) \). The domain of \( f \circ g \) is the set of \( x \) in the domain of \( g \) such that \( g(x) \) belongs to the domain of \( f \).
- The elementary functions are obtained by taking products, sums, differences, quotients, and compositions of the basic functions, which include polynomials, rational functions, algebraic functions, exponential functions, trigonometric functions, logarithmic functions, and inverse trigonometric functions.
- A piecewise-defined function is obtained by defining a function over two or more distinct domains.

### 1.3 Exercises

#### Preliminary Questions

1. Explain why both \( f(x) = x^3 + 1 \) and \( g(x) = \frac{1}{x^4+1} \) are rational functions.
2. Is \( y = |x| \) a polynomial function? What about \( y = |x^2 + 1| \)?
3. What is unusual about the domain of the composite function \( f \circ g \) for the functions \( f(x) = x^{1/2} \) and \( g(x) = 1 - |x| \)?
4. Explain why both \( f(x) = \frac{1}{1-x} \) and \( g(x) = \sqrt[3]{1-x^4} \) are algebraic functions.

5. We have \( f(x) = (x+1)^{1/2} \), \( g(x) = x^2 + 1 \), \( h(x) = 2^x \), and \( k(x) = x^2 + 1 \). Identify which of the functions may be described by each of the following.
   - (a) Transcendental
   - (b) Polynomial
   - (c) Rational but not polynomial
   - (d) Algebraic but not rational

#### Exercises

In Exercises 1–10, determine the domain of the function.

1. \( f(x) = x^{1/4} \)
2. \( g(t) = t^{2/3} \)
3. \( f(x) = x^3 + 3x - 4 \)
4. \( h(z) = z^3 + z^{-3} \)
5. \( g(t) = \frac{1}{t+2} \)
6. \( f(x) = \frac{1}{x^2+4} \)
7. \( G(u) = \frac{1}{u^2-4} \)
8. \( f(x) = \frac{\sqrt{x}}{x^2-9} \)
9. \( f(x) = x^4 + (x-1)^{-3} \)
10. \( F(t) = \sin \left( \frac{x}{x+1} \right) \)

In Exercises 11–22, identify each of the following functions as polynomial, rational, algebraic, or transcendental.

11. \( f(x) = 4x^3 + 9x^2 - 8 \)
12. \( f(x) = x^{-4} \)
13. \( f(x) = \sqrt{x} \)
14. \( f(x) = \sqrt{1-x^2} \)
15. \( f(x) = \frac{x^2}{x + \sin x} \)
16. \( f(x) = 2^x \)
17. \( f(x) = \frac{2x^3 + 3x}{9 - 7x^2} \)
18. \( f(x) = \frac{3x - 9x^{-1/2}}{9 - 7x^2} \)
19. \( f(x) = \sin(x^2) \)
20. \( f(x) = \frac{x}{\sqrt{x} + 1} \)
21. \( f(x) = x^2 + 3x^{-1} \)
22. \( f(x) = \sin(3x) \)
23. Is \( f(x) = 2^x \) a transcendental function?
24. Show that \( f(x) = x^2 + 3x^{-1} \) and \( g(x) = 3x^3 - 9x + x^{-2} \) are rational functions—that is, quotients of polynomials.

In Exercises 25–32, calculate the composite functions \( f \circ g \) and \( g \circ f \), and determine their domains.

25. \( f(x) = \sqrt{x} \), \( g(x) = x + 1 \)
26. \( f(x) = \frac{1}{x} \), \( g(x) = x^{-4} \)
27. \( f(x) = \frac{1}{\sqrt{x}} \), \( g(x) = x^2 \)
28. \( f(x) = |x| \), \( g(x) = \sin \theta \)
29. \( f(\theta) = \cos \theta \), \( g(x) = x^3 + x^2 \)
30. \( f(x) = \frac{1}{x^2 + 1} \), \( g(x) = x^{-2} \)
31. \( f(t) = \frac{1}{\sqrt{t}} \), \( g(t) = -t^2 \)

32. \( f(t) = \sqrt{t}, \ g(t) = 1 - t^3 \)

33. The volume \( V \) and surface area of a sphere (Figure 6(A)) are expressed in terms of radius \( r \) by \( V(r) = \frac{4}{3}\pi r^3 \) and \( S(r) = 4\pi r^2 \), respectively. Determine \( r(V) \), the radius as a function of volume. Then determine \( S(V) \), the surface area as a function of volume, by computing the composite \( S \circ r(V) \).

34. A tetrahedron is a polyhedron with four equilateral triangles as its faces (Figure 6(B)). The volume \( V \) and surface area of a tetrahedron are expressed in terms of the side-length \( L \) of the triangles by \( V(L) = \frac{L^3}{6\sqrt{2}} \) and \( S(L) = \sqrt{3}L^2 \), respectively. Determine \( L(V) \), the side length as a function of volume. Then determine \( S(V) \), the surface area as a function of volume, by computing the composite \( S \circ L(V) \).

![Figure 6](image)

**Figure 6** A sphere (A) and tetrahedron (B).

In Exercises 35–38, draw the graphs of each of the piecewise-defined functions.

35. \( f(x) = \begin{cases} 3 & \text{when } x < 0 \\ x^2 + 3 & \text{when } x \geq 0 \end{cases} \)

36. \( f(x) = \begin{cases} x + 1 & \text{when } x < 0 \\ 1 - x & \text{when } x \geq 0 \end{cases} \)

37. \( f(x) = \begin{cases} x^2 & \text{when } x < 0 \\ -x^2 & \text{when } x \geq 0 \end{cases} \)

38. \( f(x) = \begin{cases} 2x - 2 & \text{when } x < 0 \\ x & \text{when } x \geq 0 \end{cases} \)

39. Let \( f(x) = \frac{1}{x} \).

(a) What are the domain and range of \( f? \)

(b) Sketch the graph of \( f \).

(c) Express \( f \) as a piecewise-defined function where each of the "pieces" is a constant.

40. The Heaviside function (named after Oliver Heaviside, 1850–1925) is defined by:

\[ H(x) = \begin{cases} 0 & \text{when } x < 0 \\ 1 & \text{when } x \geq 0 \end{cases} \]

The Heaviside function can be used to "turn on" another function at a specific value in the domain, as seen in the four examples here. For each of the following, sketch the graph of \( f \).

(a) \( f(x) = H(x)x^2 \)

(b) \( f(x) = H(x)(1 - x^2) \)

(c) \( f(x) = H(x)(x - 1)x \)

(d) \( f(x) = H(x)(x + 1)x^2 \)

41. The population (in millions) of Caledonia as a function of time \( t \) (years) is \( P(t) = 30 \cdot 2^{t/10} \). Show that the population doubles every 10 years. Show more generally that for any positive constants \( a \) and \( k \), the function \( g(t) = a2^{kt} \) doubles after \( 1/k \) years.

42. Find all values of \( c \) such that \( f(x) = \frac{x + 1}{x^2 + 2cx + 4} \) has domain \( \mathbb{R} \).

43. Show that if \( f(x) = x^2 \), then \( \delta f(x) = 2x + 1 \). Calculate \( \delta f \) for \( f(x) = x \) and \( f(x) = x^3 \).

44. Show that \( \delta(10^k) = 9 \cdot 10^{k-1} \) and, more generally, that \( \delta(b^k) = (b - 1)b^{k-1} \).

45. Show that for any two functions \( f \) and \( g \), \( \delta(f + g) = \delta f + \delta g \) and \( \delta(cf) = c\delta f \), where \( c \) is any constant.

46. Suppose we can find a function \( P \) such that \( \delta P(x) = (x + 1)^k \) and \( P(0) = 0 \). Prove that \( P(1) = 1^k, P(2) = 1^k + 2^k, \) and, more generally, for every whole number \( n \),

\[ P(n) = 1^k + 2^k + \cdots + n^k \]

47. Show that if \( P(x) = \frac{x(x + 1)}{2} \), then \( \delta P = (x + 1) \). Then apply Exercise 46 to conclude that

\[ 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \]

48. Calculate \( \delta(x^3), \delta(x^2), \) and \( \delta(x) \). Then find a polynomial \( P \) of degree 3 such that \( \delta P = (x + 1)^2 \) and \( P(0) = 0 \). Conclude that \( P(n) = 1^2 + 2^2 + \cdots + n^2 \).

49. This exercise combined with Exercise 46 shows that for all whole numbers \( k \), there exists a polynomial \( P \) satisfying Eq. (1). The solution requires the Binomial Theorem and proof by induction (see Appendix C).

(a) Show that \( \delta(x^{k+1}) = (k + 1)x^k + \cdots \), where the dots indicate terms involving smaller powers of \( x \).

(b) Show by induction that there exists a polynomial of degree \( k + 1 \) with leading coefficient \( 1/(k + 1) \):

\[ P(x) = \frac{1}{k + 1}x^{k+1} + \cdots \]

such that \( \delta P = (x + 1)^k \) and \( P(0) = 0 \).
1.4 Trigonometric Functions

We begin our trigonometric review by recalling the two systems of angle measurement: radians and degrees. They are best described using the relationship between angles and rotation. As is customary, we often use the lowercase Greek letter \( \theta \) (theta) to denote angles and rotations.

FIGURE 1 The radian measure \( \theta \) of a counterclockwise rotation is the length along the unit circle of the arc traversed by \( P \) as it rotates into \( Q \).

Figure 1(A) shows a unit circle with radius \( OP \) rotating counterclockwise into radius \( OQ \). The radian measure of this rotation is the length \( \theta \) of the circular arc traversed by \( P \) as it rotates into \( Q \). On a circle of radius \( r \), the arc traversed by a counterclockwise rotation of \( \theta \) radians has length \( \theta r \) (Figure 2).

The unit circle has circumference \( 2\pi \). Therefore, a rotation through a full circle has radian measure \( \theta = 2\pi \) [Figure 1(B)]. The radian measure of a rotation through one-quarter of a circle is \( \theta = 2\pi/4 = \pi/2 \) [Figure 1(C)] and, in general, the rotation through one-\( n \)-th of a circle has radian measure \( 2\pi/n \) (Table 1). A negative rotation (with \( \theta < 0 \)) is a rotation in the clockwise direction [Figure 1(D)].

The radian measure of an angle such as \( \angle POQ \) in Figure 1(A) is defined as the radian measure of a rotation that carries \( OP \) to \( OQ \). Notice, however, that the radian measure of an angle is not unique. The rotations through \( \theta \) and \( \theta + 2\pi \) both carry \( OP \) to \( OQ \). Therefore, \( \theta \) and \( \theta + 2\pi \) represent the same angle, even though the rotation through \( \theta + 2\pi \) takes an extra trip around the circle. In general, two radian measures represent the same angle if the corresponding rotations differ by an integer multiple of \( 2\pi \). For example, \( \pi/4 \), \( 9\pi/4 \), and \( -15\pi/4 \) all represent the same angle because they differ by multiples of \( 2\pi \):

\[
\frac{\pi}{4} = \frac{9\pi}{4} - 2\pi = -\frac{15\pi}{4} + 4\pi
\]

Every angle has a unique radian measure satisfying \( 0 \leq \theta < 2\pi \). With this choice, the angle \( \theta \) subtends an arc of length \( \theta r \) on a circle of radius \( r \) (Figure 2).

Degrees are defined by dividing the circle (not necessarily the unit circle) into 360 equal parts. A degree is \( \frac{180}{\pi} \) of a circle. A rotation through \( \theta \) degrees (denoted \( \theta^\circ \)) is a rotation through the fraction \( \theta/360 \) of the complete circle. For example, a rotation through \( 90^\circ \) is a rotation through the fraction \( \frac{90}{360} \) or \( \frac{1}{4} \), of a circle.

As with radians, the degree measure of an angle is not unique. Two degree measures represent the same angle if they differ by an integer multiple of 360. For example, the angles \(-45^\circ \) and \( 675^\circ \) coincide because \( 675 = -45 + 2(360) \). Every angle has a unique degree measure \( \theta \) with \( 0 \leq \theta < 360 \).

To convert between radians and degrees, remember that \( 2\pi \) radians is equal to \( 360^\circ \). Therefore, 1 radian equals \( 360/2\pi \) or \( 180/\pi \) degrees.

- To convert from radians to degrees, multiply by \( 180/\pi \).
- To convert from degrees to radians, multiply by \( \pi/180 \).
EXAMPLE 1 Convert (a) 55° to radians and (b) 0.5 radians to degrees.

Solution

(a) \( 55^\circ \times \frac{\pi}{180^\circ} \approx 0.9599 \) radians

(b) \( 0.5 \text{ radians} \times \frac{180^\circ}{\pi} \approx 28.648^\circ \)

Convention Unless otherwise stated, we always measure angles in radians.

The trigonometric functions sine and cosine can be defined in terms of right triangles. Let \( \theta \) be an acute angle in a right triangle, and let us label the sides as in Figure 3. Then

\[
\sin \theta = \frac{b}{c} = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos \theta = \frac{a}{c} = \frac{\text{adjacent}}{\text{hypotenuse}}
\]

A disadvantage of this definition is that it makes sense only if \( \theta \) lies between 0 and \( \pi/2 \) (because an angle in a right triangle cannot exceed \( \pi/2 \)). However, sine and cosine can be defined for all angles in terms of the unit circle. Let \( P = (x, y) \) be the point on the unit circle corresponding to the angle \( \theta \), as in Figures 4(A) and (B), and define

\[
\cos \theta = x-, \text{coordinate of } P, \quad \sin \theta = y-, \text{coordinate of } P
\]

This agrees with the right-triangle definition when \( 0 < \theta < \frac{\pi}{2} \). On the circle of radius \( r \) (centered at the origin), the point corresponding to the angle \( \theta \) has coordinates

\[
(r \cos \theta, r \sin \theta)
\]

Furthermore, we see from Figure 4(C) that \( f(\theta) = \sin \theta \) is an odd function and \( f(\theta) = \cos \theta \) is an even function:

\[
\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta
\]

Although we can use a calculator to evaluate sine and cosine for general angles, the standard values listed in Figure 5 and Table 2 appear often and are worth knowing.

The graph of \( y = \sin \theta \) is the familiar "sine wave" shown in Figure 6. Observe how the graph is generated by the \( y \)-coordinate of the point \( P = (\cos \theta, \sin \theta) \) moving around the unit circle.
TABLE 2

<table>
<thead>
<tr>
<th>θ</th>
<th>0</th>
<th>π/6</th>
<th>π/4</th>
<th>π/3</th>
<th>π/2</th>
<th>2π/3</th>
<th>3π/4</th>
<th>5π/6</th>
<th>π</th>
</tr>
</thead>
<tbody>
<tr>
<td>sinθ</td>
<td>0</td>
<td>1/2</td>
<td>√2/2</td>
<td>√2/2</td>
<td>1</td>
<td>√2/2</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>cosθ</td>
<td>1</td>
<td>√2/2</td>
<td>√2/2</td>
<td>1</td>
<td>0</td>
<td>-1/2</td>
<td>-√2/2</td>
<td>-√2/2</td>
<td>-1</td>
</tr>
</tbody>
</table>

The graph of \( y = \sin \theta \) is generated as the point \( P = (\cos \theta, \sin \theta) \) moves around the unit circle.

The graph of \( y = \cos \theta \) has the same shape but is shifted to the left \( \pi/2 \) units (Figure 7). The signs of \( \sin \theta \) and \( \cos \theta \) vary as \( P = (\cos \theta, \sin \theta) \) changes quadrant.

A function \( f \) is called periodic with period \( T \) if \( f(x + T) = f(x) \) (for all \( x \)) and \( T \) is the smallest positive number with this property. The sine and cosine functions are periodic with period \( T = 2\pi \) (Figure 8) because the radian measures \( x \) and \( x + 2\pi k \) correspond to the same point on the unit circle for any integer \( k \):

\[
\sin x = \sin(x + 2\pi k), \quad \cos x = \cos(x + 2\pi k)
\]

There are four other standard trigonometric functions, each defined in terms of \( \sin x \) and \( \cos x \) or as ratios of sides in a right triangle (Figure 9):

- **Tangent:** \( \tan x = \frac{\sin x}{\cos x} = \frac{b}{a} \)
- **Cotangent:** \( \cot x = \frac{\cos x}{\sin x} = \frac{c}{a} \)
- **Secant:** \( \sec x = \frac{1}{\cos x} = \frac{c}{a} \)
- **Cosecant:** \( \csc x = \frac{1}{\sin x} = \frac{c}{b} \)

These functions are periodic (Figure 10): \( y = \tan x \) and \( y = \cot x \) have period \( \pi \), and \( y = \sec x \) and \( y = \csc x \) have period \( 2\pi \) (see Exercise 57).
**EXAMPLE 2** Computing Values of Trigonometric Functions  Find the values of the six trigonometric functions at $x = 4\pi/3$.

**Solution** The point $P$ on the unit circle corresponding to the angle $x = 4\pi/3$ lies opposite the point with angle $\pi/3$ (Figure 11). It follows that $P = (-1/2, -\sqrt{3}/2)$, and therefore

$$\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}, \quad \cos \frac{4\pi}{3} = -\frac{1}{2}$$

The remaining values are

$$\tan \frac{4\pi}{3} = \frac{\sin 4\pi/3}{\cos 4\pi/3} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3}, \quad \csc \frac{4\pi}{3} = \frac{1}{\sin 4\pi/3} = \frac{-2\sqrt{3}}{3}$$

**EXAMPLE 3** Find the angles $x$ such that $\cos x = 1/2$.

**Solution** From Figure 12, we see that $x = \pi/3$ and $x = -\pi/3$ are solutions. We may add any integer multiple of $2\pi$, so the general solution is $x = \pm\pi/3 + 2\pi k$ for any integer $k$.

**EXAMPLE 4** Sketch the graph of $f(x) = 3\cos(2\left(x + \frac{\pi}{2}\right))$ over $[0, 2\pi]$.

**Solution** The graph is obtained by scaling and shifting the graph of $y = \cos x$ in three steps (Figure 13):

- Compress horizontally by a factor of 2: $y = \cos 2x$
- Shift to the left $\pi/2$ units: $y = \cos \left(2\left(x + \frac{\pi}{2}\right)\right)$
- Expand vertically by a factor of 3: $y = 3\cos \left(2\left(x + \frac{\pi}{2}\right)\right)$
In the previous example the coefficient 3 of the cosine term is referred to as the amplitude of the oscillation. The idea is that the function oscillates by 3 above and below a central value of 0.

The sine function lends itself well to modeling oscillatory phenomena. By scaling and translating in both the vertical and horizontal directions, we can fit a sine curve to data representing many different relationships. In the next example, we use the sine function to model the varying day length throughout the year.

**Example 5 A Day-Length Model** The function \( L(t) = 12 + 3.1 \sin\left(\frac{\pi}{12} t\right) \) approximates the length of a day, in hours from sunrise to sunset, in Orange City, Iowa, where \( t \) represents the day in the year assuming \( t = 0 \) is the spring equinox on March 21 (Figure 14). What are the lengths of the longest day and the shortest day? What are the day lengths on May 1, August 1, and November 1?

**Solution** The function \( L \) oscillates with amplitude 3.1 on either side of a central value of 12. According to the model, the longest day is approximately 15.1 hours long, and the shortest is approximately 8.9 hours long.

May 1, August 1, and November 1 correspond to \( t = 41, t = 133, \) and \( t = 225 \), respectively. Evaluating \( L(t) \) at each of these values of \( t \), we find that the day lengths are approximately 14.0 hours on May 1, 14.3 hours on August 1, and 10.0 hours on November 1.

This model \( L(t) \), with amplitude 3.1, works for locations along the same latitude as Orange City, Iowa, 42 degrees north. At other latitudes, different amplitudes would need to be used. In more northern latitudes the days are longer in the summer and shorter in the winter, and a greater amplitude would be used. At more southern latitudes (in the Northern Hemisphere), smaller amplitudes would be used (see Exercises 41 and 42).

**Trigonometric Identities**

A key feature of trigonometric functions is that they satisfy a large number of identities. First and foremost, sine and cosine satisfy a fundamental identity, which is equivalent to the Pythagorean Theorem:

\[
\sin^2 x + \cos^2 x = 1
\]

Equivalent versions are obtained by dividing Eq. (1) by \( \cos^2 x \) or \( \sin^2 x \):

\[
\tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x
\]

Here is a list of some other commonly used identities. The identities for complementary angles are justified by Figure 15.

**Basic Trigonometric Identities**

**Complementary angles:** \( \sin\left(\frac{\pi}{2} - x\right) = \cos x, \quad \cos\left(\frac{\pi}{2} - x\right) = \sin x \)

**Addition formulas:** \( \sin(x + y) = \sin x \cos y + \cos x \sin y \)
\( \cos(x + y) = \cos x \cos y - \sin x \sin y \)

**Double-angle formulas:** \( \sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \)
\( \cos 2x = \cos^2 x - \sin^2 x, \quad \sin 2x = 2 \sin x \cos x \)

**Shift formulas:** \( \sin\left(x + \frac{\pi}{2}\right) = \cos x, \quad \cos\left(x + \frac{\pi}{2}\right) = -\sin x \)
EXAMPLE 6 For $\theta$ between 0 and $2\pi$, the equation $\cos \theta = \frac{2}{5}$ has a solution in $(0, \frac{\pi}{2})$ and a solution in $(\frac{3\pi}{2}, 2\pi)$. Calculate $\tan \theta$ in each case.

Solution First, using the identity $\cos^2 \theta + \sin^2 \theta = 1$, we obtain

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = \pm \sqrt{1 - \frac{4}{25}} = \pm \frac{\sqrt{21}}{5}$$

If $0 < \theta < \frac{\pi}{2}$, then $\sin \theta$ is positive and we take the positive square root:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{21}/5}{2/5} = \frac{\sqrt{21}}{2}$$

To visualize this computation, draw a right triangle with angle $\theta$ such that $\cos \theta = \frac{2}{5}$ as in Figure 16. The opposite side then has length $\sqrt{21} = \sqrt{5^2 - 2^2}$ by the Pythagorean Theorem.

If $\frac{3\pi}{2} < \theta < 2\pi$, then $\sin \theta$ is negative and $\tan \theta = -\frac{\sqrt{21}}{2}$.

We conclude this section by quoting the Law of Cosines (Figure 17), which is a generalization of the Pythagorean Theorem (see Exercise 62).

**THEOREM 1 Law of Cosines** If a triangle has sides $a$, $b$, and $c$, and $\theta$ is the angle opposite side $c$, then

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

If $\theta = \pi/2$, then $\cos \theta = 0$ and the Law of Cosines reduces to the Pythagorean Theorem.

1.4 SUMMARY

- An angle of $\theta$ radians subtends an arc of length $\theta r$ on a circle of radius $r$.
- To convert from radians to degrees, multiply by $180/\pi$.
- To convert from degrees to radians, multiply by $\pi/180$.
- Unless otherwise stated, all angles in this text are given in radians.
- The functions $f(\theta) = \cos \theta$ and $f(\theta) = \sin \theta$ are defined in terms of right triangles for acute angles and as coordinates of a point on the unit circle for general angles (Figure 18):

$$\sin \theta = \frac{b}{c} \text{ opposite}, \quad \cos \theta = \frac{a}{c} \text{ adjacent}$$

- Basic properties of sine and cosine:
  - Periodicity: $\sin(\theta + 2\pi) = \sin \theta$, $\cos(\theta + 2\pi) = \cos \theta$
  - Parity: $\sin(-\theta) = -\sin \theta$, $\cos(-\theta) = \cos \theta$
  - Basic identity: $\sin^2 \theta + \cos^2 \theta = 1$

- The four additional trigonometric functions:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}$$
1.4 EXERCISES

Preliminary Questions
1. How is it possible for two different rotations to define the same angle?
2. Give two different positive rotations that define the angle \( \pi/4 \).
3. Give a negative rotation that defines the angle \( \pi/3 \).
4. The definition of \( \cos \theta \) using right triangles applies when (choose the correct answer):
   (a) \( 0 < \theta < \frac{\pi}{2} \)  
   (b) \( 0 < \theta < \pi \)  
   (c) \( 0 < \theta < 2\pi \)
5. What is the unit circle definition of \( \sin \theta \)?
6. How does the periodicity of \( f(\theta) = \sin \theta \) and \( f(\theta) = \cos \theta \) follow from the unit circle definition?

Exercises
1. Find the angle between 0 and \( 2\pi \) equivalent to \( 13\pi/4 \).
2. Describe \( \theta = \pi/6 \) by an angle of negative radian measure.
3. Convert from radians to degrees:
   (a) 1  
   (b) \( \frac{\pi}{3} \)  
   (c) \( \frac{5\pi}{12} \)  
   (d) \( -\frac{3\pi}{4} \)
4. Convert from degrees to radians:
   (a) 1°  
   (b) 30°  
   (c) 25°  
   (d) 120°
5. Find the lengths of the arcs subtended by the angles \( \theta \) and \( \phi \) radians in Figure 19.

![Figure 19 Circle of radius 4.](image)

6. Calculate the values of the six standard trigonometric functions for the angle \( \theta \) in Figure 20.

![Figure 20 Triangle](image)

7. Fill in the remaining values of \( (\cos \theta, \sin \theta) \) for the points in Figure 21.

![Figure 21 Unit Circle](image)

8. Find the values of the six standard trigonometric functions at \( \theta = 11\pi/6 \).

In Exercises 9–14, use Figure 21 to find all angles between 0 and \( 2\pi \) satisfying the given condition.
9. \( \cos \theta = \frac{1}{2} \)  
10. \( \tan \theta = 1 \)
11. \( \tan \theta = -1 \)  
12. \( \csc \theta = 2 \)
13. \( \sin \theta = \frac{\sqrt{3}}{2} \)  
14. \( \sec \theta = 2 \)
15. Fill in the following table of values:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \frac{\pi}{6} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{2\pi}{3} )</th>
<th>( \frac{3\pi}{4} )</th>
<th>( \frac{5\pi}{6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sec \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

16. Complete the following table of signs:

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \cot \theta )</th>
<th>( \sec \theta )</th>
<th>( \csc \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; \theta &lt; \frac{\pi}{2} )</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{\pi}{2} &lt; \theta &lt; \pi )</td>
<td></td>
<td></td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \pi &lt; \theta &lt; \frac{3\pi}{2} )</td>
<td></td>
<td></td>
<td></td>
<td>+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{3\pi}{2} &lt; \theta &lt; 2\pi )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+</td>
<td></td>
</tr>
</tbody>
</table>

17. Show that if \( \tan \theta = c \) and \( 0 \leq \theta < \pi/2 \), then \( \cos \theta = \sqrt{1 + c^2} \).
   *Hint: Draw a right triangle whose opposite and adjacent sides have lengths \( c \) and 1.*
18. Suppose that \( \cos \theta = \frac{1}{2} \).
   (a) Show that if \( 0 \leq \theta < \pi/2 \), then \( \sin \theta = \sqrt{3}/2 \) and \( \tan \theta = 1 \).
   (b) Find \( \sin \theta \) and \( \tan \theta \) if \( 3\pi/2 \leq \theta < 2\pi \).

In Exercises 19–24, assume that \( 0 \leq \theta < \pi/2 \).
19. Find \( \sin \theta \) and \( \tan \theta \) if \( \cos \theta = \frac{\sqrt{3}}{2} \).
33. Use addition formulas and the values of \( \sin \theta \) and \( \cos \theta \) for \( \theta = \frac{\pi}{4}, \frac{\pi}{4} \) to compute \( \sin \frac{\pi}{2} \) and \( \cos \frac{\pi}{2} \) exactly.

34. Use addition formulas and the values of \( \sin \theta \) and \( \cos \theta \) for \( \theta = \frac{\pi}{4}, \frac{\pi}{4} \) to compute \( \sin \frac{\pi}{2} \) and \( \cos \frac{\pi}{2} \) exactly.

In Exercises 35–38, sketch the graph over \([0, 2\pi]\).

35. \( f(\theta) = 2 \sin 4\theta \)

36. \( f(\theta) = \cos \left( 2 \left( \theta - \frac{\pi}{2} \right) \right) \)

37. \( f(\theta) = \cos \left( 2 \theta - \frac{\pi}{2} \right) \)

38. \( f(\theta) = \sin \left( 2 \left( \theta - \frac{\pi}{2} \right) + \pi \right) + 2 \)

39. Determine a function that would have a graph as in Figure 25(A), stating the period and amplitude.

40. Determine a function that would have a graph as in Figure 25(B), stating the period and amplitude.

41. During a year, the length of a day, from sunrise to sunset, in Wolf Point, Montana, varies from a shortest day of approximately 8.1 hours to a longest day of approximately 15.9 hours, while in Mexico City, the day lengths vary from 10.7 hours to 13.3 hours. For each location, determine a function \( L(t) = 12 + A \sin \left( \frac{2\pi}{365} t \right) \) that approximates the length of a day, in hours, where \( t \) represents the day in the year assuming \( t = 0 \) is the spring equinox on March 21. Compare the day lengths in each location on April 1, July 15, and November 1.

42. During a year, the length of a day, from sunrise to sunset, in Tallahassee, Florida, varies from a shortest day of approximately 9.6 hours to a longest day of approximately 14.4 hours, while in Montreal, the day lengths vary from 8.3 hours to 15.7 hours. For each location, determine a function \( L(t) = 12 + A \sin \left( \frac{2\pi}{365} t \right) \) that approximates the length of a day, in hours, where \( t \) represents the day in the year assuming \( t = 0 \) is the spring equinox on March 21. Compare the day lengths in each location on April 15, July 30, and November 15.

43. How many points lie on the intersection of the horizontal line \( y = c \) and the graph of \( y = \sin x \) for \( 0 \leq x < 2\pi? \) Hint: The answer depends on \( c \).

44. How many points lie on the intersection of the horizontal line \( y = c \) and the graph of \( y = \tan x \) for \( 0 \leq x < 2\pi? \)

In Exercises 45–46, solve for \( 0 \leq \theta < 2\pi \).

45. \( \sin \theta = \sin 29 \) Hint: Use the double-angle formula for sine.
46. $\sin \theta = \cos 2\theta$ Hint: Use appropriate identities to express $\cos 2\theta$ in terms of the sine function.

In Exercises 47–55, derive the identity using the identities listed in this section.

47. $\cos 2\theta = 2 \cos^2 \theta - 1$

48. $\cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}$

49. $\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$

50. $\sin(\theta + \pi) = -\sin \theta$

51. $\cos(\theta + \pi) = -\cos \theta$

52. $\tan \frac{x}{2} = \cot \left( \frac{\pi}{2} - x \right)$

53. $\tan(\pi - \theta) = -\tan \theta$

54. $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

55. $\tan x = \frac{\sin 2x}{1 + \cos 2x}$

56. $\sin^2 x \cos^2 x = \frac{1 - \cos 4x}{8}$

57. Use Exercises 50 and 51 to show that $\tan \theta$ and $\cot \theta$ are periodic with period $\pi$.

58. Use the double-angle formulas to show that $\sin^2 \theta$ and $\cos^2 \theta$ are periodic with period $\pi$.

59. Use the identity of Exercise 48 to show that $\cos \frac{\pi}{4}$ is equal to $\frac{\sqrt{2}}{2}$.

60. Use Exercise 55 to compute $\tan \frac{\pi}{4}$.

61. Use the Law of Cosines to find the distance from $P$ to $Q$ in Figure 26.

**Further Insights and Challenges**

62. Use Figure 27 to derive the Law of Cosines from the Pythagorean Theorem.

63. Use the addition formula to prove $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

64. Use the addition formulas for sine and cosine to prove $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$ and $\cot(a - b) = \frac{\cot a \cot b + 1}{\cot b - \cot a}$.

65. Let $\theta$ be the angle between the line $y = mx + b$ and the x-axis [Figure 28(A)]. Prove that $m = \tan \theta$.

66. Let $L_1$ and $L_2$ be the lines of slope $m_1$ and $m_2$ [Figure 28(B)]. Show that the angle $\theta$ between $L_1$ and $L_2$ satisfies $\cot \theta = \frac{m_2 m_1 + 1}{m_1 m_2 - 1}$.

67. Perpendicular Lines Use Exercise 66 to prove that two lines with nonzero slopes $m_1$ and $m_2$ are perpendicular if and only if $m_2 = -1/m_1$.

68. Apply the double-angle formula to prove:

(a) $\cos \frac{\pi}{8} = \frac{1}{2} \sqrt{2 + \sqrt{2}}$

(b) $\cos \frac{\pi}{16} = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}$

Guess the values of $\cos \frac{\pi}{32}$ and of $\cos \frac{\pi}{2^n}$ for all $n$.

**1.5 Technology: Calculators and Computers**

Computer technology has vastly extended our ability to calculate and visualize mathematical relationships. In applied settings, computers are indispensable for solving complex systems of equations and analyzing data, as in weather prediction and medical imaging. Mathematicians use computers to study complex structures and relationships such as the
geometry and symmetry of cubes of dimensions higher than 3 (Figure 1). We take advantage of this technology to explore the ideas of calculus visually and numerically.

**FIGURE 1** Representations of cubes of dimension 4, 6, 8, and 10.

When we plot a function with a graphing calculator or computer algebra system, the graph is contained within a **viewing rectangle**, the region determined by the range of x- and y-values in the plot. We write $[a, b] \times [c, d]$ to denote the rectangle for which $a \leq x \leq b$ and $c \leq y \leq d$.

The appearance of the graph depends heavily on the choice of viewing rectangle. Different choices may convey very different impressions that are sometimes misleading. Compare the three viewing rectangles for the graph of $f(x) = 12 - x - x^2$ in Figure 2. Only (A) successfully displays the shape of the graph as a parabola. In (B), the graph is cut off, and no graph at all appears in (C). Keep in mind that the scales along the axes may change with the viewing rectangle. For example, the unit increment along the y-axis is larger in (B) than in (A), so the graph in (B) is steeper.

**FIGURE 2** Viewing rectangles for the graph of $f(x) = 12 - x - x^2$.

(A) $[-6, 5] \times [-18, 18]$  
(B) $[-6, 5] \times [-4, 4]$  
(C) $[-1, 2] \times [-3, 1]$

There is no single "correct" viewing rectangle. The goal is to select the viewing rectangle that displays the properties you wish to investigate. This usually requires experimentation.

**EXAMPLE 1** How Many Roots and Where? How many real roots does the function $f(x) = x^3 - 20x + 1$ have? Find their approximate locations.

**Solution** We experiment with several viewing rectangles (Figure 3). Our first attempt (A) displays a cut-off graph, so we try a viewing rectangle that includes a larger range of y-values. Plot (B) shows that the roots of $f$ probably lie somewhere in the interval $[-3, 3]$, but it does not reveal how many real roots there are. Therefore, we try the viewing rectangle in (C). Now we can see clearly that $f$ has three roots. A further zoom in (D) shows that these roots are located near $-1.5, 0.1$, and $1.5$. Further zooming would provide their locations with greater accuracy.

**FIGURE 3** Graphs of $f(x) = x^3 - 20x + 1$. 

(A) $[-12, 12] \times [-10, 10]$  
(B) $[-12, 12] \times [-10000, 10000]$  
(C) $[-2, 2] \times [-100, 100]$  
(D) $[-2, 2] \times [-20, 25]$
**EXAMPLE 2** Does a Solution Exist? Does \( \cos x = \tan x \) have a solution? Describe the set of all solutions.

**Solution** The solutions of \( \cos x = \tan x \) are the \( x \)-coordinates of the points where the graphs of \( y = \cos x \) and \( y = \tan x \) intersect. Figure 4(A) shows that there are two solutions in the interval \([0, 2\pi]\). By zooming in on the graph as in (B), we see that the first positive root lies between 0.6 and 0.7, and the second positive root lies between 2.4 and 2.5. Further zooming shows that the first root is approximately 0.67 (Figure 4(C)). Continuing this process, we find that the first two roots are \( x \approx 0.666 \) and \( x \approx 2.475 \).

Since \( f(x) = \cos x \) is periodic with period \( 2\pi \), and \( f(x) = \tan x \) is periodic with period \( \pi \), the picture repeats itself with period \( 2\pi \). All solutions are obtained by adding multiples of \( 2\pi \) to the two solutions in \([0, 2\pi]\):

\[
x \approx 0.666 + 2\pi k \quad \text{and} \quad x \approx 2.475 + 2\pi k \quad \text{for any integer} \ k
\]

**FIGURE 4** Graphs of \( y = \cos x \) and \( y = \tan x \).

**EXAMPLE 3** Functions with Asymptotes Plot the function \( f(x) = \frac{1 - 3x}{x - 2} \) and describe its asymptotic behavior.

**Solution** First, we plot \( f \) in the viewing rectangle \([-10, 20] \times [-5, 5] \), as in Figure 6(A). The vertical line \( x = 2 \) is called a **vertical asymptote**. Many graphing calculators display this line, but it is not part of the graph (and it can usually be eliminated by choosing a smaller range of \( y \)-values). We see that \( f(x) \) tends to \( \infty \) as \( x \) approaches 2 from the left, and to \(-\infty \) as \( x \) approaches 2 from the right. To display the horizontal asymptotic behavior of \( f \), we use the viewing rectangle \([-10, 20] \times [-10, 5] \) [Figure 6(B)]. Here, we see that the graph approaches the horizontal line \( y = -3 \), called a **horizontal asymptote** (which we have added as a dashed horizontal line in the figure).

**FIGURE 6** Graphs of \( f(x) = \frac{1 - 3x}{x - 2} \).

Calculators and computer algebra systems give us the freedom to experiment numerically. For instance, we can explore the behavior of a function by constructing a table of values. In the next example, we investigate a function related to the exponential function.
Example 4 Investigating the Behavior of a Function How does \( f(n) = (1 + 1/n)^n \) behave for large positive whole-number values of \( n \)?

Solution First, we make a table of values of \( f(n) \) for larger and larger positive values of \( n \). Table 1 suggests that \( f(n) \) appears to get closer to some value near 2.718. This is an example of limiting behavior that we will discuss in Chapter 2. In fact, the value of this limit is the important numerical value \( e \), the base of the natural exponential and logarithmic functions that we will examine in Chapter 7. Next, replace \( n \) by the variable \( x \) and plot the function \( f(x) = (1 + 1/x)^x \). The graphs in Figure 7 also suggest that \( f(x) \) approaches a limiting value near 2.7.

![Figure 7](image)

Example 5 Bird Flight: Finding a Minimum Graphically According to one model of bird flight, the power consumed by a pigeon flying at velocity \( v \) (in meters per second) is \( P(v) = 17v^{-1} + 10^{-3}v^3 \) (in joules per second). Use a graph of \( P \) to find the velocity that minimizes power consumption.

Solution The velocity that minimizes power consumption corresponds to the lowest point on the graph of \( P \). We plot \( P \) first in a large viewing rectangle (Figure 8). This figure reveals the general shape of the graph and shows that \( P \) takes on a minimum value for \( v \) somewhere between \( v = 8 \) and \( v = 9 \). In the viewing rectangle \([8, 9.2] \times [2.6, 2.65]\), we see that the minimum occurs at approximately \( v = 8.65 \) m/s.

![Figure 8](image)

Local linearity is an important concept in calculus that is based on the idea that many functions are nearly linear over small intervals. Local linearity can be illustrated effectively with a graphing calculator.

Example 6 Illustrating Local Linearity Illustrate local linearity for the function \( f(x) = \sin x \) at \( x = 1 \).

Solution First, we plot \( f(x) = x^\sin x \) in the viewing window of Figure 9(A). The graph moves up and down and appears very wavy. However, as we zoom in, the graph straightens out. Figures (B)–(D) show the result of zooming in on the point \((1, f(1))\). When viewed up close, the graph looks like a straight line. This illustrates the local linearity of \( f \) at \( x = 1 \).
1.5 SUMMARY

- The appearance of a graph on a graphing calculator depends on the choice of viewing rectangle. Experiment with different viewing rectangles until you find one that displays the information you want. Keep in mind that the scales along the axes may change as you vary the viewing rectangle.
- The following are some ways in which graphing calculators and computer algebra systems can be used in calculus:
  - Visualizing the behavior of a function
  - Finding solutions graphically or numerically
  - Conducting numerical or graphical experiments
  - Illustrating theoretical ideas (such as local linearity)

1.5 EXERCISES

Preliminary Questions

1. Is there a definite way of choosing the optimal viewing rectangle, or is it best to experiment until you find a viewing rectangle appropriate to the problem at hand?

2. Describe the calculator screen produced when the function \( y = 3 + x^2 \) is plotted with a viewing rectangle:
   - (a) \([-1, 1] \times [0, 2]\)
   - (b) \([0, 1] \times [0, 4]\)

3. According to the evidence in Example 4, it appears that \( f(n) = (1 + 1/n)^n \) never takes on a value greater than 3 for \( n > 0 \). Does this evidence prove that \( f(n) \leq 3 \) for \( n > 0 \)?

4. How can a graphing calculator be used to find the minimum value of a function?

Exercises

The exercises in this section should be done using a graphing calculator or computer algebra system.

1. Plot \( f(x) = 2x^4 + 3x^3 - 14x^2 - 9x + 18 \) in the appropriate viewing rectangles and determine its roots.

2. How many solutions does \( x^3 - 4x + 8 = 0 \) have?

3. How many positive solutions does \( x^3 - 12x + 8 = 0 \) have?

4. Does \( \cos x + x = 0 \) have a solution? A positive solution?

5. Find all the solutions of \( \sin x = \sqrt{x} \) for \( x > 0 \).

6. How many solutions does \( \cos x + x^2 \) have?

7. Let \( f(x) = (x - 100)^2 + 1000 \). What will the display show if you graph \( f \) in the viewing rectangle \([-10, 10] \) by \([-10, 10] \)? Find an appropriate viewing rectangle.

8. Plot \( f(x) = \frac{8x + 1}{8x - 8} \) in an appropriate viewing rectangle. What are the vertical and horizontal asymptotes?

9. Plot the graph of \( f(x) = x/(4 - x) \) in a viewing rectangle that clearly displays the vertical and horizontal asymptotes.

10. Illustrate local linearity for \( f(x) = x^2 \) by zooming in on the graph at \( x = 0.5 \) (see Example 6).

11. Plot \( f(x) = \cos(x^2) \sin x \) for \( 0 \leq x \leq 2\pi \). Then illustrate local linearity at \( x = 3.8 \) by choosing appropriate viewing rectangles.

12. By zooming in on the graph of \( f(x) = \sqrt{x} \) at \( x = 0 \), examine the local linearity. How does the resulting "line" appear?

13. By examining the graph of \( f(x) = 2x^3 - x^3 - 3x^4 \) in appropriate viewing rectangles, approximate the maximum value of \( f(x) \) and the value of \( x \) at which it occurs.

14. (a) Plot the graph of \( f(x) = \frac{2x^2 + 1}{x^2 - 4} \) in a viewing rectangle that clearly shows the two vertical asymptotes.
   (b) By examining the graph of \( f \) in appropriate viewing rectangles, approximate the minimum value of \( f(x) \) between the vertical asymptotes, and approximate the value of \( x \) at which the minimum occurs.
15. If $500 is deposited in a bank account paying 3.5% interest compounded monthly, then the account has value $V(N) = 500 \left(1 + \frac{0.035}{12}\right)^N$ dollars after $N$ months. By examining the graph of $V(N)$, find, to the nearest integer $N$, the number of months it takes for the account value to reach $5000.

16. If $1000 is deposited in a bank account paying 5% interest compounded monthly, then the account has value $V(N) = 1000 \left(1 + \frac{0.05}{12}\right)^N$ dollars after $N$ months. By examining the graph of $V(N)$, find, to the nearest integer $N$, the number of months it takes for the account value to double.

In Exercises 17-22, investigate the behavior of the function as $n$ or $x$ grows large by making a table of function values and plotting a graph (see Example 4). Describe the behavior in words.

17. $f(n) = n^{1/3}$
18. $f(n) = \frac{4n + 1}{6n - 5}$

Further Insights and Challenges

27. (CAS) Let $f_1(x) = x$ and define a sequence of functions by $f_{n+1}(x) = \frac{1}{2} \left( f_n(x) + x / f_n(x) \right)$. For example, $f_2(x) = 3x + 1$. Use a computer algebra system to compute $f_n(x)$ for $n = 3, 4, 5$ and plot $y = f_n(x)$ together with $y = \sqrt{x}$ for $x \geq 0$. What do you notice?

28. Set $P_0(x) = 1$ and $P_1(x) = x$. The Chebyshev polynomials (useful in approximation theory) are defined inductively by the formula $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$.

(a) Show that $P_2(x) = 2x^2 - 1$.
(b) Compute $P_n(x)$ for $3 \leq n \leq 6$ using a computer algebra system or by hand, and plot $y = P_n(x)$ over $[-1, 1]$.
(c) Check that your plots confirm the following important properties of the Chebyshev polynomials. (A) $P_n(x)$ has $n$ real roots in $[-1, 1]$, and (B) for $x \in [-1, 1]$, $P_n(x)$ lies between $-1$ and $1$.

CHAPTER REVIEW EXERCISES

1. Match each quantity (a)-(d) with (i), (ii), or (iii) if possible, or state that no match exists.
   (a) $2a^2b^3$
   (b) $2^{a^2}b^3$
   (c) $2^{a^2b^3} - b^2 - a$
   (d) $2^{a^2}b^3 - a$

2. Indicate which of the following are correct, and correct the ones that are not.
   (a) $5^2 \cdot 5^3 = 5^5$
   (b) $(\sqrt{8})^{4/3} = 8^{4/3}$
   (c) $\frac{3^2}{3^2} = 3^2$
   (d) $(2^4)^{-2} = 2^8$

3. Express $4\sqrt{10}$ as a set $\{x : |x - a| < c\}$ for suitable $a$ and $c$.

4. Express as an interval:
   (a) $[x : |x - 5| < 4]$
   (b) $[x : |5x + 3| \leq 2]$

5. Express $x:2 \leq |x - 1| \leq 6$ as a union of two intervals.

6. Give an example of numbers $x, y$ such that $|x| + |y| = x - y$.

7. Describe the pairs of numbers $x, y$ such that $|x + y| = x - y$.

8. Sketch the graph of $y = f(x + 2) - 1$, where $f(x) = x^2$ for $-2 \leq x \leq 2$.

In Exercises 9-12, let $f$ be the function whose graph is shown in Figure 1.

9. Sketch the graphs of $y = f(x) + 2$ and $y = f(x - 2)$.

10. Sketch the graphs of $y = \frac{1}{2}f(x)$ and $y = f\left(\frac{1}{2}x\right)$.

11. Continue the graph of $f$ to the interval $[-4, 4]$ as an even function.

12. Continue the graph of $f$ to the interval $[-4, 4]$ as an odd function.

13. $f(x) = \sqrt{x+1}$
14. $f(x) = \frac{4}{x^2 + 1}$
15. $f(x) = \frac{2}{3 - x}$
16. $f(x) = \sqrt{x^2 + 5}$

17. Determine whether the function is increasing, decreasing, or neither:
   (a) $f(x) = \sqrt{8 - x}$
   (b) $f(x) = \frac{1}{x^2 + 1}$
   (c) $g(t) = t^2 + 1$
   (d) $g(t) = t^3 + 1$

18. Determine whether the function is even, odd, or neither:
   (a) $f(x) = x^4 - 3x^2$
   (b) $g(x) = \sin(x + 1)$
In Exercises 19–26, find the equation of the line.
19. Line passing through (−1, 4) and (2, 6)
20. Line passing through (−1, 4) and (−1, 6)
21. Line of slope 6 through (9, 1)
22. Line of slope \(-\frac{1}{2}\) through (4, −12)
23. Line through (2, 1) perpendicular to the line given by \(y = 3x + 7\)
24. Line through (3, 4) perpendicular to the line given by \(y = 4x - 2\)
25. Line through (2, 3) parallel to \(y = 4 - x\)
26. Horizontal line through (−3, 5)
27. Does the following table of market data suggest a linear relationship between price and number of homes sold during a 1-year period? Explain.

<table>
<thead>
<tr>
<th>Price (thousands of $)</th>
<th>180</th>
<th>195</th>
<th>220</th>
<th>240</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of homes sold</td>
<td>127</td>
<td>118</td>
<td>103</td>
<td>91</td>
</tr>
</tbody>
</table>

28. Does the following table of revenue data for a computer manufacturer suggest a linear relation between revenue and time? Explain.

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2009</th>
<th>2011</th>
<th>2014</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>13</td>
<td>18</td>
<td>15</td>
<td>11</td>
</tr>
</tbody>
</table>

29. Suppose that a cell phone plan that is offered at a price of \(P\) dollars per month attracts \(C\) customers, where \(C(P)\) is a linear demand function for \(100 \leq P \leq 500\). Assume \(C(100) = 1,000,000\) and \(C(500) = 100,000\).
   (a) Determine the demand function \(C(P)\).
   (b) What is the slope of the graph of \(C(P)\)? Describe what the slope represents.
   (c) What is the decrease in the number of customers for each increase of $100 in the price?

30. Suppose that Internet domain names are sold at a price of \(SP\) per month for $2 \leq P \leq $100. The number of customers \(C\) who buy the domain names is a linear function of the price. Assume that 10,000 customers buy a domain name when the price is $2 per month, and 1000 customers buy when the price is $100 per month.
   (a) Determine the demand function \(C(P)\).
   (b) What is the slope of the graph of \(C(P)\)? Describe what the slope represents.
   (c) Find the roots of \(f(x) = x^4 - 4x^2\) and sketch its graph. On which intervals is \(f\) decreasing?
   (d) Let \(h(x) = -2x^2 + 12x + 3\). Complete the square and find the maximum value of \(h\).
   (e) Let \(f(x)\) be the square of the distance from the point (2, 1) to a point \((x, 3x + 2)\) on the line \(y = 3x + 2\). Show that \(f\) is a quadratic function, and find its minimum value by completing the square.
   (f) Prove that \(x^2 + 3x + 3 \geq 0\) for all \(x\).

In Exercises 35–40, sketch the graph by hand.
35. \(y = r^4\)
36. \(y = r^5\)
37. \(y = \sin \frac{\theta}{2}\)
38. \(y = |x - 3|\)
39. \(y = x^{1/3}\)
40. \(y = \frac{1}{x^2}\)

41. Show that the graph of \(y = f(\frac{1}{2}x - b)\) is obtained by shifting the graph of \(y = f(\frac{1}{2}x)\) to the right \(2b\) units. Use this observation to sketch the graph of \(y = |\frac{1}{2}x - 4|\).

42. Let \(h(x) = \cos x\) and \(g(x) = x^{-1}\). Compute the composite functions \(h \circ g\) and \(g \circ h\), and find their domains.

43. Find functions \(f\) and \(g\) such that the function 
   \(f(g(t)) = (12t + 9)^6\)
   Sketch the points on the unit circle corresponding to the following three angles, and find the values of the six standard trigonometric functions at each angle:
   (a) \(\frac{2\pi}{3}\)  (b) \(\frac{7\pi}{4}\)  (c) \(\frac{5\pi}{6}\)
45. What are the periods of these functions?
   (a) \(y = \sin 2\theta\)  (b) \(y = \sin \frac{\theta}{2}\)
   (c) \(y = \sin 2\theta + \sin \frac{\theta}{2}\)
46. Determine \(A\), \(B\), and \(C\) so that \(f(x) = A \cos(Bx) + C\) cycles once from \(8\) to \(-2\) and back to \(8\) as \(x\) goes from \(0\) to \(2\).
47. \(H(t) = A \sin(Bt) + C\) models the height (in meters) of the tide in Happy Harbor at time \(t\) (hours since midnight) in a day. Determine \(A\), \(B\), and \(C\) if the high tide of \(18\) m occurs at \(6\) AM and the subsequent low tide of \(15\) m occurs at \(6\) PM.
48. Assume that \(\sin \theta = \frac{3}{5}\), where \(\pi/2 < \theta < \pi\). Find:
   (a) \(\tan \theta\)  (b) \(\sin 2\theta\)  (c) \(\csc \frac{\theta}{2}\)
49. Give an example of values \(a\) and \(b\) such that
   (a) \(\cos(a + b) \neq \cos a + \cos b\)  (b) \(\cos \frac{a}{2} \neq \frac{\cos a}{2}\)
50. Let \(f(x) = \cos x\). Sketch the graph of \(y = 2f\left(\frac{1}{2}x - \frac{3}{4}\right)\) for \(0 \leq x \leq 6\pi\).
51. Solve \(\sin 2x + \cos x = 0\) for \(0 \leq x \leq 2\pi\).
52. How does \(h(x) = \sqrt[3]{x}\) behave for large whole-number values of \(x\)\
    Does \(h(n)\) tend to infinity?
53. Use a graphing calculator to determine whether the equation \(\cos x = 5x^2 - 8x^4\) has any solutions.
54. Using a graphing calculator, find the number of real roots and estimate the largest root to two decimal places:
   (a) \(f(x) = 1.8x^4 - x^3 - x\)
   (b) \(g(x) = 1.7x^4 - x^3 - x\)
2 LIMITS

Calculus is usually divided into two branches, differential and integral, partly for historical reasons. The subject grew out of efforts in the seventeenth century to solve two important geometric problems: finding tangent lines to curves (differential calculus) and computing areas under curves (integral calculus). However, calculus is a broad subject with no clear boundaries. It includes other topics, such as the theory of infinite series, and it has an extraordinarily wide range of applications. What makes these methods and applications part of calculus is that they all rely on the concept of a limit. We will see throughout the text how limits allow us to make computations and solve problems that cannot be solved using algebra alone.

This chapter introduces the limit concept and sets the stage for our study of the derivative in Chapter 3. The first section, intended as motivation, discusses how limits arise in the study of instantaneous velocity and tangent lines.

2.1 The Limit Idea: Instantaneous Velocity and Tangent Lines

A limit describes how the values \( f(x) \), of a function \( f \), behave as \( x \) approaches a number \( a \). Does \( f(x) \) get closer and closer to a number \( L \)? If so, we say \( L \) is the limit of \( f(x) \) as \( x \) approaches \( a \). If not, we say the limit does not exist.

Limits play a key role throughout calculus. In this section, we motivate the need for the limit concept by examining two closely related questions that are central to differential calculus:

- How do we define and compute instantaneous velocity at a particular time?
- How do we define and compute the slope of the line tangent to a graph at a particular point?

When we speak of velocity, we usually mean instantaneous velocity, which indicates the speed and direction of an object at a particular moment. The idea of instantaneous velocity makes intuitive sense, but care is required to define it precisely. We do that via the concept of a limit.

Consider an object traveling along a line (linear motion). The average velocity over a given time interval has a straightforward definition as the ratio

\[
\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}
\]

Thus, average velocity is the rate of change of position with respect to time over a particular interval. For example, if an automobile travels forward 200 km in 4 h, then its average velocity during this 4-h period is \( \frac{200}{4} = 50 \text{ km/h} \). At any given moment, the automobile may be going faster or slower than the average.

Like average velocity, instantaneous velocity is the rate of change of position with respect to time, but at a particular instant (thus the term “instantaneous”) rather than over an interval. We cannot define instantaneous velocity as a ratio, as above, because we would have to divide by a change in time equal to zero. However, we can estimate instantaneous velocity by computing average velocity over a small time interval. And we can continually improve the estimate by using smaller and smaller time intervals. Ultimately, we can obtain an exact value of the instantaneous velocity as a number, which is called the limit, that the estimates approach more and more closely as the time interval shrinks to zero.

We investigate the relationship between average velocity and instantaneous velocity for the motion of an object falling to earth under the influence of gravity (assuming no air resistance).
Galileo discovered that if the object is released at time \( t = 0 \) from a state of rest (Figure 1), then the distance traveled (in meters) after \( t \) seconds is approximately given by the formula

\[
s(t) = 4.9t^2
\]

To compute average velocity over a time interval \([t_0, t_1]\), we set

\[
\Delta s = s(t_1) - s(t_0) = \text{change in position}
\]

\[
\Delta t = t_1 - t_0 = \text{change in time (length of time interval)}
\]

The change in position \( \Delta s \) is also called the displacement, or net change in position. For \( t_1 \neq t_0 \),

\[
\text{average velocity over } [t_0, t_1] = \frac{\Delta s}{\Delta t} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}
\]

**EXAMPLE 1** A ball is dropped at time \( t = 0 \). What is the average velocity over the time interval from \( t_0 = 0 \) to \( t_1 = 0.8 \) s? Estimate the instantaneous velocity at \( t = 0.8 \) s.

**Solution** We use Galileo's formula \( s(t) = 4.9t^2 \) to compute the average velocity over the interval \([0, 0.8]\):

\[
\Delta s = s(0.8) - s(0) = 4.9(0.8)^2 - 4.9(0)^2 = 3.136 \text{ m}
\]

\[
\Delta t = 0.8 - 0 = 0.8 \text{ s}
\]

The average velocity over \([0, 0.8]\) is the ratio

\[
\frac{\Delta s}{\Delta t} = \frac{3.136}{0.8} = 3.92 \text{ m/s}
\]

To estimate the instantaneous velocity at \( t = 0.8 \) s, we examine the average velocity over the five short time intervals listed in Table 1. Consider the first interval \([0, 0.81]\):

\[
\Delta s = s(0.81) - s(0.8) = 4.9(0.81)^2 - 4.9(0.8)^2 \approx 3.2149 - 3.1360 = 0.0789 \text{ m}
\]

\[
\Delta t = 0.81 - 0.8 = 0.01 \text{ s}
\]

The average velocity over \([0.8, 0.81]\) is the ratio

\[
\frac{\Delta s}{\Delta t} = \frac{0.0789}{0.01} = 7.89 \text{ m/s}
\]

Table 1 shows the results of similar calculations for intervals of successively shorter lengths. It looks like these average velocities are getting closer to 7.84 m/s as the length of the time interval shrinks:

7.889, 7.8645, 7.8405, 7.84024, 7.840005

This suggests that 7.84 m/s is a good estimate for the instantaneous velocity at \( t = 0.8 \).

Our estimate of the instantaneous velocity is a guess at what happens to the average velocities as we shrink the time interval to 0. Formally, the limit of the average velocities is the instantaneous velocity:

Instantaneous velocity is the limit of average velocity as the length of the time interval shrinks to zero. That is,

\[
\text{instantaneous velocity} = \lim_{\Delta t \to 0} \text{(average velocity)}
\]
To employ the concept of the limit properly in calculus, we need to define the limit, establish its properties, and develop rules for computing limits. Those are the goals of the remaining sections in this chapter.

The notion that the limit of average velocities is the instantaneous velocity is vividly revealed using graphs. Notice, first, that the ratio defining the average velocity over \([t_0, t_1]\) is the slope of the line, called the secant line, through the points \((t_0, s(t_0))\) and \((t_1, s(t_1))\) on the graph of \(s\) (Figure 2).

\[
\text{average velocity} = \text{slope of secant line} = \frac{\Delta s}{\Delta t} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}
\]

By interpreting average velocity as a slope, we can visualize what happens as the time interval gets smaller. Figure 3 shows a closeup of the graph of position for the falling ball in Example 1. As the time interval shrinks, the secant lines get closer to—and seem to rotate into—a line that appears to be tangent to the graph at \(t = 0.8\).

<table>
<thead>
<tr>
<th>Time Interval (s)</th>
<th>Average Velocity (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0.8, 0.8005]</td>
<td>7.8645</td>
</tr>
<tr>
<td>[0.8, 0.8001]</td>
<td>7.8405</td>
</tr>
<tr>
<td>[0.8, 0.80005]</td>
<td>7.8402</td>
</tr>
</tbody>
</table>

And since the secant lines approach the tangent line, the slopes of the secant lines get closer and closer to the slope of the tangent line. In other words,

\[
\text{slope of tangent line} = \lim_{\Delta t \to 0} (\text{slopes of secant lines})
\]

Now, putting together the relationships between instantaneous velocity, average velocity, slope of tangent, and slope of secant, it follows that

\[
\text{slope of tangent line} = \text{instantaneous velocity}
\]

The concepts of secant lines and tangent lines carry over to graphs of any function \(f\), not just functions representing position changing in time.

**EXAMPLE 2** The graph of \(f(x) = x^3\) is shown in Figure 4.

(a) Compute the slope of the secant line from \(x = 2\) to \(x = 3\).

(b) Compute the slope of the secant line from \(x = 2\) to \(x = P\), and then investigate the slope of the tangent line at \(x = 2\) by considering what happens to the slopes of the secant lines as we let the length of the interval from 2 to \(P\) shrink to 0.

**Solution**

(a) Using Eq. (1) for \(f(x)\) [rather than \(s(t)\)], we have that the slope of the secant line from \(x = 2\) to \(x = 3\) is

\[
\frac{f(3) - f(2)}{3 - 2} = \frac{27 - 8}{1} = 19
\]
We are using the difference-of-cubes factoring formula:
\[ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \]

\[ \begin{array}{|c|c|}
\hline
 p & p^2 + 2p + 4 \\
\hline
 1.9 & 11.41 \\
 1.99 & 11.9401 \\
 1.999 & 11.994001 \\
 1.9999 & 11.99940001 \\
 2.1 & 12.61 \\
 2.01 & 12.0601 \\
 2.001 & 12.006001 \\
 2.0001 & 12.00060001 \\
\hline
\end{array} \]

(b) Similarly, the slope of the secant line from \( x = 2 \) to \( x = P \) is
\[ \frac{f(P) - f(2)}{P - 2} = \frac{P^3 - 8}{P - 2} = \frac{(P - 2)(P^2 + 2P + 4)}{P - 2} = P^2 + 2P + 4 \]

This expression for the slope of the secant line from \( x = 2 \) to \( x = P \) holds for both \( P > 2 \) and \( P < 2 \). In Table 2 we investigate these slopes for various \( P \) near 2.

The slope of the tangent line at \( x = 2 \) is obtained from these secant-line slopes by shrinking to 0 the length of the interval from 2 to \( P \). It is apparent from the table that, in the limit as \( P \) approaches 2, the slopes of the secant lines approach 12.

In the previous example we can also see that the slopes of the secant lines approach 12 by directly examining the expression for the secant slopes. That is, as \( P \) approaches 2, the expression \( P^2 + 2P + 4 \) approaches \( 2^2 + 2(2) + 4 = 12 \). This reasoning for computing limits is justified by limit "laws" introduced in Section 2.3 and essentially amounts to substituting 2 for \( P \) in the expression for the secant slope. Sometimes taking a limit can be this simple. Other times a substitution might result in an undefined expression, in which case special care needs to be taken to understand what that means for the limit. All of these points are addressed in our investigation of limits in this chapter.

We return to the study of tangent lines, velocity (the rate of change of position with respect to time), and other rates of change in Chapter 3 when we begin our development of the derivative. For now, we need to build a good understanding of limits so that we have an appropriate foundation on which we can build the theory of calculus.

**HISTORICAL PERSPECTIVE**

Philosophy is written in this grand book—I mean the universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics...

—Galileo Galilei, 1623

The scientific revolution of the sixteenth and seventeenth centuries reached its highpoint in the work of Isaac Newton (1643–1727), who was the first scientist to show that the physical world, despite its complexity and diversity, is governed by a small number of universal laws. One of Newton's great insights was that the universal laws are dynamic, describing how the world changes over time in response to forces, rather than how the world actually is at any given moment in time. These laws are expressed best in the language of calculus, which is the mathematics of change.

More than 50 years before the work of Newton, the astronomer Johannes Kepler (1571–1630) discovered his three laws of planetary motion, the most famous of which states that the path of a planet around the sun is an ellipse. Kepler arrived at these laws through a painstaking analysis of astronomical data, but he could not explain why they were true. According to Newton, the motion of any object—planet or pebble—is determined by the forces acting on it. The planets, if left undisturbed, would travel in straight lines. Since their paths are elliptical, some force—in this case, the gravitational force of the sun—must be acting to make them change direction continuously. In his magnum opus *Principia Mathematica*, published in 1687, Newton proved that Kepler's laws follow from Newton's own universal laws of motion and gravity.

For these discoveries, Newton gained widespread fame in his lifetime. His fame continued to increase after his death, assuming a nearly mythic dimension, and his ideas had a profound influence, not only in science but also in the arts and literature, as expressed in this epitaph by British poet Alexander Pope: "Nature and Nature's Laws lay hid in Night. God said, Let Newton be, and all was Light."

### 2.1 SUMMARY

- Average velocity over \([t_0, t_1]\) = \(\frac{\text{change in position}}{\text{change in time}}\) = \(\frac{x(t_1) - x(t_0)}{t_1 - t_0}\).
- Instantaneous Velocity = \(\lim_{\Delta t \to 0} \) (average velocity)
The slope of the secant line through the points \((t_0, s(t_0))\) and \((t_1, s(t_1))\) on the graph of \(s(t)\) is \(\frac{s(t_1) - s(t_0)}{t_1 - t_0}\).  
- Slope of the tangent line = \(\lim_{t \to 0} \) (slopes of the secant lines)  
- Average velocity over an interval \([t_0, t_1]\) is the slope of the secant line through the points \((t_0, s(t_0))\) and \((t_1, s(t_1))\) on the graph of \(s(t)\).  
- Instantaneous velocity at \(t_0\) is the slope of the tangent line at \(t_0\).  
- To estimate the instantaneous velocity or tangent-line slope at \(t_0\), compute the average velocity or secant-line slope over several intervals \([t_0, t_1]\) (or \([t_1, t_0]\)) for \(t_1\) close to \(t_0\) and estimate from those values.

### 2.1 EXERCISES

#### Preliminary Questions

1. Average velocity is equal to the slope of a secant line through two points on a graph. Which graph?

2. Can instantaneous velocity be defined as a ratio? If not, how is instantaneous velocity computed?

3. With \(r\) in hours, at \(t = 0\) Dale entered Highway 1. At \(t = 2\) he was 126 miles down the highway, on the side of the road with a flat tire. At \(t = 3\) he was still on the side of the road, waiting for road assistance. What was Dale’s average velocity over each of the time intervals:
   - (a) From \(t = 0\) to \(t = 2\)
   - (b) From \(t = 0\) to \(t = 3\)
   - (c) From \(t = 2\) to \(t = 3\)

4. What is the graphical interpretation of instantaneous velocity at a specific time \(t = t_0\)?

#### Exercises

1. A ball dropped from a state of rest at time \(t = 0\) travels a distance \(s(t) = 4.9t^2\) m in \(t\) seconds. Estimate the instantaneous velocity at \(t = 3\).

2. A wrench dropped from a state of rest at time \(t = 0\) travels a distance \(s(t) = 4.9t^2\) m in \(t\) seconds. Estimate the instantaneous velocity at \(t = 3\).

3. On her bicycle ride Fabiana’s position (in km) as a function of time (in hours) is \(s(t) = 22t + 17\). What was her average velocity between \(t = 2\) and \(t = 3\)?

4. Compute \(\Delta y/\Delta x\) for the interval \([2, 5]\), where \(y = 4x - 9\). What is the slope of the tangent line at \(x = 2\)?

5. On her bicycle ride Fabiana’s position (in km) as a function of time (in hours) is \(s(t) = 22t + 17\). What was her average velocity between \(t = 2\) and \(t = 3\)?

6. Compute the average velocity for the time intervals in the table and estimate the ball’s instantaneous velocity at \(t = 2\).

<table>
<thead>
<tr>
<th>Interval</th>
<th>[2, 2.001]</th>
<th>[2, 2.005]</th>
<th>[2, 2.001]</th>
<th>[2, 2.00001]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average velocity</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Compute the ball’s average velocity over the time intervals \([3, 6]\) and estimate the instantaneous velocity at \(t = 3\).

8. Compute the stone’s average velocity over the time intervals \([1, 1.01]\), \([1, 1.001]\), \([1, 1.0001]\), and \([0.999, 1]\), \([0.9999, 1]\), \([0.99999, 1]\), and then estimate the instantaneous velocity at \(t = 1\).

9. The position of a particle at time \(t\) is \(s(t) = 2t^2\). Compute the average velocity over the time interval \([2, 4]\) and estimate the instantaneous velocity at \(t = 2\).

10. The position of a particle at time \(t\) is \(s(t) = t^3 + t\). Compute the average velocity over the time interval \([1, 4]\) and estimate the instantaneous velocity at \(t = 1\).

11. \(f(x) = x^2 + x; \quad x = 0\)

12. \(P(x) = 3x^2 - 5; \quad x = 2\)

13. \(f(t) = 12t - 7; \quad t = -4\)

14. \(y(x) = \frac{1}{x + 2}; \quad x = 2\)

15. \(y(t) = \sqrt{3t + 1}; \quad t = 1\)

16. \(f(s) = \sin x; \quad x = \frac{\pi}{6}\)

17. \(f(x) = \tan x; \quad x = \frac{\pi}{4}\)

18. \(f(x) = \tan x; \quad x = 0\)

19. The height (in centimeters) at time \(t\) (in seconds) of a small mass oscillating at the end of a spring is \(h(t) = 3\sin(2\pi t)\). Estimate its instantaneous velocity at \(t = 4\).

20. The height (in centimeters) at time \(t\) (in seconds) of a small mass oscillating at the end of a spring is \(h(t) = 8\cos(12\pi t)\).
   - (a) Calculate the mass’s average velocity over the time intervals \([0.0, 0.1]\) and \([3, 3.5]\).
   - (b) Estimate its instantaneous velocity at \(t = 3\).
21. Consider the function \( f(x) = \sqrt{x} \).
   (a) Compute the slope of the secant lines from \((0,0)\) to \((x, f(x))\) for
   \( x = 1, 0.1, 0.01, 0.001, 0.0001 \).
   (b) Discuss what the secant-line slopes in (a) suggest happens to the
tangent line at 0.
   (c) \( \blacksquare \) Plot the graph of \( f \) near \( x = 0 \) and verify your observation
from (b).

22. Consider the function \( f(x) = (x - 1)^{1/2} \).
   (a) Compute the slope of the secant lines between \((1,0)\) and \((x, f(x))\) for
   \( x = 0.9, 0.99, 0.999 \) and for \( x = 1.1, 1.01, 1.001 \).
   (b) Discuss what the secant-line slopes in (a) suggest happens to the
tangent line at 1.
   (c) \( \blacksquare \) Plot the graph of \( f \) near \( x = 1 \) and verify your observation
from (b).

23. \( \square \) If an object in linear motion (but with changing velocity) covers
   \( \Delta s \) meters in \( \Delta t \) seconds, then its average velocity is \( \bar{v} = \Delta s/\Delta t \) m/s.
   Show that it would cover the same distance if it traveled at constant velocity \( \bar{v} \)
   over the same time interval. This justifies our calling \( \Delta s/\Delta t \) the
   average velocity.

24. \( \blacksquare \) Sketch the graph of \( f(x) = x(x - 1) \) over \([0, 1]\). Refer to the graph
   and, without making any computations, find:
   (a) The slope of the secant line over \([0, 1]\)
   (b) The slope of the tangent line at \( x = \frac{1}{2} \)
   (c) The values of \( x \) at which the slope of the tangent line is positive

25. \( \blacksquare \) Which graph in Figure 5 has the following property: For all \( x \),
   the slope of the secant line over \([0, x]\) is greater than the slope of the tangent
   line at \( x \). Explain.

Further Insights and Challenges

The next two exercises involve limit estimates related to the definite integral,
as an important topic introduced in Chapter 5.

26. The height of a projectile fired in the air vertically with initial velocity
   25 m/s is
   \[
   h(t) = 25t - 4.9t^2 
   \]
   (a) Compute \( h(1) \). Show that \( h(t) - h(1) \) can be factored with \( t - 1 \) as a
   factor.
   (b) Using part (a), show that the average velocity over the interval \([1, t]\) is
   \( 20.1 - 4.9t \).
   (c) Use this formula to estimate the instantaneous velocity at time \( t = 1 \).

27. Let \( G(t) = t^2 \). Find a formula for the slope of the secant line over the
   interval \([1, t]\) and use it to estimate the slope of the tangent line at \( t \).
   Repeat for the interval \([2, t]\) and for the slope of the tangent line at \( t = 2 \).

28. For \( f(x) = x^2 \), show that the slope of the secant line over \([1, x]\) is
   \( x^2 + x + 1 \), and use this to estimate the slope of the tangent line at \( x = 1 \).

29. For \( f(x) = x^3 \), show that the slope of the secant line over \([-3, x]\) is
   \( x^2 - 3x + 9 \), and use this to estimate the slope of the tangent line at \( x = -3 \).

By dividing \([0, 1]\) into more and more subintervals, you can improve your
estimate. You can use technology to carry out these computations for large
numbers of rectangles. The exact value of the area is the limit of the estimates
as the number of subintervals gets larger and larger.

Alternatively, for this example there is a formula (that we show how
to derive in Section 5.1) that gives the total area \( A(n) \) of the rectangles
formed when \([0, 1]\) is divided into \( n \) subintervals of equal width:

\[
A(n) = \frac{(n+1)^2}{6n^2} 
\]

(e) Compute \( A(n) \) for \( n = 2, 3, 5, 10 \) to verify your results from (a)-(d).

(f) Compute \( A(n) \) for \( n = 100, 1000, \) and \( 10,000 \). Use your results to
conjecture what the area \( A \) equals.

31. Let \( A \) represent the area under the graph of \( y = x^3 \) between \( x = 0 \)
   and \( x = 1 \). In this problem, we will follow the process in Exercise 30 to
   approximate \( A \).

(a) As in (a)-(d) in Exercise 30, separately divide \([0, 1]\) into \( 2, 3, 5, \) and
   \( 10 \) equal-width subintervals, and in each case compute an overestimate
   of \( A \) using rectangles on each subinterval whose height is the value of \( x^3 \)
at the right end of the subinterval.

In this case, it can be shown that if we use \( n \) equal-width subintervals, then
the total area \( A(n) \) of the \( n \) rectangles is:

\[
A(n) = \frac{(n+1)^3}{4n^2} 
\]

(b) Compute \( A(n) \) for \( n = 2, 3, 5, 10 \) to verify your results from (a).

(c) Compute \( A(n) \) for \( n = 100, 1000, \) and \( 10,000 \). Use your results to
conjecture what the area \( A \) equals.
2.2 Investigating Limits

The goal in this section is to define limits and study them. Here we primarily use numerical and graphical techniques, but in upcoming sections we will develop other means for investigating and evaluating limits. We begin with the following question: How do the values \( f(x) \) of a function \( f \) behave when \( x \) approaches a number \( c \), whether or not \( f(c) \) is defined?

To explore this question, we'll experiment with the function

\[
f(x) = \frac{\sin x}{x} \quad (x \text{ in radians})
\]

Notice that \( f(0) \) is not defined. In fact, when we set \( x = 0 \) in

\[
f(x) = \frac{\sin x}{x}
\]

we obtain the undefined expression \((\sin 0)/0\). Nevertheless, we can compute \( f(x) \) for values of \( x \) close to 0. When we do this, a clear trend emerges.

To describe the trend, we use the phrase "\( x \) approaches 0" or "\( x \) tends to 0" to indicate that \( x \) takes on values (both positive and negative) that get closer and closer to 0. The notation for this is \( x \to 0 \), and more specifically we write

\[
\begin{align*}
&\cdot \quad x \to 0^+ \quad \text{if } x \text{ approaches 0 from the right (on the number line)}. \\
&\cdot \quad x \to 0^- \quad \text{if } x \text{ approaches 0 from the left (on the number line)}. 
\end{align*}
\]

Now consider the values listed in Table 1. The table gives the unmistakable impression that \( f(x) \) gets closer and closer to 1 as \( x \to 0^+ \) and as \( x \to 0^- \).

This conclusion is supported by the graph of \( f \) in Figure 1. The point \((0, 1)\) is missing from the graph because \( f(x) \) is not defined at \( x = 0 \), but the graph approaches this missing point as \( x \) approaches 0 from the left and right. We say that the limit of \( f(x) \) as \( x \to 0 \) is equal to 1, and we write

\[
\lim_{x \to 0} f(x) = 1
\]

We also say that \( f(x) \) approaches or converges to 1 as \( x \to 0 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \sin x )</th>
<th>( x )</th>
<th>( \sin x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.841470985</td>
<td>-1</td>
<td>0.841470985</td>
</tr>
<tr>
<td>0.5</td>
<td>0.958851077</td>
<td>-0.5</td>
<td>0.958851077</td>
</tr>
<tr>
<td>0.1</td>
<td>0.998331466</td>
<td>-0.1</td>
<td>0.998331466</td>
</tr>
<tr>
<td>0.05</td>
<td>0.999583385</td>
<td>-0.05</td>
<td>0.999583385</td>
</tr>
<tr>
<td>0.01</td>
<td>0.999983333</td>
<td>-0.01</td>
<td>0.999983333</td>
</tr>
<tr>
<td>0.005</td>
<td>0.999995833</td>
<td>-0.005</td>
<td>0.999995833</td>
</tr>
<tr>
<td>0.001</td>
<td>0.999999833</td>
<td>-0.001</td>
<td>0.999999833</td>
</tr>
</tbody>
</table>

\( x \to 0^+ \) \( f(x) \to 1 \) \( x \to 0^- \) \( f(x) \to 1 \)

In Section 2.6 we will verify the numerical and graphical evidence seen here and prove that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \). Note that, if instead we use degrees for \( x \) rather than radians, then this limit is \( \frac{\pi}{180} \approx 0.01745 \) (since \( \pi \) radians \( = 180 \) degrees). Exploring \( \frac{\sin x}{x} \) numerically, with \( x \) in degrees, you can verify that the values approach 0.01745 as \( x \) approaches 0 (Exercise 6). Because this limit is 1 when \( x \) is measured in radians, we primarily work with radians when using the trigonometric functions in calculus. Important calculus formulas are much simpler as a result (see Section 3.6).
CONCEPTUAL INSIGHT  Could we arrive at \( \lim_{x \to 0} \frac{\sin x}{x} \) by simply substituting 0 for \( x \) and saying that \( \frac{\sin 0}{0} = \frac{0}{0} = 1 \)? The answer is no. We cannot divide by 0 under any circumstances, and it is not correct to say that the undefined expression 0/0 equals 1 or any other number.

This example shows that a function \( g \) may approach a limit as \( x \to c \) even if the formula for \( g(c) \) produces an undefined expression. In this example, the limit turns out to be 1. We will encounter examples of other functions \( g \) where the formula for \( g(c) \) produces the undefined expression 0/0 but the limit is a number other than 1 (or the limit does not exist).

Definition of a Limit

The formal definition of a limit is somewhat technical, and we will wait until Section 2.9 to present it. Here we present a definition that is more conceptually oriented. It will serve our purposes for the time being. To begin, let us recall that the distance between two numbers \( a \) and \( b \) is the absolute value \( |a - b| \). So we can express the idea that \( f(x) \) is close to \( L \) by saying that \( |f(x) - L| \) is small.

DEFINITION Limit  Assume that \( f(x) \) is defined for all \( x \) in an open interval containing \( c \), but not necessarily at \( c \) itself. We say that

the limit of \( f(x) \) as \( x \) approaches \( c \) is equal to the number \( L \)

if \( |f(x) - L| \) can be made arbitrarily small by taking \( x \) sufficiently close (but not equal) to \( c \). In this case, we write

\[
\lim_{x \to c} f(x) = L
\]

We also say that \( f(x) \) approaches or converges to \( L \) as \( x \to c \) [and we write \( f(x) \to L \)].

In other words, as \( x \) approaches \( c \), \( f(x) \) approaches \( L \). See Figure 2 for the graphical interpretation. If the values of \( f(x) \) do not converge to any number \( L \) as \( x \to c \), we say that \( \lim_{x \to c} f(x) \) does not exist. It is important to note that the value \( f(c) \) itself, which may or may not be defined, plays no role in the limit. All that matters are the values of \( f(x) \) for \( x \) close to \( c \). Furthermore, if \( f(x) \) approaches a limit as \( x \to c \), then the limiting value \( L \) is unique.

EXAMPLE 1  Let \( f(x) = 5 \) and \( g(x) = 3x + 1 \). Use the definition above to verify the following limits:

(a) \( \lim_{x \to 7} f(x) = 5 \)

(b) \( \lim_{x \to 4} g(x) = 13 \)

Solution

(a) To show that \( \lim_{x \to 7} f(x) = 5 \), we must show that \( |f(x) - 5| \) becomes arbitrarily small when \( x \) is sufficiently close (but not equal) to 7. But note that \( |f(x) - 5| = |5 - 5| = 0 \) for all \( x \), so what we are required to show is automatic.

(b) To show that \( \lim_{x \to 4} g(x) = 13 \), we must show that \( |g(x) - 13| \) becomes arbitrarily small when \( x \) is sufficiently close (but not equal) to 4. We have

\[
|g(x) - 13| = |(3x + 1) - 13| = |3x - 12| = 3|x - 4|
\]
The concept of a limit was not fully clarified until the nineteenth century. The French mathematician Augustin-Louis Cauchy (1789–1857, pronounced Kah-sheh) gave the following verbal definition: "When the values successively attributed to the same variable approach a fixed value indefinitely, in such a way as to end up differing from it by as little as one could wish, this last value is called the limit of all the others. So, for example, an irrational number is the limit of the various fractions which provide values that approximate it more and more closely." (Translated by J. Grabiner)

Because \(|g(x) - 13|\) is a multiple of \(|x - 4|\), we can make \(|g(x) - 13|\) arbitrarily small by taking \(x\) sufficiently close to 4.

Reasoning as in Example 1 but with arbitrary constants, we obtain the following simple but important results:

**Theorem 1** For any constants \(k\) and \(c\),

(a) \(\lim_{x \rightarrow c} k = k\),

(b) \(\lim_{x \rightarrow c} x = c\).

To deal with more complicated limits and, especially, to provide mathematically rigorous proofs, the precise version of the above limit definition, presented in Section 2.9, is needed. There inequalities are used to pin down the exact meaning of the phrases "arbitrarily small" and "sufficiently close."

**Graphical and Numerical Investigation**

Our goal in the rest of this section is to develop a better intuitive understanding of limits by investigating them graphically and numerically.

**Graphical Investigation** Use a graphing utility to produce a graph of \(f\). The graph should give a visual impression of whether or not a limit exists. It can often be used to estimate the value of the limit.

**Numerical Investigation** We write \(x \rightarrow c^-\) to indicate that \(x\) approaches \(c\) through values less than \(c\) (i.e., from the left), and we write \(x \rightarrow c^+\) to indicate that \(x\) approaches \(c\) through values greater than \(c\) (i.e., from the right). To investigate \(\lim_{x \rightarrow c} f(x)\),

(i) Make a table of values of \(f(x)\) for \(x\) close to but less than \(c\)—that is, as \(x \rightarrow c^-\).

(ii) Make a second table of values of \(f(x)\) for \(x\) close to but greater than \(c\)—that is, as \(x \rightarrow c^+\).

(iii) If both tables indicate convergence to the same number \(L\), we take \(L\) to be an estimate for the limit.

The tables should contain enough values to reveal a clear trend of convergence to a value \(L\). If \(f(x)\) approaches a limit, the successive values of \(f(x)\) will generally agree to more and more decimal places as \(x\) is taken closer to \(c\). If no pattern emerges, then the limit may not exist.

**Example 2** Investigate \(\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3}\) graphically and numerically.

**Solution** The function \(f(x) = \frac{x - 9}{\sqrt{x} - 3}\) is undefined at \(x = 9\) because the formula for \(f(9)\) leads to the undefined expression 0/0. Therefore, the graph in Figure 3 has a gap at \(x = 9\). However, the graph suggests that \(f(x)\) approaches 6 as \(x\) approaches 9.

For numerical evidence, we consider a table of values of \(f(x)\) for \(x\) approaching 9 from both the left and the right. Table 2 supports our impression that

\[\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3} = 6\]

In Section 2.5, we will revisit this limit and show how we can use algebraic simplification to prove that this limit is, in fact, 6.
EXAMPLE 3  Limit Equals Value of the Function. Investigate \( \lim_{x \to 4} x^2. \)

Solution. Figure 4 and Table 3 both suggest that \( \lim_{x \to 4} x^2 = 16. \) Furthermore, note that \( f(x) = x^2 \) is defined at \( x = 4 \) and \( f(4) = 16, \) so in this case, the limit is equal to the function value. This pleasant conclusion is valid whenever \( f \) is a continuous function, a concept treated in Section 2.4.

<table>
<thead>
<tr>
<th>( x \to 4^- )</th>
<th>( x \to 4^+ )</th>
<th>( x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.9</td>
<td>3.99</td>
<td>15.21</td>
</tr>
<tr>
<td>3.99</td>
<td>3.999</td>
<td>15.9201</td>
</tr>
<tr>
<td>3.999</td>
<td>3.9999</td>
<td>15.992001</td>
</tr>
<tr>
<td>3.9999</td>
<td>3.99999</td>
<td>16.008001</td>
</tr>
</tbody>
</table>

EXAMPLE 4  Investigate \( \lim_{x \to 0} (1 + x)^{1/x} \) numerically and graphically.

Solution. The function \( f(x) = (1 + x)^{1/x} \) is undefined at \( x = 0, \) but both Figure 5 and Table 4 suggest that a limit exists and is approximately equal to 2.71828. In Chapter 7 we will see that this limit is equal to the important numerical value \( e, \) the base of the natural exponential and logarithmic functions.

<table>
<thead>
<tr>
<th>( x \to 0^- )</th>
<th>( (1 + x)^{1/x} )</th>
<th>( x \to 0^+ )</th>
<th>( (1 + x)^{1/x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.01</td>
<td>2.731999</td>
<td>0.01</td>
<td>2.704814</td>
</tr>
<tr>
<td>-0.001</td>
<td>2.719642</td>
<td>0.001</td>
<td>2.716924</td>
</tr>
<tr>
<td>-0.0001</td>
<td>2.718418</td>
<td>0.0001</td>
<td>2.718146</td>
</tr>
<tr>
<td>-0.00001</td>
<td>2.718295</td>
<td>0.00001</td>
<td>2.718268</td>
</tr>
<tr>
<td>-0.000001</td>
<td>2.718283</td>
<td>0.000001</td>
<td>2.718280</td>
</tr>
</tbody>
</table>

EXAMPLE 5  A Limit That Does Not Exist. Investigate \( \lim_{x \to 0} \frac{\sin \frac{\pi}{x}}{x} \) graphically and numerically.

Solution. The function \( f(x) = \sin \frac{\pi}{x} \) is not defined at \( x = 0, \) but Figure 6 suggests that it oscillates between \(+1\) and \(-1\) infinitely often as \( x \to 0. \) It appears, therefore, that \( \lim_{x \to 0} \frac{\sin \frac{\pi}{x}}{x} \) does not exist. This impression is confirmed by Table 5, which shows that the values of \( f(x) \) bounce around and do not tend toward any limit \( L \) as \( x \to 0. \)
SECTION 2.2  Investigating Limits  53

**TABLE 5**  The Function \( f(x) = \sin \frac{\pi}{x} \) Does Not Approach a Limit as \( x \to 0 \)

<table>
<thead>
<tr>
<th>( x \to 0^- )</th>
<th>( \sin \frac{\pi}{x} )</th>
<th>( x \to 0^+ )</th>
<th>( \sin \frac{\pi}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-0.1)</td>
<td>(0)</td>
<td>(0.1)</td>
<td>(0)</td>
</tr>
<tr>
<td>(-0.007)</td>
<td>(-0.866)</td>
<td>(0.007)</td>
<td>(0.434)</td>
</tr>
<tr>
<td>(-0.0009)</td>
<td>(-0.342)</td>
<td>(0.0009)</td>
<td>(-0.342)</td>
</tr>
<tr>
<td>(-0.00065)</td>
<td>(-0.935)</td>
<td>(0.00065)</td>
<td>(0.935)</td>
</tr>
</tbody>
</table>

**One-Sided Limits**

The limits discussed so far are **two-sided**. To show that \( \lim_{x \to c} f(x) = L \), it is necessary to check that \( f(x) \) converges to \( L \) as \( x \) approaches \( c \) through values both greater than and less than \( c \). In some instances, \( f(x) \) may approach \( L \) from one side of \( c \) without necessarily approaching it from the other side, or \( f(x) \) may be defined on only one side of \( c \). For this reason, we define the one-sided limits

\[
\lim_{x \to c^-} f(x) \quad \text{(left-hand limit),} \quad \lim_{x \to c^+} f(x) \quad \text{(right-hand limit)}
\]

The limit itself exists if and only if both one-sided limits exist and are equal. Otherwise, the limit does not exist.

**EXAMPLE 6  Left- and Right-Hand Limits Not Equal**  Investigate the one-sided limits of \( f(x) = \frac{x}{|x|} \) as \( x \to 0 \). Does \( \lim_{x \to 0} f(x) \) exist?

**Solution**  Figure 7 shows what is going on. For \( x < 0 \),

\[
f(x) = \frac{x}{|x|} = \frac{x}{-x} = -1
\]

Therefore, the left-hand limit is \( \lim_{x \to 0^-} f(x) = -1 \). But for \( x > 0 \),

\[
f(x) = \frac{x}{|x|} = \frac{x}{x} = 1
\]

Therefore, \( \lim_{x \to 0^+} f(x) = 1 \). These one-sided limits are not equal, so \( \lim_{x \to 0} f(x) \) does not exist.

**EXAMPLE 7**  The function \( f \) in Figure 8 is not defined at \( c = 0, 2, 4 \). Investigate the one- and two-sided limits at these points.

**Solution**

- \( c = 0 \): The left-hand limit \( \lim_{x \to 0^-} f(x) \) does not seem to exist because \( f(x) \) appears to oscillate infinitely often to the left of \( x = 0 \). On the other hand, \( \lim_{x \to 0^+} f(x) = 2 \).
- \( c = 2 \): The one-sided limits exist but are not equal:

\[
\lim_{x \to 2^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 1
\]

Therefore, \( \lim_{x \to 2} f(x) \) does not exist.

- \( c = 4 \): The one-sided limits exist and both have the value 2. Therefore, the two-sided limit exists and \( \lim_{x \to 4} f(x) = 2 \).
Infinite Limits

For some functions, $f(x)$ tends to $\infty$ or $-\infty$ as $x$ approaches a value $c$. It is important to understand that $\infty$ and $-\infty$ are not numbers, and therefore $\lim_{x \to c} f(x)$ does not exist. However, we say that $f(x)$ has an infinite limit. More precisely, we write

- $\lim_{x \to c} f(x) = \infty$ if $f(x)$ is positive and becomes arbitrarily large as $x \to c$.
- $\lim_{x \to c} f(x) = -\infty$ if $f(x)$ is negative and becomes arbitrarily large as $x \to c$.

One-sided infinite limits are defined similarly.

When $f(x)$ approaches $\infty$ or $-\infty$ as $x$ approaches $c$ from one or both sides, the line $x = c$ is called a vertical asymptote. In Figure 9, the line $x = 2$ is a vertical asymptote in (A), and $x = 0$ is a vertical asymptote in (B).

In the next example, the notation $x \to c^-$ is used to indicate that the left- and right-hand limits are to be considered separately.

**EXAMPLE 8** Investigate the one-sided limits graphically:

(a) $\lim_{x \to 2^-} \frac{1}{x - 2}$

(b) $\lim_{x \to 0^+} \frac{1}{x^2}$

**Solution**

(a) Figure 9(A) suggests that

$$\lim_{x \to 2^-} \frac{1}{x - 2} = -\infty, \quad \lim_{x \to 2^+} \frac{1}{x - 2} = \infty$$

The line $x = 2$ is a vertical asymptote. Why are the one-sided limits different? Because $f(x) = \frac{1}{x - 2}$ is negative for $x < 2$ (so the limit from the left is $-\infty$) and $f(x)$ is positive for $x > 2$ (so the limit from the right is $\infty$).

(b) Figure 9(B) suggests that $\lim_{x \to 0^+} \frac{1}{x^2} = \infty$. Indeed, $f(x) = \frac{1}{x^2}$ is positive for all $x \neq 0$ and becomes arbitrarily large as $x \to 0$ from either side. The line $x = 0$ is a vertical asymptote.

**CONCEPTUAL INSIGHT** You should not think of an infinite limit as a true limit. The notation $\lim_{x \to c} f(x) = \infty$ is merely a shorthand way of saying that $f(x)$ is positive and arbitrarily large as $x$ approaches $c$. The limit itself does not exist. We must be careful when using this notation because $\infty$ and $-\infty$ are not numbers, and contradictions can arise if we try to manipulate them as numbers. For example, if $\infty$ were a number, it would be larger than any finite number, and presumably, $\infty + 1 = \infty$. But then

$$\infty + 1 = \infty$$

$$(\infty + 1) - \infty = \infty - \infty$$

$$1 = 0$$

(contradiction!)

To avoid errors like this, keep in mind the $\infty$ is not a number but rather a convenient shorthand notation.

**2.2 SUMMARY**

- By definition, $\lim_{x \to c} f(x) = L$ if $|f(x) - L|$ can be made arbitrarily small by taking $x$ sufficiently close (but not equal) to $c$. We say that
- The limit of \( f(x) \) as \( x \) approaches \( c \) is \( L \), or
- \( f(x) \) approaches (or converges to) \( L \) as \( x \) approaches \( c \).

- If \( f(x) \) approaches a limit as \( x \to c \), then the value of the limit \( L \) is unique.
- If \( f(x) \) does not approach a limit as \( x \to c \), we say that \( \lim_{x \to c} f(x) \) does not exist.
- The limit may exist even if \( f(c) \) is not defined.
- One-sided limits:
  - \( \lim_{x \to c^-} f(x) = L \) if \( f(x) \) converges to \( L \) as \( x \) approaches \( c \) through values less than \( c \).
  - \( \lim_{x \to c^+} f(x) = L \) if \( f(x) \) converges to \( L \) as \( x \) approaches \( c \) through values greater than \( c \).

- The limit exists if and only if both one-sided limits exist and are equal.
- Infinite limits: \( \lim_{x \to c} f(x) = \infty \) if \( f(x) \) is positive and becomes arbitrarily large as \( x \) approaches \( c \), and \( \lim_{x \to c} f(x) = -\infty \) if \( f(x) \) is negative and becomes arbitrarily large as \( x \) approaches \( c \).
- In the case of a one- or two-sided infinite limit at \( c \), the vertical line \( x = c \) is called a vertical asymptote.

### 2.2 Exercises

#### Preliminary Questions

1. What is the limit of \( f(x) = 1 \) as \( x \to \pi \)?

2. What is the limit of \( g(t) = t \) as \( t \to \pi \)?

3. Is \( \lim_{x \to 10} 20 \) equal to 10 or 20?

4. Can \( f(x) \) approach a limit as \( x \to c \) if \( f(c) \) is undefined? If so, give an example.

5. What does the following table suggest about \( \lim_{x \to 1^-} f(x) \) and \( \lim_{x \to 1^+} f(x) \)?

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.9</th>
<th>0.99</th>
<th>0.999</th>
<th>1.001</th>
<th>1.01</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>7</td>
<td>25</td>
<td>4317</td>
<td>3.00011</td>
<td>3.0047</td>
<td>3.0126</td>
</tr>
</tbody>
</table>

6. Can you tell whether \( \lim_{x \to 5} f(x) \) exists from a plot of \( f \) for \( x > 5 \)? Explain.

7. If you know in advance that \( \lim_{x \to 5} f(x) \) exists, can you determine its value from a plot of \( f \) for all \( x > 5 \)?

#### Exercises

In Exercises 1–5, fill in the table and guess the value of the limit.

1. \( \lim_{x \to 1} f(x) \), where \( f(x) = \frac{x^2 - 1}{x^2 - 1} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.002</td>
<td>0.998</td>
<td>1.001</td>
<td>0.999</td>
</tr>
<tr>
<td>1.0005</td>
<td>0.9995</td>
<td>1.00001</td>
<td>0.99999</td>
</tr>
</tbody>
</table>

2. \( \lim_{t \to 0} h(t) \), where \( h(t) = \frac{\cos t - 1}{t^2} \). Note that \( h \) is even; that is, \( h(t) = h(-t) \).

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \pm 0.002 )</th>
<th>( \pm 0.0001 )</th>
<th>( \pm 0.00005 )</th>
<th>( \pm 0.00001 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(t) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. \( \lim_{y \to 2} f(y) \), where \( f(y) = \frac{y^2 - y - 2}{y^2 + y - 6} \)

<table>
<thead>
<tr>
<th>( y )</th>
<th>( f(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.002</td>
<td>1.998</td>
</tr>
<tr>
<td>2.001</td>
<td>1.999</td>
</tr>
<tr>
<td>2.0001</td>
<td>1.9999</td>
</tr>
</tbody>
</table>

4. \( \lim_{\theta \to 0} f(\theta) \), where \( f(\theta) = \frac{\sin \theta - \theta}{\theta^3} \)

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \pm 0.002 )</th>
<th>( \pm 0.0001 )</th>
<th>( \pm 0.00005 )</th>
<th>( \pm 0.00001 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(\theta) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. \( \lim_{t \to 0} f(t) \), where \( f(t) = \frac{1 - \cos 2t}{t} \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.002</td>
<td>-0.002</td>
</tr>
<tr>
<td>0.001</td>
<td>-0.001</td>
</tr>
<tr>
<td>0.0005</td>
<td>-0.0005</td>
</tr>
<tr>
<td>0.00001</td>
<td>-0.00001</td>
</tr>
</tbody>
</table>

6. Numerically investigate \( \lim_{x \to 0} \frac{\sin x}{x} \), computing the values of \( \sin x \) with \( x \) in degrees. Make an estimate of the limit accurate to 5 decimal places.
7. Determine \( \lim_{x \to 0.5} f(x) \) for \( f \) as in Figure 10.

8. Determine \( \lim_{x \to 0.5} g(x) \) for \( g \) as in Figure 11.

In Exercises 9–10, evaluate the limit.

9. \( \lim_{x \to 21} x \)

10. \( \lim_{x \to 3} \sqrt[3]{x} \)

11. Show, via illustration, that the limits \( \lim_{x \to 10} x \) and \( \lim_{x \to 10} a \) are equal but the functions in each limit are different.

12. Give examples of functions \( f \) and \( g \) such that \( \lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) \), but \( f(x) \neq g(x) \) for all \( x \), including 0.

In Exercises 13–20, verify each limit using the limit definition. For example, in Exercise 13, show that \( |3x - 12| \) can be made as small as desired by taking \( a \) close to 4.

13. \( \lim_{x \to 4} 3x = 12 \)

14. \( \lim_{x \to 3} 3x = 3 \)

15. \( \lim_{x \to 2} (5x + 2) = 17 \)

16. \( \lim_{x \to 2} (3x - 4) = 10 \)

17. \( \lim_{x \to 0} x^2 = 0 \)

18. \( \lim_{x \to 0} (3x^2 - 9) = -9 \)

19. \( \lim_{x \to 0} (4x^2 + 2x + 5) = 5 \)

20. \( \lim_{x \to 0} (x^3 + 12) = 12 \)

In Exercises 21–42, estimate the limit numerically or state that the limit does not exist. If infinite, state whether the one-sided limits are \( \infty \) or \( -\infty \).

21. \( \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \)

22. \( \lim_{x \to 2} \frac{x^2 - 32}{x + 4} \)

23. \( \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} \)

24. \( \lim_{x \to 3} \frac{x^2 - 2x - 9}{x - 3} \)

25. \( \lim_{x \to 0} \frac{\sin 2x}{x} \)

26. \( \lim_{x \to 0} \frac{\sin 3x}{x} \)

27. \( \lim_{x \to 0} \frac{\sin 5x}{x} \)

28. \( \lim_{x \to 0} \frac{\cos x}{x} \)

29. \( \lim_{x \to 0} \frac{\cos x - 1}{x} \)

30. \( \lim_{x \to 0} \frac{\sin x}{x^2} \)

31. \( \lim_{x \to 4} \frac{x^2}{x - 4} - 1 \)

32. \( \lim_{x \to 4} \frac{3 - x}{x - 4} \)

33. \( \lim_{x \to 3} \frac{x + 3}{x^2 + x - 6} \)

34. \( \lim_{x \to 3} \frac{x + 1}{x - 2} \)

35. \( \lim_{x \to 9} \frac{x - 4}{x^3 - x^2 - 9} \)

36. \( \lim_{h \to 0} \frac{2h - 1}{h} \)

37. \( \lim_{h \to 0} \frac{\sin h \cos \frac{1}{h}}{h} \)

38. \( \lim_{h \to 0} \frac{\cos \frac{1}{h}}{h} \)

39. \( \lim_{x \to 0} |x|^4 \)

40. \( \lim_{x \to 0} (1 + 2x)^{1/x} \)

41. \( \lim_{\theta \to \pi/4} \frac{\tan \theta - 2 \sin \theta \cos \theta}{\theta - \pi/4} \)

42. \( \lim_{x \to 0} \frac{\tan x - x}{\sin x} \)

43. The greatest integer function, also known as the floor function, is defined by \( [x] = n \), where \( n \) is the unique integer such that \( n \leq x < n + 1 \). Sketch the graph of \( y = [x] \). Calculate for \( c \) an integer:

   (a) \( \lim_{x \to c^+} [x] \)

   (b) \( \lim_{x \to c^-} [x] \)

   (c) \( \lim_{x \to c} [x] \)

44. Determine the one-sided limits at \( c = 1, 2, \) and \( 4 \) of the function \( g \) shown in Figure 12, and state whether the limit exists at these points.

45. \( \lim_{x \to 0^+} \frac{\sin x}{x} \)

46. \( \lim_{x \to 0^-} \frac{\sin x}{x} \)

47. \( \lim_{x \to 0^+} x - \sin x \)

48. \( \lim_{x \to 0^-} \frac{x - \sin x}{x^2} \)

49. \( \lim_{x \to 2} \frac{4x^2 + 7}{x^3 + 8} \)

50. \( \lim_{x \to 3} \frac{x^2}{x^2 - 9} \)

51. \( \lim_{x \to 1} \frac{x^3 + x - 2}{x^2 + x - 2} \)

52. \( \lim_{x \to 2} \cos \left( \frac{x}{2} (x - x) \right) \)

53. Determine the one-sided limits at \( c = 2 \) and \( c = 4 \) of the function \( f \) in Figure 13. What are the vertical asymptotes of \( f \)?

54. Determine the infinite one- and two-sided limits in Figure 14.

In Exercises 55–58, sketch the graph of a function with the given limits.

55. \( \lim_{x \to 1} f(x) = 2, \lim_{x \to 3} f(x) = 0, \lim_{x \to 3} f(x) = 4 \)

56. \( \lim_{x \to 1} f(x) = \infty, \lim_{x \to 3} f(x) = 0, \lim_{x \to 3} f(x) = -\infty \)

57. \( \lim_{x \to -2} f(x) = f(2) = 3, \lim_{x \to -2} f(x) = -1, \lim_{x \to 4} f(x) = 2 \neq f(4) \)
58. \( \lim_{x \to 1^+} f(x) = \infty, \lim_{x \to 1^-} f(x) = 3, \lim_{x \to 1} f(x) = -\infty \) 

59. Determine the one-sided limits of the function \( f \) in Figure 15, at the points \( c = 1, 3, 5, 6 \). 

60. Does either of the two oscillating functions in Figure 16 appear to approach a limit as \( x \to 0 \)?

61. \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) 

62. \( \lim_{x \to 4} \frac{12x - 1}{4^x - 1} = \frac{48}{15} \) 

63. \( \lim_{x \to 0} \frac{2 \cos x}{x} = 0 \) 

64. \( \lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta^2} = 1 \) 

65. \( \lim_{\theta \to 0} \frac{\cos \theta - \cos \theta}{\theta^2} = 0 \) 

66. \( \lim_{\theta \to 0} \frac{\sin \theta - \theta}{\theta^3} = \frac{1}{6} \) 

67. Let \( n \) be a positive integer. For which \( n \) are the two infinite one-sided limits \( \lim_{x \to 0^+} \) equal?

68. Let \( L(n) = \lim_{x \to 1} \left( \frac{n}{1 - x^n} - \frac{1}{1 - x} \right) \) for \( n \) a positive integer. Investigate \( L(n) \) numerically for several values of \( n \), and then guess the value of \( L(n) \) in general.

69. In some cases, numerical investigations can be misleading. Plot \( f(x) = \cos \frac{x}{x} \).

(a) Does \( \lim_{x \to 0} f(x) \) exist?

(b) Show, by evaluating \( f(x) \) at \( x = \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \ldots \), that you might be able to trick your friends into believing that the limit exists and is equal to \( L = 1 \).

(c) Which sequence of evaluations might trick them into believing that the limit is \( L = -17 \)?

**Further Insights and Challenges**

70. Light waves of frequency \( \lambda \) passing through a slit of width \( a \) produce a Fraunhofer diffraction pattern of light and dark fringes (Figure 17). The intensity as a function of the angle \( \theta \) is 

\[
I(\theta) = I_m \left( \frac{\sin(R \sin \theta)}{R \sin \theta} \right)^2
\]

where \( R = \pi a / A \) and \( I_m \) is a constant. Show that the intensity function is not defined at \( \theta = 0 \). Then choose any two values for \( R \) and check numerically that \( I(\theta) \) approaches \( I_m \) as \( \theta \to 0 \).

71. Investigate \( \lim_{x \to 0} \left( \frac{\sin x^2}{x} \right) \) numerically for several positive integer values of \( n \). Then guess the value in general.

72. Show numerically that \( \lim_{x \to 0} \frac{b^x - 1}{x} \) is less than 2 with \( b = 7 \) and is greater than 2 with \( b = 8 \). Experiment with values of \( b \) to find an approximate value of \( b \) for which the limit is 2.

73. Investigate \( \lim_{x \to 1} \frac{x^a - 1}{x - 1} \) for \( (m, n) = (2, 1), (1, 2), (2, 3), \) and \( (3, 2) \). Then guess the value of the limit in general and check your guess for two additional pairs.

74. Find by numerical experimentation the positive integers \( k \) such that \( \lim_{x \to 0} \frac{\sin(x^k)}{x^k} \) exists.

75. (GU) Plot the graph of \( f(x) = \frac{2^x - 8}{x - 3} \).

(a) Zoom in on the graph to estimate \( L = \lim_{x \to 3} f(x) \).

(b) Explain why 

\[
f(2.99999) \leq L \leq f(3.00001)
\]

Use this to determine \( L \) to three decimal places.

76. (GU) The function \( f(x) = \frac{2^{1/x} - 2^{-1/x}}{2^{1/x} + 2^{-1/x}} \) is defined for \( x \neq 0 \).

(a) Investigate \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^-} f(x) \) numerically.

(b) Plot the graph of \( f \) and describe its behavior near \( x = 0 \).
2.3 Basic Limit Laws

In Section 2.2, we relied on graphical and numerical approaches to investigate limits and estimate their values. In the next four sections, we go beyond this intuitive approach and develop tools for computing limits in a precise way. The next theorem provides our first set of tools.

**Theorem 1 Basic Limit Laws** If \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) exist, then

(i) **Sum Law:** \( \lim_{x \to c} (f(x) + g(x)) \) exists and

\[
\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)
\]

(ii) **Constant Multiple Law:** For any number \( k \), \( \lim_{x \to c} kf(x) \) exists and

\[
\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)
\]

(iii) **Product Law:** \( \lim_{x \to c} f(x)g(x) \) exists and

\[
\lim_{x \to c} f(x)g(x) = \left( \lim_{x \to c} f(x) \right) \left( \lim_{x \to c} g(x) \right)
\]

(iv) **Quotient Law:** If \( \lim_{x \to c} g(x) \neq 0 \), then \( \lim_{x \to c} \frac{f(x)}{g(x)} \) exists and

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}
\]

(v) **Powers and Roots:** If \( n \) is a positive integer, then

\[
\lim_{x \to c} [f(x)]^n = \left( \lim_{x \to c} f(x) \right)^n, \quad \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}
\]

In the second limit, assume that \( \lim_{x \to c} f(x) \geq 0 \) if \( n \) is even.

If \( p, q \) are integers with \( q \neq 0 \), then \( \lim_{x \to c} [f(x)]^{p/q} \) exists and

\[
\lim_{x \to c} [f(x)]^{p/q} = \left( \lim_{x \to c} f(x) \right)^{p/q}
\]

Assume that \( \lim_{x \to c} f(x) \geq 0 \) if \( q \) is even, and that \( \lim_{x \to c} f(x) \neq 0 \) if \( p/q < 0 \).

Before proceeding to the examples, we make some useful remarks.

* The Sum and Product Laws are valid for any number of functions. For example,

\[
\lim_{x \to c} (f_1(x) + f_2(x) + f_3(x)) = \lim_{x \to c} f_1(x) + \lim_{x \to c} f_2(x) + \lim_{x \to c} f_3(x)
\]

* The Sum Law has a counterpart for differences:

\[
\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)
\]

This follows from the Sum and Constant Multiple Laws (with \( k = -1 \)):

\[
\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} (-g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)
\]

* Recall two basic limits from Theorem 1 in Section 2.2:

\[
\lim_{x \to c} k = k, \quad \lim_{x \to c} x = c
\]
Applying Law (v) to $f(x) = x$, we obtain

$$\lim_{x \to c} x^{p/q} = c^{p/q}$$

for integers $p$ and $q$ such that $q \neq 0$. Note, in Eq. (1) we need to assume that $c \geq 0$ if $q$ is even and that $c \neq 0$ if $p/q < 0$.

**EXAMPLE 1** Use the Basic Limit Laws to evaluate:

(a) $\lim_{x \to 2} x^3$

(b) $\lim_{x \to 2} (x^3 + 5x + 7)$

(c) $\lim_{x \to 2} \sqrt{x^3 + 5x + 7}$

**Solution**

(a) By Eq. (1), $\lim_{x \to 2} x^3 = 2^3 = 8$.

(b) $\lim_{x \to 2} (x^3 + 5x + 7) = \lim_{x \to 2} x^3 + \lim_{x \to 2} 5x + \lim_{x \to 2} 7$ (Sum Law)

$= \lim_{x \to 2} x^3 + 5 \lim_{x \to 2} x + \lim_{x \to 2} 7$ (Constant Multiple Law)

$= 8 + 5(2) + 7 = 25$

(c) By Law (v) for roots and (b),

$$\lim_{x \to 2} \sqrt{x^3 + 5x + 7} = \sqrt{\lim_{x \to 2} (x^3 + 5x + 7)} = \sqrt{25} = 5$$

**EXAMPLE 2** Evaluate (a) $\lim_{t \to -1} \frac{t + 6}{2t^4}$ and (b) $\lim_{t \to 3} t^{-1/4}(t + 5)^{1/3}$.

**Solution**

(a) Use the Quotient, Sum, and Constant Multiple Laws:

$$\lim_{t \to -1} \frac{t + 6}{2t^4} = \frac{\lim_{t \to -1} (t + 6)}{\lim_{t \to -1} 2t^4} = \frac{\lim_{t \to -1} t + \lim_{t \to -1} 6}{2 \lim_{t \to -1} t^4} = \frac{-1 + 6}{2(-1)^4} = \frac{5}{2}$$

(b) Use the Product, Powers, and Sum Laws:

$$\lim_{t \to 3} t^{-1/4}(t + 5)^{1/3} = \left(\lim_{t \to 3} t^{-1/4}\right) \left(\lim_{t \to 3} (t + 5)^{1/3}\right) = \left(3^{-1/4}\right) \left(\sqrt[3]{3 + 5}\right)$$

$= 3^{-1/4} \sqrt[3]{8} = 3^{-1/4}(2) = \frac{2}{3^{1/4}}$

You may have noticed that each of the limits in Examples 1 and 2 could have been evaluated by a simple substitution. For example, set $t = -1$ to evaluate

$$\lim_{t \to -1} \frac{t + 6}{2t^4} = \frac{-1 + 6}{2(-1)^4} = \frac{5}{2}$$

Substitution is valid when the function is continuous, a concept we shall study in the next section.

The next example reminds us that the Basic Limit Laws apply only when the limits of both $f(x)$ and $g(x)$ exist.

**EXAMPLE 3** Assumptions Matter Show that the Product Law cannot be applied to $\lim_{x \to 0} f(x)g(x)$ if $f(x) = x$ and $g(x) = x^{-1}$.

**Solution** For all $x \neq 0$, we have $f(x)g(x) = x \cdot x^{-1} = 1$, so the limit of the product exists:

$$\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} 1 = 1$$

However, there is an issue with the product of the limits because $\lim_{x \to 0} x^{-1}$ does not exist (since $g(x) = x^{-1}$ becomes infinite as $x \to 0$). Therefore, the Product Law cannot be
applied and its conclusion does not hold even though the limit of the products does exist. Specifically, \( \lim_{x \to 0} f(x)g(x) = 1 \), but the product of the limits is not defined:

\[
1 \neq \left( \lim_{x \to 0} f(x) \right) \left( \lim_{x \to 0} g(x) \right) = \left( \lim_{x \to 0} f(x) \right) \left( \lim_{x \to 0} g(x) \right)
\]

Does not exist

2.3 SUMMARY

- The Basic Limit Laws: If \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) both exist, then

  1. \( \lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \)
  2. \( \lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) \)
  3. \( \lim_{x \to c} f(x)g(x) = \left( \lim_{x \to c} f(x) \right) \left( \lim_{x \to c} g(x) \right) \)
  4. If \( \lim_{x \to c} g(x) \neq 0 \), then \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \)
  5. If \( p, q \) are integers with \( q \neq 0 \),

\[
\lim_{x \to c} [f(x)]^{p/q} = \left( \lim_{x \to c} f(x) \right)^{p/q}
\]

For \( n \) a positive integer,

\[
\lim_{x \to c} [f(x)]^n = \left( \lim_{x \to c} f(x) \right)^n, \quad \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}
\]

- If \( \lim_{x \to c} f(x) \) or \( \lim_{x \to c} g(x) \) does not exist, then the Basic Limit Laws cannot be applied.

2.3 EXERCISES

Preliminary Questions

1. State the Sum Law and Quotient Law.
2. Which of the following is a verbal version of the Product Law (assuming the limits exist)?
   (a) The product of two functions has a limit.
   (b) The limit of the product is the product of the limits.
   (c) The product of a limit is a product of functions.
   (d) A limit produces a product of functions.
   (e) The Quotient Law does not hold if
      (a) The limit of the denominator is zero
      (b) The limit of the numerator is zero

Exercises

In Exercises 1–26, evaluate the limit using the Basic Limit Laws and the limits \( \lim_{x \to a} x^p = a^p \) and \( \lim_{x \to a} x = a \).

1. \( \lim_{x \to 2} x^9 \)
2. \( \lim_{x \to 3} 14 \)
3. \( \lim_{x \to 2} x^4 \)
4. \( \lim_{x \to 2} 2x^{2/3} \)
5. \( \lim_{x \to 2} x^{-2} \)
6. \( \lim_{x \to 2} (3x + 4) \)
7. \( \lim_{x \to 2} (3x^3 + 2x^2) \)
8. \( \lim_{x \to 2} (2x^2 - 2x + 4x) \)
9. \( \lim_{x \to 1} (3x^4 - 2x^3 + 4x^2) \)
10. \( \lim_{x \to 1} (3x^2 - 16x^{-1}) \)
11. \( \lim_{x \to 2} (x + 1)(3x^2 - 9) \)
12. \( \lim_{x \to 2} (4x + 1)(6x - 1) \)
13. \( \lim_{x \to 2} \frac{1}{x + 4} \)
14. \( \lim_{x \to 2} \frac{3}{x - 1} \)
15. \( \lim_{x \to 1} \frac{3x - 14}{x + 1} \)
16. \( \lim_{x \to 2} \frac{3}{x - 2} \)
17. \( \lim_{x \to 4} \frac{16x + 1)(2x^{1/2} + 1)}{x - 4} \)
18. \( \lim_{x \to 2} \frac{x + 1}(x + 2) \)
19. \( \lim_{y \to 4} \frac{1}{\sqrt[3]{y} + 6} \)
20. \( \lim_{y \to 6} \frac{1}{\sqrt[3]{y} + 1} \)
21. \( \lim_{x \to 0} \frac{x}{x^2 + 4x} \)
22. \( \lim_{u \to -1} \frac{u^2 + 1}{u^{-1} + (u^2 + 1)} \)
23. \( \lim_{t \to 25} \frac{3t^3 - t^2}{(t - 20)^2} \)  
24. \( \lim_{y \to 3} (18y^2 - 4)^4 \)  
25. \( \lim_{t \to 2} (4t^2 + 8t - 5)^{1/2} \)  
26. \( \lim_{t \to -1} (t^2 + 7t)^{2/3} \)

27. Use the Quotient Law to prove that if \( \lim_{x \to c} f(x) \) exists and is nonzero, then \( \lim_{x \to c} \frac{1}{f(x)} = \frac{1}{\lim_{x \to c} f(x)} \).

28. Assuming that \( \lim_{x \to 6} f(x) = 4 \), compute:
(a) \( \lim_{x \to 6} f(x)^2 \)  
(b) \( \lim_{x \to 6} \frac{1}{f(x)} \)  
(c) \( \lim_{x \to 6} \sqrt[3]{f(x)} \)

In Exercises 29–32, evaluate the limit assuming that \( \lim_{x \to 4} f(x) = 3 \) and \( \lim_{x \to 4} g(x) = 1 \).

29. \( \lim_{x \to 4} (f(x) + 3g(x)) \)  
30. \( \lim_{x \to 4} (2f(x) + 3g(x)) \)

31. \( \lim_{x \to 4} \frac{g(x)}{x^2} \)  
32. \( \lim_{x \to 4} \frac{f(x) + 1}{3g(x) - 5} \)

33. Can the Quotient Law be applied to evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \)? Explain.

34. Show that the Product Law cannot be used to evaluate the limit \( \lim_{x \to 0} x \tan x \).

35. Assume that if \( \lim f(x) = L \), then \( \lim \frac{f(x)}{x} = \frac{L}{x} \). In each case evaluate the limit or indicate that the limit does not exist.
(a) \( \lim_{x \to 0} \frac{x}{x - 1} \)  
(b) \( \lim_{x \to \pi/2} \frac{\sin x}{x} \)

(c) \( \lim_{x \to 1} \frac{3x}{\sin(1 - x)} \)  
(d) \( \lim_{x \to 1} x^2 \sin(\pi x^2) \)

36. Assume that if \( \lim f(x) = L \), then \( \lim \cos f(x) = \cos L \). In each case evaluate the limit or indicate that the limit does not exist.
(a) \( \lim_{x \to 0} \frac{2x}{1 - 2x} \)  
(b) \( \lim_{x \to \pi/2} \frac{\cos x}{x} \)

(c) \( \lim_{x \to 1} x^3 \cos(1 - x) \)  
(d) \( \lim_{x \to 1} \frac{1 - x^2}{1 - \cos^2 x} \)

37. Give an example where \( \lim(f(x) + g(x)) \) exists but neither \( \lim f(x) \) nor \( \lim g(x) \) exists.
38. Give an example where \( \lim(f(x) \cdot g(x)) \) exists but neither \( \lim f(x) \) nor \( \lim g(x) \) exists.
39. Give an example where \( \lim \frac{f(x)}{g(x)} \) exists but neither \( \lim f(x) \) nor \( \lim g(x) \) exists.

40. Show that if both \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) exist and \( \lim_{x \to c} g(x) \neq 0 \), then \( \lim_{x \to c} f(x) \) exists. Hint: Write \( f(x) = \frac{f(x)g(x)}{g(x)} \).

41. Suppose that \( \lim_{t \to 0} t^4 = 12 \). Show that \( \lim_{t \to 0} t^4 \) exists and equals 4.

42. Prove that if \( \lim_{t \to 3} t^2 = 5 \), then \( \lim_{t \to 3} h(t) = 15 \).

43. Assuming that \( \lim_{x \to 0} \frac{f(x)}{x} = 1 \), which of the following statements is necessarily true? Why?
(a) \( \lim_{x \to 0} f(x) = 0 \)  
(b) \( \lim_{x \to 0} f(x) = 0 \)

44. Prove that if \( \lim_{x \to 0} f(x) = L \neq 0 \) and \( \lim_{x \to 0} g(x) = 0 \), then the limit \( \lim_{x \to 0} \frac{f(x)}{g(x)} \) does not exist.

45. Suppose that \( \lim_{h \to 0} g(h) = L \).
(a) Explain why \( \lim_{h \to 0} g(ah) = L \) for any constant \( a \neq 0 \).
(b) If we assume instead that \( \lim_{h \to 0} a = L \), is it still necessarily true that \( \lim_{h \to 0} g(h) = L \)?
(c) Illustrate (a) and (b) with the function \( f(x) = x^2 \).

46. Assume that \( L(a) = \lim_{x \to a} f(x) \) exists for all \( x > 0 \). Assume also that \( \lim_{x \to a} f(x) = 1 \).
(a) Prove that \( L(ab) = L(a) + L(b) \) for \( a, b > 0 \). Hint: \( (ab)^2 - 1 = a^2b^2 - 1 = a^2 + a^2 - 1 = a^2(b^2 - 1) + (a^2 - 1) \). This shows that \( L(a) \) behaves like a logarithm, in the sense that \( \log(ab) = \log(a) + \log(b) \). In fact, it can be shown that \( L(a) \) is equal to what is known as the natural logarithm function.
(b) Verify numerically that \( L(12) = L(3) + L(4) \).

2.4 Limits and Continuity

In everyday speech, the word “continuous” means having no breaks or interruptions. In calculus, continuity is used to describe functions whose graphs have no breaks. If we imagine the graph of a function \( f \) as a wavy metal wire, then \( f \) is continuous if its graph consists of a single piece of wire as in Figure 1.

Many physical phenomena can be considered as continuous. Our position and velocity vary continuously with time. Barometric pressure varies continuously with altitude above the earth. The current in a simple circuit varies continuously with the voltage applied to it. Ultimately, when we determine the rate of change of a function as we do in the next chapter, we will need the function to be continuous for the mathematics to work properly.
A break in the wire as in Figure 2 is called a discontinuity. Observe in Figure 2 that the break in the graph occurs because the left- and right-hand limits as \( x \) approaches \( c \) are not equal and thus \( \lim_{{x \to c}} g(x) \) does not exist. By contrast, in Figure 1, \( \lim_{{x \to c}} f(x) \) exists and is equal to the function value \( f(c) \). This suggests the following definition of continuity in terms of limits.

**DEFINITION Continuity at a Point**  
Assume that \( f(x) \) is defined on an open interval containing \( x = c \). Then \( f \) is continuous at \( x = c \) if

\[
\lim_{{x \to c}} f(x) = f(c)
\]

If the limit does not exist, or if it exists but is not equal to \( f(c) \), we say that \( f \) has a discontinuity (or is discontinuous) at \( x = c \).

Note that for \( f \) to be continuous at \( c \), three conditions must hold:

1. \( f(c) \) is defined.
2. \( \lim_{{x \to c}} f(x) \) exists.
3. They are equal.

A function \( f \) may be continuous at some points and discontinuous at others. If \( f \) is continuous at all points in its domain, then \( f \) is simply called continuous.

**EXAMPLE 1**  
Show that the following functions are continuous:

(a) \( f(x) = k \)  \( (k \) any constant)  
(b) \( g(x) = x^n \)  \( (n \) a whole number)

**Solution**

(a) We have \( \lim_{{x \to c}} f(x) = \lim_{{x \to c}} k = k \) and \( f(c) = k \). The limit exists and is equal to the function value for all \( c \), so \( f \) is continuous (Figure 3).

(b) By Eq. (1) in Section 2.3, \( \lim_{{x \to c}} g(x) = \lim_{{x \to c}} x^n = c^n \) for all \( c \). Also \( g(c) = c^n \), so again, the limit exists and is equal to the function value. Therefore, \( g \) is continuous. (Figure 4 illustrates the case \( n = 1 \).)

**Examples of Discontinuities**

To understand continuity better, let's consider some ways in which a function can fail to be continuous. Keep in mind that continuity at a point \( x = c \) requires that:

1. \( f(c) \) is defined.  
2. \( \lim_{{x \to c}} f(x) \) exists.  
3. They are equal.

If \( \lim_{{x \to c}} f(x) \) exists, but either the limit is not equal to \( f(c) \), or \( f(c) \) is not defined, then we say that \( f \) has a removable discontinuity at \( x = c \). The function in Figure 5(A) has a removable discontinuity at \( c = 2 \) because

\[
\frac{f(2)}{x-2} = \frac{10}{x-2} \quad \text{but} \quad \lim_{{x \to 2}} \frac{f(x)}{x-2} = 5
\]

Limit exists but is not equal to function value

**Figure 5**  
Removable discontinuity: The discontinuity can be removed by redefining \( f(2) \).
A removable discontinuity at $x = c$ that occurs because $f(c)$ is not defined is sometimes referred to as a removable singularity.

Removable discontinuities are mild in the following sense: We can make $f$ continuous at $x = c$ by redefining $f(c)$ (in the case $\lim_{x \to c} f(x) \neq f(c)$) or defining $f(c)$ (in the case $f(c)$ is not defined) so that $f(c) = \lim_{x \to c} f(x)$. In Figure 5(B), $f(2)$ has been redefined as $f(2) = 5$, and this makes $f$ continuous at $x = 2$.

**Example 2**  Show that $g(x) = \frac{x^3 - 8}{x - 2}$ has a removable discontinuity at $x = 2$. How should $g(2)$ be defined so that $g$ is continuous at $x = 2$?

**Solution**  First note that $g$ is not defined at $x = 2$ since evaluating $g$ at 2 involves division by 0. Also,

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 4) = 12$$

where the Basic Limit Laws are used to determine the value of the limit. Since $\lim_{x \to 2} g(x)$ exists, but $g(2)$ is not defined, $g$ has a removable discontinuity at $x = 2$. If we define $g(2) = 12$, then $g$ would be continuous at $x = 2$.

A worse type of discontinuity is a **jump discontinuity**, which occurs if the one-sided limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist but are not equal. In this case $f$ is not continuous at $c$ because $\lim_{x \to c} f(x)$ does not exist. Figure 6 shows two functions with jump discontinuities at $c = 2$. Unlike the removable case, we cannot make $f$ continuous simply by redefining $f$ at the single point $c$.

**Figure 6**  These functions have jump discontinuities at $x = 2$.

In connection with jump discontinuities, it is convenient to define **one-sided continuity**.

**Definition**  **One-Sided Continuity**  A function $f$ is called

- **Left-continuous at** $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$
- **Right-continuous at** $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$

In Figure 6 above, the function in (A) is left-continuous at $x = 2$ but the function in (B) is neither left- nor right-continuous at $x = 2$.

Many theorems in calculus apply to functions that are continuous on an interval, a concept that is defined as follows:

**Definition**  **Continuity on an Interval**  Assume that $I$ is an interval in the form $(a, b)$, $[a, b)$, $(a, b)$, or $[a, b]$. Then $f$ is **continuous on** $I$ if $f$ is continuous at each point in $(a, b)$, $f$ is right-continuous at $b$ if $a$ is in $I$, and $f$ is left-continuous at $a$ if $b$ is in $I$.

The next example explores one-sided continuity using a piecewise-defined function—that is, a function defined by different formulas on different intervals.
EXAMPLE 3  Piecewise-Defined Function

Discuss the continuity of

\[ F(x) = \begin{cases} 
  x & \text{for } x < 1 \\
  3 & \text{for } 1 \leq x \leq 3 \\
  x & \text{for } x > 3 
\end{cases} \]

Solution  The functions \( f(x) = x \) and \( g(x) = 3 \) are continuous, so \( F \) is also continuous, except possibly at the transition points \( x = 1 \) and \( x = 3 \), where the formula for \( F(x) \) changes (Figure 7).

- At \( x = 1 \), the one-sided limits exist but are not equal:
  \[ \lim_{{x \to 1^-}} F(x) = \lim_{{x \to 1^-}} x = 1, \quad \lim_{{x \to 1^+}} F(x) = \lim_{{x \to 1^+}} 3 = 3 \]
  Thus, \( F \) has a jump discontinuity at \( x = 1 \). However, the right-hand limit is equal to the function value \( F(1) = 3 \), so \( F \) is right-continuous at \( x = 1 \).
- At \( x = 3 \), the left- and right-hand limits exist and both are equal to \( F(3) = 3 \), so \( F \) is continuous at \( x = 3 \):
  \[ \lim_{{x \to 3^-}} F(x) = \lim_{{x \to 3^-}} 3 = 3, \quad \lim_{{x \to 3^+}} F(x) = \lim_{{x \to 3^+}} x = 3 \]

We say that \( f \) has an infinite discontinuity at \( x = c \) if one or both of the one-sided limits are infinite [even if \( f(x) \) itself is not defined at \( x = c \)]. Like with a jump discontinuity, in this case \( f \) is not continuous at \( c \) because \( \lim f(x) \) does not exist. Figure 8 illustrates three types of infinite discontinuities occurring at \( x = 2 \). Notice that \( x = 2 \) does not belong to the domain of the function in cases (A) and (B).

EXAMPLE 4  The Intensity of a Light Source

A standard model for the intensity \( I \) of a light source at varying distances \( d \) from the light is an inverse-square law, \( I(d) = k/d^2 \) for a constant \( k > 0 \) depending on the light source (Figure 9). Show that \( I \) has an infinite discontinuity at \( d = 0 \).

Solution  Regardless of the value of \( k > 0 \), as \( d \) approaches 0 from the right, the values of \( I(d) = k/d^2 \) are positive and become arbitrarily large. Therefore \( \lim_{{d \to 0^+}} I(d) = \infty \), and it follows that \( I \) has an infinite discontinuity at 0.
Note that this does not mean that the intensity of the light is actually unbounded as we get closer and closer to it. The relationship \( f(d) = k/d^2 \) is a model for the true behavior. The model does, however, properly reflect the fact that the intensity rises rapidly as we approach the source.

Finally, we note that some functions have more severe types of discontinuities than those discussed above. For example, \( f(x) = \sin \frac{1}{x} \) oscillates infinitely often between \(+1\) and \(-1\) as \( x \to 0 \) (Figure 10). Neither the left- nor the right-hand limit exists at \( x = 0 \), so this discontinuity is not a jump discontinuity. See Exercises 94 and 95 for even stranger examples.

### Building Continuous Functions

Having studied some examples of discontinuities, we focus again on continuous functions. How can we show that a function is continuous? One way is to use the Laws of Continuity, which state, roughly speaking, that a function is continuous if it is built out of functions that are known to be continuous.

#### Theorem 1 Basic Laws of Continuity

If \( f \) and \( g \) are continuous at \( x = c \), then the following functions are also continuous at \( x = c \):

1. \( f + g \) and \( f - g \)
2. \( kf \) for any constant \( k \)
3. \( fg \)
4. \( f/g \) if \( g(c) \neq 0 \)

**Proof** These laws follow directly from the corresponding Basic Limit Laws (Theorem 1, Section 2.3). We illustrate by proving the first part of (i) in detail. The remaining laws are proved similarly. By definition, we must show that \( \lim_{x \to c} (f(x) + g(x)) = f(c) + g(c) \).

Because \( f \) and \( g \) are both continuous at \( x = c \), we have

\[
\lim_{x \to c} f(x) = f(c), \quad \lim_{x \to c} g(x) = g(c)
\]

The Sum Law for limits yields the desired result:

\[
\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)
\]

In Section 2.3, we noted that the Basic Limit Laws for Sums and Products are valid for an arbitrary number of functions. The same is true for continuity; that is, if \( f_1, \ldots, f_n \) are continuous, then so are the functions

\[
f_1 + f_2 + \cdots + f_n, \quad f_1 f_2 \cdots f_n
\]

The next two theorems assert that the indicated basic functions are continuous on their domains. Recall (Section 1.3) that the term “basic function” refers to polynomials, rational functions, \( n \)th-root and algebraic functions, trigonometric functions and their inverses, and exponential and logarithmic functions.

#### Theorem 2 Continuity of Polynomial and Rational Functions

Let \( P \) and \( Q \) be polynomials. Then:

- \( P \) and \( Q \) are continuous on the real line.
- \( P/Q \) is continuous on its domain [at all values \( x = c \) such that \( Q(c) \neq 0 \)].

**Proof** The function \( f(x) = x^m \) is continuous for all whole numbers \( m \) by Example 1. By Continuity Law (ii), \( f(x) = ax^m \) is continuous for every constant \( a \). A polynomial

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0
\]
is a sum of continuous functions, so it too is continuous. By Continuity Law (iv), a quotient function \( P/Q \) is continuous at \( x = c \), provided that \( Q(c) \neq 0 \).

This result shows, for example, that \( f(x) = 3x^4 - 2x^3 + 8x \) is continuous for all \( x \), and that

\[
g(x) = \frac{x + 3}{x^2 - 1}
\]

is continuous for \( x \neq \pm 1 \). Note that if \( n \) is a positive integer, then \( f(x) = x^{-n} \) is continuous for \( x \neq 0 \) because \( f(x) = \frac{1}{x^n} \) is a rational function.

The continuity of the \( n \)th-root, sine, cosine, and exponential functions should not be surprising because their graphs have no visible breaks (Figure 11). However, complete proofs of continuity are somewhat technical and are omitted.

**THEOREM 3 Continuity of Some Basic Functions**

- \( y = x^{1/n} \) is continuous on its domain for \( n \) a natural number.
- \( y = \sin x \) and \( y = \cos x \) are continuous on the real line.
- \( y = b^x \) is continuous on the real line (for \( b > 0, b \neq 0 \)).

**FIGURE 11** As the graphs suggest, these functions are continuous on their domains.

Because \( f(x) = \sin x \) and \( f(x) = \cos x \) are continuous, the Continuity Law (iv) for Quotients implies that the other standard trigonometric functions are continuous on their domains, consisting of the values of \( x \) where the denominators, in the following quotient expressions for them, are nonzero:

\[
\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}
\]

They have infinite discontinuities at points where the denominators are zero. For example, as illustrated in Figure 12, \( f(x) = \tan x \) has infinite discontinuities at the points

\[
x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots
\]

Finally, it is important to know that a composition of continuous functions is again continuous. The following theorem is proved in Appendix D.

**THEOREM 4 Continuity of Composite Functions** If \( g \) is continuous at \( x = c \), and \( f \) is continuous at \( x = g(c) \), then the composite function \( F(x) = f(g(x)) \) is continuous at \( x = c \).
For example, \( f(x) = (x^2 + 9)^{1/3} \) is continuous because it is the composite of the continuous functions \( f(x) = x^{1/3} \) and \( g(x) = x^2 + 9 \). Similarly, \( F(x) = \cos(x^{-1}) \) is continuous for all \( x \neq 0 \).

More generally, an elementary function is a function that is constructed out of basic functions using the operations of addition, subtraction, multiplication, division, and composition. Since the basic functions are continuous (on their domains), an elementary function is also continuous on its domain by the Laws of Continuity. An example of an elementary function is

\[
  f(x) = \sqrt{\frac{x^2 + \cos^4(2x + 4)}{x + 8}}
\]

This function is continuous on its domain \( \{x : x \neq 8\} \).

**Substitution: Evaluating Limits Using Continuity**

It is easy to evaluate a limit when the function in question is known to be continuous. In this case, by definition, the limit is equal to the function value:

\[
  \lim_{x \to c} f(x) = f(c)
\]

We call this the **Substitution Method** because the limit is evaluated by substituting \( x = c \) in \( f(x) \).

**EXAMPLE 5** Evaluate (a) \( \lim_{y \to \frac{\pi}{3}} \sin y \) and (b) \( \lim_{x \to 8} \frac{2x^{1/3}}{\sqrt{x + 1}} \).

**Solution**

(a) We can use substitution because \( f(y) = \sin y \) is continuous.

\[
  \lim_{y \to \frac{\pi}{3}} \sin y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}
\]

(b) The function \( f(x) = \frac{2x^{1/3}}{\sqrt{x + 1}} \) is continuous at \( x = 8 \) because the numerator and denominator are both continuous at \( x = 8 \) and the denominator \( \sqrt{x + 1} \) is nonzero at \( x = 8 \). Therefore, we can use substitution:

\[
  \lim_{x \to 8} \frac{2x^{1/3}}{\sqrt{x + 1}} = \frac{2(8)^{1/3}}{\sqrt{8 + 1}} = \frac{4}{3}
\]

The greatest integer function \( f(x) = \lfloor x \rfloor \) is the function defined by \( \lfloor x \rfloor = n \), where \( n \) is the unique integer such that \( n \leq x < n + 1 \) (Figure 13). For example, \( \lfloor 4.7 \rfloor = 4 \) and \( \lfloor -2.3 \rfloor = -3 \). This function is called the greatest integer function because \( \lfloor x \rfloor \) represents the greatest integer less than or equal to \( x \).

**EXAMPLE 6** Assumptions Matter

Can we evaluate \( \lim_{x \to 2} \lfloor x \rfloor \) using substitution?

**Solution** Substitution cannot be applied because \( f(x) = \lfloor x \rfloor \) is not continuous at \( x = 2 \). In fact, \( \lim_{x \to 2} \lfloor x \rfloor \) does not exist, and that follows because the one-sided limits are not equal:

\[
  \lim_{x \to 2^-} \lfloor x \rfloor = 2 \quad \text{and} \quad \lim_{x \to 2^+} \lfloor x \rfloor = 1
\]
CONCEPTUAL INSIGHT Real-World Modeling by Continuous Functions Continuous functions are often used to model relationships between physical quantities such as position and time, temperature and altitude, and voltage and resistance. This reflects our everyday experience that change in the physical world tends to occur continuously rather than through abrupt transitions. However, mathematical models are approximations to reality and are based on assumptions about the phenomenon being studied. It is always important to be aware of a model's assumptions and the limitations they impose.

In Figure 14, atmospheric temperature is represented as a continuous function of altitude. At such a large scale the assumption of continuity is reasonable because it is consistent with our experience. However, at smaller, less familiar scales the situation can be different. In fact, in 2002 scientists McGaughey and Ward reported observing temperature discontinuities at the surface of evaporating water droplets, suggesting that it may not be appropriate to assume temperature is continuous at such small scales.

The size of a population is often treated as a continuous function of time. Strictly speaking, population size is a whole number that changes by ±1 when an individual is born or dies, so it really is not continuous. At the scale of the size of your family, it does not make sense to consider the number of people as a continuous variable. However, if a population is large, the effect of an individual birth or death is small, and at such a scale it is both reasonable and convenient to treat population as a continuous function of time. Ultimately, the test of a model is how well it enables us to understand and predict the behavior of the actual system. When it fails to do so, the assumptions need to be examined, and possibly adjusted, to try to find a better fit between model and reality.

![Figure 14: Atmospheric temperature and world population are modeled by continuous functions.](image)

2.4 SUMMARY

- Definition: \( f \) is continuous at \( x = c \) if \( \lim_{x \to c} f(x) = f(c) \). This means that \( f(c) \) exists, \( \lim_{x \to c} f(x) \) exists, and they are equal.
- If \( \lim_{x \to c} f(x) \) does not exist, or if it exists but does not equal \( f(c) \), then \( f \) is discontinuous at \( x = c \).
- If \( f \) is continuous at all points in its domain, \( f \) is simply called continuous.
- Right-continuous at \( x = c \): \( \lim_{x \to c^+} f(x) = f(c) \).
- Left-continuous at \( x = c \): \( \lim_{x \to c^-} f(x) = f(c) \).
• Three common types of discontinuities:
  - **Removable discontinuity:** \( \lim_{x \to c} f(x) \) exists, but either the limit does not equal \( f(c) \) or \( f(c) \) is not defined.
  - **Jump discontinuity:** The one-sided limits both exist but are not equal.
  - **Infinite discontinuity:** The limit is infinite as \( x \) approaches \( c \) from one or both sides.

• Laws of Continuity: Sums, products, multiples, inverses, and composites of continuous functions are continuous. The same holds for a quotient \( f/g \) at points where \( g(x) \neq 0 \).

• The following basic functions are continuous on their domains: polynomials, rational functions, \( n \)-th root and algebraic functions, trigonometric functions, and exponential functions.

• Substitution Method: If \( f \) is known to be continuous at \( x = c \), then the value of the limit \( \lim_{x \to c} f(x) \) is \( f(c) \).

### 2.4 Exercises

#### Preliminary Questions

1. Which property of \( f(x) = x^2 \) allows us to conclude that \( \lim_{x \to 2} x^2 = 4 \)?

2. What can be said about \( f(3) \) if \( f \) is continuous and \( \lim_{x \to 2} f(x) = \frac{1}{2} \)?

3. Suppose that \( f(x) < 0 \) if \( x \) is positive and \( f(x) > 1 \) if \( x \) is negative. Can \( f \) be continuous at \( x = 0 \)?

4. Is it possible to determine \( f(3) \) if \( f(x) = 3 \) for all \( x < 7 \) and \( f \) is right-continuous at \( x = 7 \)? What if \( f \) is left-continuous?

5. Are the following true or false? If false, then draw or give a counterexample, and state a correct version.
   (a) \( f \) is continuous at \( x = a \) if the left- and right-hand limits of \( f(x) \) as \( x \to a \) exist and are equal.
   (b) \( f \) is continuous at \( x = a \) if the left- and right-hand limits of \( f(x) \) as \( x \to a \) exist and equal \( f(a) \).
   (c) If the left- and right-hand limits of \( f(x) \) as \( x \to a \) exist, then \( f \) has a removable discontinuity at \( x = a \).
   (d) If \( f \) and \( g \) are continuous at \( x = a \), then \( f + g \) is continuous at \( x = a \).
   (e) If \( f \) and \( g \) are continuous at \( x = a \), then \( f/g \) is continuous at \( x = a \).

#### Exercises

1. Referring to Figure 15, state whether \( f \) is left- or right-continuous (or neither) at each point of discontinuity. Does \( f \) have any removable discontinuities?

   *Exercises 2–4 refer to the function \( g \) whose graph appears in Figure 16.*

2. State whether \( g \) is left- or right-continuous (or neither) at each of its points of discontinuity.

3. At which point \( c \) does \( g \) have a removable discontinuity? How should \( g(c) \) be redefined to make \( g \) continuous at \( x = c \)?

4. Find the point \( c_1 \) at which \( g \) has a jump discontinuity but is left-continuous. How should \( g(c_1) \) be redefined to make \( g \) right-continuous at \( x = c_1 \)?

5. In Figure 17, determine the one-sided limits at the points of discontinuity. Which discontinuity is removable and how should \( f \) be redefined to make it continuous at this point?

6. Suppose that \( f(x) = 2 \) for \( x < 3 \) and \( f(x) = -4 \) for \( x > 3 \).
   (a) What is \( f(3) \) if \( f \) is left-continuous at \( x = 3 \)?
   (b) What is \( f(3) \) if \( f \) is right-continuous at \( x = 3 \)?

   *In Exercises 7–16, use Theorems 1–4 to show that the function is continuous.*

7. \( f(x) = x + \sin x \)
8. \( f(x) = x \sin x \)
9. \( f(x) = 3x + 4 \sin x \)
10. \( f(x) = 3x^3 + 8x^2 - 20x \)
In Exercises 17–38, determine the points of discontinuity. State the type of discontinuity (removable, jump, infinite, or none of these) and whether the function is left- or right-continuous.

17. \( f(x) = \frac{1}{x} \)  
18. \( f(x) = |x| \)

19. \( f(x) = \frac{x - 2}{|x - 1|} \)  
20. \( f(x) = [x] \)

21. \( f(x) = \left\lfloor \frac{x}{2} \right\rfloor \)  
22. \( g(t) = \frac{1}{t^2 - 1} \)

23. \( h(x) = \frac{1}{2 - |x|} \)  
24. \( k(x) = \frac{x - 2}{|2 - x|} \)

25. \( f(x) = \frac{x + 1}{4x - 2} \)  
26. \( h(z) = \frac{1 - 2z}{2z - 6} \)

27. \( f(x) = 3x^{2/3} - 9x^3 \)  
28. \( g(t) = 3t^{-2/3} - 9t^3 \)

29. \( f(x) = \begin{cases} \frac{x - 2}{|x - 2|} & x \neq 2 \\ 1 & x = 2 \end{cases} \)  
30. \( f(x) = \begin{cases} \cos \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \)

31. \( f(x) = 2x^2 - \frac{50}{x + 5} \)  
32. \( u(t) = \frac{t + 1}{t^2 - 1} \)

33. \( g(t) = \tan 2t \)  
34. \( f(x) = \cos(x^2) \)

35. \( f(x) = \tan(x) \)  
36. \( f(x) = \cos(x[x]) \)

37. \( f(x) = [x + 3] + [2x] \)  
38. \( f(x) = 2[x/2] + 4[x/4] \)

In Exercises 39–52, determine the domain of the function and prove that it is continuous on its domain using Theorems 1–4.

39. \( f(x) = 2 \sin x + 3 \cos x \)  
40. \( f(x) = \sqrt{x^2 + 9} \)

41. \( f(x) = \sqrt{x} \sin x \)  
42. \( f(x) = \frac{x^2}{x + x^{3/2}} \)

43. \( f(x) = x^2 - 3x^{1/2} \)  
44. \( f(x) = x^{1/3} + x^{3/2} \)

45. \( f(x) = x^{4/3} \)  
46. \( f(x) = \cos^3 x \)

47. \( f(x) = \tan^2 x \)  
48. \( f(x) = \cos(x^{1/2} + 1) \)

49. \( f(x) = (x^2 + 1)^{3/2} \)  
50. \( f(x) = (x^3 + 3)^{3/2} \)

51. \( f(x) = \frac{\cos(x^2)}{x^2 - 1} \)  
52. \( f(x) = \frac{\tan(x - 2)}{9x^2 + 2} \)

53. The graph of the following function is shown in Figure 18.

\[
f(x) = \begin{cases} 
  x^2 + 3 & \text{for } x < 1 \\
  10 - x & \text{for } 1 \leq x \leq 2 \\
  6x - x^2 & \text{for } x > 2 
\end{cases}
\]

Show that \( f \) is continuous for \( x \neq 1, 2 \). Then compute the right- and left-hand limits at \( x = 1, 2, \) and determine whether \( f \) is left-continuous, right-continuous, or continuous at these points.

54. Sawtooth Function 

Draw the graph of \( f(x) = x - [x] \). At which points is \( f \) discontinuous? Is it left- or right-continuous at those points?

In Exercises 55–56, \([x]\) refers to the least integer function. It is defined by \([x] = n\), where \( n \) is the unique integer such that \( n - 1 < x \leq n \). In each case, provide the graph of \( f \), indicate the points of discontinuity and type of each (removable, jump, infinite, or none of these), and indicate whether \( f \) is left- or right-continuous.

55. \( f(x) = [x] \)

56. \( f(x) = [x] - [x] \)

In Exercises 57–60, sketch the graph of \( f \). At each point of discontinuity, state whether \( f \) is left- or right-continuous.

57. \( f(x) = \begin{cases} 
  x^2 & \text{for } x \leq 1 \\
  2 - x & \text{for } x > 1 
\end{cases} \)

58. \( f(x) = \begin{cases} 
  x + 1 & \text{for } x < 1 \\
  \frac{1}{x} & \text{for } x \geq 1 
\end{cases} \)

59. \( f(x) = \begin{cases} 
  x^2 - 3x + 2 & \text{for } x \neq 2 \\
  0 & \text{for } x = 2 
\end{cases} \)

60. \( f(x) = \begin{cases} 
  x^3 + 1 & \text{for } -\infty < x \leq 0 \\
  -x + 1 & \text{for } 0 < x < 2 \\
  -x^2 + 10x - 15 & \text{for } x \geq 2 
\end{cases} \)

61. Show that the function

\[
f(x) = \begin{cases} 
  \frac{x^2 - 16}{x - 4} & \text{for } x \neq 4 \\
  10 & \text{for } x = 4 
\end{cases}
\]

has a removable discontinuity at \( x = 4 \).

62. (Continuity) 

Define \( f(x) = x \sin \frac{1}{x} + 2 \) for \( x \neq 0 \). Plot \( f \). How should \( f(0) \) be defined so that \( f \) is continuous at \( x = 0 \)?

In Exercises 63–64, \( H \) is the Heaviside function, defined by

\[
H(x) = \begin{cases} 
  0 & \text{when } x < 0 \\
  1 & \text{when } x \geq 0 
\end{cases}
\]

63. In each case, sketch the graph of \( f \), indicate whether or not \( f \) is continuous, and—if \( f \) is not continuous—identify the points of discontinuity.

(a) \( f(x) = H(x)(x^2 + 1) \)  
(b) \( f(x) = H(x)x \)

(c) \( f(x) = H(x - 2)\sqrt{x} \)  
(d) \( f(x) = H(1 + x)H(1 - x)(1 - x^2) \)

64. Assume that a function \( f \) is defined and continuous for all \( x \). Under what condition on \( f \) are we assured that the function \( g(x) \), defined by \( g(x) = H(x - a)f(x) \), is continuous?
In Exercises 65–67, find the value of the constant \( a, b, \) or \( c \) that makes the function continuous.

65. \( f(x) = \begin{cases} x^2 - c & \text{for } x < 5 \\ 4x + 2e & \text{for } x \geq 5 \end{cases} \) for \( x < 5 \)

66. \( f(x) = \begin{cases} \frac{2x + 9c}{x^2} & \text{for } x \leq 3 \\ -4x + c & \text{for } x > 3 \end{cases} \) for \( x \leq 3 \)

67. \( f(x) = \begin{cases} x^2 - c & \text{for } x < -1 \\ ax + b & \text{for } -1 \leq x \leq \frac{1}{2} \\ x^2 - 1 & \text{for } x > \frac{1}{2} \end{cases} \) for \( x < -1 \)

68. Define \( g(x) = \begin{cases} x + 3 & \text{for } x < -1 \\ cx & \text{for } -1 \leq x \leq 2 \\ x + 2 & \text{for } x > 2 \end{cases} \)

Find a value of \( c \) such that \( g \) is

(a) left-continuous
(b) right-continuous

In each case, sketch the graph of \( g \).

69. Define \( g(t) = \frac{t^2 - 1}{-2t - 1} \) for \( t \neq \pm 1 \). Answer the following questions, using a table if necessary.

(a) Can \( g(1) \) be defined so that \( g \) is continuous at \( t = 1 \)? If yes, how?
(b) Can \( g(-1) \) be defined so that \( g \) is continuous at \( t = -1 \)? If so, how?

70. Each of the following statements is false. For each statement, sketch the graph of a function that provides a counterexample.

(a) If \( \lim_{x \to a} f(x) \) exists, then \( f \) is continuous at \( x = a \).
(b) If \( f \) has a jump discontinuity at \( x = a \), then \( f(a) \) is equal to either \( \lim_{x \to a^-} f(x) \) or \( \lim_{x \to a^+} f(x) \).

In Exercises 71–74, draw the graph of a function on \([0, 5]\) with the given properties.

71. \( f \) is not continuous at \( x = 1 \), but \( \lim_{x \to 1^-} f(x) \) and \( \lim_{x \to 1^+} f(x) \) exist and are equal.

72. \( f \) is left-continuous but not continuous at \( x = 2 \), and right-continuous but not continuous at \( x = 3 \).

73. \( f \) has a removable discontinuity at \( x = 1 \), a jump discontinuity at \( x = 2 \), and \( \lim_{x \to 1^-} f(x) = -\infty \), \( \lim_{x \to 1^+} f(x) = 2 \)

74. \( f \) is right- but not left-continuous at \( x = 1 \), left- but not right-continuous at \( x = 2 \), and neither left- nor right-continuous at \( x = 3 \).

In Exercises 75–86, evaluate using substitution.

75. \( \lim_{x \to 2} (2x^3 - 4) \)

76. \( \lim_{x \to 0} (5x - 12x^2) \)

77. \( \lim_{x \to 0} \frac{x + 2}{x^2 + 2x} \)

78. \( \lim_{x \to \pi} \sin\left(\frac{x}{2} - \pi\right) \)

79. \( \lim_{x \to \frac{\pi}{2}} \tan(3x) \)

80. \( \lim_{x \to 0} \frac{1}{x^2 + \cos x} \)

81. \( \lim_{x \to 3} x^{-2} \)

82. \( \lim_{x \to 2} \sqrt{x^2 + 4x} \)

83. \( \lim_{x \to 1} (1 - 8x^3)^{\frac{2}{3}} \)

84. \( \lim_{x \to 2} \frac{7x + 2}{4 - x} \)

85. \( \lim_{x \to 3} 10x^2 - 2x \)

86. \( \lim_{x \to 3} \frac{3x}{x - 3} \)

87. Suppose that \( f \) and \( g \) are discontinuous at \( x = c \). Does it follow that \( f + g \) is discontinuous at \( x = c \)? If not, give a counterexample. Does this contradict Theorem 10?

88. Prove that \( f(x) = \lfloor x \rfloor \) is continuous for all \( x \). Hint: To prove continuity at \( x = 0 \), consider the one-sided limits.

89. Use the result of Exercise 88 to prove that if \( g \) is continuous, then \( f(x) = \lfloor g(x) \rfloor \) is also continuous.

90. Which of the following quantities would be represented by the continuous functions of time and which would have one or more discontinuities?

(a) Velocity of an airplane during a flight
(b) Temperature in a room under ordinary conditions
(c) Value of a bank account with interest paid yearly
(d) Salary of a teacher
(e) Population of the world

91. \( \square \) In 2017, the federal income tax \( T \) on income of \( x \) dollars (up to \$91,900) was determined by the formula

\[ T(x) = \begin{cases} 0.10x & \text{for } 0 \leq x \leq 9325 \\ 0.15x - 466.25 & \text{for } 9325 \leq x \leq 37,950 \\ 0.25x - 4221.25 & \text{for } 37,950 \leq x \leq 91,900 \end{cases} \]

Sketch the graph of \( T \). Does \( T \) have any discontinuities? Explain why, if \( T \) had a jump discontinuity, it might be advantageous in some situations to earn less money.

Further Insights and Challenges

92. \( \square \) If \( f \) has a removable discontinuity at \( x = c \), then it is possible to redefine \( f(c) \) so that \( f \) is continuous at \( x = c \). Can this be done in more than one way? Explain.

93. Give an example of functions \( f \) and \( g \) such that \( f(g(x)) \) is continuous but \( g \) has at least one discontinuity.

94. Continuous at Only One Point \( \square \) Show that the following function is continuous only at \( x = 0 \):

\[ f(x) = \begin{cases} x & \text{for } x \text{ rational} \\ -x & \text{for } x \text{ irrational} \end{cases} \]

95. Show that \( f \) is a discontinuous function for all \( x \), where \( f(x) \) is defined as follows:

\[ f(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ -1 & \text{for } x \text{ irrational} \end{cases} \]

Show that \( f^2 \) is continuous for all \( x \).
2.5 Indeterminate Forms

Substitution can be used to evaluate limits when the function in question is known to be continuous. For example, \( f(x) = x^{-2} \) is continuous at \( x = 3 \), and therefore,

\[
\lim_{x \to 3} x^{-2} = 3^{-2} = \frac{1}{9}
\]

When we study derivatives in Chapter 3, we will be faced with limits \( \lim_{x \to c} f(x) \), where \( f(c) \) is not defined. In such cases, substitution cannot be used directly. However, many of these limits can be evaluated if we use algebra to rewrite the formula for \( f(x) \).

To illustrate, consider this limit (Figure 1):

\[
\lim_{x \to 4} \frac{x^2 - 16}{x - 4}
\]

The function \( f(x) = \frac{x^2 - 16}{x - 4} \) is not defined at \( x = 4 \) because the formula for \( f(4) \) produces the undefined expression \( 0/0 \). However, the numerator of \( f(x) \) factors:

\[
\frac{x^2 - 16}{x - 4} = \frac{(x + 4)(x - 4)}{x - 4} = x + 4 \quad \text{(valid for } x \neq 4)\]

This shows that \( f \) coincides with the continuous function \( y = x + 4 \) for all \( x \neq 4 \). Since the limit depends only on the values of \( f(x) \) for \( x \neq 4 \), we have

\[
\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} (x + 4) = 8
\]

Evaluate by substitution

Some other indeterminate forms are \( \frac{0}{0} \), \( \infty / \infty \), \( \frac{\infty}{0} \), and \( 0 \cdot \infty \). These are treated in Section 7.5.

An indeterminate form, as the name suggests, indicates that the limit cannot be determined from the form. It does not mean that the limit does not exist. Instead, we think of it as a warning sign that tells us more work needs to be done to evaluate the limit. One strategy, when \( f(x) \) has an indeterminate form at \( x = c \), is to transform \( f(x) \) algebraically, if possible, into a new expression that is defined and continuous at \( x = c \), and then evaluate the limit by substitution. As you study the following examples, notice that the critical step is to cancel a common factor from the numerator and denominator at the appropriate moment, thereby removing the indeterminacy.

**EXAMPLE 1** Calculate \( \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12} \).

**Solution** The function has the indeterminate form \( 0/0 \) at \( x = 3 \) because

Numerator at \( x = 3 \):
\[ x^2 - 4x + 3 = 3^2 - 4(3) + 3 = 0 \]

Denominator at \( x = 3 \):
\[ x^2 + x - 12 = 3^2 + 3 - 12 = 0 \]

**Step 1.** Transform algebraically and cancel.

\[
\frac{x^2 - 4x + 3}{x^2 + x - 12} = \frac{(x - 3)(x - 1)}{(x - 3)(x + 4)} = \frac{x - 1}{x + 4} \quad \text{(if } x \neq 3)\]

Cancel common factor Continuous at \( x = 3 \)
 SECTION 2.5 Indeterminate Forms 73

Step 2. Substitute (evaluate using continuity).
Because the expression on the right in Eq. (1) is continuous at \( x = 3 \),
\[
\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12} = \lim_{x \to 3} \frac{x - 1}{x + 4} = \frac{2}{7}
\]
Evaluate by substitution

The next example illustrates the algebraic technique of multiplying by the conjugate, which can be used to treat some indeterminate forms involving square roots.

**EXAMPLE 2** Multiplying by the Conjugate Evaluate \( \lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} \).

Solution We check that \( f(x) = \frac{x - 9}{\sqrt{x} - 3} \) has the indeterminate form \( 0/0 \) at \( x = 9 \):

Numerator at \( x = 9 \):
\[
x - 9 = 9 - 9 = 0
\]
Denominator at \( x = 9 \):
\[
\sqrt{x} = 3 = \sqrt{9} - 3 = 0
\]

Step 1. Multiply by the conjugate and cancel.
\[
\left( \frac{x - 9}{\sqrt{x} - 3} \right) \left( \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \right) = \frac{(x - 9)(\sqrt{x} + 3)}{x - 9} = \sqrt{x} + 3 \quad \text{(if } x \neq 9\text{)}
\]

Step 2. Substitute (evaluate using continuity).
Because \( g(x) = \sqrt{x} + 3 \) is continuous at \( x = 9 \), we can now evaluate the limit by substitution:
\[
\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} = \lim_{x \to 9} (\sqrt{x} + 3) = 6
\]

**EXAMPLE 3** Calculate \( \lim_{x \to 1} \frac{x^3 - 1}{(x - 1)^3} \).

Solution The function has the indeterminate form \( 0/0 \) at \( x = 1 \) because

Numerator at \( x = 1 \):
\[
x^3 - 1 = 1^3 - 1 = 0
\]
Denominator at \( x = 1 \):
\[
(x - 1)^3 = (1 - 1)^3 = 0
\]

Step 1. Transform algebraically and cancel.
\[
\frac{x^3 - 1}{(x - 1)^3} = \frac{(x - 1)(x^2 + x + 1)}{(x - 1)^3} = \frac{x^2 + x + 1}{(x - 1)^2} \quad \text{(if } x \neq 1\text{)}
\]

Step 2. Determine the limit.
Because \( x^2 + x + 1 \) approaches \( 3 \) as \( x \to 1 \) and \( (x - 1)^2 \) is positive and approaches \( 0 \) as \( x \to 1 \), it follows that \( \frac{x^2 + x + 1}{(x - 1)^2} \) is positive and becomes arbitrarily large as \( x \to 1 \). Therefore, \( \lim_{x \to 1} \frac{x^3 - 1}{(x - 1)^3} \) does not exist, but we can say \( \lim_{x \to 1} \frac{x^3 - 1}{(x - 1)^3} = \infty \).

Note that in this example, we obtained the undefined expression \( 3/0 \) for the limit form after simplifying. This is not an indeterminate form, and from it we were able to evaluate the limit (determining that it does not exist). A limit form \( a/0 \) with nonzero \( a \) is not an indeterminate form. If \( \lim f(x) \) is in the form \( a/0 \) with \( a \neq 0 \), then \( f \) takes on arbitrarily large values arbitrarily close to \( c \), and we can conclude that the limit does not exist.
The previous three examples involve limits in the indeterminate form 0/0. In the first case the limit is 2/7, in the second it is 6, and in the third the limit does not exist. This underscores the meaning of indeterminate. We do not know yet whether or not the limit exists. Indeterminate means further work is needed to evaluate the limit.

**Example 4** The Form \( \infty/\infty \) Calculate \( \lim_{x \to \frac{\pi}{2}} \tan x \). 

**Solution** As we see in Figure 2, both \( f(x) = \tan x \) and \( f(x) = \sec x \) have infinite discontinuities at \( x = \frac{\pi}{2} \), so this limit has the indeterminate form \( \infty/\infty \) at \( x = \frac{\pi}{2} \).

\[ y = \sec x \]
\[ y = \tan x \]

**Figure 2**

*Step 1.* Transform algebraically and cancel.

\[ \frac{\tan x}{\sec x} = \frac{(\sin x)}{\cos x} \cdot \frac{\cos x}{\cos x} = \sin x \quad (\text{if } \cos x \neq 0) \]

*Step 2.* Substitute (evaluate using continuity).

Because \( f(x) = \sin x \) is continuous,

\[ \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \sin x = \sin \frac{\pi}{2} = 1 \]

**Example 5** The Form \( \infty - \infty \) Calculate \( \lim_{x \to 1} \left( \frac{1}{x-1} - \frac{2}{x^2-1} \right) \).

**Solution** As we see in Figure 3, \( y = \frac{1}{x-1} \) and \( y = \frac{2}{x^2-1} \) both have infinite discontinuities at \( x = 1 \). This limit has the indeterminate form \( \infty - \infty \).

*Step 1.* Transform algebraically and cancel.

Combine the fractions and simplify (for \( x \neq 1 \)):

\[ \frac{1}{x-1} - \frac{2}{x^2-1} = \frac{x+1}{x^2-1} - \frac{2}{x^2-1} = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1} \]

*Step 2.* Substitute (evaluate using continuity).

\[ \lim_{x \to 1} \left( \frac{1}{x-1} - \frac{2}{x^2-1} \right) = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2} \]

As preparation for work we will do computing derivatives in Chapter 3, we evaluate a limit involving a symbolic constant.

**Example 6** Symbolic Constant Calculate \( \lim_{h \to 0} \frac{(h+a)^2 - a^2}{h} \), where \( a \) is a constant.

**Solution** We have the indeterminate form 0/0 at \( h = 0 \) because

Numerator at \( h = 0 \):
\[ (h+a)^2 - a^2 = (0+a)^2 - a^2 = 0 \]

Denominator at \( h = 0 \):
\[ h = 0 \]
Expand the numerator and simplify (for \( h \neq 0 \)):

\[
\frac{(h + a)^2 - a^2}{h} = \frac{(h^2 + 2ah + a^2) - a^2}{h} = \frac{h^2 + 2ah}{h} = \frac{h(h + 2a)}{h} = h + 2a
\]

The function \( f(h) = h + 2a \) is continuous (for any constant \( a \)), so

\[
\lim_{h \to 0} \frac{(h + a)^2 - a^2}{h} = \lim_{h \to 0} (h + 2a) = 2a
\]

Some expressions that yield an indeterminate form cannot be simplified algebraically to determine the limit. In the next two examples we consider limits that cannot be found with the rules we have developed so far. Nevertheless, we can investigate the limits numerically to find estimates for them. In the next section and in Section 7.5 we introduce important theorems that provide other tools for evaluating limits in indeterminate forms.

The limit in the next example is important when determining derivative formulas of the sine and cosine functions in the next chapter. We investigate it numerically here and determine the exact value in the next section.

**EXAMPLE 7** Show that

\[
\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta}
\]

is in an indeterminate form and estimate its value by examining it numerically.

**Solution** Note that at \( \theta = 0 \), the denominator is 0, and the numerator is \( 1 - \cos 0 = 1 - 1 = 0 \). Thus the limit is in the indeterminate form 0/0. The values in the table suggest that

\[
\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0
\]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \frac{1 - \cos \theta}{\theta} )</th>
<th>( \theta )</th>
<th>( \frac{1 - \cos \theta}{\theta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>-0.04995835</td>
<td>0.1</td>
<td>0.04995835</td>
</tr>
<tr>
<td>-0.01</td>
<td>-0.00499996</td>
<td>0.01</td>
<td>0.00499996</td>
</tr>
<tr>
<td>-0.001</td>
<td>-0.00050000</td>
<td>0.001</td>
<td>0.00050000</td>
</tr>
<tr>
<td>-0.0001</td>
<td>-0.00005000</td>
<td>0.0001</td>
<td>0.00005000</td>
</tr>
</tbody>
</table>

**CONCEPTUAL INSIGHT** In our work with functions and limits so far, we have encountered three expressions that are similar but have different meanings: undefined, does not exist, and indeterminate. It is important to understand the meanings of these expressions so that you can use them correctly to describe functions and limits.

- The word "undefined" is used for a mathematical expression that is not defined, such as \( 2/0 \) or \( \ln 0 \).
- The phrase "does not exist" means \( \lim_{x \to c} f(x) \) does not exist, that is, \( f(x) \) does not approach a particular numerical value as \( x \) approaches \( c \).
- The term "indeterminate" is used when, upon substitution, a function or limit has one of the indeterminate forms.

**2.5 SUMMARY**

- When \( f \) is known to be continuous at \( x = c \), the limit can be evaluated by substitution: \( \lim_{x \to c} f(x) = f(c) \).
- If the formula for \( f(c) \) yields an undefined expression of the type

\[
\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty
\]

then we say that \( f(x) \) is indeterminate (or has an indeterminate form) at \( x = c \).
• If \( f(x) \) is indeterminate at \( x = c \):
  - Try to transform \( f(x) \) algebraically into a new expression that is defined and continuous at \( x = c \), and then evaluate by substitution.
  - Examine it numerically or graphically, or evaluate it with specialized theorems (see Sections 2.6 and 7.5).

### 2.5 EXERCISES

**Preliminary Questions**

1. Which of the following is indeterminate at \( x = 1 \)?
   - \( \frac{x^2 + 1}{x - 1} \)
   - \( \frac{x^2 - 1}{x + 2} \)
   - \( \frac{\sqrt{x} + 3 - 2}{\sqrt{x} + 3 - 2} \)
   - \( \frac{x^2 + 1}{\sqrt{x} + 3 - 2} \)

2. Give counterexamples to show that these statements are false:
   - (a) If \( f(x) \) is indeterminate, then the right- and left-hand limits as \( x \to c \) are not equal.
   - (b) If \( \lim_{x \to c} f(x) \) exists, then \( f(c) \) is not indeterminate.
   - (c) If \( f(x) \) is undefined at \( x = c \), then \( f(x) \) has an indeterminate form at \( x = c \).

3. The method for evaluating limits discussed in this section is sometimes called simplify and substitute. Explain how it actually relies on the property of continuity.

**Exercises**

In Exercises 1–4, show that the limit leads to an indeterminate form. Then carry out the two-step procedure: Transform the function algebraically and evaluate using continuity.

1. \( \lim_{x \to 0} \frac{x^2 - 36}{x - 6} \)
2. \( \lim_{x \to 3} \frac{9 - h^2}{h - 3} \)
3. \( \lim_{x \to 1} \frac{x^2 + 2x + 1}{x + 1} \)
4. \( \lim_{x \to 5} \frac{2r - 18}{5i - 45} \)

In Exercises 5–34, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

5. \( \lim_{x \to 7} \frac{x - 7}{x^2 - 49} \)
6. \( \lim_{x \to 8} \frac{x^2 - 64}{x - 8} \)
7. \( \lim_{x \to 2} \frac{x^2 + 2x + 2}{x - 2} \)
8. \( \lim_{x \to 8} \frac{x^2 - 64x}{x - 8} \)
9. \( \lim_{x \to 5} \frac{2x^2 - 9x - 5}{x^2 - 25} \)
10. \( \lim_{h \to 0} \frac{h + (1 + h)^3 - 1}{h} \)
11. \( \lim_{x \to -1} \frac{2x + 1}{2x^2 + 3x + 1} \)
12. \( \lim_{x \to -9} \frac{x^2 - x}{x^3 - 9} \)
13. \( \lim_{x \to 2} \frac{3x^2 - 4x - 4}{x^2 - 8} \)
14. \( \lim_{h \to 0} \frac{(h + 2)^3 - 27}{h} \)
15. \( \lim_{x \to -3} \frac{4x^2 - 1}{x - 1} \)
16. \( \lim_{x \to 3} \frac{h + 2x^2 - 9h}{h - 4} \)
17. \( \lim_{x \to 4} \frac{\sqrt{x} - 4}{x - 16} \)
18. \( \lim_{t \to 2} \frac{2t + 4}{12 - 3t^3} \)
19. \( \lim_{x \to -2} \frac{1}{(x + 2x^2 - 4)} \)
20. \( \lim_{t \to 4} \frac{y^2 + y - 12}{3y^3 - 10y + 3} \)
21. \( \lim_{x \to 0} \frac{\sqrt{x} + 2 - 2}{x} \)
22. \( \lim_{x \to 8} \frac{\sqrt{x} - 4 - 2}{x - 8} \)
23. \( \lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 6} \)
24. \( \lim_{x \to 4} \frac{\sqrt{5 - x} - 1}{2 - \sqrt{x}} \)
25. \( \lim_{x \to 1} \frac{1}{\sqrt{x} - 2 - x - 4} \)
26. \( \lim_{x \to 0} \frac{1}{\sqrt{x} - \sqrt{x^3 + x}} \)
27. \( \lim_{x \to \frac{1}{2}} \frac{\cos x}{\sin x} \)
28. \( \lim_{x \to \frac{1}{2}} \frac{\cos x}{\sin x} \)
29. \( \lim_{x \to \frac{1}{2}} \frac{1}{\cos x} - \frac{2}{1 - x^2} \)
30. \( \lim_{x \to \frac{1}{2}} \frac{\sin x - \cos x}{\tan x - 1} \)
31. \( \lim_{x \to \frac{1}{2}} \frac{2x^2 + 2 - 20}{2x - 8} \)
32. \( \lim_{x \to \frac{1}{2}} \frac{(1 + \tan \theta - 1 - \tan^2 \theta)}{(\tan \theta - \tan \theta)} \)
33. \( \lim_{x \to \frac{1}{2}} \frac{2x^3 + 2 - 20}{2x - 8} \)
34. \( \lim_{x \to \frac{1}{2}} \frac{2x^3 + 2 - 20}{2x - 8} \)
35. \( \lim_{x \to \frac{1}{2}} \frac{2x^3 + 2 - 20}{2x - 8} \)
36. \( \lim_{x \to \frac{1}{2}} \frac{2x^3 + 2 - 20}{2x - 8} \)
37. \( \lim_{x \to \frac{1}{2}} \frac{2x^3 + 2 - 20}{2x - 8} \)
38. \( \lim_{x \to \frac{1}{2}} \frac{2x^3 + 2 - 20}{2x - 8} \)

In Exercises 37 and 38, show that the limit is in an indeterminate form, then investigate the limit numerically to estimate the value.
39. **GU** Use a plot of \( f(x) = \frac{x - 4}{\sqrt{x^2 - x - 4}} \) to estimate \( \lim_{x \to 4} f(x) \) to two decimal places. Compare with the answer obtained algebraically in Exercise 23.

40. **GU** Use a plot of \( f(x) = \frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \) to estimate \( \lim_{x \to 4} f(x) \) numerically. Compare with the answer obtained algebraically in Exercise 25.

In Exercises 41–46, evaluate using the identity

\[ a^3 - b^3 = (a - b)(a^2 + ab + b^2) \]

41. \( \lim_{x \to 2} \frac{x^3 - 8}{x - 2} \)
42. \( \lim_{x \to 3} \frac{x^3 - 27}{x^2 - 9} \)
43. \( \lim_{x \to 1} \frac{x^2 + 5x + 4}{x^3 - 1} \)
44. \( \lim_{x \to 2} \frac{x^2 + 5x + 8}{x^3 + 6x + 8} \)
45. \( \lim_{x \to 4} \frac{x^2 - 1}{x^3 - 1} \)
46. \( \lim_{x \to 27} \frac{x - 27}{x^{1/3} - 3} \)

In Exercises 47–54, evaluate in terms of the constant \( a \).

47. \( \lim_{x \to 0} (2a + x) \)
48. \( \lim_{x \to 0} (4ah + 7a) \)
49. \( \lim_{x \to 1} (4t - 2at + 3a) \)
50. \( \lim_{x \to a} \frac{(x + a)^2 - 4x^2}{x - a} \)
51. \( \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \)
52. \( \lim_{h \to 0} \frac{\sqrt{a + 2h} - \sqrt{a}}{h} \)
53. \( \lim_{x \to -a} \frac{(x + a)^3 - a^3}{x} \)
54. \( \lim_{h \to 0} \frac{1}{h} - \frac{a}{h + a - a} \)

55. Evaluate \( \lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} \). Hint: Set \( x = \sqrt{1 + h} \), express \( h \) as a function of \( x \), and rewrite as a limit as \( x \to 1 \).

56. Evaluate \( \lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} \). Hint: Set \( x = \sqrt{1 + h} \), express \( h \) as a function of \( x \), and rewrite as a limit as \( x \to 1 \).

Further Insights and Challenges

In Exercises 57–60, find all values of \( c \) such that the limit exists.

57. \( \lim_{x \to c} \frac{x^2 - 5x - 6}{x - c} \)
58. \( \lim_{x \to 1} \frac{x^2 + 5x + c}{x - 1} \)
59. \( \lim_{x \to 1} \left( \frac{1}{x - 1} - \frac{c}{x^3 - 1} \right) \)
60. \( \lim_{x \to 0} \left( \frac{1 + cx^2 - x + x^2}{x^4} \right) \)

61. For which sign, + or −, does the following limit exist?

\[ \lim_{x \to 0} \left( \frac{1}{x} ± \frac{1}{x(x - 1)} \right) \]

2.6 The Squeeze Theorem and Trigonometric Limits

In our study of the derivative, we will need to evaluate certain limits involving transcendental functions such as sine and cosine. The algebraic techniques of the previous section are often ineffective for such functions, and other tools are required. In this section, we discuss one such tool—the Squeeze Theorem—and use it to evaluate the trigonometric limits needed in Section 3.6.

The Squeeze Theorem

Consider a function \( f \) that is "trapped" between two functions \( I \), for lower bound, and \( u \), for upper bound, on an interval \( I \). In other words,

\[ I(x) \leq f(x) \leq u(x) \quad \text{for all } x \in I \]

Thus, the graph of \( f \) lies between the graphs of \( I \) and \( u \) (Figure 1).

The Squeeze Theorem applies when \( f \) is not just trapped but squeezed at a point \( x = c \) (Figure 2). By this we mean that for all \( x \neq c \) in some open interval containing \( c \),

\[ I(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c} I(x) = \lim_{x \to c} u(x) = L \]

We do not require that \( f(x) \) be defined at \( x = c \), but it is clear graphically that \( f(x) \) must approach the limit \( L \), as stated in the next theorem. See Appendix D for a proof.
THEOREM 1 Squeeze Theorem Assume that for \( x \neq c \) (in some open interval containing \( c \)),
\[
I(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c} I(x) = \lim_{x \to c} u(x) = L
\]
Then \( \lim_{x \to c} f(x) \) exists and \( \lim_{x \to c} f(x) = L \).

EXAMPLE 1 Show that \( \lim_{x \to 0} x \sin \frac{1}{x} = 0 \).

Solution Although \( f(x) = x \sin \frac{1}{x} \) is a product of two functions, we cannot use the Product Law because \( \lim_{x \to 0} \frac{1}{x} \) does not exist. However, the sine function takes on values between 1 and -1, and therefore \( |\sin \frac{1}{x}| \leq 1 \) for all \( x \neq 0 \). Multiplying by \( |x| \), we obtain
\[
|x| \leq x \sin \frac{1}{x} \leq |x|
\]
and conclude that (Figure 3)
\[
-|x| \leq x \sin \frac{1}{x} \leq |x|
\]
Furthermore, we have
\[
\lim_{x \to 0} |x| = 0 \quad \text{and} \quad \lim_{x \to 0} (-|x|) = 0
\]
and therefore it follows that \( f(x) = x \sin \frac{1}{x} \) is squeezed between \( I(x) = -|x| \) and \( u(x) = |x| \) at \( x = 0 \). The Squeeze Theorem now applies, and we can conclude that \( \lim_{x \to 0} x \sin \frac{1}{x} = 0 \).

In Section 2.2, we found numerical and graphical evidence suggesting that the limit
\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta}
\]
is equal to 1. The Squeeze Theorem will allow us to prove this fact.

THEOREM 2 Important Trigonometric Limits

To apply the Squeeze Theorem to prove that \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \), we must find functions that squeeze \( \frac{\sin \theta}{\theta} \) at \( \theta = 0 \). These are illustrated in Figure 4 and provided by the next theorem.

THEOREM 3

\[
\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1 \quad \text{for} \quad \frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad \theta \neq 0
\]

Proof Assume first that \( 0 < \theta < \frac{\pi}{2} \). Our proof is based on the following relation between the areas in Figure 5:
\[
\text{area of } \triangle OAB < \text{area of sector } BOA < \text{area of } \triangle OAC
\]
Let's determine these three areas. First, $\triangle OAB$ has base 1 and height $\sin \theta$, so its area is $\frac{1}{2} \sin \theta$. Next, recall that a sector of angle $\theta$ has area $\frac{1}{2} \theta$. Finally, to compute the area of $\triangle OAC$, we observe that

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{AC}{OA} = \frac{AC}{1} = AC$$

Thus, $\triangle OAC$ has base 1, height $\tan \theta$, and area $\frac{1}{2} \tan \theta$. We have shown, therefore, that

$$\frac{1}{2} \sin \theta \leq \frac{1}{2} \theta \leq \frac{1}{2} \sin \theta \quad \text{Area } \triangle OAB \quad \text{Area } \triangle OAC$$

The first inequality yields $\sin \theta \leq \theta$, and because $\theta > 0$, we obtain

$$\frac{\sin \theta}{\theta} \leq 1$$

Next, multiply the second inequality in (3) by $\frac{2 \cos \theta}{\theta}$ to obtain

$$\cos \theta \leq \frac{\sin \theta}{\theta}$$

The combination of (4) and (5) gives us (1) when $0 < \theta < \frac{\pi}{2}$. However, the functions in (1) do not change value when $\theta$ is replaced by $-\theta$ because both $f(x) = \cos \theta$ and $f(x) = \frac{\sin \theta}{\theta}$ are even functions. Indeed, $\cos(-\theta) = \cos \theta$ and

$$\frac{\sin(-\theta)}{-\theta} = -\frac{\sin \theta}{-\theta} = \frac{\sin \theta}{\theta}$$

Therefore, (1) holds for $-\frac{\pi}{2} < \theta < 0$ as well. This completes the proof of Theorem 3.

**Proof of Theorem 2** According to Theorem 3,

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1$$

Since $\lim_{\theta \to 0} \cos \theta = \cos 0 = 1$ and $\lim_{\theta \to 0} 1 = 1$, the Squeeze Theorem yields $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, as required. It then follows that

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = \lim_{\theta \to 0} \frac{1 + \cos \theta}{1 + \cos \theta} \frac{1 - \cos \theta}{\theta} = \lim_{\theta \to 0} \frac{1 - \cos^2 \theta}{(1 + \cos \theta)\theta} = \lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \frac{\sin \theta}{\theta} = 0 \cdot 1 = 0$$
In the next example, we evaluate another trigonometric limit. The key idea is to rewrite the function of \( h \) in terms of the new variable \( \theta = 4h \).

**EXAMPLE 2** Evaluating a Limit by Changing Variables

Investigate \( \lim_{h \to 0} \frac{\sin 4h}{h} \) numerically and then evaluate it exactly.

Solution
The values in Table 1 suggest that the limit is equal to 4. To evaluate the limit exactly, we rewrite it in terms of the limit of \( \frac{\sin \theta}{\theta} \) so that Theorem 2 can be applied. Thus, we set \( \theta = 4h \) and write

\[
\frac{\sin 4h}{h} = 4 \left( \frac{\sin 4h}{4h} \right) = 4 \frac{\sin \theta}{\theta}
\]

The new variable \( \theta \) tends to zero as \( h \to 0 \) because \( \theta \) is a multiple of \( h \). Therefore, we may change the limit as \( h \to 0 \) into a limit as \( \theta \to 0 \) to obtain

\[
\lim_{h \to 0} \frac{\sin 4h}{h} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 4 \left( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \right) = 4(1) = 4
\]

Note that the change of variables \( \theta = kx \) demonstrates that \( \frac{\sin kx}{kx} \) approaches 1 as \( x \to 0 \). We use this limit to our advantage in the next example.

**EXAMPLE 3** Find \( \lim_{x \to 0} \frac{\tan 3x}{\tan 2x} \).

Solution

\[
\lim_{x \to 0} \frac{\tan 3x}{\tan 2x} = \lim_{x \to 0} \frac{\sin 3x}{\cos 3x} \cdot \frac{\cos 2x}{\sin 2x} = \lim_{x \to 0} \frac{\sin 3x}{\cos 3x} \cdot \frac{\cos 2x}{\sin 2x} \cdot \frac{x}{x}
\]

\[
= \lim_{x \to 0} \frac{3}{2} \left( \frac{\sin 3x}{3x} \right) \left( \frac{2x}{\sin 2x} \right) \left( \frac{\cos 2x}{\cos 3x} \right) = \frac{3}{2} \cdot 1 \cdot 1 \cdot \frac{1}{2} = \frac{3}{2}
\]

**2.6 SUMMARY**

- We say that a function \( f \) is squeezed at \( x = c \) if there exist functions \( l \) and \( u \) such that \( l(x) \leq f(x) \leq u(x) \) for all \( x \neq c \) in an open interval \( I \) containing \( c \), and

\[
\lim_{x \to c} l(x) = \lim_{x \to c} u(x) = L
\]

The Squeeze Theorem states that in this case, \( \lim_{x \to c} f(x) = L \).

- Two important trigonometric limits:

\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0
\]

**2.6 EXERCISES**

**Preliminary Questions**

1. Assume that \( -x^2 \leq f(x) \leq x^2 \). What is \( \lim_{x \to 1} f(x) \)? Is there enough information to evaluate \( \lim_{x \to 1} f(x) \)? Explain.

3. If you want to evaluate \( \lim_{h \to 0} \frac{\sin 5h}{3h} \), it is a good idea to rewrite the limit in terms of the variable (choose one):

\( a \) \( \theta = 5h \) \( b \) \( \theta = 3h \) \( c \) \( \theta = \frac{5h}{3} \)

2. State the Squeeze Theorem carefully.
Exercises

In Exercises 1–10, evaluate using the Squeeze Theorem.

1. \( \lim_{x \to 0} x^2 \cos \frac{1}{x} \)

2. \( \lim_{x \to 0} x \sin \frac{1}{x^2} \)

3. \( \lim_{x \to 1} (x - 1) \sin \frac{\pi}{x - 1} \)

4. \( \lim_{x \to 3} (x^2 - 9) \frac{x - 3}{|x - 3|} \)

5. \( \lim_{t \to 0} (t^2 - 1) \cos \frac{1}{t} \)

6. \( \lim_{x \to 6} \sqrt{x} \cos \frac{\pi}{x} \)

7. \( \lim_{t \to 2} (t^2 - 4) \cos \frac{1}{t - 2} \)

8. \( \lim_{x \to 6} \tan x \cos \left( \frac{1}{x} \right) \)

9. \( \lim_{\theta \to \frac{\pi}{2}} \cos \theta \cos(\tan \theta) \)

10. \( \lim_{x \to 0} (3x - 1) \sin^2 \left( \frac{1}{x} \right) \)

11. State precisely the hypothesis and conclusions of the Squeeze Theorem for the situation in Figure 6.

![Figure 6](image)

12. In Figure 7, is \( f \) squeezed by \( u \) and \( l \) at \( x = 3 \)? At \( x = 2 \)?

![Figure 7](image)

13. What does the Squeeze Theorem say about \( \lim f(x) \) if the limits \( \lim f(x) = \lim u(x) = 6 \) and \( f, u, \) and \( l \) are related as in Figure 8? The inequality \( f(x) \leq u(x) \) is not satisfied for all \( x \). Does this affect the validity of your conclusion?

![Figure 8](image)

15. State whether the inequality provides sufficient information to determine \( \lim f(x) \), and if so, find the limit.

(a) \( 4x - 5 \leq f(x) \leq x^2 \)

(b) \( 2x - 1 \leq f(x) \leq x^2 \)

(c) \( 4x - x^2 \leq f(x) \leq x^2 + 2 \)

16. \( \text{(GJ)} \) Plot the graphs of \( u(x) = 1 + |x - \frac{\pi}{2}| \) and \( f(x) = \sin x \) on the same set of axes. What can you say about \( \lim f(x) \) if \( f \) is squeezed by \( l \) and \( u \) at \( x = \frac{\pi}{2} \)?

In Exercises 17–26, evaluate using Theorem 2 as necessary.

17. \( \lim_{t \to \infty} \frac{\tan x}{x} \)

18. \( \lim_{x \to 0} \frac{\sin x \sec x}{x} \)

19. \( \lim_{t \to \infty} \frac{\sqrt{t^2 + 9} \sin t}{t} \)

20. \( \lim_{x \to 0} \frac{\sin^2 t}{t} \)

21. \( \lim_{x \to 0} \frac{x^3}{\sin x} \)

22. \( \lim_{x \to \infty} \frac{1 - \cos x}{x} \)

23. \( \lim_{x \to 0} \frac{\sec \theta - 1}{\theta} \)

24. \( \lim_{x \to 0} \frac{1 - \cos \theta}{\sin \theta} \)

25. \( \lim_{x \to 0} \frac{\sin t}{t} \)

26. \( \lim_{x \to 0} \frac{\cos t - \cos \theta}{\theta} \)

27. Evaluate \( \lim_{x \to 0} \frac{\sin 11x}{x} \) using a substitution \( \theta = 11x \).

28. Evaluate \( \lim_{x \to 0} \frac{\sin 7t}{x} \). Hint: Multiply the numerator and denominator by \( (7)(11)x \).

In Exercises 29–48, evaluate the limit.

29. \( \lim_{x \to \infty} \frac{\sin 9h}{h} \)

30. \( \lim_{h \to 0} \frac{\sin 4h}{h} \)

31. \( \lim_{h \to \infty} \frac{\sin h}{5h} \)

32. \( \lim_{x \to \frac{\pi}{2}} \frac{x}{\sin 3x} \)

33. \( \lim_{x \to 0} \frac{\sin 7x}{9x} \)

34. \( \lim_{x \to 0} \frac{\tan 4x}{x} \)

35. \( \lim_{x \to 0} \frac{x \csc 25x}{x} \)

36. \( \lim_{x \to 0} \frac{\tan 4t}{t \sec 7t} \)

37. \( \lim_{x \to 0} \frac{\sin 2x \sin 3x}{x^2} \)

38. \( \lim_{x \to 0} \frac{\sin(3x/3)}{x \sin x} \)

39. \( \lim_{x \to 0} \frac{\sin(-3x)}{x^2} \)

40. \( \lim_{x \to 0} \frac{\tan x}{\sin 9x} \)

41. \( \lim_{x \to 0} \frac{\csc 8x}{\csc 4x} \)

42. \( \lim_{x \to 0} \frac{5x \sin 2x}{x \sin 3x} \)

43. \( \lim_{x \to 0} \frac{3x \sin 2x}{x \sin 5x} \)

44. \( \lim_{x \to 0} \frac{1 - \cos 2x}{h} \)

45. \( \lim_{x \to 0} \frac{\sin(2x)(1 - \cos 2x)}{h^2} \)

46. \( \lim_{x \to 0} \frac{1 - \cos 2x}{h} \)

47. \( \lim_{x \to 0} \frac{\cos 2x - \cos \theta}{\theta} \)

48. \( \lim_{x \to 0} \frac{1 - \cos 2(\theta \cos 2x)}{\theta^2} \)

49. Use the identity \( \sin 2\theta = 2 \sin \theta \cos \theta \) to evaluate \( \lim_{x \to 0} \frac{\sin 2\theta - 2 \sin \theta}{\theta^2} \).
50. Use the identity \( \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \) to evaluate the limit, \( \lim_{\theta \to 0} \frac{\sin 3\theta - 3 \sin \theta}{\theta^3} \).

51. Explain why \( \lim_{\theta \to 0} (\sec \theta - \cot \theta) \) involves an indeterminate form, and then prove that the limit equals 0.

52. Explain why \( \lim_{\theta \to \frac{\pi}{2}} (2 \sin \theta - \sec \theta) \) involves an indeterminate form, and then evaluate the limit.

53. \( \text{(GU)} \) Investigate \( \lim_{h \to 0} \frac{1 - \cos 2h}{h^2} \) numerically or graphically. Then evaluate the limit using the double angle formula \( \cos 2h = 1 - 2 \sin^2 h \).

54. \( \text{(GU)} \) Investigate \( \lim_{h \to 0} \frac{1 - \cos h}{h^2} \) numerically or graphically. Then prove that the limit is equal to \( \frac{1}{2} \). *Hint: See the proof of Theorem 2.*

In Exercises 55-57, evaluate using the result of Exercise 54.

55. \( \lim_{h \to 0} \frac{\cos 3h - 1}{h^2} \)

56. \( \lim_{h \to 0} \frac{\cos 3h - 1}{\cos 2h - 1} \)

57. \( \lim_{t \to 0} \frac{\sqrt{1 - \cos t}}{t} \)

58. Use the Squeeze Theorem to prove that if \( \lim_{x \to c} |f(x)| = 0 \), then \( \lim_{x \to c} f(x) = 0 \).

Further Insights and Challenges

59. Use the result of Exercise 54 to prove that for \( m \neq 0 \),

\[
\lim_{x \to 0} \frac{\cos mx - 1}{x^2} = -\frac{m^2}{2}
\]

60. Using a diagram of the unit circle and the Pythagorean Theorem, show that

\[
\sin^2 \theta \leq (1 - \cos \theta)^2 \quad \text{and} \quad \sin^2 \theta \leq \theta^2
\]

2.7 Limits at Infinity

So far we have considered limits as \( x \) approaches a number \( c \). It is also important to consider limits where \( x \) approaches \( \infty \) or \( -\infty \), which we refer to as limits at infinity. In applications, limits at infinity arise naturally when we describe the "long-term" behavior of a system as in Figure 1.

The notation \( x \to \infty \) indicates that \( x \) increases without bound, and \( x \to -\infty \) indicates that \( x \) decreases (through negative values) without bound. We write

- \( \lim_{x \to \infty} f(x) = L \) if \( f(x) \) gets closer and closer to \( L \) as \( x \to \infty \)
- \( \lim_{x \to -\infty} f(x) = L \) if \( f(x) \) gets closer and closer to \( L \) as \( x \to -\infty \)

As before, "closer and closer" means that \( |f(x) - L| \) becomes arbitrarily small. In either case, the line \( y = L \) is called a horizontal asymptote. We use the notation \( x \to \pm \infty \) to indicate that we are considering both infinite limits, as \( x \to \infty \) and as \( x \to -\infty \).

Infinite limits describe the asymptotic behavior of a function, which is determined by the behavior of the graph as we move out indefinitely to the right or the left.

**EXAMPLE 1** Discuss the asymptotic behavior in Figure 2.

**Solution** The function \( g \) approaches \( L = 7 \) as we move to the right and it approaches \( L = 3 \) as we move to the left, so

\[
\lim_{x \to \infty} g(x) = 7 \quad \text{and} \quad \lim_{x \to -\infty} g(x) = 3
\]

Accordingly, the lines \( y = 7 \) and \( y = 3 \) are horizontal asymptotes of \( g \).

A function may approach an infinite limit as \( x \to \pm \infty \). We write

\[
\lim_{x \to \infty} f(x) = \infty \quad \text{or} \quad \lim_{x \to -\infty} f(x) = \infty
\]
Figure 3

If \( f(x) \) is positive and becomes arbitrarily large as \( x \to \infty \) or \( -\infty \). Similar notation is used if \( f(x) \) approaches \( -\infty \) as \( x \to \pm \infty \). For example, we see in Figure 3(A) that

\[
\lim_{x \to \infty} 2^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} 2^x = 0
\]

However, limits at infinity do not always exist. For example, \( f(x) = \sin x \) oscillates indefinitely [Figure 3(B)], so the following limits do not exist:

\[
\lim_{x \to \infty} \sin x \quad \text{and} \quad \lim_{x \to -\infty} \sin x
\]

The limits at infinity of the power functions \( f(x) = x^n \) are easily determined. If \( n > 0 \), then \( x^n \) increases without bound as \( x \to \infty \), so (Figure 4)

\[
\lim_{x \to \infty} x^n = \infty \quad \text{and} \quad \lim_{x \to -\infty} x^{-n} = \lim_{x \to -\infty} \frac{1}{x^n} = 0
\]

Figure 4

(A) \( n \) even: \( \lim_{x \to \infty} x^n = \lim_{x \to -\infty} x^n = \infty \)
(B) \( n \) odd: \( \lim_{x \to \infty} x^n = \infty \), \( \lim_{x \to -\infty} x^n = -\infty \)
(C) \( \lim_{x \to \infty} \frac{1}{x^n} = \lim_{x \to -\infty} \frac{1}{x^n} = 0 \)

**CAUTION** \( \lim_{x \to -\infty} x^{1/2} \) does not exist, since the square root of a negative number is not a real number.

To describe the limits as \( x \to -\infty \), assume that \( n \) is a whole number so that \( x^n \) is defined for \( x < 0 \). If \( n \) is even, then \( x^n \) becomes large and positive as \( x \to -\infty \), and if \( n \) is odd, it becomes large and negative. We summarize these limits in the following theorem:

**THEOREM 1** For all \( n > 0 \),

\[
\lim_{x \to \infty} x^n = \infty \quad \text{and} \quad \lim_{x \to -\infty} x^{-n} = \lim_{x \to -\infty} \frac{1}{x^n} = 0
\]

If \( n \) is a positive whole number,

\[
\lim_{x \to -\infty} x^n = \begin{cases} 
\infty & \text{if } n \text{ is even} \\
-\infty & \text{if } n \text{ is odd}
\end{cases} \quad \text{and} \quad \lim_{x \to -\infty} x^{-n} = \lim_{x \to -\infty} \frac{1}{x^n} = 0
\]

Note also that if \( p \) and \( q \) are positive integers and \( q \) is odd, then \( \lim_{x \to -\infty} x^{p/q} = \infty \) if \( p \) is even and \( \lim_{x \to -\infty} x^{p/q} = -\infty \) if \( p \) is odd. In the case that \( q \) is even, \( x^{p/q} \) is not defined for negative \( x \), so it does not make sense to address the limit as \( x \to -\infty \).
The Basic Limit Laws (Theorem 1 in Section 2.3) are valid for limits at infinity. For example, the Sum and Constant Multiple Laws yield
\[
\lim_{x \to \infty} \left( 3 - 4x^{-3} + 5x^{-5} \right) = \lim_{x \to \infty} 3 - \lim_{x \to \infty} 4x^{-3} + \lim_{x \to \infty} 5x^{-5} = 3 + 0 + 0 = 3
\]

**Example 2** Calculate \( \lim_{x \to \infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5} \).

**Solution** It would be nice if we could apply the Quotient Law directly, but this law is valid only if the denominator has a finite, nonzero limit. Our limit has the indeterminate form \( \infty/\infty \) because
\[
\lim_{x \to \infty} (20x^2 - 3x) = \infty \quad \text{and} \quad \lim_{x \to \infty} (3x^5 - 4x^2 + 5) = \infty
\]
The way around this difficulty is to divide the numerator and denominator by \( x^5 \) (the highest power of \( x \) in the denominator):
\[
\frac{20x^2 - 3x}{3x^5 - 4x^2 + 5} = \frac{x^{-5}(20x^2 - 3x)}{x^{-5}(3x^5 - 4x^2 + 5)} = \frac{20x^{-3} - 3x^{-4}}{3 - 4x^{-3} + 5x^{-5}}
\]
Now we can use the Quotient Law:
\[
\lim_{x \to \infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5} = \lim_{x \to \infty} \frac{20x^{-3} - 3x^{-4}}{3 - 4x^{-3} + 5x^{-5}} = 0
\]
In general, if
\[
f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0}
\]
where \( a_n \neq 0 \) and \( b_m \neq 0 \), divide the numerator and denominator by \( x^m \):
\[
f(x) = \frac{a_n x^{-m} + a_{n-1} x^{-m+1} + \ldots + a_0 x^{-m}}{b_m x^{-m} + b_{m-1} x^{-m+1} + \ldots + b_0 x^{-m}}
\]
\[
x^{-m} \left( \frac{a_n + a_{n-1} x^{-1} + \ldots + a_0 x^{-m}}{b_m + b_{m-1} x^{-1} + \ldots + b_0 x^{-m}} \right)
\]
The quotient in parentheses approaches the finite limit \( a_n/b_m \) because
\[
\lim_{x \to \infty} \left( \frac{a_n + a_{n-1} x^{-1} + \ldots + a_0 x^{-m}}{b_m + b_{m-1} x^{-1} + \ldots + b_0 x^{-m}} \right) = \frac{a_n}{b_m}
\]
This also holds true for \( x \to -\infty \), and therefore,
\[
\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} x^{-m} \lim_{x \to \pm \infty} \frac{a_n + a_{n-1} x^{-1} + \ldots + a_0 x^{-m}}{b_m + b_{m-1} x^{-1} + \ldots + b_0 x^{-m}} = \frac{a_n}{b_m} \lim_{x \to \pm \infty} x^{-m}
\]

**Theorem 2** Limits at Infinity of a Rational Function The asymptotic behavior of a rational function depends only on the leading terms of its numerator and denominator. If \( a_n, b_m \neq 0 \), then
\[
\lim_{x \to \pm \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0} = \frac{a_n}{b_m} \lim_{x \to \pm \infty} x^{n-m}
\]
Here are some examples:

* \( n = m: \)
  \[
  \lim_{{x \to \infty}} \frac{{3x^4 - 7x + 9}}{{7x^4 - 4}} = \frac{3}{7} \lim_{{x \to \infty}} x^0 = \frac{3}{7}
  \]

* \( n < m: \)
  \[
  \lim_{{x \to \infty}} \frac{{3x^3 - 7x + 9}}{{7x^4 - 4}} = \frac{3}{7} \lim_{{x \to \infty}} x^{-1} = 0
  \]

* \( n > m, n - m \text{ odd}: \)
  \[
  \lim_{{x \to \infty}} \frac{{3x^8 - 7x + 9}}{{7x^3 - 4}} = \frac{3}{7} \lim_{{x \to \infty}} x^5 = -\infty
  \]

* \( n > m, n - m \text{ even}: \)
  \[
  \lim_{{x \to \infty}} \frac{{3x^7 - 7x + 9}}{{7x^3 - 4}} = \frac{3}{7} \lim_{{x \to \infty}} x^4 = \infty
  \]

Our method can be adapted to noninteger exponents and algebraic functions.

**Example 3** Calculate the limits

(a) \( \lim_{{x \to \infty}} \frac{{3x^{7/2} + 7x^{-1/2}}}{{x^2 - x^{1/2}}} \)

(b) \( \lim_{{x \to \pm\infty}} \frac{{4x}}{{\sqrt{x^2 + 1}}} \)

**Solution**

(a) As before, divide the numerator and denominator by \( x^{7/2} \), which is the highest power of \( x \) occurring in the denominator (this means multiply by \( x^{-7/2} \)):

\[
\frac{{3x^{7/2} + 7x^{-1/2}}}{{x^2 - x^{1/2}}} = \frac{{(x^{-2}) \left(3x^{7/2} + 7x^{-1/2}\right)}}{{(x^{-2}) \left(x^2 - x^{1/2}\right)}} = \frac{{3x^{3/2} + 7x^{-5/2}}}{{1 - x^{-3/2}}}
\]

\[
\lim_{{x \to \infty}} \frac{{3x^{3/2} + 7x^{-5/2}}}{{1 - x^{-3/2}}} = \lim_{{x \to \infty}} \frac{{(3x^{3/2} + 7x^{-5/2})}}{{(1 - x^{-3/2})}} = \infty = \infty
\]

(b) First, consider \( x \to \infty \). The key is to observe that the denominator of \( \frac{{4x}}{{\sqrt{x^2 + 1}}} \) behaves like \( x^1 = x \):

\[
\sqrt{x^2 + 1} = \sqrt{x^2(1 + x^{-2})} = x \sqrt{1 + x^{-2}} \quad (\text{for } x > 0)
\]

This suggests that we divide the numerator and denominator by \( x \):

\[
\frac{{4x}}{{\sqrt{x^2 + 1}}} = \frac{{4x}}{x \sqrt{1 + x^{-2}}} = \frac{4}{x} \sqrt{1 + x^{-2}}
\]

Then apply the Quotient Law:

\[
\lim_{{x \to \infty}} \frac{4x}{\sqrt{x^2 + 1}} = \lim_{{x \to \infty}} \frac{4}{\sqrt{1 + x^{-2}}} = \lim_{{x \to \infty}} \frac{4}{\sqrt{1 + x^{-2}}} = \frac{4}{1} = 4
\]

For the limit as \( x \to -\infty \), one approach is to replace \( x \) with \( -t \). Since \( x = -t \) and \( x \to -\infty \), then \( t \to \infty \). So we have

\[
\lim_{{x \to -\infty}} \frac{4x}{\sqrt{x^2 + 1}} = \lim_{{t \to \infty}} \frac{4(-t)}{\sqrt{(-t)^2 + 1}} = \lim_{{t \to \infty}} \frac{-4t}{\sqrt{t^2 + 1}} = -\lim_{{t \to \infty}} \frac{4t}{\sqrt{t^2 + 1}} = -4
\]

where the last equality holds by the limit we previously calculated.

The limits in (b) indicate that the graph of \( f(x) = \frac{4x}{\sqrt{x^2 + 1}} \) has horizontal asymptotes at \( y = 4 \) and \( y = -4 \), which is confirmed in Figure 5.

**2.7 Summary**

- **Limits at infinity:**
  \[
  \lim_{{x \to \infty}} f(x) = L \text{ if } |f(x) - L| \text{ becomes arbitrarily small as } x \text{ increases without bound.}
  \]
2.7 EXERCISES

Preliminary Questions

1. Assume that
\[ \lim_{{x \to \infty}} f(x) = L \quad \text{and} \quad \lim_{{x \to -\infty}} g(x) = \infty \]
Which of the following statements are correct?
(a) \( x = L \) is a vertical asymptote of \( g \).
(b) \( y = L \) is a horizontal asymptote of \( g \).
(c) \( x = L \) is a vertical asymptote of \( f \).
(d) \( y = L \) is a horizontal asymptote of \( f \).

2. What are the following limits?
(a) \( \lim_{{x \to \infty}} x^3 \)  
(b) \( \lim_{{x \to -\infty}} x^3 \)  
(e) \( \lim_{{x \to \infty}} x^4 \)

Exercises

1. What are the horizontal asymptotes of the function in Figure 6?

2. Sketch the graph of a function \( f \) that has both \( y = -1 \) and \( y = 5 \) as horizontal asymptotes.

3. Sketch the graph of a function \( f \) with a single horizontal asymptote \( y = 3 \).

4. Sketch the graphs of functions \( f \) and \( g \) that have both \( y = 2 \) and \( y = 4 \) as horizontal asymptotes but \( \lim_{{x \to \infty}} f(x) \neq \lim_{{x \to \infty}} g(x) \).

5. [GU] Investigate the asymptotic behavior of \( f(x) = \frac{x^2}{x^2 + 1} \) numerically and graphically:
   (a) Make a table of values of \( f(x) \) for \( x = \pm 50, \pm 100, \pm 500, \pm 1000 \).
   (b) Plot the graph of \( f \).
   (c) What are the horizontal asymptotes of \( f \)?

6. [GU] Investigate \( \lim_{{x \to \infty}} \frac{12x + 1}{\sqrt{4x^2 + 9}} \) numerically and graphically:
   (a) Make a table of values of \( f(x) = \frac{12x + 1}{\sqrt{4x^2 + 9}} \) for the following:
      \( x = \pm 100, \pm 500, \pm 1000, \pm 10,000 \).
In Exercises 7–16, evaluate the limits.

7. \( \lim_{x \to -\infty} \frac{3x^3}{x + 9} \)
8. \( \lim_{x \to -\infty} \frac{3x^2 + 20x}{4x^2 + 9} \)
9. \( \lim_{x \to -\infty} \frac{3x^2 + 20x}{2x^3 + 3x^2 - 29} \)
10. \( \lim_{x \to -\infty} \frac{4}{x + 5} \)
11. \( \lim_{x \to -\infty} \frac{7x - 9}{4x + 3} \)
12. \( \lim_{x \to -\infty} \frac{9x^2 - 2}{x + 6 - 29x} \)
13. \( \lim_{x \to -\infty} \frac{7x^2 - 9}{4x + 3} \)
14. \( \lim_{x \to -\infty} \frac{5x}{4x + 3} \)
15. \( \lim_{x \to -\infty} \frac{3x^3 - 10}{x + 4} \)
16. \( \lim_{x \to -\infty} \frac{2x^2 + 3x^2 - 31x}{8x^2 - 18x^2 + 12} \)

In Exercises 17–24, find the horizontal asymptotes.

17. \( f(x) = \frac{2x^2 - 3x}{8x^2 + 8} \)
18. \( f(x) = \frac{8x^3 - 2x^2}{7 + 11x - 4x^4} \)
19. \( f(x) = \frac{\sqrt{6x^2 + 7}}{9x - 4} \)
20. \( f(x) = \frac{\sqrt{6x^2 + 7}}{9x^2 + 4} \)
21. \( f(t) = \frac{3t^{1/3}}{1 + 2t^{1/3}} \)
22. \( f(t) = \frac{t^{1/3}}{(6t^2 + 9)^{1/5}} \)
23. \( g(t) = \frac{10}{1 + 2t^3} \)
24. \( p(t) = 2 - t^2 \)

The following statement is incorrect: "If \( f \) has a horizontal asymptote \( y = L \) at \( x = \infty \), then the graph of \( f \) approaches the line \( y = L \) as \( x \) gets greater and greater, but never touches it." In Exercises 25 and 26, determine \( \lim_{x \to \infty} f(x) \) and indicate how \( f \) demonstrates that the statement is incorrect.

25. \( f(x) = \frac{2x + |x|}{x} \)
26. \( f(x) = \frac{\sin x}{x} \)

In Exercises 27–34, evaluate the limits.

27. \( \lim_{x \to \infty} \sqrt{2x^2 + 3x + 2} \)
28. \( \lim_{x \to \infty} \sqrt{x^2 + 20x + x} \)
29. \( \lim_{x \to \infty} \frac{x^2 + 7x + 1}{x^2 + 1} \)
30. \( \lim_{x \to \infty} \frac{4x - 3}{\sqrt{2x^2 + 4x}} \)
31. \( \lim_{x \to \infty} \frac{x^{1/3} - 1}{x^{1/3} + 1} \)
32. \( \lim_{x \to \infty} \frac{4x^2 + 2x^2 - 1}{(8 + 2x)^2/3} \)
33. \( \lim_{x \to \infty} \frac{\sqrt{x} + x}{x - 5 - 9x} \)
34. \( \lim_{x \to \infty} \frac{4x + 6x^2}{x^2 + 1} \)
35. \( \lim_{x \to \infty} \frac{\sqrt{x^2 + 1} - 1}{x^2 + 1} \)
36. Show that \( \lim_{x \to \infty} \sqrt{x^2 + 1} - x = 0 \). Hint: Observe that \( \sqrt{x^2 + 1} - x = \frac{1}{\sqrt{x^2 + 1} + x} \).

Further Insights and Challenges

46. Every limit as \( x \to \infty \) can be expressed alternatively as a one-sided limit as \( r \to 0^+ \), where \( r = x^{-1} \). Setting \( g(t) = f(0^+) \), we have

\[ \lim_{x \to \infty} f(x) = \lim_{r \to 0^+} g(r) \]

In Exercises 37–44, calculate the limits.

37. \( \lim_{x \to \infty} \sqrt{4x^2 + 9x - 2x^3} \)
38. \( \lim_{x \to \infty} (\sqrt{9x^3 + x} - x^{3/2}) \)
39. \( \lim_{x \to \infty} (2\sqrt{x} - \sqrt{x + 2}) \)
40. \( \lim_{x \to \infty} \left( \frac{1}{x} - \frac{1}{x + 2} \right) \)
41. \( \lim_{r \to 0^+} \frac{4 + 5r^2}{5 - 5^2} \)
42. \( \lim_{r \to 0^+} \frac{5r + x}{x + 1} \)

43. Let \( P(n) \) be the perimeter of an \( n \)-gon inscribed in a unit circle (Figure 7).

(a) Explain, intuitively, why \( P(n) \) approaches \( 2\pi \) as \( n \to \infty \).
(b) Show that \( P(n) = 2n \sin \left( \frac{\pi}{n} \right) \).
(c) Combine (a) and (b) to conclude that \( \lim_{n \to \infty} \frac{2n \sin \left( \frac{\pi}{n} \right)}{n} = 1 \).
(d) Use this to give another argument that \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \).

44. Physicists have observed that Einstein’s theory of special relativity reduces to Newtonian mechanics in the limit as \( c \to \infty \), where \( c \) is the speed of light. This is illustrated by a stone tossed up vertically from ground level so that it returns to Earth 1 s later. Using Newton’s Laws, we find that the stone’s maximum height is \( h = \frac{g}{8} \) m (\( g = 9.8 \text{ m/s}^2 \)). According to special relativity, the stone’s mass depends on its velocity divided by \( c \), and the maximum height is \( h(c) = c \sqrt{c^2/g^2 + 1/4 - c^2/g} \).

Prove that \( \lim_{c \to \infty} h(c) = g/8 \).

45. According to the Michaelis–Menten equation, when an enzyme is combined with a substrate of concentration \( s \) (in millimolars) the reaction rate (in micromolars/min) is

\[ R(s) = \frac{A}{K + s} \quad (A, K \text{ constants}) \]

(a) Show, by computing \( \lim_{s \to 0} R(s) \), that \( A \) is the limiting reaction rate as the concentration \( s \) approaches \( 0 \).
(b) Show that the reaction rate \( R(s) \) attains one-half of the limiting value \( A \) when \( s = K \).
(c) For a certain reaction, \( K = 1.25 \text{ mM} \) and \( A = 0.1 \). For which concentration \( s \) is \( R(s) \) equal to 75% of its limiting value?
47. Rewrite the following as one-sided limits as in Exercise 46 and evaluate.

(a) \( \lim_{x \to \infty} \frac{3 - 12x^3}{4x^3 + 3x + 1} \)
(b) \( \lim_{x \to \infty} 3^{1/x} \)
(c) \( \lim_{x \to \infty} \frac{1}{x} \)

48. Let \( G(b) = \lim_{x \to \infty} (1 + b^x)^{1/x} \) for \( b \geq 0 \). Investigate \( G(b) \) numerically and graphically for \( b = 0.2, 0.8, 2, 3, 5 \) (and additional values if necessary). Then make a conjecture for the value of \( G(b) \) as a function of \( b \). Draw a graph of \( y = G(b) \). Does \( G \) appear to be continuous? We will evaluate \( G(b) \) using L'Hôpital's Rule in Section 7.5 (see Exercise 69 there).

2.8 The Intermediate Value Theorem

The Intermediate Value Theorem (IVT) says, roughly speaking, that a continuous function cannot skip values. Consider a plane that takes off and climbs from 0 to 10,000 m in 20 min. The plane must reach every altitude between 0 and 10,000 m during this 20-min interval. Thus, at some moment, the plane’s altitude must have been exactly 8371 m. Of course, this assumes that the plane’s motion is continuous, so its altitude cannot jump abruptly from, say, 8000 to 9000 m.

To state this conclusion formally, let \( A(t) \) be the plane’s altitude at time \( t \). The IVT asserts that for every altitude \( M \) between 0 and 10,000, there is a time \( t_0 \) between 0 and 20 min such that \( A(t_0) = M \). In other words, the graph of \( A \) must intersect the horizontal line \( y = M \) (Figure 1(A)).

![Figure 1](image_url)

By contrast, a discontinuous function can skip values. The greatest integer function \( f(x) = \lfloor x \rfloor \) in Figure 1(B) satisfies \( \lfloor 1 \rfloor = 1 \) and \( \lfloor 2 \rfloor = 2 \), but it does not take on the value \( 1.5 \) (or any other value between 1 and 2).

**Theorem 1 Intermediate Value Theorem** If \( f \) is continuous on a closed interval \([a, b]\), then for every value \( M \), strictly between \( f(a) \) and \( f(b) \), there exists at least one value \( c \in (a, b) \) such that \( f(c) = M \).

Graphically, as in Figure 2, the result appears obvious. For a continuous function, every horizontal line at height \( M \) between \( f(a) \) and \( f(b) \) is forced to hit the graph, and therefore there must be at least one value \( c \) in \((a, b)\) such that \( f(c) = M \). The proof appears in Appendix B.

**Example 1** Prove that the equation \( \sin x = 0.3 \) has at least one solution in the interval \( \left(0, \frac{\pi}{2}\right)\).

**Solution** We may apply the IVT because \( f(x) = \sin x \) is continuous. The desired value 0.3 lies between the values of the function at the endpoints of the interval:

\[
\sin 0 = 0 \quad \text{and} \quad \sin \frac{\pi}{2} = 1
\]

as illustrated in (Figure 3). The IVT tells us that \( \sin x = 0.3 \) has at least one solution in \( \left(0, \frac{\pi}{2}\right)\).
The IVT can be used to show the existence of zeros of functions. If \( f \) is continuous and takes on both nonpositive and nonnegative values (say, \( f(a) \leq 0 \) and \( f(b) \geq 0 \)) then the IVT guarantees that \( f(c) = 0 \) for some \( c \) in \([a, b]\). This is extremely useful when we cannot explicitly solve for the zero but would like to know that there is one in the interval.

**Corollary 2 Existence of Zeros**  If \( f \) is continuous on \([a, b]\), and if one of \( f(a) \) and \( f(b) \) is nonnegative and the other is nonpositive, then \( f \) has a zero in \([a, b]\).

We can locate zeros of functions to arbitrary accuracy using the **Bisection Method**. The idea is to find an interval \([a, b]\) such that the function has opposite signs at the endpoints. Then Corollary 2 tells us that there is a zero on this interval. To find its location more precisely, we cut the interval into two equal subintervals. Then, check the signs at the endpoints of each of these intervals to see which one Corollary 2 tells us has a zero. (But keep in mind that there may be more than one zero, so both could contain a zero.) Next, we repeat the process on this smaller interval. Eventually, we narrow down on a zero. This is illustrated in the next example.

**Example 2**  The Bisection Method Show that \( f(x) = \cos^2 x - 2 \sin \frac{x}{2} \) has a zero in \((0, 2)\). Then, using the Bisection Method, find a subinterval of \((0, 2)\) of length 1/8 that contains a zero of \( f \).

Solution  To begin, we note that \( f \) is continuous on \([0, 2]\). Calculating \( f(0) \) and \( f(2) \), we find that they have opposite signs:

\[
 f(0) = 1 > 0, \quad f(2) = -0.786 < 0
\]

Corollary 2 guarantees that \( f(x) = 0 \) has a solution in \((0, 2)\) (Figure 4).

To locate a zero more accurately, divide \([0, 2]\) into two intervals \([0, 1]\) and \([1, 2]\). At least one of these intervals must contain a zero of \( f \). To determine which, we evaluate \( f \) at the midpoint \( m = 1 \), obtaining \( f(1) \approx -0.203 < 0 \). Since \( f(0) = 1 \), we see that \( f(x) \) takes on opposite signs at the endpoints of \([0, 1]\).

Therefore, \((0, 1)\) must contain a zero. Note that from the function values \( f(1) \) and \( f(2) \) alone, we cannot conclude that \( f \) does not have a zero in the interval \([1, 2]\), but it is clear from the graph in Figure 4 that it does not.

The Bisection Method consists of continuing this process until we narrow down the location of a zero to any desired accuracy. In the following table, the process is carried out three times to find an interval of length 1/8 containing a zero of \( f \).

<table>
<thead>
<tr>
<th>Interval</th>
<th>Midpoint of interval</th>
<th>Function values</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 1])</td>
<td>(\frac{1}{2})</td>
<td>(f(\frac{1}{2}) \approx 0.521)</td>
<td>(f(1) \approx -0.203)</td>
</tr>
<tr>
<td>([\frac{1}{2}, 1])</td>
<td>(\frac{3}{4})</td>
<td>(f(\frac{3}{4}) \approx 0.163)</td>
<td>(f(1) \approx -0.203)</td>
</tr>
<tr>
<td>([\frac{3}{4}, 1])</td>
<td>(\frac{7}{8})</td>
<td>(f(\frac{7}{8}) \approx -0.0231)</td>
<td>(f(\frac{7}{8}) \approx 0.163)</td>
</tr>
</tbody>
</table>

We conclude that \( f \) has a zero \( c \) satisfying \( 0.75 < c < 0.875 \).

**Example 3**  A Meteorological Consequence of the IVT  Take a map and draw a circle anywhere on it such as in Figure 5. With the IVT we can show that there must
be a pair of points that have the same temperature and lie opposite each other through the
center of the circle.

Let \( T(\theta) \) be the temperature of the location on the circle that is at angle \( \theta \) from the
horizontal (as shown in Figure 5). We assume that \( T \) is a continuous function. Define
another function \( f \) by \( f(\theta) = T(\theta) - T(\theta + \pi) \). Thus, \( f(\theta) \) is the difference between
the temperature at location \( \theta \) and the temperature at the location opposite it on the circle.
Since \( T \) is continuous, it follows that \( f \) is as well.

Show that \( f(\pi) = -f(0) \), and use Corollary 2 and the relationship between \( f \) and
\( T \) to argue that there are opposite points on the circle with the same temperature.

Solution For \( f \) we have

\[
f(0) = T(0) - T(\pi)
\]

\[
f(\pi) = T(\pi) - T(\pi + \pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) = -f(0)
\]

Since \( f(\pi) = -f(0) \) by Eq. (1), it follows that one of \( f(0) \) and \( f(\pi) \) is nonnegative
and the other is nonpositive. Corollary 2 then implies that there is \( c \) between 0 and \( \pi \) such
that \( f(c) = 0 \). So \( T(c) - T(c + \pi) = 0 \), implying that \( c \) and \( c + \pi \) are opposite points
with the same temperature.

Note that the argument in Example 3 works with any continuous variable defined
over the map. Temperature could be replaced with barometric pressure, relative humidity,
elevation, and so on, and the same type of conclusion could be drawn.

CONCEPTUAL INSIGHT The IVT seems to state the obvious, namely that a continuous
function cannot skip values. Yet its proof (given in Appendix B) is subtle because
it depends on the completeness property of real numbers. To highlight the subtlety,
obscrve that the IVT is false for functions defined only on the rational numbers. For
example, \( f(x) = x^2 \) is continuous, but the Intermediate Value Theorem does not apply
if we restrict its domain to the rational numbers. Indeed, \( f(0) = 0, f(2) = 4, \) and 3 is
between 0 and 4 but \( f(c) = 3 \) has no solution for \( c \) rational. The solution \( c = \sqrt{3} \) is
missing from the set of rational numbers because it is irrational. No doubt the IVT was
always regarded as obvious, but it was not possible to give a correct proof until the
completeness property was clarified in the second half of the nineteenth century.

2.8 SUMMARY

- The Intermediate Value Theorem (IVT) says that a continuous function cannot skip
values.
- More precisely, if \( f \) is continuous on \([a, b]\) with \( f(a) \neq f(b) \), and if \( M \) is a number
strictly between \( f(a) \) and \( f(b) \), then \( f(c) = M \) for some \( c \in (a, b) \).
- Existence of zeros: If \( f \) is continuous on \([a, b]\), and if one of \( f(a) \) and \( f(b) \) is non-
negative and the other is nonpositive, then \( f \) has a zero in \([a, b]\).
• Bisected Method: Assume $f$ is continuous and that $f(a)$ and $f(b)$ have opposite signs, so that $f$ has a zero in $(a, b)$. Then $f$ has a zero in $[a, m]$ or $[m, b]$, where $m = (a + b)/2$ is the midpoint of $[a, b]$. A zero lies in $(a, m)$ if $f(a)$ and $f(m)$ have opposite signs and a zero lies in $(m, b)$ if $f(m)$ and $f(b)$ have opposite signs. Continuing the process, we can locate zeros with arbitrary accuracy.

2.8 EXERCISES

Preliminary Questions
1. Prove that $f(x) = x^2$ takes on the value 0.5 in the interval $[0, 1]$.

2. The temperature in Vancouver was 8°C at 6 AM and rose to 20°C at noon. Which assumption about temperature allows us to conclude that the temperature was 15°C at some point of time between 6 AM and noon?

3. What is the graphical interpretation of the IVT?

4. Show that the following statement is false by drawing a graph that provides a counterexample:

   If $f$ is continuous and has a root in $[a, b]$, then $f(a)$ and $f(b)$ have opposite signs.

5. Assume that $f$ is continuous on $[1, 5]$ and that $f(1) = 20$, $f(5) = 100$. Determine whether each of the following statements is always true, never true, or sometimes true.
   
   (a) $f(c) = 3$ has a solution with $c \in [1, 5]$.
   
   (b) $f(c) = 75$ has a solution with $c \in [1, 5]$.

   (c) $f(c) = 50$ has a solution with $c \in [1, 5]$.

   (d) $f(c) = 30$ has exactly one solution with $c \in [1, 5]$.

Exercises

1. Use the IVT to show that $f(x) = x^2 + x$ takes on the value 9 for some $x$ in $[1, 2]$.

2. Show that $g(t) = \frac{t}{t+1}$ takes on the value 0.499 for some $t$ in $[0, 1]$.

3. Show that $g(t) = t^2 \tan t$ takes on the value $\frac{1}{2}$ for some $t$ in $[0, \frac{\pi}{2}]$.

4. Show that $f(x) = \frac{x^2}{x^2 + 1}$ takes on the value 0.4.

5. Show that $\cos x = x$ has a solution in the interval $[0, 1]$. Hint: Show that $f(x) = x - \cos x$ has a zero in $[0, 1]$.

6. Use the IVT to find an interval of length $\frac{1}{3}$ containing a root of $f(x) = x^2 + 2x + 1$.

In Exercises 7–16, prove using the IVT.

7. $\sqrt{2} + \sqrt{3} + 2 = 3$ has a solution.

8. For all integers $n$, $\sin nx = \cos x$ for some $x \in [0, \pi]$.

9. $\sqrt{2}$ exists. Hint: Consider $f(x) = x^2$.

10. A positive number $x$ has an $nth$ root for all positive integers $n$.

11. For all positive integers $k$, $\cos x = x^k$ has a solution.

12. $2^x = bx$ has a solution if $b > 2$.

13. $2^x + 3^x = 4^x$ has a solution.

14. $\cos x = \tan 2x$ has a solution in $[0, 1)$.

15. $2^x + 1/x = -4$ has a solution.

16. $x^{1/3} = 1/(x - 1)$ has a solution in $(1, 2)$.

17. Use the Intermediate Value Theorem to show that the equation $x^9 - 8x^6 + 10x^3 - 1 = 0$ has at least six distinct solutions.

In Exercises 18–20, determine whether or not the IVT applies to show that the given function takes on all values between $f(a)$ and $f(b)$ for $x \in [a, b]$. If it does not apply, determine any values between $f(a)$ and $f(b)$ that the function does not take on for $x \in [a, b]$.

18. $f(x) = \begin{cases} x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$ for the interval $[-1, 1]$.

19. $f(x) = \begin{cases} -x & \text{for } x < 0 \\ x^3 + 1 & \text{for } x \geq 0 \end{cases}$ for the interval $[-1, 1]$.

20. $f(x) = \begin{cases} -x^2 & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ x^2 & \text{for } x > 0 \end{cases}$ for the interval $[-2, 2]$.

21. Carry out three steps of the Bisection Method for $f(x) = 2^x - x^2$ as follows:
   (a) Show that $f$ has a zero in $[1, 1.5]$.
   (b) Show that $f$ has a zero in $[1.25, 1.5]$.
   (c) Determine whether $[1.25, 1.375]$ or $[1.375, 1.5]$ contains a zero.

22. Figure 6 shows that $f(x) = x^2 - 8x - 1$ has a root in the interval $[2.75, 3]$. Apply the Bisection Method twice to find an interval of length $\frac{1}{4}$ containing this root.

![Figure 6](image)

23. Find an interval of length $\frac{1}{4}$ in $[1, 2]$ containing a root of the equation $x^7 + 3x - 10 = 0$.

24. Show that $\tan^3 \theta - 8 \tan^2 \theta + 17 \tan \theta - 8 = 0$ has a root in $[0.5, 0.6]$. Apply the Bisection Method twice to find an interval of length 0.025 containing this root.
In Exercises 25–28, draw the graph of a function \( f \) on \([0, 4]\) with the given property.

25. Jump discontinuity at \( x = 2 \) and does not satisfy the conclusion of the IVT
26. Jump discontinuity at \( x = 2 \) and satisfies the conclusion of the IVT
27. Infinite one-sided limits at \( x = 2 \) and does not satisfy the conclusion of the IVT
28. Infinite one-sided limits at \( x = 2 \) and satisfies the conclusion of the IVT

29. Can Corollary 2 be applied to \( f(x) = x^2 \) on \([-1, 1]\)? Does \( f \) have any roots?

30. (a) Assume that \( g \) and \( h \) are continuous on \([a, b]\). Use Corollary 2 to show that if \( g(a) < h(a) \) and \( h(b) < g(b) \), then there exists \( c \in [a, b] \) such that \( g(c) = h(c) \).

31. At 1:00 PM, Jacqueline began to climb up Waterfall Hill from the bottom. At the same time, Giles began to climb down from the top. Giles reached the bottom at 2:20 PM, when Jacqueline was 95% of the way up. Jacqueline reached the top at 2:50. Use the result in Exercise 30 to prove that there was a time when they were at the same elevation on the hill.

32. On Wednesday at noon the weather was fair in Boston with a barometric pressure of 1014 mb. At the same time, a low-pressure storm system was passing by Buffalo, where the pressure was 995 mb. On Thursday the storm was approaching Boston, where the pressure was 1002 mb, while the weather was clearing in Buffalo and the pressure had risen to 1014 mb. Use the result in Exercise 30 to prove that there was a time between noon Wednesday and noon Thursday when Boston and Buffalo had the same barometric pressure.

Further Insights and Challenges

Exercises 33 and 34, address the 1-Dimensional Brouwer Fixed Point Theorem. It indicates that every continuous function \( f \) mapping the closed interval \([0, 1]\) to itself must have a fixed point: that is, a point \( c \) such that \( f(c) = c \).

33. Show that if \( f \) is continuous and \( 0 \leq f(x) \leq 1 \) for \( 0 \leq x \leq 1 \), then \( f \) has a fixed point \( c \).

34. (a) Give an example showing that if \( f \) is continuous and \( 0 < f(x) < 1 \) for \( 0 < x < 1 \), then there does not need to be a \( c \) in \((0, 1)\) such that \( f(c) = c \).

(b) Give an example showing that if \( 0 \leq f(x) \leq 1 \) for \( 0 \leq x \leq 1 \), but \( f \) is not necessarily continuous, then there does not need to be a \( c \) in \((0, 1)\) such that \( f(c) = c \).

35. Use the IVT to show that if \( f \) is continuous and one-to-one on an interval \([a, b]\), then \( f \) is either an increasing or a decreasing function.

36. Ham Sandwich Theorem Figure 8(A) shows a slice of ham. Prove that for any angle \( \theta \) \((0 \leq \theta \leq \pi)\), it is possible to cut the slice in half with a cut of angle \( \theta \). Hint: The lines of inclination \( \theta \) are given by the equations \( y = (\tan \theta)x + b \), where \( b \) varies from \(-\infty\) to \(\infty\). Each such line divides the slice into two pieces (one of which may be empty). Let \( A(\theta) \) be the amount of ham to the left of the line minus the amount to the right, and let \( B \) be the total area of the ham. Show that \( A(\pi - \theta) = A \) if \( B \) is large enough and \( A(\pi - \theta) = -A \) if \( B \) is smaller enough. Then use the IVT. This works if \( 0 \neq \theta \) or \( \pi - \theta \).

37. Figure 8(B) shows a slice of ham on a piece of bread. Prove that it is possible to slice this open-faced sandwich so that each part has equal amounts of ham and bread. Hint: By Exercise 36, for all \( 0 \leq \theta \leq \pi \) there is a line \( L(\theta) \) of inclination \( \theta \) which divides the ham into two equal pieces. Let \( h(\theta) \) denote the amount of ham to the left of (or above) \( L(\theta) \) minus the amount to the right (or below). Notice that \( L(\pi - \theta) \) is the same line, but \( B(\pi - \theta) \) since left and right get interchanged as the angle moves from \( 0 \) to \( \pi \). Assume that \( B \) is continuous and apply the IVT. By a further extension of this argument, one can prove the full Ham Sandwich Theorem, which states that if you allow the knife to cut at a slant, then it is possible to cut a sandwich consisting of a slice of ham and two slices of bread so that all three layers are divided in half.

2.9 The Formal Definition of a Limit

In this section, we reexamine the definition of a limit in order to state it in a more rigorous and precise fashion. Why is this necessary? In Section 2.2, we defined limits by saying that \( \lim_{x \to c} f(x) = L \) if \( |f(x) - L| \) becomes arbitrarily small when \( x \) is sufficiently close
A rigorous proof in mathematics is an argument based on a complete chain of logic where each step follows unambiguously from what proceeds it. The formal definition of a limit is a key ingredient of rigorous proofs in calculus. A few such proofs are included in Appendix D. More complete developments can be found in textbooks on the branch of mathematics called analysis.

The Size of the Gap

Recall that the distance from \( f(x) \) to \( L \) is \( |f(x) - L| \). It is convenient to refer to the quantity \( |f(x) - L| \) as the gap between the value \( f(x) \) and the limit \( L \).

Let's reexamine the trigonometric limit

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

In this example, \( f(x) = \frac{\sin x}{x} \) and \( L = 1 \), so Eq. (1) tells us that the gap \( |f(x) - L| \) gets arbitrarily small when \( x \) is sufficiently close, but not equal, to 0 [Figure 1(A)].

Suppose we want the gap \( |f(x) - L| \) to be less than 0.2. How close to 0 must \( x \) be? Figure 1(B) shows that \( f(x) \) lies within 0.2 of \( L = 1 \) for all values of \( x \) in the interval \([-1, 1]\). In other words, the following statement is true:

\[
\text{If } 0 < |x| < 1, \text{ then } \left| \frac{\sin x}{x} - 1 \right| < 0.2
\]

If we insist instead that the gap be smaller than 0.004, we can check by zooming in on the graph, as in Figure 1(C), that

\[
\text{If } 0 < |x| < 0.15, \text{ then } \left| \frac{\sin x}{x} - 1 \right| < 0.004
\]

It would seem that this process can be continued: Given any positive number, no matter how small, by zooming in on the graph we can find a small interval around \( c = 0 \) where the gap \( |f(x) - L| \) is smaller than the given number.

To express this in a precise fashion, we follow time-honored tradition of using the Greek letters \( \epsilon \) (epsilon) and \( \delta \) (delta) to denote small numbers specifying the sizes of the gap and the quantity \( |x - c| \), respectively. In our case, \( c = 0 \) and \( |x - c| = |x| \). The precise meaning of Eq. (1) is that for every choice of \( \epsilon > 0 \), there exists some \( \delta \) (depending on \( \epsilon \)) such that

\[
\text{If } 0 < |x| < \delta, \text{ then } \left| \frac{\sin x}{x} - 1 \right| < \epsilon
\]
The number δ pins down just how close is “sufficiently close” for a given ε. With this motivation, we are ready to state the formal definition of the limit.

**FORMAL DEFINITION OF A LIMIT**

Suppose that \( f(x) \) is defined for all \( x \) in an open interval containing \( c \) (but not necessarily at \( x = c \)). Then

\[
\lim_{x \to c} f(x) = L
\]

if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|f(x) - L| < \epsilon \quad \text{if } 0 < |x - c| < \delta.
\]

The condition \( 0 < |x - c| < \delta \) in this definition excludes \( x = c \). In other words, the limit depends only on values of \( f(x) \) near \( c \) but not on \( f(c) \) itself. As we have seen in previous sections, the limit may exist even when \( f(c) \) is not defined.

**EXAMPLE 1**

Let \( f(x) = 8x + 3 \).

(a) Prove that \( \lim_{x \to 3} f(x) = 27 \) using the formal definition of the limit.

(b) Find values of \( \delta \) that work for \( \epsilon = 0.2 \) and 0.001.

**Solution**

(a) We break the proof into two steps.

**Step 1. Relate the gap to \( |x - c| \).**

We must find a relation between two absolute values: \( |f(x) - L| \) for \( L = 27 \) and \( |x - c| \) for \( c = 3 \). Observe that

\[
|f(x) - 27| = |(8x + 3) - 27| = |8x - 24| = 8|x - 3|
\]

Thus, the gap is 8 times as large as \( |x - 3| \).

**Step 2. Choose \( \delta \) (in terms of \( \epsilon \)).**

We can now see how to make the gap small: if \( |x - 3| < \frac{\epsilon}{8} \), then the gap is less than \( 8 \left( \frac{\epsilon}{8} \right) = \epsilon \). Therefore, for any \( \epsilon > 0 \), we choose \( \delta = \frac{\epsilon}{8} \). With this choice, the following statement holds:

\[
\text{If } 0 < |x - 3| < \delta, \quad \text{then } |f(x) - 27| < \epsilon, \quad \text{where } \delta = \frac{\epsilon}{8}
\]

Since we have specified \( \delta \) for all \( \epsilon > 0 \), we have fulfilled the requirements of the formal definition, thus proving rigorously that \( \lim_{x \to 3} (8x + 3) = 27 \).

(b) For the particular choice \( \epsilon = 0.2 \), we may take \( \delta = \frac{\epsilon}{8} = \frac{0.2}{8} = 0.025 \):

\[
\text{If } 0 < |x - 3| < 0.025, \quad \text{then } |f(x) - 27| < 0.2
\]

This statement is illustrated in Figure 2. But note that any positive \( \delta \) smaller than 0.025 will also work. For example, the following statement is also true, although it places an unnecessary restriction on \( x \):

\[
\text{If } 0 < |x - 3| < 0.019, \quad \text{then } |f(x) - 27| < 0.2
\]

**FIGURE 2**

To make the gap less than 0.2, we may take \( \delta = 0.025 \) (not drawn to scale).
Similarly, to make the gap less than \( \varepsilon = 0.001 \), we may take
\[
\delta = \frac{\varepsilon}{8} = \frac{0.001}{8} = 0.000125
\]

The difficulty in applying the limit definition lies in trying to relate \( |f(x) - L| \) to \( |x - c| \). The next two examples illustrate how this can be done in special cases.

**EXAMPLE 2** Prove that \( \lim_{x \to 2} x^2 = 4 \).

**Solution** Let \( f(x) = x^2 \).

**Step 1. Relate the gap to \( |x - c| \).**

In this case, we must relate the gap \( |f(x) - 4| = |x^2 - 4| \) to the quantity \( |x - 2| \) (Figure 3). This is more difficult than in the previous example because the gap is not a constant multiple of \( |x - 2| \). To proceed, consider the factorization
\[
|x^2 - 4| = |x + 2||x - 2|
\]

Because we are going to require that \( |x - 2| \) be small, we may as well assume from the outset that \( |x - 2| < 1 \), which means that \( 1 < x < 3 \). In this case, \( |x + 2| \) is less than 5 and the gap satisfies
\[
\text{If } |x - 2| < 1, \text{ then } |x^2 - 4| = |x + 2||x - 2| < 5|x - 2|
\]

**Step 2. Choose \( \delta \) (in terms of \( \varepsilon \)).**

We see from Eq. (2) that if \( |x - 2| \) is smaller than both \( \frac{\varepsilon}{5} \) and 1, then the gap satisfies
\[
|x^2 - 4| < 5|x - 2| < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon
\]

Therefore, the following statement holds for all \( \varepsilon > 0 \):

\[
\text{If } 0 < |x - 2| < \delta, \text{ then } |x^2 - 4| < \varepsilon, \text{ where } \delta \text{ is the smaller of } \frac{\varepsilon}{5} \text{ and 1}
\]

We have specified \( \delta \) for all \( \varepsilon > 0 \), so we have fulfilled the requirements of the formal limit definition, thus proving that \( \lim_{x \to 2} x^2 = 4 \).

**EXAMPLE 3** Prove that \( \lim_{x \to 3} \frac{1}{x} = \frac{1}{3} \).

**Solution**

**Step 1. Relate the gap to \( |x - c| \).**

The gap is equal to
\[
\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{3-x}{3x}\right| = \left|x - 3\right| \left|\frac{1}{3x}\right|
\]
Because we are going to require that \(|x - 3|\) be small, we may as well assume from the outset that \(|x - 3| < 1\), or equivalently, 2 < x < 4. Now observe that if x > 2, then 3x > 6 and \(\frac{1}{3x} < \frac{1}{6}\), so the following inequality is valid if \(|x - 3| < 1\):

\[
\left| f(x) - \frac{1}{3} \right| = \left| \frac{3-x}{2x} \right| = \left| \frac{1}{3x} \right| \left| x - 3 \right| < \frac{1}{6} |x - 3|
\]

Step 2. Choose \(\delta\) (in terms of \(\epsilon\)).

By Eq. (3), if \(|x - 3| < 1\) and \(|x - 3| < 6\epsilon\), then

\[
\frac{1}{x} - \frac{1}{3} < \frac{1}{6} |x - 3| < \frac{1}{6}(6\epsilon) = \epsilon
\]

Therefore, given any \(\epsilon > 0\), we let \(\delta\) be the smaller of the numbers 6\(\epsilon\) and 1. Then we have

If \(0 < |x - 3| < \delta\), then \(\left| \frac{1}{x} - \frac{1}{3} \right| < \epsilon\), where \(\delta\) is the smaller of 6\(\epsilon\) and 1.

Again, we have fulfilled the requirements of the formal limit definition, thus proving rigorously that \(\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}\).

**GRAPHICAL INSIGHT** Keep the graphical interpretation of limits in mind. In Figure 4(A), \(f(x)\) approaches \(L\) as \(x \to c\) because for any \(\epsilon > 0\), we can make the gap less than \(\epsilon\) by taking \(\delta\) sufficiently small. By contrast, consider the function \(g\) in Figure 4(B). It has a jump discontinuity at \(x = c\). Because of the jump, the gap at \(c\) cannot be made less than half the distance between \(a\) and \(b\). Therefore if \(\epsilon < \frac{b - a}{2}\), then no matter how small we choose \(\delta\), the gap corresponding to \((c - \delta, c + \delta)\) cannot be made smaller than \(\epsilon\). Thus, the limit as \(x \to c\) of \(g(x)\) does not exist.

![Figure 4](image)

(A) The function \(f\) is continuous at \(x = c\). By taking \(\delta\) sufficiently small, we can make the gap smaller than \(\epsilon\).

(B) The function \(g\) is not continuous at \(x = c\). The gap is always larger than \((b - a)/2\), no matter how small \(\delta\) is.

**Proving Limit Theorems**

In practice, the formal definition of the limit is rarely used to evaluate limits. Most limits are evaluated using the Basic Limit Laws or other techniques such as the Squeeze Theorem. However, the formal definition allows us to prove these laws in a rigorous fashion and thereby ensure that calculus is built on a solid foundation. We illustrate by proving the Sum Law. Other proofs are given in Appendix D.
Proof of the Sum Law  
Assume that  

\[ \lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M. \]

We must prove that  \( \lim_{x \to c} (f(x) + g(x)) = L + M \).  

Apply the Triangle Inequality (see margin) with \( a = f(x) - L \) and \( b = g(x) - M \):  

\[ |(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| \]

Each term on the right in (4) can be made small by the limit definition. More precisely, given \( \epsilon > 0 \), we can choose \( \delta \) such that if \( 0 < |x - c| < \delta \), then \( |f(x) - L| < \frac{\epsilon}{2} \) and \( |g(x) - M| < \frac{\epsilon}{2} \) (in principle, we might choose different \( \delta \)'s for \( f \) and \( g \), but we may then use the smaller of the two \( \delta \)'s). Thus, Eq. (4) gives

\[ \text{If } 0 < |x - c| < \delta, \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

This proves that

\[ \lim_{x \to c} (f(x) + g(x)) = L + M = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \]

\[ \square \]

2.9 SUMMARY

- Informally speaking, the statement \( \lim_{x \to c} f(x) = L \) means that the gap \( |f(x) - L| \) tends to 0 as \( x \) approaches \( c \).
- The formal definition (called the \( \epsilon-\delta \) definition): \( \lim_{x \to c} f(x) = L \) if, for all \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that
  
  \[ \text{if } 0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \epsilon \]

2.9 EXERCISES

Preliminary Questions

1. Given that \( \lim_{x \to 0} \cos x = 1 \), which of the following statements is true?

(a) If \( |\cos x - 1| \) is very small, then \( x \) is close to 0.

(b) There is an \( \epsilon > 0 \) such that if \( 0 < |\cos x - 1| < \epsilon \), then \( |x| < 10^{-3} \).

(c) There is a \( \delta > 0 \) such that if \( 0 < |x| < \delta \), then \( |\cos x - 1| < 10^{-3} \).

(d) There is a \( \delta > 0 \) such that if \( 0 < |x - 1| < \delta \), then \( |\cos x - 1| < 10^{-3} \).

2. Suppose it is known that for a given \( \epsilon \) and \( \delta \), if \( 0 < |x - 3| < \delta \), then \( |f(x) - 2| < \epsilon \). Which of the following statements must also be true?

(a) If \( 0 < |x - 3| < 2\delta \), then \( |f(x) - 2| < \epsilon \).

(b) If \( 0 < |x - 3| < \delta \), then \( |f(x) - 2| < 2\epsilon \).

(c) If \( 0 < |x - 3| < \frac{\delta}{2} \), then \( |f(x) - 2| < \frac{\epsilon}{2} \).

(d) If \( 0 < |x - 3| < \frac{\delta}{2} \), then \( |f(x) - 2| < \epsilon \).

Exercises

1. Based on the information conveyed in Figure 5(A), find values of \( L \), \( \epsilon \), and \( \delta > 0 \) such that the following statement holds: If \( |x| < \delta \), then \( |f(x) - L| < \epsilon \).

2. Based on the information conveyed in Figure 5(B), find values of \( c \), \( L \), \( \epsilon \), and \( \delta > 0 \) such that the following statement holds: If \( 0 < |x - c| < \delta \), then \( |f(x) - L| < \epsilon \).

3. Make a sketch illustrating the following statement: To prove \( \lim_{x \to a} x = a \), given \( \epsilon > 0 \), we can take \( \delta = \epsilon \) to have the gap be small enough.

4. Make a sketch illustrating the following statement: To prove \( \lim_{x \to a} f(x) = a \), given \( \epsilon > 0 \), we can choose any \( \delta > 0 \) to have the gap be small enough.
5. Consider \( \lim_{x \to 2} f(x) \), where \( f(x) = 8x + 3 \).
   (a) Show that \( |f(x) - 35| = 8|x - 4| \).
   (b) Show that for any \( \epsilon > 0 \), if \( 0 < |x - 4| < \delta \), then \( |f(x) - 35| < \epsilon \), where \( \delta = \frac{\epsilon}{8} \). Explain how this proves rigorously that \( \lim_{x \to 2} f(x) = 35 \).

6. Consider \( \lim_{x \to 4} f(x) \), where \( f(x) = 4x - 1 \).
   (a) Show that if \( 0 < |x - 2| < \delta \), then \( |f(x) - 7| < \epsilon \).
   (b) Find a \( \delta \) such that
   \[ |f(x) - 7| < 0.01 \]
   (c) Prove rigorously that \( \lim_{x \to 2} f(x) = 7 \).

7. Consider \( \lim_{x \to 4} x^2 = 4 \) (refer to Example 2).
   (a) Show that if \( 0 < |x - 2| < 0.01 \), then \( |x^2 - 4| < 0.05 \).
   (b) Show that if \( 0 < |x - 2| < 0.0002 \), then \( |x^2 - 4| < 0.00009 \).
   (c) Find a value of \( \delta \) such that if \( 0 < |x - 2| < \delta \), then \( |x^2 - 4| < 0.000001 \).

8. Consider the limit \( \lim_{x \to \infty} x^2 = 25 \).
   (a) Show that if \( 4 < x < 6 \), then \( |x^2 - 25| < |x - 5| \). \( \text{Hint: Write} x^2 - 25 = (x - 5)(x + 5) \).
   (b) Find a \( \delta \) such that if \( 0 < |x - 5| < \delta \), then \( |x^2 - 25| < 10^{-3} \).
   (c) Give a rigorous proof of the limit by showing that if \( 0 < |x - 5| < \delta \), then \( |x^2 - 25| < \epsilon \), where \( \delta \) is the smaller of \( \frac{\epsilon}{10} \) and 1.

9. Refer to Example 3 to find a value of \( \delta > 0 \) such that
   \[ |\frac{1}{x} - 3| < \delta \] then \( |\frac{1}{x} - 3 - \frac{1}{x}| < 10^{-4} \).

10. Use Figure 6 to find a value of \( \delta > 0 \) such that the following statement holds: if \( 0 < |x - 2| < \delta \), then \( \frac{1}{x^2} - \frac{1}{4} < \epsilon \) for \( \epsilon = 0.03 \). Then find a value of \( \delta \) that works for \( \epsilon = 0.01 \).

11. Plot the function \( f(x) = \sqrt{2x - 1} \) together with the horizontal lines \( y = 2.9 \) and \( y = 3.1 \). Use this plot to find a value of \( \delta > 0 \) such that \( 0 < |x - 1.5| < \delta \), then \( |\sqrt{2x - 1} - 2.9| < 0.1 \).

12. Plot the function \( f(x) = \tan x \) together with the horizontal lines \( y = 0.99 \) and \( y = 1.01 \). Use this plot to find a value of \( \delta > 0 \) such that \( 0 < |x - \frac{\pi}{4}| < \delta \), then \( |\tan x - 1| < 0.01 \).

13. The function \( f(x) = 2^{-x^2} \) satisfies \( \lim_{x \to \infty} f(x) = 1 \). Use a plot of \( f \) to find a value of \( \delta > 0 \) such that \( |f(x) - 1| < 0.001 \) if \( 0 < |x| < \delta \).

14. Let \( f(x) = \frac{4}{x^2 + 1} \) and \( \epsilon = 0.5 \). Using a plot of \( f \), find a value of \( \delta > 0 \) such that if \( 0 < |x - \frac{1}{2}| < \delta \), then \( |f(x) - \frac{8}{3}| < \epsilon \). Repeat for \( \epsilon = 0.2 \) and 0.1.

15. Consider \( \lim_{x \to 2} \frac{1}{x} \).
   (a) Show that if \( 0 < |x - 2| < 1 \), then
   \[ \left| \frac{1}{x} - \frac{1}{2} \right| < \frac{1}{2} |x - 2| \]
   (b) Find a \( \delta > 0 \) such that if \( 0 < |x - 2| < \delta \), then \( \left| \frac{1}{x} - \frac{1}{2} \right| < 0.01 \).
   (c) Let \( \delta \) be the smaller of 1 and 2\( \epsilon \). Prove the following:
   \[ 0 < |x - 2| < \delta, \quad \left| \frac{1}{x} - \frac{1}{2} \right| < \epsilon \]
   Then explain why this proves that \( \lim_{x \to 2} \frac{1}{x} = \frac{1}{2} \).

16. Consider \( \lim_{x \to 1} \sqrt{x + 3} \).
   (a) Show that if \( |x - 1| < 4 \), then \( |\sqrt{x + 3} - 2| < \frac{1}{4} |x - 1| \). \( \text{Hint: Multiply the inequality by} \ |\sqrt{x + 3} + 2| \) and observe that \( |\sqrt{x + 3} + 2| > 2 \).
   (b) Find \( \delta > 0 \) such that if \( 0 < |x - 1| < \delta \), then \( |\sqrt{x + 3} - 2| < 10^{-4} \).
   (c) Prove rigorously that the limit is equal to 2.

17. Let \( f(x) = \sin x \). Using a calculator, find
   \[ f \left( \frac{\pi}{4} - 0.1 \right) \approx 0.633, \quad f \left( \frac{\pi}{4} + 0.1 \right) \approx 0.707, \quad f \left( \frac{\pi}{4} + 0.1 \right) \approx 0.774 \]
   Use these values and the fact that \( f \) is increasing on \( [0, \frac{\pi}{4}] \) to justify the statement
   \[ 0 < |x - f(x)| < 0.01, \quad \text{then} \quad f(x) < f \left( \frac{\pi}{4} \right) < 0.08 \]
   Then draw a figure like Figure 3 to illustrate this statement.

18. Adapt the argument in Example 1 to prove rigorously that \( \lim_{x \to c} (ax + b) = ac + b \), where \( a, b, c \) are arbitrary.

19. Adapt the argument in Example 2 to prove rigorously that \( \lim_{x \to c} e^x = e^c \) for all \( c \).

20. Adapt the argument in Example 3 to prove rigorously that \( \lim_{x \to c} \frac{1}{e^x} = \frac{1}{e^c} \) for all \( c \neq 0 \).

In Exercises 21–26, use the formal definition of the limit to prove the statement rigorously.

21. \( \lim_{x \to 4} \sqrt{x} = 2 \)
22. \( \lim_{x \to 1} (3x^2 + x) = 4 \)
23. \( \lim_{x \to 1} x^3 = 1 \)
24. \( \lim_{x \to 0} (x^2 + x^3) = 0 \)
25. \( \lim_{x \to 2} x^2 = 4 \)
26. \( \lim_{x \to 0} x \sin \frac{1}{x} = 0 \)

27. Let \( f(x) = \frac{x}{|x|} \). Prove rigorously that \( \lim_{x \to 0} f(x) \) does not exist. \( \text{Hint: Show that for any} \ L, \text{there always exists some} \ x \text{such that} \ |x| < \delta \text{but} \ |f(x) - L| \geq \frac{1}{2}, \text{no matter how small} \ \delta \text{is taken.} \)

28. Prove rigorously that \( \lim_{x \to 0} |x| = 0 \).

29. Let \( f(x) = \min(x, x^2) \), where \( \min(a, b) \) is the minimum of \( a \) and \( b \). Prove rigorously that \( \lim_{x \to 0} f(x) = 1 \).
30. Prove rigorously that \( \lim_{x \to a} \frac{1}{x} \) does not exist.

31. Use the identity
\[
\sin x + \sin y = 2 \sin \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right)
\]
to prove that
\[
\frac{\sin(a + h) - \sin a}{h} = \frac{\sin\left(\frac{h}{2}\right)}{h/2} \cos\left(a + \frac{h}{2}\right)
\]
Then use the inequality \( \left| \frac{\sin x}{x} \right| \leq 1 \) for \( x \neq 0 \) to show that
\[
|\sin(a + h) - \sin a| < |h| \text{ for all } a. \text{ Finally, prove rigorously that } \lim_{x \to a} \sin x = \sin a.
\]

32. **Uniqueness of the Limit** Prove that a function converges to at most one limiting value. In other words, use the limit definition to prove that if \( \lim_{x \to a} f(x) = L_1 \) and \( \lim_{x \to a} f(x) = L_2 \), then \( L_1 = L_2 \).

In Exercises 33–35, prove the statement using the formal limit definition.

33. The Constant Multiple Law (Theorem 1, part (ii) in Section 2.3)

34. The Squeeze Theorem (Theorem 1 in Section 2.6)

35. The Product Law (Theorem 1, part (iii) in Section 2.3). *Hint:* Use the identity.
\[
f(x)g(x) - LM = (f(x) - L)g(x) + L(g(x) - M)
\]

36. Let \( f(x) = 1 \) if \( x \) is rational and \( f(x) = 0 \) if \( x \) is irrational. Prove that \( \lim_{x \to a} f(x) \) does not exist for any \( c \). *Hint:* There exist rational and irrational numbers arbitrarily close to any \( c \).

37. Here is a function with strange continuity properties:
\[
f(x) = \begin{cases} 
1 & \text{if } x \text{ is the rational number } p/q \text{ in lowest terms} \\
0 & \text{if } x \text{ is an irrational number}
\end{cases}
\]

### CHAPTER REVIEW EXERCISES

1. The position of a particle at time \( t \) is \( s(t) = \sqrt{t^2 + 1} \). Compute its average velocity over \([2, 5]\) and estimate its instantaneous velocity at \( t = 2 \).

2. A rock dropped from a state of rest at time \( t = 0 \) on the planet Ginnormon travels a distance \( s(t) = 15.2t^2 \) in \( t \) seconds. Estimate the instantaneous velocity at \( t = 5 \).

3. For \( f(x) = \sqrt{x} \), compute the slopes of the secant lines from 16 to each of 16 ± 0.01, 16 ± 0.001, 16 ± 0.0001 and use those values to estimate the slope of the tangent line at \( x = 16 \).

4. Show that the slope of the secant line for \( f(x) = x^2 - 2x \) over \([5, x]\) is equal to \( x^2 + 5x + 23 \). Use this to estimate the slope of the tangent line at \( x = 5 \).

In Exercises 5–10, estimate the limit numerically to two decimal places or state that the limit does not exist.

5. \( \lim_{x \to 0} \frac{1 - \cos^5(x)}{x^2} \)

6. \( \lim_{x \to 1} x^{1/(x-1)} \)

7. \( \lim_{x \to 2} x^2 - 4 \)

8. \( \lim_{x \to 2} \frac{x - 2}{x^2 - 4} \)

9. \( \lim_{x \to 1} \frac{7}{x^2} - \frac{3}{1-x^3} \)

10. \( \lim_{x \to 2} \frac{3x^2 - 9}{x^2 - 2x + 5} \)

11. \( \lim_{x \to 4} (3 + x^{1/2}) \)

12. \( \lim_{x \to 1} \frac{5 - x^2}{4x + 7} \)

13. \( \lim_{x \to -2} \frac{4}{x^3} \)

14. \( \lim_{x \to 1} \frac{3x^2 + 4x + 1}{x + 1} \)

15. \( \lim_{x \to -9} \sqrt[3]{x} - 3 \)

16. \( \lim_{x \to 3} \sqrt[3]{x + 1} - 2 \)

17. \( \lim_{x \to 1} \frac{x^3 - x}{x - 1} \)

18. \( \lim_{h \to 0} \frac{2(a + h)^2 - 2a^2}{h} \)

19. \( \lim_{x \to 0} \frac{1}{x^2 + 1} \)

20. \( \lim_{x \to 0} \frac{1 - \sqrt{x^2 + 1}}{x^3} \)

21. \( \lim_{y \to \frac{1}{4}} \frac{3y^2 + 5y - 2}{6y^2 - 5y + 1} \)

22. \( \lim_{a \to b} \frac{a^2 - 3ab + 2b^2}{a - b} \)

23. \( \lim_{a \to b} \frac{\sin 5b}{\theta} \)

24. \( \lim_{x \to 1} \frac{1}{x^4} \)

25. \( \lim_{x \to 1} \frac{1}{x} \)

26. \( \lim_{\theta \to \frac{1}{4}} \sec \theta \)

In Exercises 11–50, evaluate the limit if it exists. If not, determine whether the one-sided limits exist. For limits that don't exist indicate whether they can be expressed as \( -\infty \) or \( +\infty \).
29. \( \lim_{x \to -3} \frac{z + 3}{z^2 + 4z + 3} \)
30. \( \lim_{x \to 0} \frac{x^3 - ax^2 + ax - 1}{x - 1} \)
31. \( \lim_{x \to b} \frac{x^3 - b^3}{x - b} \)
32. \( \lim_{x \to 0} \frac{\sin 4x}{\sin 3x} \)
33. \( \lim_{x \to 0} \left( \frac{1}{3x} - \frac{1}{x(x + 3)} \right) \)
34. \( \lim_{x \to 1} \frac{\tan(x^2)}{x} \)
35. \( \lim_{x \to 0^+} \frac{|x|}{x} \)
36. \( \lim_{y \to 0^+} \left( \frac{\pi}{y} \right)^{1/2} \)
37. \( \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} \)
38. \( \lim_{x \to 0} \frac{1}{x^2} \)
39. \( \lim_{x \to -2} \frac{x^3 - 2x}{x^2 - 2} \)
40. \( \lim_{x \to 0} \frac{\sin^2 t}{t^2} \)
41. \( \lim_{t \to 0} \frac{\sqrt{2} \cos t - 1}{t} \)
42. \( \lim_{x \to 0} \frac{\tan x}{x} \)
43. \( \lim_{x \to 1^+} \frac{\frac{1}{\sqrt{x - 1}}}{\frac{1}{\sqrt{x^2 + 1}}} \)
44. \( \lim_{x \to 0} \frac{\pi}{y} \)
45. \( \lim_{x \to 0} \frac{\cos x}{x} \)
46. \( \lim_{x \to 0} \frac{\sin x}{x} \)
47. \( \lim_{x \to 0} \frac{\sqrt{x} \cos x}{x} \)
48. \( \lim_{x \to 0} \frac{x^3 + 2x - 24}{x^3 + 25} \)
49. \( \lim_{x \to 0} \frac{\ln x}{x} \)
50. \( \lim_{x \to 0} \frac{\ln x}{x} \)

51. Find the left- and right-hand limits of the function \( f \) in Figure 1 at \( x = 0, 2, 4 \). State whether \( f \) is left- or right-continuous (or both) at these points.

\[ \text{FIGURE 1} \]

52. Sketch the graph of a function \( f \) such that
   (a) \( \lim_{x \to 3} f(x) = 1 \), \( \lim_{x \to 4} f(x) = 3 \)
   (b) \( \lim_{x \to 3} f(x) \) exists but does not equal \( f(4) \)

53. Graph \( h \) and describe the discontinuity:
   \( h(x) = \begin{cases} 
   2x & \text{for } x \leq 0 \\
   x^{-1/2} & \text{for } x > 0 
   \end{cases} \)
   Is \( h \) left- or right-continuous?

54. Sketch the graph of a function \( g \) such that
   \( \lim_{x \to -3} g(x) = \infty \), \( \lim_{x \to -3} g(x) = -\infty \), \( \lim_{x \to 4} g(x) = \infty \)

55. Find the points of discontinuity of
   \[ g(x) = \begin{cases} 
   \cos \left( \frac{\pi x}{2} \right) & \text{for } |x| < 1 \\
   |x - 1| & \text{for } |x| \geq 1 
   \end{cases} \]
   Determine the type of discontinuity and whether \( g \) is left- or right-continuous.

56. Find a constant \( b \) such that \( h \) is continuous at \( x = 2 \), where
   \[ h(x) = \begin{cases} 
   x + 1 & \text{for } |x| < 2 \\
   b - x^2 & \text{for } |x| \geq 2 
   \end{cases} \]
   With this choice of \( b \), find all points of discontinuity.

In Exercises 57–64, find the horizontal asymptotes of the function by computing the limits at infinity.

57. \( f(x) = \frac{9x^2 - 4}{2x^2 - x} \)
58. \( f(x) = \frac{x^2 - 3x^4}{x - 1} \)
59. \( f(u) = \frac{8u^3 - 3}{\sqrt[6]{u^4 + 6}} \)
60. \( f(u) = \frac{2u^2 - 1}{\sqrt[6]{u^4 + 6}} \)
61. \( f(x) = \frac{3x^2 + 9x^3}{7x^4/5 - 4x - 1/3} \)
62. \( f(t) = \frac{t^{1/3} - t^{-1/3}}{(t - 1)^{1/3}} \)
63. \( f(t) = \frac{17}{1 + 2t} \)
64. \( g(x) = \frac{6}{1 - 3x^2} \)

65. Calculate (a)-(d), assuming that
   \[ \lim_{x \to 3} f(x) = 6, \quad \lim_{x \to 3} g(x) = 4 \]
   (a) \( \lim_{x \to 3} (f(x) - 2g(x)) \)
   (b) \( \lim_{x \to 3} x^2 f(x) \)
   (c) \( \lim_{x \to 3} \frac{f(x)}{x - 3} g(x) + x \)
   (d) \( \lim_{x \to 3} (2g(x)^3 - g(x)^{3/2}) \)

66. Assume that the following limits exist:
   \[ A = \lim_{x \to a} f(x), \quad B = \lim_{x \to a} g(x), \quad L = \lim_{x \to a} f(x) g(x) \]
   Prove that if \( L = 1 \), then \( A = B \). Hint: You cannot use the Quotient Law if \( B = 0 \), so apply the Product Law to \( L \) and \( B \) instead.

67. In the notation of Exercise 66, give an example where \( L \) exists but neither \( A \) nor \( B \) exists.

68. True or false?
   (a) If \( \lim_{x \to 3} f(x) \) exists, then \( \lim_{x \to 3} f(x) = f(3) \).
   (b) If \( \lim_{x \to 0} f(x) = 1 \), then \( f(0) = 1 \).
   (c) If \( \lim_{x \to -7} f(x) = 8 \), then \( \lim_{x \to -7} \frac{1}{f(x)} = \frac{1}{8} \).
   (d) If \( \lim_{x \to 4} f(x) = 4 \) and \( \lim_{x \to 4} f(x) = 8 \), then \( \lim_{x \to 4} f(x) = 6 \).
   (e) If \( \lim_{x \to 0} f(x) = 1 \), then \( \lim_{x \to 0} f(x) = 0 \).
   (f) If \( \lim_{x \to 5} f(x) = 2 \), then \( \lim_{x \to 5} f(x)^3 = 8 \).
69. Let \( f(x) = \left\lfloor \frac{1}{x} \right\rfloor \), where \( \lfloor x \rfloor \) is the greatest integer function. Show that for \( x \neq 0 \),
\[
\frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}
\]
Then use the Squeeze Theorem to prove that
\[
\lim_{x \to 0} \left\lfloor \frac{1}{x} \right\rfloor = 1
\]
*Hint:* Treat the one-sided limits separately.

70. Let \( r_1 \) and \( r_2 \) be the roots of \( f(x) = ax^2 - 2x + 20 \). Observe that \( f \) "approaches" the linear function \( L(x) = -2x + 20 \) as \( a \to 0 \). Because \( r = 10 \) is the unique root of \( L \), we might expect one of the roots of \( f \) to approach 10 as \( a \to 0 \) (Figure 2). Prove that the roots can be labeled so that \( \lim_{a \to 0} r_1 = 10 \) and \( \lim_{a \to 0} r_2 = \infty \).

71. Use the IVT to prove that the curves \( y = x^2 \) and \( y = \cos x \) intersect.

72. Use the IVT to prove that \( f(x) = x^3 - \frac{x^2 + 2}{\cos x + 2} \) has a root in the interval \([0, 2]\).

73. Use the IVT to show that \( 2^{-x^2} = x \) has a solution on \((0, 1)\).

74. Use the Bisection Method to locate a solution of \( x^2 - 7 = 0 \) to two decimal places.

75. Give an example of a (discontinuous) function that does not satisfy the conclusion of the IVT on \([-1, 1]\). Then show that the function
\[
f(x) = \begin{cases} 
\sin \frac{1}{x} & x \neq 0 \\
0 & x = 0
\end{cases}
\]
satisfies the conclusion of the IVT on every interval \([-a, a]\).

76. Let \( f(x) = \frac{1}{x + 2} \).

(a) Show that if \( |x - 2| < 1 \), then \( |f(x) - \frac{1}{4}| < \frac{|x - 2|}{12} \). *Hint:* Observe that if \( |x - 2| < 1 \), then \( |4(x + 2)| > 12 \).

(b) Find \( \delta > 0 \) such that if \( |x - 2| < \delta \), then \( |f(x) - \frac{1}{4}| < 0.01 \).

(c) Prove rigorously that \( \lim_{x \to 2} f(x) = \frac{1}{4} \).

77. Plot the function \( f(x) = x^{1/3} \). Use the zoom feature to find a \( \delta > 0 \) such that if \( |x - 8| < \delta \), then \( |x^{1/3} - 2| < 0.05 \).

78. Use the fact that \( f(x) = 2^x \) is increasing to find a value of \( \delta \) such that \( |2^{x - 8}| < 0.001 \) if \( |x - 2| < \delta \). *Hint:* Find \( c_1 \) and \( c_2 \) such that \( 7.999 < f(c_1) < f(c_2) < 8.001 \).

79. Prove rigorously that \( \lim_{x \to -1} (4 + 8x) = -4 \).

80. Prove rigorously that \( \lim_{x \to 3} (x^2 - x) = 6 \).
DIFFERENTIATION

Differential calculus is the study of the derivative, and differentiation is the process of computing derivatives. What is a derivative? There are three equally important answers: A derivative is a rate of change, it is the slope of a tangent line, and (more formally) it is the limit of a difference quotient, as we will explain shortly. In this chapter, we explore all three facets of the derivative and develop the basic rules of differentiation. When you master these techniques, you will possess one of the most useful and flexible tools that mathematics has to offer.

3.1 Definition of the Derivative

We begin with two questions: What is the precise definition of a tangent line? And how can we compute its slope? To answer these questions, let’s return to the relationship between tangent and secant lines first mentioned in Section 2.1.

The secant line through distinct points \( P = (a, f(a)) \) and \( Q = (x, f(x)) \) on the graph of a function \( f \) has slope \( \Delta f/\Delta x \) [Figure 1(A)]

\[
\frac{\Delta f}{\Delta x} = \frac{f(x) - f(a)}{x - a}
\]

where

\( \Delta f = f(x) - f(a) \) and \( \Delta x = x - a \)

The expression \( \frac{f(x) - f(a)}{x - a} \) is called the difference quotient. We can think of the secant line through \( P \) and \( Q \) as a rough approximation to the tangent line at \( P \) [Figure 1(B)].

\[\text{Figure 1} \quad \text{The secant line has slope } \Delta f/\Delta x. \text{ Our goal is to compute the slope of the tangent line at } (a, f(a)).\]

\[\text{REMINDER} \quad \text{A secant line is any line through two points on a curve or graph.}\]

We can improve the secant-line approximation to the tangent line by moving point \( Q \) closer to point \( P \), equivalently by moving \( x \) closer to \( a \) [Figure 2(A–C)]. As \( Q \) approaches \( P \), the secant lines get progressively closer to the tangent line as in Figure 2(D). Therefore, we may expect the slopes of the secant lines to approach the slope of the tangent line; that is, we expect that as \( x \) approaches \( a \), the limit of the secant-line slopes is equal to the slope of the tangent line.

\[\text{Figure 2} \quad \text{The secant lines approach the tangent line as } Q \text{ approaches } P.\]
Based on this intuition, we define the derivative $f'(a)$ (which is read "$f$ prime of $a$") as the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Another way of writing the difference quotient is to use the variable $h = x - a$ (Figure 3). We have $x = a + h$ and, for $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = \frac{f(a + h) - f(a)}{h}$$

The variable $h$ approaches 0 as $x \to a$, so we can rewrite the derivative as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Each way of writing the derivative is useful. In examples and proofs in this and the upcoming sections, we will use the formula for the derivative that is most appropriate for the situation.

**DEFINITION** The Derivative  
The derivative of $f$ at a point $a$ is the limit of the difference quotients (if it exists):

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

or equivalently:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

When the limit exists, we say that $f$ is **differentiable** at $a$.

---

**REMINDER** The equation of the line through $P = (a, b)$ of slope $m$ in point-slope form:

$$y - b = m(x - a)$$

**EXAMPLE 1**  
**Equation of a Tangent Line**  
Find an equation of the tangent line to the graph of $f(x) = x^2$ at $x = 5$.

**Solution**  
First, we must compute $f'(5)$. We are free to use either Eq. (1) or Eq. (2). Using Eq. (2), we have

$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \to 5} \frac{x^2 - 25}{x - 5}$$
This limit is in the indeterminate form 0/0. We can simplify and then evaluate by substitution:

\[
f'(5) = \lim_{x \to 5} \frac{(x - 5)(x + 5)}{x - 5} = \lim_{x \to 5} (x + 5) = 10
\]

Next, we apply Eq. (3) with \(a = 5\). Because \(f(5) = 25\), an equation of the tangent line is \(y - 25 = 10(x - 5)\), or in slope-intercept form, \(y = 10x - 25\) (Figure 4).

**CONCEPTUAL INSIGHT** In the previous example we encountered an indeterminate limit in the form 0/0. In general, if \(f\) is continuous at \(a\), then the difference-quotient limits in Eq. (1) and Eq. (2) are guaranteed to be in the form 0/0.

Although we do not always indicate that we have an indeterminate form when we compute derivatives via the limit definition, it is important to realize that the approach we take in evaluating these limits is usually motivated by the fact that we are working with an indeterminate form.

In the next two examples, we perform the differentiation (the process of computing the derivative) using Eq. (1). For clarity, we break up the computations into three steps.

**EXAMPLE 2** Compute \(f'(3)\), where \(f(x) = x^2 - 8x\).

**Solution** Using Eq. (1), we write the difference quotient at \(a = 3\) as

\[
\frac{f(a + h) - f(a)}{h} = \frac{f(3 + h) - f(3)}{h} \quad (h \neq 0)
\]

**Step 1. Write out the numerator of the difference quotient.**

\[
f(3 + h) - f(3) = ((3 + h)^2 - 8(3 + h)) - (3^2 - 8(3))
\]

\[
= (9 + 6h + h^2) - (24 + 8h) - (9 - 24)
\]

\[
= h^2 - 2h
\]

**Step 2. Divide by \(h\) and simplify.**

\[
\frac{f(3 + h) - f(3)}{h} = \frac{h^2 - 2h}{h} = \frac{h(h - 2)}{h} = h - 2
\]

Cancel \(h\)

**Step 3. Compute the limit.**

\[
f'(3) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} (h - 2) = -2
\]

**EXAMPLE 3** Sketch the graph of \(f(x) = \frac{1}{x}\) and the tangent line at \(x = 2\).

(a) Based on the sketch, do you expect \(f'(2)\) to be positive or negative?

(b) Find \(f'(2)\).

**Solution** The graph and tangent line at \(x = 2\) are shown in Figure 5.

(a) We see that the tangent line has negative slope, so \(f'(2)\) must be negative.

(b) We compute \(f'(2)\) in three steps as before.

**Step 1. Write out the numerator of the difference quotient.**

\[
f(2 + h) - f(2) = \frac{1}{2 + h} - \frac{1}{2} = \frac{2}{2(2 + h)} - \frac{2 + h}{2(2 + h)} = -\frac{h}{2(2 + h)}
\]

**Step 2. Divide by \(h\) and simplify.**

\[
\frac{f(2 + h) - f(2)}{h} = \frac{1}{h} \left( -\frac{h}{2(2 + h)} \right) = -\frac{1}{2(2 + h)}
\]
Step 3. Compute the limit.

\[
f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}
\]

The graph of a linear function \( f(x) = mx + b \) (where \( m \) and \( b \) are constants) is a line of slope \( m \). The tangent line at any point coincides with the line itself (Figure 6), so we should expect that \( f'(a) = m \) for all \( a \). Let's check this by computing the derivative:

\[
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(ma + h) + b - (ma + b)}{h} = \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m
\]

If \( m = 0 \), then \( f(x) = b \) is constant and \( f'(a) = 0 \) (Figure 7). In summary:

**Theorem 1** Derivative of Linear and Constant Functions

- If \( f(x) = mx + b \) is a linear function, then \( f'(a) = m \) for all \( a \).
- If \( f(x) = b \) is a constant function, then \( f'(a) = 0 \) for all \( a \).

**Example 4** Find the derivative of \( f(x) = 9x - 5 \) at \( x = 2 \) and \( x = 5 \).

**Solution** We have \( f'(a) = 9 \) for all \( a \). Hence, \( f'(2) = f'(5) = 9 \).

**Example 5** Failure to be Differentiable

Show that the functions \( f(x) = |x| \) and \( g(x) = x^{1/3} \) are not differentiable at \( x = 0 \).

**Solution** First, note that

\[
|0 + h| - |0| = |h| = \begin{cases} 
1 & \text{if } h > 0 \\
-1 & \text{if } h < 0 
\end{cases}
\]

Since

\[
\frac{|h|}{h} = \begin{cases} 
1 & \text{if } h > 0 \\
-1 & \text{if } h < 0 
\end{cases}
\]

we have the one-sided limits

\[
\lim_{h \to 0^+} \frac{|h|}{h} = 1 \quad \text{and} \quad \lim_{h \to 0^-} \frac{|h|}{h} = -1
\]

These one-sided limits are not equal; therefore, \( \lim_{h \to 0} \frac{|h|}{h} \) does not exist. Thus, \( f \) is not differentiable at \( x = 0 \).

Figure 8 reveals the problem we have here. To the left of \( x = 0 \) the secant lines all have slope \(-1\), and to the right they all have slope \( 1 \). The one-sided limits on either side of \( 0 \) equal the corresponding secant-line slopes and therefore are not equal.

Now consider \( g(x) = x^{1/3} \). The limit defining \( g'(0) \) is infinite:

\[
\lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{h^{1/3} - 0}{h} = \lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}} = \infty
\]

Therefore, \( g'(0) \) does not exist, and it follows that \( g \) is not differentiable at \( x = 0 \).

In this case, the problem is that the secant lines on either side of \( x = 0 \) become infinitely steep as \( h \) approaches \( 0 \). The tangent line is therefore vertical, having an undefined slope (see Figure 9).
Estimating the Derivative

Approximations to the derivative are useful in situations where we cannot evaluate $f'(a)$ exactly. Since the derivative is the limit of difference quotients, the difference quotient should give a good numerical approximation when $h$ is sufficiently small:

$$f'(a) \approx \frac{f(a + h) - f(a)}{h} \quad \text{if } h \text{ is small}$$

We refer to this estimate as the **difference quotient approximation**. Graphically, this says that for small $h$, the slope of the secant line is nearly equal to the slope of the tangent line (Figure 10).

**EXAMPLE 6** Estimate the derivative of $f(x) = \sin x$ at $x = \frac{\pi}{6}$.

**Solution** We calculate the difference quotient for several small values of $h$:

$$\frac{\sin\left(\frac{\pi}{6} + h\right) - \sin\frac{\pi}{6}}{h} = \frac{\sin\left(\frac{\pi}{6} + h\right) - 0.5}{h}$$

This difference quotient represents the slope of a secant line through the graph at $\frac{\pi}{6}$. The resulting secant-line slopes are shown in Table 1. Note that Figure 11 indicates that the slope of the tangent line at $\frac{\pi}{6}$ lies between the slopes of the secant lines for $h > 0$ and those for $h < 0$. Thus, we can infer that

$$0.8660229 < f'\left(\frac{\pi}{6}\right) < 0.8660279$$

It follows that $f'\left(\frac{\pi}{6}\right) \approx 0.8660$, accurate to four decimal places.

<table>
<thead>
<tr>
<th>$h &gt; 0$</th>
<th>$\frac{\sin\left(\frac{\pi}{6} + h\right) - 0.5}{h}$</th>
<th>$h &lt; 0$</th>
<th>$\frac{\sin\left(\frac{\pi}{6} + h\right) - 0.5}{h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.863511</td>
<td>-0.01</td>
<td>0.868511</td>
</tr>
<tr>
<td>0.001</td>
<td>0.865775</td>
<td>-0.001</td>
<td>0.866275</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.866000</td>
<td>-0.0001</td>
<td>0.866050</td>
</tr>
<tr>
<td>0.00001</td>
<td>0.8660229</td>
<td>-0.00001</td>
<td>0.8660279</td>
</tr>
</tbody>
</table>

In Section 3.6 we develop a derivative formula for $f(x) = \sin x$ and will see that $f'\left(\frac{\pi}{6}\right) = 0.5\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \approx 0.8660254$.

In the previous example, because we had a specific formula for $f(x)$, we were able to choose $h$ as small as we like to obtain an accurate approximation to $f'(x)$. In applied situations, as in the following example, we might have only a data set of values available for approximating a derivative. We can use the difference quotient approximation, but accuracy is limited because we cannot choose $h$ to be arbitrarily small.

**EXAMPLE 7** The Internet of Things

The Internet of things (IoT) refers to the collection of devices that are connected to the Internet, such as computers, cell phones, smart watches, home security systems, and so on. Table 2 gives estimates of the size of the IoT for the years 2012–2019. Assume that $I(t)$ represents the size of the IoT (in billions of devices) $t$ years after January 1, 2012, and that the data in the table represent the size of the IoT at the start of the indicated year. Approximate $I'(3)$ and $I'(6)$.
<table>
<thead>
<tr>
<th>$t$ (years after 2012)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_t$ (billions of devices)</td>
<td>8.7</td>
<td>11.2</td>
<td>14.4</td>
<td>18.2</td>
<td>22.9</td>
<td>28.4</td>
<td>34.8</td>
<td>42.1</td>
</tr>
</tbody>
</table>

**Solution** We use the difference quotient approximation with $h = 1$, the time interval for the data set. We have $I'(3) \approx \frac{I(4) - I(3)}{1} = 4.7$ and $I'(6) \approx \frac{I(7) - I(6)}{1} = 7.3$

The derivative of a function defines the slope of the function’s graph. As with the slope of a line, the derivative is a rate of change of the dependent variable with respect to the independent variable. In applications, when units are associated with each of the variables, the derivative has units

$$\text{units of derivative} = \frac{\text{units of dependent variable}}{\text{units of independent variable}}$$

When we include the units in the previous example, the derivatives are the rates of change $I'(3) \approx 4.7$ billion devices per year and $I'(6) \approx 7.3$ billion devices per year. These rates of change represent the rate of growth of the Internet of things at the start of 2015 and 2018, respectively.

**CONCEPTUAL INSIGHT The Significance of Limits in the Definition of the Derivative**

With the introduction of the derivative in this section, we have arrived at an important point in the development of the calculus concepts in this text. We summarize how the important ideas of slopes of lines and limits brought us here.

The slope of a line can be computed if the coordinates of two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on the line are known:

$$\text{slope of line} = \frac{y_2 - y_1}{x_2 - x_1}$$

This formula cannot be applied to the tangent line to the graph of a function $f$ at $x = a$ because we know only one point that it passes through, $P = (a, f(a))$. Limits provide an ingenious way around this obstacle. We choose a point $Q = (a + h, f(a + h))$ on the graph near $P$ and form the secant line. The slope of this secant line is just an approximation to the slope of the tangent line:

$$\text{slope of secant line} = \frac{f(a + h) - f(a)}{h} \approx \text{slope of tangent line}$$

This approximation improves as $h$ is made small. By taking the limit as $h \to 0$, we convert our approximations into the exact slope for the tangent line, and that slope is $f'(a)$, the derivative of $f$ at $a$.

**3.1 SUMMARY**

- **The difference quotient** is the slope of the secant line through the points $P$ and $Q$ on the graph of $f$ and equals

$$\frac{f(a + h) - f(a)}{h} \quad \text{with} \quad P = (a, f(a)) \quad \text{and} \quad Q = (a + h, f(a + h))$$

- **The derivative** $f'(a)$ is defined by the following equivalent limits:

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

If the limit exists, we say that $f$ is **differentiable** at $x = a$. 

Scanned with CamScanner
3.1 EXERCISES

Preliminary Questions

1. Which of the lines in Figure 12 are tangent to the curve?

2. What are the two ways of writing the difference quotient?

3. Find a and h such that \( \frac{f(a + h) - f(a)}{h} \) is equal to the slope of the secant line between \((3, f(3))\) and \((5, f(5))\).

Exercises

1. Let \( f(x) = 5x^2 \). Show that \( f(3 + h) = 5h^2 + 30h + 45 \). Then show that

\[
\frac{f(3 + h) - f(3)}{h} = 5h + 30
\]

and compute \( f'(3) \) by taking the limit as \( h \to 0 \).

2. Let \( f(x) = 2x^2 - 3x - 5 \). Show that the secant line through \((2, f(2))\) and \((2 + h, f(2 + h))\) has slope \(2a + 5\). Then use this formula to compute:
   (a) The slope of the secant line through \((2, f(2))\) and \((3, f(3))\)
   (b) The slope of the tangent line at \(x = 2\) (by taking a limit)

In Exercises 3–8, compute \( f'(a) \) in two ways using Eq. (1) and Eq. (2).

3. \( f(x) = x^2 + 9x \), \( a = 0 \)
4. \( f(x) = x^2 + 9x \), \( a = 2 \)
5. \( f(x) = 3x^2 + 4x + 2 \), \( a = -1 \)
6. \( f(x) = x^3 \), \( a = 2 \)
7. \( f(x) = x^3 + 2x \), \( a = 1 \)
8. \( f(x) = \frac{1}{x} \), \( a = 2 \)

In Exercises 9–12, refer to Figure 13.

9. Find the slope of the secant line through \((2, f(2))\) and \((2.5, f(2.5))\). Is it greater than or less than \( f'(2) \)? Explain.

4. Which derivative is approximated by \( \frac{\tan(\frac{\pi}{4} + 0.0001) - 1}{0.0001} \)?

5. What do the following quantities represent in terms of the graph of \( f(x) = \sin x \)?
   (a) \( \sin 1.3 - \sin 0.9 \)
   (b) \( \frac{\sin 1.3 - \sin 0.9}{0.4} \)
   (c) \( f'(0.9) \)

6. Choose (a) or (b). The derivative at a point is zero if the tangent line at that point is (a) horizontal (b) vertical.

7. Choose (a) or (b). The derivative at a point does not exist if the tangent line at that point is (a) horizontal (b) vertical.

10. Estimate \( \frac{f(2 + h) - f(2)}{h} \) for \( h = -0.5 \). What does this quantity represent? Is it greater than or less than \( f'(2) \)? Explain.

11. Estimate \( f'(1) \) and \( f'(2) \).

12. Find a value of \( h \) for which \( \frac{f(2 + h) - f(2)}{h} = 0 \).

In Exercises 13–16, refer to Figure 14.

13. Determine \( f'(a) \) for \( a = 1, 2, 4, 7 \).

14. For which values of \( x \) is \( f'(x) < 0 \)?
15. Which is larger, \( f'(5.5) \) or \( f'(6.5) \)?

16. Show that \( f'(3) \) does not exist.

![Graph of \( f \)](image)

17. \( f(x) = 7x - 9 \)
18. \( f(x) = 12 \)
19. \( g(x) = 8 - 3t \)
20. \( k(t) = 14t + 12 \)
21. Find an equation of the tangent line at \( x = 3 \), assuming that \( f(3) = 5 \) and \( f'(3) = 2 \).
22. Find \( f(3) \) and \( f'(3) \), assuming that the tangent line to \( y = f(x) \) at \( a = 3 \) has equation \( y = 5x + 2 \).
23. Describe the tangent line at an arbitrary point on the graph of \( y = 2x + 8 \).
24. Suppose that \( f(2 + h) - f(2) = 3h^2 + 5h \). Calculate:
   (a) The slope of the secant line through \((2, f(2))\) and \((6, f(6))\)
   (b) \( f'(2) \)
25. Let \( f(x) = \frac{1}{x} \). Does \( f(-2 + h) = \frac{1}{-2 + h} \) or \( \frac{1}{-2 + \frac{1}{h}} \)? Compute the difference quotient at \( a = -2 \) with \( h = 0.5 \).
26. Let \( f(x) = \sqrt{x} \). Does \( f(5 + h) = \sqrt{5 + h} \) or \( \sqrt{5} + \sqrt{h} \)? Compute the difference quotient at \( a = 5 \) with \( h = 1 \).
27. Let \( f(x) = \frac{1}{\sqrt{x}} \). Compute \( f'(5) \) by showing that
   \[
   \frac{f(5 + h) - f(5)}{h} = \frac{1}{\sqrt{5 + h} \sqrt{5 + h + h}}
   \]
28. Find an equation of the tangent line to the graph of \( f(x) = \frac{1}{\sqrt{x}} \) at \( x = 9 \).

In Exercises 29–46, use the limit definition to compute \( f'(a) \) and find an equation of the tangent line.
29. \( f(x) = 2x^2 + 10x \), \( a = 3 \)
30. \( f(x) = 4 - x^2 \), \( a = -1 \)
31. \( f(t) = t - 2t^2 \), \( a = 3 \)
32. \( f(x) = 8x^3 \), \( a = 1 \)
33. \( f(x) = x^3 + x \), \( a = 0 \)
34. \( f(t) = 2t^3 + 4t \), \( a = 4 \)
35. \( f(x) = x^{11} \), \( a = 8 \)
36. \( f(x) = x + x^{-1} \), \( a = 4 \)
37. \( f(x) = \frac{1}{x + 3} \), \( a = -2 \)
38. \( f(t) = \frac{2}{1 - t} \), \( a = -1 \)
39. \( f(x) = \sqrt{x + 4} \), \( a = 1 \)
40. \( f(t) = \sqrt{3t + 5} \), \( a = -1 \)
41. \( f(x) = \frac{1}{\sqrt{x}} \), \( a = 4 \)
42. \( f(x) = \frac{1}{\sqrt{2x + 1}} \), \( a = 4 \)
43. \( f(t) = \sqrt{t^2 + 1} \), \( a = 3 \)
44. \( f(x) = x^{-2} \), \( a = -1 \)
45. \( f(x) = \frac{1}{x^2 + 1} \), \( a = 0 \)
46. \( f(t) = t^{-3} \), \( a = 1 \)
47. Show that \( f \) is not differentiable at \( x = 1 \) and has a corner in its graph there.
   \[
   f(x) = \begin{cases} 1 & x \leq 1 \\ x^2 & x > 1 \end{cases}
   \]
48. Show that \( f \) is not differentiable at \( x = 0 \) and has a corner in its graph there.
   \[
   f(x) = \begin{cases} x^3 & x \leq 0 \\ x & x > 0 \end{cases}
   \]

In Exercises 49–51, sketch a graph of \( f \) and identify the points \( c \) such that \( f'(c) \) does not exist. In which cases is there a corner at \( c \)?
49. \( f(x) = [x + 3] \)
50. \( f(x) = x^{2/5} \)
51. \( f(x) = [x^2 - 4] \)

52. Figure 15(A) shows the graph of \( f(x) = \sqrt{x} \). The close-up in Figure 15(B) shows that the graph is nearly a straight line near \( x = 16 \). Estimate the slope of this line and take it as an estimate for \( f'(16) \). Then compute \( f'(16) \) with the limit definition and compare with your estimate.

![Close-up view near (16, 4)](image)

53. (GU) Let \( f(x) = \frac{4}{1 + 2x} \). Plot \( f \) over \([-2, 2]\). Then zoom in near \( x = 0 \) until the graph appears straight, and estimate the slope \( f'(0) \).
54. (GU) Let \( f(x) = \cot x \). Estimate \( f'(\frac{\pi}{2}) \) graphically by zooming in on a plot of \( f \) near \( x = \frac{\pi}{2} \).
55. Determine the intervals along the \( x \)-axis on which the derivative in Figure 16 is positive.

![Graph of \( f'(x) \)](image)
56. Sketch the graph of \( f(x) = \sin x \) on \([0, \pi]\) and guess the value of \( f'(\frac{\pi}{2}) \). Then calculate the difference quotient at \( x = \frac{\pi}{4} \) for two small positive and negative values of \( h \). Are these calculations consistent with your guess?

In Exercises 57–62, each limit represents a derivative \( f'(a) \). Find \( f(x) \) and \( a \).

57. \( \lim_{h \to 0} \frac{(5 + h)^3 - 125}{h} \)

58. \( \lim_{x \to 3} \frac{x^3 - 125}{x - 5} \)

59. \( \lim_{h \to 0} \frac{\sin (\frac{\pi}{2} + h) - 0.5}{h} \)

60. \( \lim_{x \to 4} \frac{x^4 - 16}{x - 4} \)

61. \( \lim_{h \to 0} \frac{5h + 2h^2 - 25}{h} \)

62. \( \lim_{h \to 0} \frac{5h - 1}{h} \)

63. Apply the method of Example 6 to \( f(x) = \sin x \) to determine \( f'(\frac{\pi}{2}) \) accurately to four decimal places.

64. Apply the method of Example 6 to \( f(x) = \cos x \) to determine \( f'(\frac{\pi}{2}) \) accurately to four decimal places. Use a graph of \( f \) to explain how the method works in this case.

65. For each graph in Figure 17, determine whether \( f'(1) \) is larger or smaller than the slope of the secant line between \( x = 1 \) and \( x = 1 + h \) for \( h > 0 \). Explain.

66. Refer to the graph of \( f(x) = 2^x \) in Figure 18.

(a) Explain graphically why, for \( h > 0 \),
\[
\frac{f(-h) - f(0)}{-h} \leq f'(0) \leq \frac{f(h) - f(0)}{h}
\]

(b) Use (a) to show that \( 0.69314 \leq f'(0) \leq 0.69315 \).

(c) Similarly, compute \( f'(x) \) to four decimal places for \( x = 1, 2, 3, 4 \).

(d) Now compute the ratios \( f'(x)/f'(0) \) for \( x = 1, 2, 3, 4 \). Can you guess an approximate formula for \( f'(x) \)?

67. Sketch the graph of \( f(x) = x^{3/2} \) on \([0, 6]\).

(a) Use the sketch to justify the inequalities for \( h > 0 \):
\[
\frac{f(4) - f(4 - h)}{h} \leq f'(4) \leq \frac{f(4 + h) - f(4)}{h}
\]

(b) Use (a) to compute \( f'(4) \) to four decimal places.

(c) Use a graphing utility to plot \( y = f(x) \) and the tangent line at \( x = 4 \), utilizing your estimate for \( f'(4) \).

68. Verify that \( P = (1, \frac{1}{2}) \) lies on the graphs of both
\[
f(x) = \frac{1}{1 + x^2} \text{ and } L(x) = \frac{1}{2} + m(x - 1)
\]
for every slope \( m \). Plot \( y = f(x) \) and \( y = L(x) \) on the same axes for several values of \( m \) until you find a value of \( m \) for which \( y = L(x) \) appears tangent to the graph of \( f \). What is your estimate for \( f'(1) \)?

69. Use a plot of \( f(x) = x^2 \) to estimate the value \( c \) such that \( f'(c) = 0 \). Find \( c \) to sufficient accuracy so that
\[
\left| \frac{f(c + h) - f(c)}{h} \right| \leq 0.006 \quad \text{for} \quad h = \pm 0.001
\]

70. Plot \( f(x) = x^2 \) and \( y = 2x + a \) on the same set of axes for several values of \( a \) until the line becomes tangent to the graph. Then estimate the value \( c \) such that \( f'(c) = 2 \).

The vapor pressure of water at temperature \( T \) (in kelvins) is the atmospheric pressure \( P \) at which no net evaporation takes place. In Exercises 71–72, use the following table to estimate the indicated derivatives using the difference quotient approximation.

<table>
<thead>
<tr>
<th>( T ) (K)</th>
<th>293</th>
<th>303</th>
<th>313</th>
<th>323</th>
<th>333</th>
<th>343</th>
<th>353</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P ) (atm)</td>
<td>0.0278</td>
<td>0.0482</td>
<td>0.0680</td>
<td>0.1311</td>
<td>0.2067</td>
<td>0.3173</td>
<td>0.4754</td>
</tr>
</tbody>
</table>

71. Estimate \( P'(T) \) for \( T = 293, 313, 333 \). (Include proper units on the derivative.)

72. Estimate \( P'(T) \) for \( T = 303, 323, 343 \). (Include proper units on the derivative.)

Let \( P(t) \) represent the U.S. ethanol production as shown in Figure 19. In Exercises 73–74, estimate the indicated derivatives using the difference quotient approximation.

<table>
<thead>
<tr>
<th>Year</th>
<th>Production (billions of gallons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>1.30</td>
</tr>
<tr>
<td>1998</td>
<td>1.40</td>
</tr>
<tr>
<td>1999</td>
<td>1.47</td>
</tr>
<tr>
<td>2000</td>
<td>1.53</td>
</tr>
<tr>
<td>2001</td>
<td>1.77</td>
</tr>
<tr>
<td>2002</td>
<td>2.12</td>
</tr>
<tr>
<td>2003</td>
<td>2.81</td>
</tr>
<tr>
<td>2004</td>
<td>4.00</td>
</tr>
<tr>
<td>2005</td>
<td>6.69</td>
</tr>
<tr>
<td>2006</td>
<td>9.31</td>
</tr>
<tr>
<td>2007</td>
<td>10.94</td>
</tr>
<tr>
<td>2008</td>
<td>13.30</td>
</tr>
<tr>
<td>2009</td>
<td>13.93</td>
</tr>
<tr>
<td>2010</td>
<td>13.30</td>
</tr>
</tbody>
</table>

**FIGURE 17**

**FIGURE 18** Graph of \( f(x) = 2^x \).

**FIGURE 19** U.S. ethanol production.


In the remaining exercises, SDQ refers to the symmetric difference quotient derivative approximation that is based on the slope of the secant line between $(x - h, f(x - h))$ and $(x + h, f(x + h))$:

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h}$$

75. With $P(T)$ as in Exercises 71 and 72, estimate $P'(T)$ for $T = 303, 313, 333, 343$, now using the SDQ.

76. With $P(t)$ as in Exercises 73 and 74, estimate $P'(t)$ for $t = 1999, 2001, 2005, 2011$, now using the SDQ.

In Exercises 77–78, traffic speed $S$ along Katyman Road (in kilometers per hour) varies as a function of traffic density $q$ (number of cars per kilometer of road) according to the data:

<table>
<thead>
<tr>
<th>$q$ (density)</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$ (speed)</td>
<td>72.5</td>
<td>67.5</td>
<td>63.5</td>
<td>60</td>
<td>56</td>
</tr>
</tbody>
</table>

77. Estimate $S'(80)$ using the SDQ. (Include proper units on the derivative.)

78. Explain why $V = qS$, called traffic volume, is equal to the number of cars passing a point per hour. Use the data and the SDQ to estimate $V'(80)$. (Include proper units on the derivative.)

Exercises 79–81: The current (in amperes) at time $t$ (in seconds) flowing in the circuit in Figure 20 is given by Kirchhoff’s Law:

$$i(t) = C\frac{dv(t)}{dt} + R\frac{v(t)}{dv(t)}$$

where $v(t)$ is the voltage (in volts, $V$), $C$ the capacitance (in farads, $F$), and $R$ the resistance (in ohms, $\Omega$).

79. Calculate the current at $t = 3$ if

$$v(t) = 0.5t + 4 \text{ V}$$

where $C = 0.01 \text{ F}$ and $R = 100 \text{ } \Omega$.

80. Use the following data and the SDQ to estimate $v'(10)$. Then estimate $i'(10)$, assuming $C = 0.03$ and $R = 1000$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>9.8</th>
<th>9.9</th>
<th>10</th>
<th>10.1</th>
<th>10.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$</td>
<td>256.52</td>
<td>257.32</td>
<td>258.11</td>
<td>258.9</td>
<td>259.69</td>
</tr>
</tbody>
</table>

81. Assume that $R = 200 \text{ } $ but $C$ is unknown. Use the following data and the SDQ to estimate $v'(4)$, and deduce an approximate value for the capacitance $C$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>3.8</th>
<th>3.9</th>
<th>4</th>
<th>4.1</th>
<th>4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(t)$</td>
<td>388.8</td>
<td>404.2</td>
<td>420</td>
<td>436.2</td>
<td>452.8</td>
</tr>
<tr>
<td>$i(t)$</td>
<td>32.34</td>
<td>33.22</td>
<td>34.1</td>
<td>34.98</td>
<td>35.86</td>
</tr>
</tbody>
</table>

**Further Insights and Challenges**

82. The SDQ usually approximates the derivative much more closely than does the ordinary difference quotient. Let $f(x) = \sqrt{x}$ and $a = 3$. Compute the SDQ with $h = 0.01$ and the ordinary difference quotients with $h = \pm 0.01$. Compare with the actual value, which is $f'(3) = 1/(2\sqrt{3})$.

83. (a) Show that the symmetric difference quotient $\frac{f(x+h) - f(x-h)}{2h}$ is the slope of the secant line to the graph of $f$ from $x - h$ to $x + h$. (Include an illustration.)

(b) Prove that the symmetric difference quotient is the average of the slopes of the secant lines from $x$ to $x + h$ and from $x - h$ to $x$.

84. Which of the two functions in Figure 21 satisfies the inequality

$$\frac{f(a+h) - f(a-h)}{2h} \leq \frac{f(a+h) - f(a)}{h}$$

for $h > 0$? Explain in terms of secant lines.

85. Show that if $f$ is a quadratic polynomial, then the SDQ at $x = a$ (for any $h \neq 0$) is equal to $f'(a)$. Explain the graphical meaning of this result.

86. Let $f(x) = x^2$. Compute $f'(1)$ by taking the limit of the SDQs (with $a = 1$) as $h \to 0$.

3.2 The Derivative as a Function

In the previous section, we computed the derivative $f'(a)$ for specific values of $a$. It is also useful to view the derivative as a function $f'$ whose values $f'(x)$ are defined by the limit definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If $y = f(x)$, we also write $y'$ or $y'(x)$ for $f'(x)$. 
The domain of \( f' \) consists of all values of \( x \) in the domain of \( f \) for which the limit in Eq. (1) exists. We say that \( f \) is differentiable on \((a, b)\) if \( f'(x) \) exists for all \( x \) in \((a, b)\). When \( f'(x) \) exists for all \( x \) in the interval or intervals on which \( f(x) \) is defined, we say simply that \( f \) is differentiable.

**Example 1** Prove that \( f(x) = x^3 - 12x \) is differentiable. Compute \( f'(x) \) and find \( f'(-3) \), \( f'(0) \), \( f'(2) \), and \( f'(3) \).

**Solution** We compute \( f'(x) \) in three steps as in the previous section.

**Step 1.** Write out the numerator of the difference quotient.

\[
\frac{f(x + h) - f(x)}{h} = \frac{(x + h)^3 - 12(x + h) - (x^3 - 12x)}{h}
\]

\[
= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 12x - 12h - x^3 + 12x}{h}
\]

\[
= \frac{3x^2h + 3xh^2 + h^3 - 12h}{h}
\]

\[
= h(3x^2 + 3xh + h^2 - 12) \quad \text{(factor out } h)\]

**Step 2.** Divide by \( h \) and simplify.

\[
\frac{f(x + h) - f(x)}{h} = \frac{h(3x^2 + 3xh + h^2 - 12)}{h} = 3x^2 + 3xh + h^2 - 12 \quad (h \neq 0)
\]

**Step 3.** Compute the limit.

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 12) = 3x^2 - 12
\]

In this limit, \( x \) is treated as a constant because it does not change as \( h \to 0 \). We see that the limit exists for all \( x \), so \( f \) is differentiable and \( f'(x) = 3x^2 - 12 \).

Now evaluate:

- \( f'(-3) = 3(-3)^2 - 12 = 15 \)
- \( f'(0) = 3(0)^2 - 12 = -12 \)
- \( f'(2) = 3(2)^2 - 12 = 0 \)
- \( f'(3) = 3(3)^2 - 12 = 15 \)

These derivatives indicate the slope of the graph of \( f \) (and the tangent line to the graph) at the corresponding points, as shown in Figure 1.

In the previous example, we used the definition of \( f'(x) \) to find an equation for the derivative as a function of \( x \). Using the formula for \( f'(x) \) we computed the derivative at specific values of \( x \) rather than computing each derivative separately using the limit definition. As we proceed with the development of the derivative, one goal will be to develop formulas and rules for the derivatives of functions so that we do not have to return to the limit definition for every derivative computation.

**Example 2** Prove that \( y = x^{-2} \) is differentiable and calculate \( y' \).

**Solution** The domain of \( f(x) = x^{-2} \) is \( \{x : x \neq 0\} \), so assume that \( x \neq 0 \). We compute \( f'(x) \) directly, without the separate steps of the previous example:

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{(x + h)^2} - \frac{1}{x^2}
\]

\[
= \lim_{h \to 0} \frac{x^2 - (x + h)^2}{x^2(x + h)^2} = \lim_{h \to 0} \frac{1}{h} \left(\frac{x^2 - (x + h)^2}{x^2(x + h)^2}\right)
\]
\[ \lim_{h \to 0} \frac{1}{h} \left( \frac{-h(2x + h)}{x^2(x + h)^2} \right) = \lim_{h \to 0} \frac{2x + h}{x^2(x + h)^2} \quad (\text{cancel } h) \]

\[ = \frac{-2x + 0}{x^2(x + 0)^2} = \frac{-2x}{x^4} = -2x^{-3} \]

The limit exists for all \( x \neq 0 \), so \( y = x^{-2} \) is differentiable and \( y' = -2x^{-3} \).

**Leibniz Notation**

The "prime" notation \( y' \) and \( f'(x) \) was introduced by the French mathematician Joseph Louis Lagrange (1736–1813). There is another standard notation for the derivative that we owe to Gottfried Wilhelm Leibniz:

\[ \frac{df}{dx} \quad \text{or} \quad \frac{dy}{dx} \]

In Example 2, we showed that the derivative of \( y = x^{-2} \) is \( y' = -2x^{-3} \). In Leibniz notation, we would write

\[ \frac{dy}{dx} = -2x^{-3} \quad \text{or} \quad \frac{d}{dx} x^{-2} = -2x^{-3} \]

To specify the value of the derivative for a fixed value of \( x \), say, \( x = 4 \), we write

\[ \frac{df}{dx} \bigg|_{x=4} \quad \text{or} \quad \frac{dy}{dx} \bigg|_{x=4} \]

You should not think of \( \frac{dy}{dx} \) as the fraction "\( dy \) divided by \( dx \)." Separately, the expressions \( dy \) and \( dx \) are called **differentials**. They play a role in linear approximation (Section 4.1), and relationships between them are used as a guide for "substitutions" we do later when working with integrals.

**CONCEPTUAL INSIGHT** Leibniz notation is widely used for several reasons. First, it reminds us that the derivative \( df/dx \), although not itself a ratio, is in fact a limit of ratios \( \Delta f/\Delta x \). Second, the notation specifies the independent variable. This is useful when variables other than \( x \) are used. For example, if the independent variable is \( t \), we write \( df/dt \). Third, we often think of \( d/dx \) as an "operator" that performs differentiation on functions. In other words, we apply the operator \( d/dx \) to \( f \) to obtain the derivative \( df/dx \). We will see other advantages of Leibniz notation when we discuss the Chain Rule in Section 3.7.

Now we are ready to start assembling a collection of derivative rules and formulas that will enable us to compute derivatives of the most common functions in mathematics, the sciences, and engineering. We begin with two simple formulas that are consequences of Theorem 1 in the previous section:

\[ \frac{d}{dx} x = 1 \quad \text{and} \quad \frac{d}{dx} c = 0 \quad \text{for any constant } c \]

The first indicates that the derivative with respect to \( x \) of \( x \) is 1, reflecting that the slope of the line \( y = x \) is 1. The second, known as the **Constant Rule**, indicates that the derivative of a constant is 0. This makes sense, of course, since a constant does not change and therefore has a rate of change of zero. As simple as the latter is, we will find it quite useful as we work with derivatives throughout the book.

The next theorem will prove to be very valuable for differentiating polynomial functions and many other types of functions involving constant powers.
THEOREM 1 The Power Rule

For all exponents \( n \):

\[
\frac{d}{dx} x^n = nx^{n-1}
\]

The Power Rule is valid for all exponents. We prove it here for positive integers \( n \). See Exercise 93 for a proof for negative integers \( n \) and the marginal note on page 365 for arbitrary \( n \).

Proof

We have

\[
f'(x) = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}
\]

To simplify the difference quotient, we need to expand the \((x + h)^n\) term. However, we do not need to write all the terms to work through the limit. The binomial expansion formula helps here (see Section 1.1); it indicates that expanding \((x + h)^n\) results in a sum of terms \(\binom{n}{p}(x^{n-p}h^p)\), one for each \( p \) from 0 to \( n \). For \( p = 0 \) we have \( x^n \), for \( p = 1 \) we have \( nx^{n-1}h \), and for the rest of the terms, it is enough to observe that they are terms containing \( x^q h^p \) for \( p \geq 2 \). Thus, for our purposes, we can express the expansion of \((x + h)^n\) as

\[
(x + h)^n = x^n + nx^{n-1}h + [\text{terms with } x^q h^p, p \geq 2]
\]

Now, we have

\[
f'(x) = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h} = \lim_{h \to 0} \frac{(x^n + nx^{n-1}h + [\text{terms with } x^q h^p, p \geq 2]) - x^n}{h} = \lim_{h \to 0} \frac{nx^{n-1}h + [\text{terms with } x^q h^p, p \geq 2]}{h} = \lim_{h \to 0} \frac{nx^{n-1} + [\text{terms with } x^q h^p, p \geq 1]}{h} = nx^{n-1}
\]

At the last stage, the limit of the terms containing \( x^q h^p \) equal 0 as \( h \to 0 \) because those terms include a factor of \( h \) to at least the first power.

This proves that \( f'(x) = nx^{n-1} \) for the case where \( n \) is a positive integer.

We make a few remarks before proceeding:

- The Power Rule in words: To differentiate \( x \) to a power, multiply by the power and reduce the power by one.

\[
\frac{d}{dx} x^{\text{power}} = (\text{power}) x^{\text{power}-1}
\]

- The Power Rule is valid for all exponents, whether positive, negative, fractional, or irrational:

\[
\frac{d}{dx} x^{-3/5} = -\frac{3}{5} x^{-8/5}, \quad \frac{d}{dx} x^{\sqrt{2}} = \sqrt{2} x^{\sqrt{2}-1}
\]

- The Power Rule can be applied with any variable, not just \( x \). For example,

\[
\frac{d}{dz} z^2 = 2z, \quad \frac{d}{dt} t^{20} = 20t^{19}, \quad \frac{d}{dr} r^{1/2} = \frac{1}{2} r^{-1/2}
\]
Next, we state the Linearity Rules for derivatives, which are analogous to the Linearity Laws for limits:

**Theorem 2** Linearity Rules. Assume that $f$ and $g$ are differentiable. Then

Sum and Difference Rules: $f + g$ and $f - g$ are differentiable, and

$$ (f + g)' = f' + g', \quad (f - g)' = f' - g' $$

Constant Multiple Rule: For any constant $c$, $cf$ is differentiable, and

$$ (cf)' = cf' $$

**Proof** To prove the Sum Rule, we use the definition

$$ (f + g)'(x) = \lim_{h \to 0} \frac{(f(x + h) + g(x + h)) - (f(x) + g(x))}{h} $$

This difference quotient is equal to a sum ($h \neq 0$):

$$ \frac{(f(x + h) + g(x + h)) - (f(x) + g(x))}{h} = \frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} $$

Therefore, by the Sum Law for limits,

$$ (f + g)'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} $$

$$ = f'(x) + g'(x) $$

as claimed. The Difference and Constant Multiple Rules are proved similarly.

In words, these rules state:

- The derivative of a sum is the sum of the derivatives.
- The derivative of a difference is the difference of the derivatives.
- The derivative of a constant times a function is the constant times the derivative of the function.

**Example 3** Find the points on the graph of $f(t) = t^3 - 12t + 4$ where the tangent line is horizontal.

**Solution** We calculate the derivative:

$$ \frac{df}{dt} = \frac{d}{dt} (t^3 - 12t + 4) $$

$$ = \frac{d}{dt} t^3 - \frac{d}{dt} (12t) + \frac{d}{dt} 4 $$  (Sum and Difference Rules)

$$ = \frac{d}{dt} t^3 - 12 \frac{d}{dt} t + 0 $$  (Constant Multiple Rule and Constant Rule)

$$ = 3t^2 - 12 $$  (Power Rule)

The tangent line is horizontal at points where the slope $f'(t)$ is zero, so we solve

$$ 3t^2 - 12 = 0 \quad \Rightarrow \quad t = \pm 2 $$

Now $f(2) = -12$ and $f(-2) = 20$. Hence, the tangent lines are horizontal at $(2, -12)$ and $(-2, 20)$, as shown in Figure 2.
EXAMPLE 4 Calculate \( \frac{dg}{dt} \bigg| _{t=1} \), where \( g(t) = t^{-3} + 2\sqrt{t} - t^{-4/5} \).

Solution We differentiate term-by-term using the Power Rule and the Linearity Rules. Writing \( \sqrt{t} \) as \( t^{1/2} \), we have

\[
\frac{dg}{dt} = \frac{d}{dt}(t^{-3} + 2t^{1/2} - t^{-4/5}) = -3t^{-4} + 2\left(\frac{1}{2}\right)t^{-1/2} - \left(-\frac{4}{5}\right)t^{-9/5}
\]

\[
= -3t^{-4} + t^{-3/2} + \frac{4}{5}t^{-9/5}
\]

So \( \frac{dg}{dt} \bigg| _{t=1} = -3 + 1 + \frac{4}{5} = -\frac{6}{5} \)

EXAMPLE 5 A power-law model relating the pulse rate \( P \) (in beats per minute) in mammals to body mass \( m \) (in kilograms) is given by \( P = 200m^{-1/4} \) (see Figure 3). It is clear from the graph that, from species to species, as the mass increases, the pulse rate drops off. Furthermore the pulse rate drops off quickly for small mammals but relatively slowly for large mammals. Determine \( P'(m) \) at the mass of a guinea pig (1 kg) and at the mass of cattle (500 kg).

![Pulse (beats/min) vs. Mass (kg) graph]

Solution Applying the Power Rule and the Constant Multiple Rule to \( P(m) = 200m^{-1/4} \), we obtain \( P'(m) = -50m^{-5/4} \). So,

- \( P'(1) = -50 \) beats per minute per kilogram
- \( P'(500) = -50(500^{-5/4}) \approx -0.02 \) beats per minute per kilogram

These values confirm our observation that \( P \) decreases rapidly for small \( m \) and decreases slowly for large \( m \).

The Derivative and Behavior of the Graph

The derivative \( f' \) gives us important information about the graph of \( f \). For example, the sign of \( f'(x) \) tells us whether the tangent line has positive or negative slope. When the tangent line has positive slope, it slopes upward and the graph must be increasing. When the tangent line has negative slope, it slopes downward and the graph must be decreasing. The magnitude of \( f'(x) \) reveals how steep the slope is.

EXAMPLE 6 \( f' \) and the Graph of \( f \) How is the graph of \( f(x) = x^3 - 12x^2 + 36x - 16 \) related to the derivative \( f'(x) = 3x^2 - 24x + 36 \)?

Solution The derivative \( f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6) \) is negative for \( 2 < x < 6 \) and positive for \( x < 2 \) and \( x > 6 \) (Figure 4). The following table summarizes this sign information:

![Graph of f(x) and its derivative]


### Property of \( f'(x) \) of the Graph of \( f \)

- **Property of \( f'(x) \):**
  - \( f'(x) < 0 \) for \( 2 < x < 6 \)
  - \( f'(2) = f'(6) = 0 \)
  - \( f'(x) > 0 \) for \( x < 2 \) and \( x > 6 \)

- **Property of the Graph of \( f \):**
  - Tangent has negative slope for \( 2 < x < 6 \) (graph is decreasing).
  - Tangent is horizontal at \( x = 2 \) and \( x = 6 \).
  - Tangent has positive slope for \( x < 2 \) and \( x > 6 \) (graph is increasing).

Note also that \( f'(x) \to \infty \) as \( |x| \) becomes large. This corresponds to the fact that the tangent lines to the graph of \( f \) get steeper as \( |x| \) grows large.

---

**EXAMPLE 7** Identifying the Derivative

The graph of \( f \) (with some tangent lines included) is shown in Figure 5(A). Which graph, (B) or (C), is the graph of \( f'' \)?

![Graph of \( f \) with tangent lines](Figure5)

**Solution** In Figure 5(A), we see that on the intervals \((0, 1)\) and \((4, 7)\), the graph is decreasing, and therefore the tangent lines to the graph have negative slope. Thus, \( f'(x) \) is negative on these intervals. Similarly, on the intervals \((1, 4)\) and \((7, \infty)\), the graph is increasing, and therefore the tangent lines have positive slope and \( f'(x) \) is positive (see the table in the margin). Only (C) has these properties, so (C) is the graph of \( f'' \).

---

**Differentiability, Continuity, and Local Linearity**

In the rest of this section, we examine the concept of **differentiability** more closely. We begin by proving that a differentiable function is necessarily continuous. In particular, a function with a jump discontinuity cannot be differentiable. Figure 6 shows why: Although the secant lines from the right approach the line \( L \) (which is tangent to the right half of the graph), the secant lines from the left approach the vertical (and their slopes tend to \( \infty \)).

So the limit of the slopes of the secant lines does not exist and therefore the function is not differentiable at the point where the jump discontinuity occurs.

**THEOREM 3** Differentiability Implies Continuity

If \( f \) is differentiable at \( x = c \), then \( f \) is continuous at \( x = c \).

**Proof** By definition, if \( f \) is differentiable at \( x = c \), then the following limit exists:

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

We must prove that \( \lim_{x \to c} f(x) = f(c) \), because this is the definition of continuity at \( x = c \).

To relate the two limits, consider the equation (valid for \( x \neq c \))

\[
f(x) - f(c) = (x - c) \frac{f(x) - f(c)}{x - c}
\]
Both factors on the right approach a limit as \( x \rightarrow c \), so

\[
\lim_{{x \to c}} (f(x) - f(c)) = \lim_{{x \to c}} \left( (x - c) \cdot \frac{f(x) - f(c)}{x - c} \right) \\
= \left( \lim_{{x \to c}} (x - c) \right) \left( \lim_{{x \to c}} \frac{f(x) - f(c)}{x - c} \right) \\
= 0 \cdot f'(c) = 0
\]

by the Product Law for limits. The Sum Law now yields the desired conclusion:

\[
\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (f(x) - f(c)) + \lim_{{x \to c}} f(c) = 0 + f(c) = f(c)
\]

Most of the functions encountered in this text are differentiable, but exceptions exist. As we saw in Example 5 in Section 3.1, the functions \( f(x) = |x| \) and \( g(x) = x^{1/3} \) are not differentiable at \( x = 0 \). Note that both of these functions are continuous at \( x = 0 \), and therefore they demonstrate that continuity at a point does not imply differentiability at the point (i.e., the converse of Theorem 3 does not hold).

Example 5 in Section 3.1 showed that \( f(x) = |x| \) is not differentiable at 0 due to the corner in its graph, and \( g(x) = x^{1/3} \) is not differentiable at \( x = 0 \) due to the vertical tangent. The function

\[
h(x) = \begin{cases} 
  x \sin \frac{1}{x} & \text{if } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases}
\]

is also continuous and not differentiable at \( x = 0 \), but its failure to be differentiable is more complicated than the situation with \( f \) and \( g \) (see Figure 7). In this case the secant lines from \((0, 0)\) to nearby points \( Q \) on the curve oscillate and do not settle down to a limiting tangent as \( Q \) approaches the origin (see Exercise 95).

**Figure 7**

As we have seen with \( f(x) = |x| \), \( g(x) = x^{1/3} \), and \( h \) above, a function may not be differentiable at a point because of unusual behavior of the function near the point. On the other hand, as we see in the following Graphical Insight, differentiability at a point implies that a function behaves quite nicely near the point.

**Graphical Insight**

Differentiability has an important graphical interpretation in terms of local linearity. We say that \( f \) is **locally linear** at \( x = a \) if the graph looks more and more like a straight line as we zoom in on the point \((a, f(a))\). In this context, the adjective *linear* means "resembling a line," and *local* indicates that we are concerned only with the behavior of the graph near \((a, f(a))\). The graph of a locally linear function may be very wavy or *nonlinear*, as in Figure 8. But as soon as we zoom in on a sufficiently small piece of the graph, it begins to appear straight.
Not only does the graph look like a line as we zoom in on a point, but as Figure 8 suggests, the "zoom line" is the tangent line. Thus, the relation between differentiability and local linearity can be expressed as follows:

If \( f'(a) \) exists, then \( f \) is locally linear at \( x = a \). That is, as we zoom in on the point \((a, f(a))\), the graph becomes nearly indistinguishable from its tangent line.

Local linearity gives us a graphical way to understand why \( f(x) = |x| \) is not differentiable at \( x = 0 \). Figure 9 shows that the graph of \( f(x) = |x| \) has a corner at \( x = 0 \), and this corner does not disappear, no matter how closely we zoom in on the origin. Since the graph does not straighten out under zooming, \( f \) is not locally linear at \( x = 0 \), reflecting that \( f \) is not differentiable at \( x = 0 \).

**FIGURE 8** Local linearity: The graph looks more and more like the tangent line as we zoom in on a point.

**FIGURE 9** The graph of \( f(x) = |x| \) is not locally linear at \( x = 0 \). The corner does not disappear when we zoom in on the origin.

### 3.2 SUMMARY

- The derivative \( f' \) is the function whose value at \( x \) is the derivative \( f'(x) \).
- We have several different notations for the derivative of \( y = f(x) \):
  
  \[
  y', \quad y'(x), \quad f'(x), \quad \frac{dy}{dx}, \quad \frac{df}{dx}
  \]

  The value of the derivative at a particular point \( x = a \) is written
  
  \[
  y'(a), \quad f'(a), \quad \left. \frac{dy}{dx} \right|_{x=a}, \quad \left. \frac{df}{dx} \right|_{x=a}
  \]

- Derivative Rules

  The Constant Rule: \( \frac{d}{dx} c = 0 \)

  The Power Rule: \( \frac{d}{dx} x^n = nx^{n-1} \)

  The Linearity Rules: \((f + g)' = f' + g'\) and \((cf)' = cf'\)
3.2 EXERCISES

Preliminary Questions
1. What is the slope of the tangent line through the point (2, f(2)) if \( f'(x) = 3x^2 \)?
2. Evaluate \( f - g'(1) \) and \( (3f + 2g)'(1) \), assuming that \( f'(1) = 3 \) and \( g'(1) = 5 \).
3. To which of the following does the Power Rule apply?
   (a) \( f(x) = x^3 \)  \hspace{1cm} (b) \( f(x) = 2^x \)
   (c) \( f(x) = x^4 \)  \hspace{1cm} (d) \( f(x) = x^{-4/5} \)
4. State whether each claim is true or false. If false, give an example demonstrating that it is false.
   (a) If \( f \) is continuous at \( a \), then \( f \) is differentiable at \( a \).
   (b) If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

Exercises
In Exercises 1–6, compute \( f'(x) \) using the limit definition.
1. \( f(x) = 3x - 7 \)
2. \( f(x) = x^2 + 3x \)
3. \( f(x) = x^3 \)
4. \( f(x) = 1 - x^{-1} \)
5. \( f(x) = x - \sqrt{x} \)
6. \( f(x) = x^{-1/2} \)

In Exercises 7–14, use the Power Rule to compute the derivative.
7. \( \frac{d}{dx} x^4 \) | \( x = -2 \)
8. \( \frac{d}{dt} t^{-1} \) | \( t = 4 \)
9. \( \frac{d}{dt} t^{2/3} \) | \( t = 8 \)
10. \( \frac{d}{dt} t^{-2} \) | \( t = 1 \)
11. \( \frac{d}{dx} x^{0.35} \)
12. \( \frac{d}{dx} x^{14/3} \)
13. \( \frac{d}{dt} t^{\sqrt{7}} \)
14. \( \frac{d}{dt} t^{\pi} \)

In Exercises 15–18, compute \( f'(x) \) and find an equation of the tangent line to the graph at \( x = a \).
15. \( f(x) = x^a \), \( a = 2 \)
16. \( f(x) = x^{-1} \), \( a = 5 \)
17. \( f(x) = 5x - 32\sqrt{x} \), \( a = 4 \)
18. \( f(x) = \sqrt[3]{x} \), \( a = 8 \)
19. Find an equation of the tangent line to \( y = \sqrt{x} \) at \( x = 9 \).
20. Find a point on the graph of \( y = \sqrt{x} \) where the tangent line has slope 10.

In Exercises 21–32, calculate the derivative.
21. \( f(x) = 2x^3 - 3x^2 + 5 \)
22. \( f(x) = 2x^3 - 3x^2 + 2x \)
23. \( f(x) = 4x^{5/3} - 3x^{2/3} - 12 \)
24. \( f(x) = x^{2/3} + 4x^{1/3} + 11x \)
25. \( g(z) = 7e^{x/14} + x^{-5} + 9 \)
26. \( h(t) = 6\sqrt{t} + \frac{1}{\sqrt{t}} \)
27. \( f(y) = \sqrt{5} + \sqrt{y} \)
28. \( W(y) = 6y^4 + 7y^{2/3} \)
29. \( g(x) = \pi \)
30. \( f(x) = x^\pi \)
31. \( h(t) = \sqrt{2t^2} \)
32. \( R(z) = \frac{z^{3/2} - 4z^{3/2}}{z} \) \hspace{1cm} Hint: Simplify.

In Exercises 33–38, expand or simplify the function, and then calculate the derivative.
33. \( P(s) = (4s - 3)^2 \)
34. \( Q(r) = (1 - 2r)(3r + 5) \)
35. \( f(x) = (2 - x)(2 + x) \)
36. \( g(w) = (1 + 2w)^3 \)
37. \( g(x) = \frac{x^2 + 4x^{1/2}}{x^2} \)
38. \( x(t) = \frac{1 - 2t}{t^{1/2}} \)

In Exercises 39–44, calculate the derivative indicated.
39. \( \frac{dT}{dC} \) | \( C = 8 \), \( T = 3C^{2/3} \)
40. \( \frac{dP}{dV} \) | \( V = 2 \), \( P = \frac{7}{V} \)
41. \( \frac{ds}{dz} \) | \( z = 2 \), \( s = 4z - 16z^2 \)
42. \( \frac{dR}{dW} \) | \( W = 1 \), \( R = W^2 \)
43. \( \frac{dr}{dt} \) | \( r = t^2 + 1 \), \( t^{1/2} \)
44. \( \frac{dp}{dh} \) | \( h = 32 \), \( p = 16h^{0.2} + 8h^{-0.8} \)
45. Match the functions in graphs (A)–(D) with their derivatives (I)–(III) in Figure 10. Note that two of the functions have the same derivative. Explain why.

![Figure 10](image-url)
46. Of the two functions \( f \) and \( g \) in Figure 11, which is the derivative of the other? Justify your answer.

![Figure 11](image)

47. Assign the labels \( y = f(x) \), \( y = g(x) \), and \( y = h(x) \) to the graphs in Figure 12 in such a way that \( f'(x) = g(x) \) and \( g'(x) = h(x) \).

![Figure 12](image)

48. Prove each of the following using the definition of the derivative.

(a) The First-Power Rule: \( \frac{d}{dx} x^1 = 1 \)

(b) The Constant Rule: \( \frac{d}{dx} c = 0 \)

49. Use the rules in Exercise 48 and the Linearity Rules to prove the first part of Theorem 1 in Section 3.1.

50. According to the Peak Oil Theory, first proposed in 1956 by geophysicist M. Hubbert, the total amount of crude oil \( Q(t) \) produced worldwide up to time \( t \) has a graph like that in Figure 13.

(a) Sketch the derivative \( Q'(t) \) for 1900 \( \leq t \leq 2150 \). What does \( Q'(t) \) represent?

(b) In which year (approximately) does \( Q'(t) \) take on its maximum value?

(c) What is \( L = \lim_{t \to \infty} Q(t) \)? And what is its interpretation?

(d) What is the value of \( \lim_{t \to \infty} Q'(t) \)?

![Figure 13](image)

51. Use the table of values of \( f \) to determine which of (A) or (B) in Figure 14 is the graph of \( f' \). Explain.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>10</td>
<td>55</td>
<td>98</td>
<td>139</td>
<td>177</td>
<td>210</td>
<td>257</td>
<td>257</td>
<td>268</td>
</tr>
</tbody>
</table>

![Figure 14](image)

52. Let \( R \) be a variable and \( r \) a constant. Compute the derivatives:

(a) \( \frac{d}{dR} R \)

(b) \( \frac{d}{dR} r \)

(c) \( \frac{d}{dR} r^2 R^3 \)

53. Compute the derivatives, where \( c \) is a constant.

(a) \( \frac{d}{dt} e^{ct} \)

(b) \( \frac{d}{dz} (5z + 4cz^2) \)

(c) \( \frac{d}{dy} (9c^2 - 9yz - 24cy) \)

54. Find the points on the graph of \( f(x) = 12x - x^3 \) where the tangent line is horizontal.

55. Find the points on the graph of \( y = x^2 + 3x - 7 \) at which the slope of the tangent line is equal to 4.

56. Find the values of \( x \) where \( y = x^3 \) and \( y = x^2 + 5x \) have parallel tangent lines.

57. Determine \( a \) and \( b \) such that

\[ p(x) = x^2 + ax + b \]

satisfies \( p(1) = 0 \) and \( p'(1) = 4 \).

58. Find all values of \( x \) such that the tangent line to

\[ y = 4x^2 + 11x + 2 \]

is steeper than the tangent line to \( y = x^3 \).

59. Let \( f(x) = x^3 - 3x + 1 \). Show that \( f'(x) \geq -3 \) for all \( x \) and that, for every \( m > -3 \), there are precisely two points where \( f'(x) = m \). Indicate the position of these points and the corresponding tangent lines for one value of \( m \) in a sketch of the graph of \( f \).

60. Show that the tangent lines to

\[ y = \frac{1}{3} x^3 - x^2 \]

at \( x = a \) and at \( x = b \)

are parallel if \( a = b \) or \( a + b = 2 \).
61. Compute the derivative of \( f(x) = x^{3/2} \) using the limit definition. 
   Hint: Show that 
   \[
   \lim_{{h \to 0}} \frac{f(x + h) - f(x)}{h} = \frac{1}{\sqrt{x}} \left( \frac{1}{\sqrt{x + h} + \sqrt{x}} \right)
   \]

62. Compute the derivative of \( f(x) = x^{1/3} \) using the limit definition. 
   Hint: Multiply the numerator and denominator in the difference quotient \( \frac{f(x+h) - f(x)}{h} \) by 
   \((x + h)^{2/3} + (x + h)^{1/3}x^{1/3} + x^{2/3}\)

63. Show using the limit definition of the derivative that \( f(x) = |x^2 - 4| \) 
   is not differentiable at \( x = 2 \).

64. The average speed (in meters per second) of a gas molecule is

   \[ v_{\text{avg}} = \sqrt{\frac{8RT}{\pi M}} \]

   where \( T \) is the temperature (in kelvins), \( M \) is the molar mass (in kilograms per mole), and \( R = 8.31 \). Calculate \( dv_{\text{avg}}/dT \) at \( T = 300 \text{ K} \) for oxygen, which has a molar mass of 0.032 kg/mol.

65. The brightness \( b \) of the sun (in watts per square meter) at a distance of \( d \) meters from the sun is expressed as an inverse-square law in the form 
   \[ b = \frac{L}{4d^2} \]

   where \( L \) is the luminosity of the sun and equals \( 3.9 \times 10^{26} \) watts. What is the derivative of \( b \) with respect to \( d \) at the earth’s distance from the sun (1.5 \times 10^{11} \text{ m})?

66. A power law model relating the kidney mass \( K \) in mammals (in kilograms) to the body mass \( m \) (in kilograms) is given by \( K = 0.007m^{0.85} \).

   Calculate \( dK/dm \) at \( m = 68 \). Then calculate the derivative with respect to \( m \) of the relative kidney-to-mass ratio \( K/m \) at \( m = 68 \).

67. The Clausius–Clapeyron Law relates the vapor pressure of water \( P \) (in atmospheres) to the temperature \( T \) (in kelvins):

   \[ \frac{dP}{dT} = k \frac{P}{T^2} \]

   where \( k \) is a constant. Estimate \( dP/dT \) for \( T = 303, 313, 323, 333, 343 \) using the data and the symmetric difference approximation

   \[ \frac{dP}{dT} \approx \frac{P(T + 10) - P(T - 10)}{20} \]

<table>
<thead>
<tr>
<th>( T ) (K)</th>
<th>293</th>
<th>303</th>
<th>313</th>
<th>323</th>
<th>333</th>
<th>343</th>
<th>353</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P ) (atm)</td>
<td>0.0278</td>
<td>0.0482</td>
<td>0.0808</td>
<td>0.1311</td>
<td>0.2067</td>
<td>0.3173</td>
<td>0.4754</td>
</tr>
</tbody>
</table>

Do your estimates seem to confirm the Clausius–Clapeyron Law? What is the approximate value of \( k \)?

68. Let \( L \) be the tangent line to the hyperbola \( xy = 1 \) at \( x = a \), where \( a > 0 \). Show that the area of the triangle bounded by \( L \) and the coordinate axes does not depend on \( a \).

69. In the setting of Exercise 68, show that the point of tangency is the midpoint of the segment of \( L \) lying in the first quadrant.

70. Match functions (A)-(C) with their derivatives (I)-(III) in Figure 15.

71. Make a rough sketch of the graph of the derivative of the function in Figure 16(A).

72. Graph the derivative of the function in Figure 16(B), omitting points where the derivative is not defined.

73. Sketch the graph of \( f(x) = x|\text{x}| \). Then show that \( f'(0) \) exists.

74. Determine the values of \( x \) at which the function in Figure 17 is: (a) discontinuous and (b) nondifferentiable.
In Exercises 75–80, zoom in on a plot of $f$ at the point $(a, f(a))$ and state whether or not $f$ appears to be differentiable at $x = a$. If it is nondifferentiable, state whether the tangent line appears to be vertical or does not exist.

75. $f(x) = (x - 1)(x), \quad a = 0$
76. $f(x) = (x - 3)^{3/2}, \quad a = 3$
77. $f(x) = (x - 3)^{1/3}, \quad a = 3$
78. $f(x) = \sin(x^{1/3}), \quad a = 0$
79. $f(x) = |\sin x|, \quad a = 0$
80. $f(x) = |x - \sin x|, \quad a = 0$

81. Find the coordinates of the point $P$ in Figure 18 at which the tangent line passes through $(5,0)$.

![Figure 18](image)

82. Plot the derivative $f'$ of $f(x) = 2x^3 - 10x^{-1}$ for $x > 0$ and observe that $f'(x) > 0$. What does the positivity of $f'(x)$ tell us about the graph of $f$ itself? Plot $f$ and confirm this conclusion.

Exercises 83–86 refer to Figure 19. Length $QR$ is called the subtangent at $P$, and length $RT$ is called the subnormal.

83. Calculate the subtangent of $f(x) = x^3 + 3x$ at $x = 2$.
84. Show that for $n \neq 0$, the subtangent of $f(x) = x^n$ at $x = c$ is equal to $c/n$.
85. Prove in general that the subnormal at $P$ is $|f'(x)f(x)|$.
86. Show that $PQ$ has length $|f(x)|\sqrt{1 + f'(x)^2}$.

![Figure 19](image)

87. Prove the following theorem of Apollonius of Perga (the Greek mathematician born in 262 BCE who gave the parabola, ellipse, and hyperbola their names): The subtangent of the parabola $y = ax^2$ at $x = a$ is equal to $a/2$.
88. Show that the subtangent to $y = x^3$ at $x = a$ is equal to $a/3$.
89. Formulate and prove a generalization of Exercises 87 and 88 for $y = x^n$.

Further Insights and Challenges

90. Two small arches have the shape of parabolas. The first is the graph of $f(x) = 1 - x^2$ for $-1 \leq x \leq 1$ and the second is the graph of $g(x) = 4 - (x - 4)^2$ for $2 \leq x \leq 6$. A board is placed on top of these arches so it rests on both (Figure 20). What is the slope of the board? Hint: Find the tangent line to $y = f(x)$ that intersects $y = g(x)$ in exactly one point.

![Figure 20](image)

91. A vase is formed by rotating $y = x^2$ around the $y$-axis. If we drop a marble, it will either touch the bottom point of the vase or be suspended above the bottom by touching the sides (Figure 21). How small must the marble be to touch the bottom?

![Figure 21](image)

92. Let $f$ be a differentiable function, and set the function $g(x) = f(x + c)$, where $c$ is a constant. Use the limit definition to show that $g'(x) = f'(x + c)$. Explain this result graphically, recalling that the graph of $g$ is obtained by shifting the graph of $f$ $c$ units to the left (if $c > 0$) or right (if $c < 0$).

93. Negative Exponents Let $n$ be a whole number. Calculate the derivative of $f(x) = x^{-n}$ by showing that

$$\frac{f(x+h) - f(x)}{h} = \frac{-1}{x^n(x+h)^n - x^n}$$
94. Verify the Power Rule for the exponent $1/n$, where $n$ is a positive integer, using the following trick: Rewrite the difference quotient for $y = x^{1/n}$ at $x = b$ in terms of

$u = (b + h)^{1/n}$ and $a = b^{1/n}$

95. Infinitely Rapid Oscillations Define

$$h(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that $h$ is continuous at $x = 0$ but $h'(0)$ does not exist (see Figure 7).

96. For which values of $c$ does the equation $x^2 + 4 = cx$ have a unique solution? *Hint:* Draw a graph.

97. If

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h}$$

exist but are not equal, then $f$ is not differentiable at $c$, and the graph of $f$ has a corner at $c$. Prove that $f$ is continuous at $c$.

### 3.3 Product and Quotient Rules

This section covers the **Product Rule** and **Quotient Rule** for computing derivatives. These two rules, together with the Chain Rule and implicit differentiation (covered in later sections), make up an extremely effective differentiation toolkit.

**Theorem 1** **Product Rule** If $f$ and $g$ are differentiable functions, then $fg$ is differentiable and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

It may be helpful to remember the Product Rule in words: The derivative of a product of terms is equal to the derivative of the first term times the second plus the first term times the derivative of the second:

$$(\text{first}) \cdot \text{second} + \text{first} \cdot (\text{second})'$$

Be careful when taking the derivative of products. The product rule is not $(fg)' = f'g'$; that is, it does not say that the derivative of a product is the product of the derivatives.

We prove the Product Rule after presenting some examples.

**Example 1** Find the derivative of $h(x) = x^2(9x + 2)$.

**Solution** This function is a product:

$$h(x) = \frac{\text{First}}{x^2} \cdot \frac{\text{Second}}{9x + 2}$$

By the Product Rule (in Leibniz notation),

$$h'(x) = \frac{\text{First}}{dx} \frac{d}{dx} (9x + 2) + \frac{\text{Second}}{dx} \frac{d}{dx} (9x + 2)$$

$$= (2x)(9x + 2) + (x^2)(9) = 27x^2 + 4x$$

**Example 2** Find the derivative of $y = (2 + x^{-1})(x^{3/2} + 1)$.

**Solution** Use the Product Rule:

$$(\text{First}) \cdot \text{Second} + \text{First} \cdot (\text{Second})'$$

$$y' = (2 + x^{-1})(x^{3/2} + 1) + (2 + x^{-1})(x^{3/2} + 1)'$$

$$= (-x^{-2})(x^{3/2} + 1) + (2 + x^{-1}) \left( \frac{3}{2} x^{1/2} \right)$$

(compute the derivatives)

$$= -x^{-1/2} - x^{-2} + 3x^{1/2} + \frac{3}{2} x^{-1/2} = \frac{1}{2} x^{-1/2} - x^{-2} + 3x^{1/2}$$

(simplify)
In the previous two examples, we could have avoided the Product Rule by expanding the function. Thus, the result of Example 2 can be obtained as follows:
\[ y = (2 + x^{-1})(x^{3/2} + 1) = 2x^{3/2} + 2 + x^{1/2} + x^{-1} \]
\[ y' = \frac{d}{dx} (2x^{3/2} + 2 + x^{1/2} + x^{-1}) = 3x^{1/2} + \frac{1}{2}x^{-1/2} - x^{-2} \]

In many cases, the function cannot be expanded and we must use the Product Rule. One such function is \( f(x) = x \cos x \) whose derivative we find in Section 3.6.

**Example 3** Figure 1 depicts a rectangle whose length \( L(t) \) and width \( W(t) \) (measured in inches) are varying in time \( t \), in minutes. At \( t = 5 \), the length is 8, the width is 5, and they are changing according to \( L'(5) = -4 \) and \( W'(5) = 3 \). Compute \( A'(5) \).

**Solution** Since the area is given by \( A(t) = L(t)W(t) \), we can use the Product Rule to compute \( A'(t) \). We have \( A'(t) = L'(t)W(t) + L(t)W'(t) \). Therefore,
\[ A'(5) = (-4)(5) + (8)(3) = 4 \]

It follows that the area of the rectangle in the example is increasing at a rate of 4 in.\(^2\)/min at \( t = 5 \). This may appear counterintuitive, given that the length is decreasing at a faster rate than the width is increasing. What is important, as the Product Rule demonstrates, is that the decreasing length acts over a short width of 5, contributing 20 to the rate of change of area, while the increasing width acts over a long length of 8, contributing 24 to the rate of change of area, resulting in an increasing area.

**Proof of the Product Rule** According to the limit definition of the derivative,
\[ (fg)'(x) = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h} \]

We can interpret the numerator as the area of the shaded region in Figure 2: the area of the larger rectangle \( f(x + h)g(x + h) \) minus the area of the smaller rectangle \( f(x)g(x) \). This shaded region is the union of two rectangular strips, so we obtain the following identity (which we can also obtain algebraically by adding and subtracting the term \( f(x + h)g(x) \) from the left-hand side and then manipulating the result algebraically):
\[ f(x + h)g(x + h) - f(x)g(x) = (f(x + h) - f(x))g(x) + f(x + h)(g(x + h) - g(x)) \]

Now use this identity to write \( (fg)'(x) \) as a sum of two limits:
\[ (fg)'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} g(x) + \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h} \]

We show that this equals \( f'(x)g(x) \). We show that this equals \( f(x)g'(x) \).

The use of the Sum Law is valid, provided that each limit on the right exists. To check that the first limit exists and to evaluate it, we note that \( f \) is differentiable. Thus,
\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} g(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \lim_{h \to 0} g(x) \]

The second limit is similar, but using the facts that \( f \) is continuous (because it is differentiable) and \( g \) is differentiable:
\[ \lim_{h \to 0} f(x + h) \frac{g(x + h) - g(x)}{h} = \lim_{h \to 0} f(x + h) \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x)g'(x) \]

Using Eq. (2) and Eq. (3) in Eq. (1), we conclude that \( fg \) is differentiable and that \( (fg)'(x) = f'(x)g(x) + f(x)g'(x) \) as claimed.
CONCEPTUAL INSIGHT The Product Rule was first stated by the 29-year-old Leibniz in 1675, the year he developed some of his major ideas on calculus. To document his process of discovery for posterity, he recorded his thoughts and struggles, the moments of inspiration as well as the mistakes. In a manuscript dated November 11, 1675, Leibniz suggests incorrectly that $(fg)'$ equals $f'g'$. He then catches his error by taking $f(x) = x^2 = x$ and noticing that

$$(fg)'(x) = (x^2)' = 2x \quad \text{is not equal to} \quad f'(x)g'(x) = 1 \cdot 1 = 1$$

Ten days later, on November 21, Leibniz writes down the correct Product Rule and comments, "Now this is a really noteworthy theorem."

The next theorem states the rule for differentiating quotients. Note, in particular, that $(f/g)'$ is not equal to the quotient $f'/g'$; the derivative of the quotient is not the quotient of the derivatives.

**Theorem 2: Quotient Rule** If $f$ and $g$ are differentiable functions, then $f/g$ is differentiable for all $x$ such that $g(x) \neq 0$, and

$$\left( \frac{f}{g} \right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

The numerator in the Quotient Rule is the bottom times the derivative of the top minus the top times the derivative of the bottom. The denominator is the bottom squared:

$$\frac{\text{bottom} \cdot (\text{top}') - \text{top} \cdot (\text{bottom}')}{\text{bottom}^2}$$

**Proof of the Quotient Rule** Let $Q(x) = f(x)/g(x)$. Our goal is to find the formula for $Q'(x)$. First, we multiply the equation for $Q(x)$ through by $g(x)$, then use the product rule on the result, and finally solve for $Q'(x)$. So, multiplying by $g(x)$ we have $f(x) = Q(x) \cdot g(x)$. Differentiating both sides, utilizing the Product Rule for the right side, we obtain $f'(x) = Q'(x) \cdot g(x) + Q(x) \cdot g'(x)$. Solving for $Q'(x)$, we obtain

$$Q'(x) = \frac{f'(x) - Q(x) \cdot g'(x)}{g(x)} = \frac{f'(x) - \frac{df}{dx} \cdot g'(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

as we wanted to show.

An alternative proof appears in Exercises 66–68.

**Example 4** Compute the derivative of $f(x) = \frac{x}{1+x^2}$.

**Solution** Apply the Quotient Rule:

$$f'(x) = \frac{\text{Bottom} \cdot (\text{Top}')} - \frac{\text{Top} \cdot (\text{Bottom}')}{\text{Bottom}^2}$$

$$f'(x) = \frac{(1+x^2)(1) - (x)2x}{(1+x^2)^2} = \frac{(1+x^2) - x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2}$$

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}$$
EXAMPLE 5 Find the tangent line to the graph of \( f(x) = \frac{3x^2 + x - 2}{4x^3 + 1} \) at \( x = 1 \).

Solution

\[
\frac{d}{dx} \left( \frac{3x^2 + x - 2}{4x^3 + 1} \right) = \frac{(4x^3 + 1) (3x^2 + x - 2)' - (3x^2 + x - 2) (4x^3 + 1)'}{(4x^3 + 1)^2}
\]

\[
= \frac{(4x^3 + 1)(6x + 1) - (3x^2 + x - 2)(12x^2)}{(4x^3 + 1)^2}
\]

\[
= \frac{(24x^4 + 4x^2 + 6x + 1) - (36x^4 + 12x^3 - 24x^2)}{(4x^3 + 1)^2}
\]

\[
= \frac{-12x^4 - 8x^3 + 24x^2 + 6x + 1}{(4x^3 + 1)^2}
\]

At \( x = 1 \),

\[
f(1) = \frac{3 + 1 - 2}{4 + 1} = \frac{2}{5}
\]

\[
f'(1) = \frac{-12 - 8 + 24 + 6 + 1}{5^2} = \frac{11}{25}
\]

An equation of the tangent line at \((1, \frac{2}{5})\) is

\[
y - \frac{2}{5} = \frac{11}{25}(x - 1) \quad \text{or} \quad y = \frac{11}{25}x - \frac{1}{25}
\]

EXAMPLE 6 Power Delivered by a Battery The power that a battery supplies to an apparatus such as a laptop depends on the \textit{internal resistance} of the battery. For a battery of voltage \( V \) and internal resistance \( r \), the total power delivered to an apparatus of resistance \( R \) (Figure 3) is

\[
P = \frac{V^2 R}{(R + r)^2}
\]

(a) Calculate \( \frac{dP}{dR} \), assuming that \( V \) and \( r \) are constants.

(b) Where, in the graph of \( P \) versus \( R \), is the tangent line horizontal?

Solution

(a) Using the Constant Multiple Rule (\( V \) is a constant) and the Quotient Rule, we obtain

\[
\frac{dP}{dR} = V^2 \frac{d}{dR} \left( \frac{R}{(R + r)^2} \right) = V^2 \left( R + r \right)^2 \frac{R - R \frac{d}{dR}(R + r)^2}{(R + r)^4}
\]

We have \( \frac{d}{dR} R = 1 \), and \( \frac{d}{dR} r = 0 \) because \( r \) is a constant. Therefore,

\[
\frac{d}{dR}(R + r)^2 = \frac{d}{dR}(R^2 + 2rR + r^2) = \frac{d}{dR} R^2 + 2r \frac{d}{dR} R + \frac{d}{dR} r^2 = 2R + 2r + 0 = 2(R + r)
\]

Using Eq. (5) in Eq. (4), we obtain

\[
\frac{dP}{dR} = V^2 \left( R + r \right)^2 - 2R(R + r) \quad \text{or} \quad \frac{dP}{dR} = V^2 \frac{R - R}{(R + r)^3} = V^2 \frac{r - R}{(R + r)^3}
\]

(b) The tangent line is horizontal when the derivative is zero. We see from Eq. (6) that the derivative is zero when \( r - R = 0 \); that is, when \( R = r \).
3.3 SUMMARY

- Two basic rules of differentiation:
  
  **Product Rule:** \((fg)' = fg' + g'f\)
  
  **Quotient Rule:** \(\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}\)

- Remember: The derivative of \(fg\) is **not** equal to \(f'g\). Similarly, the derivative of \(f/g\) is **not** equal to \(f'/g'\).

### 3.3 EXERCISES

#### Preliminary Questions

1. Are the following statements true or false? If false, state the correct version.
   
   (a) \(fg\) denotes the function whose value at \(x\) is \(f(g(x))\).
   
   (b) \(f/g\) denotes the function whose value at \(x\) is \(f(x)/g(x)\).
   
   (c) The derivative of the product is the product of the derivatives.

2. Find \((fg)'(1)\) if \(f(1) = f'(1) = g(1) = 2\) and \(g'(1) = 4\).

3. Find \(g(1)\) if \(f(1) = 0, f'(1) = 2, \) and \((fg)'(1) = 10\).

#### Exercises

In Exercises 1–6, use the Product Rule to calculate the derivative.

1. \(f(x) = x^3(2x^2 + 1)\)
2. \(f(x) = (3x - 5)(2x^2 - 3)\)
3. \(f(x) = \sqrt{x}(1 - x^2)\)
4. \(f(x) = (3x^4 + 2x^2)(x - 2)\)
5. \(\frac{dh}{ds}\bigg|_{s=0}, h(s) = (s^{-1/2} + 2s)(7 - s^{-1})\)
6. \(y = (t^3 - 8t^{-1})(t^2 + 3t)\)

In Exercises 7–12, use the Quotient Rule to calculate the derivative.

7. \(f(x) = \frac{x}{x^2 - 2}\)
8. \(f(x) = \frac{x + 4}{x^2 + x + 1}\)
9. \(g(t) = \frac{t^2 + 1}{t^2 - 1}\)
10. \(\frac{dw}{dx}\bigg|_{x=2}, w = \frac{x^2}{\sqrt{x} + 2}\)
11. \(g(x) = \frac{1}{1 + x^{1/2}}\)
12. \(h(x) = \frac{3x^{3/2}}{x^2 + 1}\)

In Exercises 13–18, calculate the derivative in two ways. First use the Product or Quotient Rule; then rewrite the function algebraically and directly calculate the derivative.

13. \(f(x) = x^{2/3}x^{-3}\)
14. \(h(x) = \frac{x^2}{x+2}\)
15. \(f(t) = (2t + 1)(t^2 - 2)\)
16. \(f(x) = x^2(3 + x^{-1})\)
17. \(h(t) = \frac{t^2 - 1}{t - 1}\)
18. \(g(x) = \frac{x^3 + 2x^2 + 3x^{-1}}{x}\)

In Exercises 19–40, calculate the derivative.

19. \(f(x) = (x^5 + 5)(x^3 + x + 1)\)
20. \(f(x) = \left(\frac{1}{x^2}\right)(x^3 + 1)\)
21. \(\frac{dy}{dx}\bigg|_{x=2}, y = \frac{1}{x + 10}\)
22. \(\frac{dz}{dx}\bigg|_{x=2}, z = \frac{x}{3x^2 + 1}\)
23. \(f(x) = (\sqrt{x} + 1)(\sqrt{x} - 1)\)
24. \(f(x) = \frac{9x^{5/2} - 2}{x}\)
25. \(\frac{dy}{dx}\bigg|_{x=2}, y = \frac{x^4 - 4}{x^2 - 5}\)
26. \(f(x) = \frac{x^5 + x^{-1}}{x + 1}\)
27. \(\frac{dz}{dx}\bigg|_{x=1}, z = \frac{1}{x^3 + 1}\)
28. \(f(x) = \frac{3x^3 - x^2 + 2}{\sqrt{x}}\)
29. \(h(x) = \frac{1}{(t + 1)(t^2 + 1)}\)
30. \(f(x) = x^{3/2}(2x^4 - 3x + x^{-1/2})\)
31. \(f(x) = x^{2/3}(x^2 - 1)\)
32. \(h(x) = \pi^2(x - 1)\)
33. \(f(x) = (x + 3)(x - 1)(x - 5)\)
34. \(h(s) = s(s + 4)(x^2 + 1)\)
35. \( f(x) = \frac{\sqrt{3}(x^2 + 1)}{x + 1} \) 
36. \( g(z) = \frac{(z - 2)(z^2 + 1)}{z} \)

37. \( g(z) = \left( \frac{z^2 - 4}{z - 1} \right) \left( \frac{z^2 - 1}{z + 2} \right) \) Hint: Simplify first.

38. \( \frac{d}{dx} \left( (ax + b)(bx^2 + 1) \right) \) (a, b constants)

39. \( \frac{d}{dt} \left( \frac{x + 4}{t^2 - x} \right) \) (x constant)

40. \( \frac{d}{dx} \left( \frac{ax + b}{cx + d} \right) \) (a, b, c, d constants)

In Exercises 41–44, calculate \( f'(x) \) in terms of \( P(x), Q(x), \) and \( R(x), \) assuming that \( P'(x) = Q(x), \) \( Q'(x) = -R(x), \) and \( R'(x) = P(x). \)

41. \( f(x) = xR(x) + Q(x) \)

42. \( f(x) = Q(x)P(x) \)

43. \( f(x) = \frac{P(x)}{Q(x)} \)

44. \( f(x) = \frac{Q(x)R(x)}{P(x)} \)

In Exercises 45–48, calculate the derivative using the values:

\[
\begin{array}{cccc}
  f(4) & f'(4) & g(4) & g'(4) \\
  10 & -2 & 5 & -1 \\
\end{array}
\]

45. \( (fg)'(4) \) and \( (f/g)'(4) \)

46. \( F'(4), \) where \( F(x) = x^2f(x) \)

47. \( G'(4), \) where \( G(x) = (g(x))^2 \)

48. \( H'(4), \) where \( H(x) = \frac{x}{g(x)f(x)} \)

In Exercises 49 and 50, a rectangle's length \( L(t) \) and width \( W(t) \) (measured in inches) are varying in time \( t \) (in minutes). Determine \( A(t) \) in each case. Is the area increasing or decreasing at that time?

49. At \( t = 3, \) we have \( L(3) = 4, \) \( W(3) = 6, \) \( L'(3) = -4, \) and \( W'(3) = 5. \)

50. At \( t = 6, \) we have \( L(6) = 6, \) \( W(6) = 3, \) \( L'(6) = 5, \) and \( W'(6) = -2. \)

51. Calculate \( F'(0), \) where

\[
F(x) = \frac{x^9 + x^8 + 4x^5 - 7x}{x^4 - 3x^2 + 2x + 1}
\]

Hint: Do not calculate \( F'(x). \) Instead, write \( F(x) = f(x)g(x) \) and express \( F'(0) \) directly in terms of \( f(0), f'(0), g(0), g'(0). \)

52. Proceed as in Exercise 51 to calculate \( F'(0), \)

\[
F(x) = \left( 1 + x + x^4/3 + x^{3/2} \right) \frac{3x^2 + 5x^4 + 5x + 1}{8x^9 - 7x^4 + 1}
\]

53. Verify the formula \( x^3y = 3x^2y \) by writing \( x^2 = x \cdot x \cdot x \) and applying the Product Rule.

54. \( \text{GU} \) Plot the derivative of \( f(x) = \sqrt{x^2 + 1} \) over \([-4, 4]. \) Use the graph to determine the intervals on which \( f'(x) > 0 \) and \( f'(x) < 0. \) Then plot \( f \) and describe how the sign of \( f'(x) \) is reflected in the graph of \( f. \)

55. \( \text{GU} \) Plot \( f(x) = x/(x^2 - 1). \) Use the plot to determine whether \( f'(x) \) is positive or negative on its domain \([x : x \neq \pm 1]. \) Then compute \( f'(x) \) and confirm your conclusion algebraically.

56. Let \( P = \sqrt{R} + R^2 \) as in Example 6. Calculate \( dP/dr, \) assuming that \( r \) is variable and \( R \) is constant.

57. Find all values of \( a \) such that the tangent line to

\[
f(x) = \frac{x - 1}{x + 8}
\]

passes through the origin (Figure 5).

58. Current \( I \) (amperes), voltage \( V \) (volts), and resistance \( R \) (ohms) in a circuit are related by Ohm's Law, \( I = V/R. \)

(a) Calculate \( \frac{dI}{dR} \) if \( V \) is constant with value \( V = 24. \)

(b) Calculate \( \frac{dI}{dR} \) if \( I \) is constant with value \( I = 4. \)

59. The revenue per month earned by the Couture clothing chain at time \( t \) is \( R(t) = N(t)/S(t), \) where \( N(t) \) is the number of stores and \( S(t) \) is average revenue per store per month. Couture embarks on a two-part campaign: (A) to build new stores at a rate of \( 5 \) stores per month, and (B) to use advertising to increase average revenue per store at a rate of \( 10 \) thousand per month. Assume that \( N(0) = 50 \) and \( S(0) = 150,000. \)

(a) Show that total revenue will increase at the rate

\[
\frac{dR}{dt} = 55(t + 10,000)N(t)
\]

Note that the two terms in the Product Rule correspond to the separate effects of increasing the number of stores and the average revenue per store.

(b) Calculate \( \frac{dR}{dt} \) when \( t = 0. \)

(c) If Couture can implement only one leg (A or B) of its expansion at \( t = 0, \) which choice will grow revenue most rapidly?

60. The tip speed ratio of a turbine is the ratio \( T = W/T, \) where \( T \) is the speed of the tip of a blade and \( W \) is the speed of the wind. Engineers have found empirically that a turbine with a blades extracts maximum power from the wind when \( T = 2 \) or \( n. \) Calculate \( dR/dt \) (in minutes) if \( W = 35 \) km/h and \( W \) decreases at a rate of \( 4 \) km/h per minute, and the tip speed has constant value \( T = 150 \) km/h.

61. The curve \( y = 1/(x^2 + 1) \) is called the witch of Agnesi (Figure 6) after the Italian mathematician Maria Agnesi (1718–1799). This strange name is the result of a mistranslation of the Italian word \( la \) versiera, meaning "that which turns." Find equations of the tangent lines at \( x = \pm 1. \)

62. Let \( f(x) = g(x) = x. \) Show that \( (gf)' \neq f'g'. \)

63. Use the Product Rule to show that \( (f^3)' = 3ff'. \)

64. Show that \( (f^3)' = 3f^2f'. \)
Further Insights and Challenges

65. Let \( f, g, h \) be differentiable functions. Show that \((fgh)'(x)\) is equal to 
\[ f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \]

Hint: Write \( fgh \) as \( g(fh) \).

66. Prove the Quotient Rule using the limit definition of the derivative.

67. Derivative of the Reciprocal  Use the limit definition to prove 
\[
\frac{d}{dx} \left( \frac{1}{f(x)} \right) = -\frac{f'(x)}{f^2(x)}
\]

Hint: Show that the difference quotient for \(1/f(x)\) is equal to 
\[
\frac{f(x) - f(x + h)}{h} / f(x)(f(x + h))
\]

68. Prove the Quotient Rule using Eq. (7) and the Product Rule.

69. Use the limit definition of the derivative to prove the following special case of the Product Rule:
\[
\frac{d}{dx} (xf(x)) = f(x) + xf'(x)
\]

70. Use the limit definition of the derivative to prove the following special case of the Quotient Rule:
\[
\frac{d}{dx} \left( \frac{f(x)}{x} \right) = \frac{xf'(x) - f(x)}{x^2}
\]

71. The Power Rule Revisited  If you are familiar with proof by induction, use induction to prove the Power Rule for all whole numbers \( n \). Show that the Power Rule holds for \( n = 1 \), then write \( x^n = x \cdot x^{n-1} \) and use the Product Rule.

Exercises 72 and 73: A basic fact of algebra states that \( c \) is a root of a polynomial \( f \) if and only if \( f(x) = (x - c)g(x) \) for some polynomial \( g \). We say that \( c \) is a multiple root if \( f(x) = (x - c)^2 h(x) \), where \( h \) is a polynomial.

72. Show that \( c \) is a multiple root of \( f \) if and only if \( c \) is a root of both \( f \) and \( f' \).

73. Use Exercise 72 to determine whether \( c = -1 \) is a multiple root.
(a) \( x^5 + 2x^4 - 4x^3 - 8x^2 - x + 2 \)
(b) \( x^4 + x^3 - 5x^2 - 3x + 2 \)

74. Figure 7 is the graph of a polynomial with roots at \( A, B, \) and \( C \). Which of these is a multiple root? Explain your reasoning using Exercise 72.

3.4 Rates of Change

In this section, we pause from building tools for computing the derivative and instead focus on the derivative as a rate of change, particularly in applied settings.

Recall the notation for the average rate of change of a function \( y = f(x) \) over an interval \([x_0, x_1] \):
\[
\Delta y = \text{change in } y = f(x_1) - f(x_0) \\
\Delta x = \text{change in } x = x_1 - x_0 \\
'\text{average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}
\]

In our prior discussion in Section 2.1, limits and derivatives had not yet been introduced. Now that we have them at our disposal, we can define the **instantaneous** rate of change of \( y \) with respect to \( x \) at \( x = x_0 \):
\[
\text{instantaneous rate of change} = f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}
\]

Keep in mind the geometric interpretations: The average rate of change is the slope of the secant line (Figure 1), and the instantaneous rate of change is the slope of the tangent line (Figure 2).
Leibniz notation \( \frac{dy}{dx} \) is particularly convenient because it specifies that we are considering the rate of change of \( y \) with respect to the independent variable \( x \). The rate \( \frac{dy}{dx} \) is measured in units of \( y \) per unit of \( x \). For example, the rate of change of temperature with respect to time has units such as degrees per minute, whereas the rate of change of temperature with respect to altitude has units such as degrees per kilometer. In applications, it is important to be mindful of the units on rates of change and to interpret properly what the rate of change is communicating about the variables.

**EXAMPLE 1** Surface Temperatures During an Eclipse Since the moon has no atmosphere to help moderate the temperature, its surface experiences large extremes in temperature (typically between \(-150^\circ C\) and \(120^\circ C\)). Furthermore, the primary source of heat is direct radiation from the sun, so a location experiences its highest temperatures when in direct sunlight and lowest when in darkness. With a "day" on the moon being 29.5 Earth days, a location on the moon will cycle between hottest and coldest temperatures over such a period of time. The only situation where this temperature-change cycle is disrupted is when the earth blocks the sun (that is, when a lunar eclipse is seen from Earth, as in Figure 3). During an eclipse, the temperature on the moon drops quickly, but then rebounds once the earth passes by the sun.

Table 1 contains data on the temperature (\( T \), in \( ^\circ C \)) at a location on the moon \( t \) minutes into an eclipse.

<table>
<thead>
<tr>
<th>( t ) (min)</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
<th>140</th>
<th>160</th>
<th>180</th>
<th>200</th>
<th>220</th>
<th>240</th>
<th>260</th>
<th>280</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ) (( ^\circ C ))</td>
<td>7</td>
<td>45</td>
<td>37</td>
<td>33</td>
<td>61</td>
<td>78</td>
<td>86</td>
<td>91</td>
<td>95</td>
<td>97</td>
<td>99</td>
<td>101</td>
<td>84</td>
<td>33</td>
<td>12</td>
<td>37</td>
</tr>
</tbody>
</table>

(a) Calculate the average rate of change of temperature \( T \) from the start of the eclipse to the time \( t^* \) when the temperature was coldest and the average rate of change from \( t^* \) to the end of the eclipse.

(b) Use the difference quotient approximation to estimate the rate of change of the temperature 60, 160, and 260 min into the eclipse.

**Solution**

(a) From the table, we use \( t^* = 220 \). The average rate of change from eclipse start to \( t^* \) is \( \frac{\Delta T}{\Delta t} = \frac{61-(-33)}{220} \approx -0.74^\circ C/min \). The average rate of change from \( t^* \) to eclipse end is \( \frac{\Delta T}{\Delta t} = \frac{-33-(-99)}{100} \approx 0.76^\circ C/min \).

(b) The rate of change at \( t = 60 \) is approximately \( \frac{-61-(-33)}{20} = -1.40^\circ C/min \).

The rate of change at \( t = 160 \) is approximately \( \frac{-33-(-99)}{20} = -0.10^\circ C/min \).

The rate of change at \( t = 260 \) is approximately \( \frac{-99-(-33)}{20} = 2.25^\circ C/min \).

For comparison, note that under normal circumstances, if the moon heats from \(-150^\circ C\) to \(120^\circ C\) in one-half of a lunar day (21,240 min), then the average rate of change of temperature is \( \frac{120-(-150)}{21,240} \approx 0.013^\circ C/min \).
EXAMPLE 2 Let \( A = \pi r^2 \) be the area of a circle of radius \( r \).

(a) Compute \( dA/dr \) at \( r = 2 \) and \( r = 5 \).

(b) Explain geometrically why \( dA/dr \) is greater at \( r = 5 \) than at \( r = 2 \).

**Solution** The rate of change of area with respect to radius is the derivative

\[
\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r
\]

(a) We have

\[
\left. \frac{dA}{dr} \right|_{r=2} = 2\pi(2) \approx 12.57 \quad \text{and} \quad \left. \frac{dA}{dr} \right|_{r=5} = 2\pi(5) \approx 31.42
\]

(b) The derivative \( dA/dr \) measures how the area of the circle changes when \( r \) increases. Figure 4 shows that when the radius increases by \( \Delta r \), the area increases by a band of thickness \( \Delta r \). The area of the band is greater at \( r = 5 \) than at \( r = 2 \). Therefore, the derivative is larger (and the tangent line is steeper) at \( r = 5 \). In general, for a fixed \( \Delta r \), the change in area \( \Delta A \) is greater when \( r \) is larger.

**Marginal Cost in Economics**

Let \( C(x) \) denote the dollar cost (including labor and parts) of producing \( x \) units of a particular product. The number \( x \) of units manufactured is called the production level. To study the relation between costs and production, economists define the marginal cost at production level \( x_0 \) as the cost of producing one additional unit:

\[
\text{marginal cost} = C(x_0 + 1) - C(x_0)
\]

Note that if we use a difference quotient approximation for the derivative \( C'(x_0) \) with \( h = 1 \), we obtain

\[
C'(x_0) \approx \frac{C(x_0 + 1) - C(x_0)}{1} = C(x_0 + 1) - C(x_0)
\]

and therefore we can use the derivative at \( x_0 \) as an approximation to the marginal cost.

EXAMPLE 3 Cost of an Air Flight

Company data suggest that when there are 50 or more passengers, the total dollar cost of a certain flight is approximately \( C(x) = 0.0005x^3 - 0.38x^2 + 120x \), where \( x \) is the number of passengers (Figure 5).

(a) Estimate the marginal cost of an additional passenger if the flight already has 150 passengers.

(b) Compare your estimate with the actual cost of an additional passenger.

(c) Is it more expensive to add a passenger when \( x = 150 \) or when \( x = 200 \)?

**Solution** The derivative is \( C'(x) = 0.0015x^2 - 0.76x + 120 \).

(a) We estimate the marginal cost at \( x = 150 \) by the derivative

\[
C'(150) = 0.0015(150)^2 - 0.76(150) + 120 = 39.75
\]

Thus, it costs approximately $39.75 to add one additional passenger.

(b) The actual cost of adding one additional passenger is

\[
C(151) - C(150) \approx 11,177.10 - 11,137.50 = 39.60
\]

Our estimate of $39.75 is close enough for practical purposes.
(c) The marginal cost at \( x = 200 \) is approximately
\[
C'(200) = 0.0015(200)^2 - 0.76(200) + 120 = 28
\]
Since 39.75 > 28, it is more expensive to add a passenger when \( x = 150 \) than when \( x = 200 \).

### Linear Motion

Recall that linear motion is motion along a straight line. This includes horizontal motion along a straight highway and vertical motion of a falling object. Let \( s(t) \) denote the position on a line, relative to the origin, at time \( t \). Velocity is the rate of change of position with respect to time:

\[
v(t) = \text{velocity} = \frac{ds}{dt}
\]

The sign of \( v(t) \) indicates the direction of motion. For example, if \( s(t) \) is the height above ground, then \( v(t) > 0 \) indicates that the object is rising. Speed is defined as the absolute value of velocity, \( |v(t)| \).

**EXAMPLE 4** A truck enters the off-ramp of a highway at \( t = 0 \). Its position on the off-ramp after \( t \) seconds is \( s(t) = 25t - 0.3t^3 \) m for \( 0 \leq t \leq 5 \).

(a) How fast is the truck going at the moment it enters the off-ramp?
(b) Is the truck speeding up or slowing down?

**Solution** The truck’s velocity at time \( t \) is \( v(t) = \frac{ds}{dt} = (25t - 0.3t^3) = 25 - 0.9t^2 \).

(a) The truck enters the off-ramp with velocity \( v(0) = 25 \) m/s.
(b) Since \( v(t) = 25 - 0.9t^2 \) is decreasing and positive (Figure 6), the speed is decreasing and the truck is slowing down.

When we say “speeding up” or “slowing down” we typically are referring to the speed of an object, not its velocity. The relationship between speed and velocity is simple: Speed is the absolute value of velocity. Use care to apply velocity and speed properly when describing an object’s motion. For instance, as the next example demonstrates, an object’s velocity can increase while its speed decreases.

**EXAMPLE 5** Velocity and Speed Figure 7 shows graphs of an object in linear motion whose position \( s \) is changing in time \( t \) in four different circumstances.

(a) In which cases is the velocity increasing? Decreasing?
(b) In which cases is the speed increasing (so the object is speeding up)? Decreasing (so the object is slowing down)?

**Solution**

(a) In Figure 7(A), the slope is positive and getting larger, so the velocity (or rate change) is increasing.
(b) In Figure 7(B), the slope is positive and getting smaller, so the velocity is decreasing.
• In Figure 7(C), the slope is negative and getting smaller; that is, getting closer to zero. Since the slope values are negative and approaching zero, the slope is increasing, and therefore the velocity is increasing.

• In Figure 7(D), the slope is negative and is getting larger in the negative direction, so the velocity is decreasing.

(b) Now we are considering the absolute value of the velocity; that is, the absolute value of the slope of the graph. It (and therefore speed) increases when the slope gets steeper, and that occurs in both Figures 7(A) and 7(D). Thus, in both of those cases the object is speeding up. On the other hand, in Figures 7(B) and 7(C), the slopes are getting less steep and therefore are getting smaller. Thus, in those cases the speed is decreasing and the object is slowing down.

Notice that Figure 7(C) depicts a situation where the velocity is increasing but the object is slowing down.

Suppose s is the distance between a car and a wall during a crash test, and assume that during the test the car continued to speed up until it hit the wall. Which of the four graphs above best represents s(t)? This question, and others like it, is addressed in Exercises 15–16.

EXAMPLE 6 Describe the motion and velocities of a shuttle train that runs on a straight track at the airport, ferrying passengers from Terminal 1 to Terminal 2 according to the graph given in Figure 8. Assume that s represents the distance from Terminal 1 in meters, t represents time in minutes, and the terminals are 800 m apart.

Solution  Note that the graph has portions resembling each of the four graphs in Example 5. Analyzing the motion:

• The train starts at rest, but then speeds up with increasing positive velocity for the first 2 min.
• Over the interval [2, 4], the velocity remains positive, but begins decreasing as the graph becomes less steep. The train is slowing down as it approaches Terminal 2.
• In the interval [4, 6], the graph is flat with slope 0. In this interval the train is stopped at Terminal 2.
• The train speeds up again at t = 6, now with negative velocity since the distance to Terminal 1 is decreasing. Furthermore, since the graph has a negative slope and is getting steeper, the velocity is decreasing and getting larger, indicating that the train is speeding up.
• Over the interval [8, 10], the velocity remains negative, but gets smaller as the graph become less steep. The train is slowing down as it approaches and arrives back at Terminal 1.

Motion Under the Influence of Gravity

Galileo discovered that the height s(t) and velocity v(t) at time t (seconds) of an object tossed vertically in the air near the earth's surface are accurately represented by the formulas

\[s(t) = s_0 + v_0 t - \frac{1}{2} g t^2, \quad v(t) = \frac{ds}{dt} = v_0 - gt\]

The constants s_0 and v_0 are the initial values:

• s_0 = s(0), the position at time t = 0.
• v_0 = v(0), the velocity at t = 0.
• −g is the acceleration due to gravity on the surface of the earth (negative because the up direction is positive), where
  \[g \approx 9.8 \text{ m/s}^2 \text{ or } g \approx 32 \text{ ft/s}^2\]
A simple observation enables us to find the object's maximum height. Since velocity is positive as the object rises and negative as it falls back to Earth, the object reaches its maximum height at the moment of transition, when it is no longer rising and has not yet begun to fall. At that moment, its velocity is zero. In other words, the maximum height is attained when $v(t) = 0$. At this moment, the tangent line to the graph of $s$ is horizontal (Figure 9).

**EXAMPLE 7 Finding the Maximum Height** A projectile is launched upward from ground level with an initial velocity of 30 m/s.

(a) Find the velocity at $t = 2$ and at $t = 4$. Explain the change in sign.

(b) What is the projectile's maximum height and when does it reach that height?

**Solution** Apply Eq. (2) with $s_0 = 0$, $v_0 = 30$, and $g = 9.8$:

$$s(t) = s_0 + v_0t - \frac{1}{2}gt^2, \quad v(t) = 30 - 9.8t$$

(a) Therefore,

$$v(2) = 30 - 9.8(2) = 10.4 \text{ m/s}, \quad v(4) = 30 - 9.8(4) = -9.2 \text{ m/s}$$

At $t = 2$, the projectile is rising, and its velocity $v(2)$ is positive (Figure 9). At $t = 4$, the projectile is on the way down, and its velocity $v(4)$ is negative.

(b) Maximum height is attained when the velocity is zero, so we solve

$$30 - 9.8t = 0 \quad \Rightarrow \quad t = \frac{150}{49} \approx 3.06$$

The projectile reaches maximum height at $t = 150/49$ s. Its maximum height is

$$s(150/49) = 30(150/49) - 4.9(150/49)^2 \approx 45.92 \text{ m}$$

---

**HISTORICAL PERSPECTIVE**

Galileo Galilei (1564–1642) discovered the laws of motion for falling objects on the earth’s surface around 1600. This paved the way for Newton’s general laws of motion. How did Galileo arrive at his formulas? The motion of a falling object is too rapid to measure directly, without modern photographic or electronic apparatus. To get around this difficulty, Galileo experimented with balls rolling down an incline (Figure 10). For a sufficiently flat incline, he was able to measure the motion with a water clock and found that the velocity of the rolling ball is proportional to time. He then reasoned that motion in free-fall is just a faster version of motion down an incline and deduced the formula $v(t) = -gt$ for falling objects (assuming zero initial velocity).

Prior to Galileo, it had been assumed incorrectly that heavy objects fall more rapidly than lighter ones. Galileo realized that this was not true (as long as air resistance is negligible), and indeed, the formula $v(t) = -gt$ shows that the velocity depends on time but not on the weight of the object. Interestingly, 300 years later, another great physicist, Albert Einstein, was deeply puzzled by Galileo’s discovery that all objects fall at the same rate regardless of their weight. He called this the Principle of Equivalence and sought to understand why it was true. In 1916, after a decade of intensive work, Einstein developed the General Theory of Relativity, which finally gave a full explanation of the Principle of Equivalence in terms of the geometry of space and time.
3.4 SUMMARY

- The (instantaneous) rate of change of \( y = f(x) \) with respect to \( x \) at \( x = x_0 \) is defined as the derivative

\[
 f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}
\]

- The rate \( dy/dx \) is measured in units of \( y \) per unit of \( x \).
- Marginal cost is the cost of producing one additional unit. If \( C(x) \) is the cost of producing \( x \) units, then the marginal cost at production level \( x_0 \) is \( C(x_0 + 1) - C(x_0) \). The derivative \( C'(x_0) \) is often a good estimate for marginal cost.
- For linear motion, velocity \( v(t) \) is the rate of change of position \( s(t) \) with respect to time—that is, \( v(t) = s'(t) \).
- Galileo’s formulas for an object rising or falling under the influence of gravity near Earth’s surface ignoring air resistance (\( v_0 = \) initial velocity):

\[
 s(t) = s_0 + v_0 t - \frac{1}{2} gt^2, \quad v(t) = v_0 - gt
\]

where \( g \approx 9.8 \text{ m/s}^2 \), or \( g \approx 32 \text{ ft/s}^2 \). Maximum height is attained when \( v(t) = 0 \).

3.4 EXERCISES

Preliminary Questions

1. Which units might be used for each rate of change?
   
   (a) Pressure (in atmospheres) in a water tank with respect to depth
   
   (b) The rate of a chemical reaction (change in concentration with respect to time with concentration in moles per liter)

2. Two trains travel from New Orleans to Memphis in 4 h. The first train travels at a constant velocity of 90 mph, but the velocity of the second train varies. What was the second train’s average velocity during the trip?

Exercises

In Exercises 1–8, find the rate of change.

1. Area of a square with respect to its side \( s \) when \( s = 5 \)

2. Volume of a cube with respect to its side \( s \) when \( s = 5 \)

3. Cube root \( \sqrt[3]{x} \) with respect to \( x \) when \( x = 1, 8, 27 \)

4. The reciprocal \( 1/x \) with respect to \( x \) when \( x = 1, 2, 3 \)

5. The diameter of a circle with respect to radius

6. Surface area \( A \) of a sphere with respect to radius \( r \) (\( A = 4\pi r^2 \))

7. Volume \( V \) of a cylinder with respect to radius if the height is equal to the radius

8. Speed of sound \( v \) (in m/s) with respect to air temperature \( T \) (in kelvins), where \( v = 20\sqrt{T} \)

In Exercises 9–11, refer to Figure 11, the graph of distance \( s \) from the origin as a function of time for a car trip.

3. Discuss how it is possible to be speeding up with a velocity that is decreasing.

4. Sketch the graph of a function that has an average rate of change equal to zero over the interval \([0,1]\) but has instantaneous rates of change at 0 and 1 that are positive.

9. Find the average velocity over each interval.
   
   (a) \([0,0.5]\)  \hspace{1cm} (b) \([0.5,1]\)  \hspace{1cm} (c) \([1,1.5]\)  \hspace{1cm} (d) \([1,2]\)

10. At what time is velocity at a maximum?

11. Match the descriptions (i)–(iii) with the intervals (a)–(c) in Figure 11.
   
   (i) Velocity increasing
   
   (ii) Velocity decreasing
   
   (iii) Velocity negative
   
   (a) \([0,0.5]\)  \hspace{1cm} (b) \([2.5,3]\)  \hspace{1cm} (c) \([1.5,2]\)

**FIGURE 11** Distance from the origin versus time for a car trip.
Exercises 12 and 13 refer to the data in Example 1. Approximate the derivative with the symmetric difference quotient (SDQ) approximation:

\[ T'(t) \approx \frac{T(t + 20) - T(t - 20)}{40} \]

12. (a) At what \( t \) does the SDQ approximation give the fastest rate of increase of temperature? What is the rate of change?
(b) At what \( t \) does the SDQ approximation give the fastest rate of decrease of temperature? What is the rate of change?
13. At what \( t \) does the SDQ approximation give the smallest (i.e., closest to 0) rate of change of temperature? What is the rate of change?
Exercises 14–16 refer to the four graphs of \( s \) as a function of \( t \) in Figure 7.
14. Sketch \( s' \) for each of the four graphs of \( s \).
15. Match each situation with the graph that best represents it.
(a) Rocky slowed down his car as it approached the moose in the road. The distance from the car to the moose is \( s \) and the time since he spotted the moose is \( t \).
(b) The rocket's speed increased after liftoff until the fuel was used up. The distance from the rocket to the launchpad is \( s \) and the time since liftoff is \( t \).
(c) The increase in college costs slowed for the fourth year in a row. The cost of college is \( s \) and the time since the start of the 4-year period is \( t \).
16. Match each situation with the graph that best represents it.
(a) Dusty’s batting average increased over the first 10 games of the season but then came back to the same amount of increase the season went down. Dusty’s batting average is \( s \) and the time since the beginning of the season is \( t \).
(b) In performing a crash test, the car continued to speed up until it hit the wall. The distance between the car and the wall is \( s \) and the time since the car started moving is \( t \).
(c) The hurricane strengthened at an increasing rate over the first day of its development. The strength of the hurricane is \( s \), and the time since it started developing is \( t \).
17. Sketch a graph of velocity as a function of time for the shuttle train in Example 6.
18. Figure 12 shows the height \( y \) of a mass oscillating at the end of a spring, through one cycle of the oscillation. Sketch the graph of velocity as a function of time.

![](image1)

19. Fred X has to make a book delivery from his warehouse, 15 mi north of the city, to the Amazing Book Store 10 mi south of the city. Traffic is usually congested within 5 mi of the city. He leaves at noon, traveling due south through the city, and arrives at the store at 12:50. After 15 min at the store, he makes the return trip north to his warehouse, arriving at 2:00. Let \( s \) represent the distance from the warehouse in miles and \( t \) represent time in minutes since noon. Make sketches of the graphs of \( s \) and \( s' \) as functions of \( t \) for Fred’s trip.

20. At the start of the 27th century, the population of Zosania was approximately 40 million. Early-century prosperity saw the population nearly double in the first three decades, but the growth slowed in the 30s and 40s and then leveled off completely during the war years in the 50s. A postwar boom saw another rapid population increase, but that turned around in a major decline resulting from the great famine of the 70s. A slow end-century rebound resulted in an increase of the population to approximately 90 million at century's end. Let \( P \) represent the population in millions and \( t \) represent time in years since the start of the century. Make sketches of the graphs of \( P \) and \( P' \) as functions of \( t \) for Zosania's population during the century.
21. The velocity (in centimeters per second) of blood molecules flowing through a capillary of radius 0.008 cm is \( v = 6.4 \times 10^{-8} - 0.001 \), where \( r \) is the distance from the molecule to the center of the capillary. Find the rate of change of velocity with respect to \( r \) when \( r = 0.004 \) cm.
22. Figure 13 displays the voltage \( V \) across a capacitor as a function of time while the capacitor is being charged. Estimate the rate of change of voltage at \( t = 20 \) s. Indicate the values in your calculation and include proper units. Does voltage change more quickly or more slowly as time goes on? Explain in terms of tangent lines.

![](image2)

23. Use Figure 14 to estimate \( dV/dt \) at \( h = 30 \) and 70, where \( T \) is atmospheric temperature (in degrees Celsius) and \( h \) is altitude (in kilometers). Where is \( dV/dh \) equal to zero?

![](image3)

24. The earth exerts a gravitational force of \( F(r) = (2.99 \times 10^{16})/r^2 \) newtons on an object with a mass of 75 kg located \( r \) meters from the center of the earth. Find the rate of change of force with respect to distance \( r \) at the surface of the earth.
25. For the escape velocity relationship, \( v_\text{esc} = (2.82 \times 10^3)r^{-1/2} \) m/s, calculate the rate of change of the escape velocity with respect to distance \( r \) from the center of the earth.
26. The power delivered by a battery to an apparatus of resistance \( R \) (in ohms) is \( P = 2.25R/(R + 0.5)^2 \) watts (W). Find the rate of change of power with respect to resistance for \( R = 3 \) \( \Omega \) and \( R = 5 \) \( \Omega \).

27. A particle moving along a line has position \( s(t) = t^4 - 18t^2 + 1 \) m at time \( t \) seconds. At which times does the particle pass through the origin? At which times is the particle instantaneously motionless (i.e., has zero velocity)?

28. Graph the position of the particle in Exercise 27. What is the farthest distance to the left of the origin attained by the particle?

29. A projectile is launched in the air from the ground with an initial velocity \( v_0 = 60 \text{ m/s} \). What is the maximum height that the projectile reaches?

30. Find the velocity of an air conditioner accidentally dropped from a height of 300 m at the moment it hits the ground.

31. A ball tossed in the air vertically from ground level returns to Earth 4 s later. Find the initial velocity and maximum height of the ball.

32. Olivia is gazing out a window from the 10th floor of a building when a bucket (dropped by a window washer) passes by. She notes that it hits the ground 1.5 s later. Determine the floor from which the bucket was dropped if each floor is 5 m high and the window is in the middle of the 10th floor. Neglect air friction.

33. Show that for an object falling according to Galileo’s formula, the average velocity over any time interval \([t_1, t_2]\) is equal to the average of the instantaneous velocities at \( t_1 \) and \( t_2 \).

34. An object falls under the influence of gravity near the earth’s surface. Which of the following statements is true? Explain.
   (a) Distance traveled increases by equal amounts in equal time intervals.
   (b) Velocity increases by equal amounts in equal time intervals.
   (c) The derivative of velocity increases with time.

35. By Faraday’s Law, if a conducting wire of length \( L \) meters moves at velocity \( v \) m/s perpendicular to a magnetic field of strength \( B \) (in teslas), a voltage of size \( E = BLv \) is induced in the wire. Assume that \( B = 2 \) and \( L = 0.5 \).
   (a) Calculate \( dV/dt \).
   (b) Find the rate of change of \( V \) with respect to time \( t \) if \( v(t) = 4t + 9 \).

36. The voltage \( V \), current \( I \), and resistance \( R \) in a circuit are related by Ohm’s Law: \( V = IR \), where the units are volts, amperes, and ohms. Assume that voltage is constant with \( V = 12 \) volts (V). Calculate (specifying units):
   (a) The average rate of change of \( I \) with respect to \( R \) for the interval from \( R = 8 \) to \( R = 8.1 \).
   (b) The rate of change of \( I \) with respect to \( R \) when \( R = 8 \).
   (c) The rate of change of \( R \) with respect to \( I \) when \( I = 1.5 \).

37. Ethan finds that with 4 hours of tutoring, he is able to answer correctly 85% of the problems on a math exam. Which would you expect to be larger: \( S'(3) \) or \( S'(30) \)? Explain.

38. Suppose \( \theta(t) \) measures the angle between a clock’s minute and hour hands. What is \( \theta'(t) \) at 3 o’clock?

39. To determine drug dosages, doctors estimate a person’s body surface area (BSA) (in meters squared) using the formula \( \text{BSA} = \sqrt{hm/60} \), where \( h \) is the height in centimeters and \( m \) the mass in kilograms. Calculate the rate of change of BSA with respect to mass for a person of constant height \( h = 180 \). What is this rate at \( m = 70 \) and \( m = 80 \)? Express your result in the correct units. Does BSA increase more rapidly with respect to mass at lower or higher body mass?

40. The atmospheric CO2 level \( A(t) \) at Mauna Loa, Hawaii, at time \( t \) (in parts per million by volume) is recorded by the Scripps Institution of Oceanography. Reading across, the annual values for the 4-year intervals are

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>316.91</td>
<td>319.20</td>
<td>323.05</td>
<td>327.45</td>
<td>332.15</td>
<td>338.69</td>
<td>344.42</td>
</tr>
<tr>
<td>Value</td>
<td>351.48</td>
<td>356.37</td>
<td>362.64</td>
<td>369.48</td>
<td>377.38</td>
<td>385.34</td>
<td>393.87</td>
</tr>
</tbody>
</table>


(b) In which of the years from (a) did the approximation to \( A'(t) \) take on its largest and smallest values?

(c) In which of these years does the approximation suggest that the CO2 level was the most constant?

41. The tangent lines to the graph of \( f(x) = x^2 \) grow steeper as \( x \) increases. At what rate do the slopes of the tangent lines increase?

42. According to Kleiber's Law, the metabolic rate \( P \) (in kilocalories per day) and body mass \( m \) (in kilograms) of an animal are related by a three-quarter-power law \( P = 73.3m^{3/4} \). Estimate the increase in metabolic rate when body mass increases from 60 to 61 kg.

43. The dollar cost of producing \( x \) bagels is given by the function \( C(x) = 300 + 0.25x - 0.5(x/1000)^2 \). Determine the cost of producing 2000 bagels and estimate the cost of the 2001st bagel. Compare your estimate with the actual cost of the 2001st bagel.

44. Suppose that for \( x > 1000 \), the dollar cost of producing \( x \) video cameras is \( C(x) = 500x - 0.003x^2 + 10^4x^2 \). Estimate the marginal cost at production level \( x = 5000 \) and compare it with the actual cost \( C(5001) - C(5000) \).

45. According to Stevens's Law in psychology, the perceived magnitude of a stimulus is proportional (approximately) to a power of the actual intensity \( I \) of the stimulus. Experiments show that the perceived brightness \( B \) of a light satisfies \( B = kI^{1/2} \), where \( I \) is the light intensity, whereas the perceived heaviness \( H \) of a weight \( W \) satisfies \( H = kW^{3/2} \) \( (k \) is a constant that is different in the two cases). Compute \( dH/dW \) and \( dB/dI \) and state whether they are increasing or decreasing functions. Then explain the following statements:
   (a) An increase in intensity is felt more strongly when \( I \) is small than when \( I \) is large.
   (b) An increase in load \( W \) is felt more strongly when \( W \) is large than when \( W \) is small.

46. Let \( M(t) \) be the mass (in kilograms) of a plant as a function of time (in years). Recent studies by Niklas and Enquist have suggested that a remarkably wide range of plants (from algae and grass to palm trees) obey a three-quarter-power growth law—that is, \( \frac{dM}{dt} = CM^{3/4} \) for some constant \( C \).

(a) If a tree has a growth rate of 6 kg/yr when \( M = 100 \) kg, what is its growth rate when \( M = 125 \) kg?

(b) If \( M = 0.5 \) kg, how much more mass must the plant acquire to double its growth rate?
Further Insights and Challenges

Exercises 47–49: The Lorenz curve \( y = F(r) \) is used by economists to study income distribution in a given country (see Figure 15). By definition, \( F(r) \) is the fraction of the total income that goes to the bottom \( r \)th part of the population, where \( 0 \leq r \leq 1 \). For example, if \( F(0.4) = 0.245 \), then the bottom 40% of households receive 24.5% of the total income. Note that \( F(0) = 0 \) and \( F(1) = 1 \).

48. The following table provides values of \( F(r) \) for the United States in 2010. Assume that the national average income was \( A = \$66,000 \).

<table>
<thead>
<tr>
<th>r</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(r) )</td>
<td>0.033</td>
<td>0.118</td>
<td>0.264</td>
<td>0.480</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

(a) What was the average income in the lowest 40% of households?
(b) Show that the average income of the households belonging to the interval \([0.4, 0.6] \) was \$48,180.
(c) Estimate \( F'(0.5) \). Estimate the income of households in the 50th percentile. Was it greater or less than the national average?

49. Use Exercise 47(c) to prove:
(a) \( F'(r) \) is an increasing function of \( r \).
(b) Income is distributed equally (all households have the same income) if and only if \( F(r) = r \) for \( 0 \leq r \leq 1 \).

In Exercises 50 and 51, the average cost per unit at production level \( x \) is defined as
\[
C_{avg}(x) = \frac{C(x)}{x}
\]
where \( C(x) \) is the cost of producing \( x \) units. Average cost is a measure of the efficiency of the production process.

50. The cost in dollars of producing alarm clocks is given by
\[
C(x) = 50x^3 - 750x^2 + 3740x + 3750
\]
where \( x \) is in units of 1000.
(a) Calculate the average cost at \( x = 4, 6, 8, \) and 10.
(b) Use the graphical interpretation of average cost to find the production level \( x_0 \) at which average cost is lowest. What is the relationship between average cost and marginal cost at \( x_0 \) (see Figure 16)?

51. Show that \( C_{avg}(x) \) is equal to the slope of the line through the origin and the point \((x, C(x))\) on the graph of \( y = C(x) \). Using this interpretation, determine whether average cost or marginal cost is greater at points A, B, C, D in Figure 17.
3.5 Higher Derivatives

Higher derivatives are obtained by repeatedly differentiating a function \( y = f(x) \). If \( f' \) is differentiable, then the second derivative, denoted \( f'' \) or \( y'' \), is the derivative

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)
\]

For example, for \( f(x) = x^2 + \frac{1}{x} + \sqrt{x} \), we have

\[
f'(x) = 2x - x^{-2} + \frac{1}{2}x^{-1/2}
\]

\[
f''(x) = 2 + 2x^{-3} - \frac{1}{4}x^{-3/2}
\]

The second derivative is the rate of change of \( f'(x) \), so it is the rate of change of the rate of change of \( f \).

The next example highlights the difference between the first and second derivatives.

**EXAMPLE 1** Figure 1 and Table 1 show the number of cell phone subscribers \( C(t) \) in the United States in year \( t \). Discuss \( C'(t) \) and \( C''(t) \).

<table>
<thead>
<tr>
<th>Year</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number in millions</td>
<td>277</td>
<td>301</td>
<td>316</td>
<td>326</td>
</tr>
<tr>
<td>Change in ( C )</td>
<td>24</td>
<td>15</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Change in the change in ( C )</td>
<td>-9</td>
<td>-5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Solution** We will show that \( C'(t) \) is positive but \( C''(t) \) is negative. According to Table 1, the number of cell phone subscribers each year was greater than the previous year, so the rate of change \( C'(t) \) is certainly positive. However, the amount of increase declined from 24 million in 2010 to 15 million in 2011 to 10 million in 2012. Thus, \( C'(t) \) is positive, but \( C'(t) \) decreases from one year to the next, and therefore its rate of change \( C''(t) \) is negative. Figure 1 supports this conclusion: The slopes of the segments in the graph are positive [\( C'(t) \) is positive], but the slopes decrease going from one segment to the next [\( C''(t) \) is negative].

The process of differentiation can be continued, provided that the derivatives exist. The third derivative, denoted \( f'''(x) \) or \( f^{(3)}(x) \), is the derivative of \( f''(x) \). More generally, the \( n \)th derivative \( f^{(n)}(x) \) is the derivative of the \((n-1)\)st derivative. We use parentheses on the superscript for the derivative to distinguish \( f^{(n)} \), the \( n \)th derivative of \( f \), from \( f^n \), the \( n \)th power of \( f \). We call \( f(x) \) the zeroth derivative and \( f'(x) \) the first derivative. In Leibniz notation, we write

\[
\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \ldots
\]

**EXAMPLE 2** Calculate \( f'''(-1) \) for \( f(x) = 3x^5 - 2x^2 + 7x^{-2} \).

**Solution** We must calculate the first three derivatives:

\[
f'(x) = \frac{d}{dx} (3x^5 - 2x^2 + 7x^{-2}) = 15x^4 - 4x - 14x^{-3}
\]

\[
f''(x) = \frac{d}{dx} (15x^4 - 4x - 14x^{-3}) = 60x^3 - 4 + 42x^{-4}
\]

\[
f'''(x) = \frac{d}{dx} (60x^3 - 4 + 42x^{-4}) = 180x^2 - 168x^{-5}
\]

At \( x = -1 \), \( f'''(-1) = 180 + 168 = 348 \).
Polynomials have a special property: Once \( n \) is large enough, the \( n \)th derivative is the zero function, and therefore so are all higher derivatives. More precisely, if \( f \) is a polynomial of degree \( k \), then \( f^{(n)}(x) \) is zero for \( n > k \). Table 2 illustrates this property for \( f(x) = x^2 \). By contrast, the higher derivatives of a nonpolynomial function are never the zero function (see Exercise 85, Section 5.3).

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
<th>( f'''(x) )</th>
<th>( f^{(4)}(x) )</th>
<th>( f^{(5)}(x) )</th>
<th>( f^{(6)}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^5 )</td>
<td>5( x^4 )</td>
<td>20( x^3 )</td>
<td>60( x^2 )</td>
<td>120( x )</td>
<td>120</td>
<td>0</td>
</tr>
</tbody>
</table>

**EXAMPLE 3** Calculate the first four derivatives of \( y = x^{-1} \). Then find the pattern and determine a general formula for \( y^{(n)} \).

**Solution** By the Power Rule,

\[
\begin{align*}
y'(x) &= -x^{-2}, \\
y''(x) &= 2x^{-3}, \\
y'''(x) &= -2(3)x^{-4}, \\
y^{(4)}(x) &= 2(3)(4)x^{-5}
\end{align*}
\]

First note that we have a leading negative sign on each odd derivative, but not the even derivatives. In a formula for the \( n \)th derivative, a factor of \((-1)^n\) will provide this alternating sign. Ignoring the negative sign, the coefficients can be seen to follow the pattern: 1, (1)(2), (1)(2)(3), (1)(2)(3)(4), and so on. This can be expressed with a factor of \( n! \) in the \( n \)th derivative. Finally, we see that the power on \( x \) is \(-n - 1\) in the \( n \)th derivative. In general, therefore, \( y^{(n)}(x) = (-1)^n n! x^{-n-1} \).

**EXAMPLE 4** Find an equation of the tangent line to \( y = f'(x) \) at \( x = 4 \) where \( f(x) = x^{3/2} \).

**Solution** The slope of the tangent line to \( y = f'(x) \) at \( x = 4 \) is the derivative \( f''(4) \). So we compute the first two derivatives and their values at \( x = 4 \):

\[
\begin{align*}
f'(x) &= \frac{3}{2} x^{1/2}, \\
f'(4) &= \frac{3}{2} (4)^{1/2} = 3 \\
f''(x) &= \frac{3}{4} x^{-1/2}, \\
f''(4) &= \frac{3}{4} (4)^{-1/2} = \frac{3}{8}
\end{align*}
\]

Therefore, an equation of the tangent line is

\[
y - f'(4) = f''(4)(x - 4) \quad \Rightarrow \quad y - 3 = \frac{3}{8}(x - 4)
\]

In slope-intercept form, the equation is \( y = \frac{9}{8}x + \frac{15}{8} \).

A second derivative that you might be familiar with is acceleration. An object that is in linear motion with position \( s(t) \) at time \( t \) has velocity \( v(t) = s'(t) \) and acceleration \( a(t) = v'(t) = s''(t) \). Thus, acceleration is the rate at which velocity changes and is measured in units of velocity per unit of time or “distance per time squared,” such as m/s².

**EXAMPLE 5** **Acceleration Due to Gravity** Find the acceleration \( a(t) \) of a ball tossed vertically in the air from ground level with an initial velocity of 12 m/s. How does \( a(t) \) describe the change in the ball’s velocity as it rises and falls?

**Solution** The ball’s height at time \( t \) is \( s(t) = s_0 + v_0 t - 4.9t^2 \) m by Galileo’s formula. In our case, \( s_0 = 0 \) and \( v_0 = 12 \), so \( s(t) = 12t - 4.9t^2 \) m [Figure 2(A)]. Therefore, \( v(t) = s'(t) = 12 - 9.8t \) m/s and the ball’s acceleration is \( a(t) = s''(t) = \frac{d}{dt}(12 - 9.8t) = -9.8 \) m/s².
The acceleration is constant with value $-9.8 = -g$, where $g$ (in m/s$^2$) is the acceleration due to gravity, as introduced in Section 3.4. As the ball rises and falls, its velocity decreases from 12 to $-12$ m/s at the constant rate $-g$ [Figure 2(B)].

**GRAPHICAL INSIGHT** Can we visualize the rate represented by $f''(x)$? The second derivative is the rate at which $f'(x)$ is changing, so $f''(x)$ is large if the slopes of the tangent lines change rapidly, as in Figure 3(A). Similarly, $f''(x)$ is small if the slopes of the tangent lines change slowly—in this case, the curve is relatively flat, as in Figure 3(B). If $f$ is a linear function [Figure 3(C)], then the tangent line does not change at all and $f''(x) = 0$. Thus, $f''(x)$ measures the “bending” or concavity of the graph.

(A) Large second derivative: Tangent lines turn rapidly.
(B) Smaller second derivative: Tangent lines turn slowly.
(C) Second derivative is zero: Tangent line does not change.

**EXAMPLE 6** Identify curves I and II in Figure 4(B) as the graphs of $f'$ or $f''$ for the function $f$ in Figure 4(A).

![Graph of $f$](image)

(A) Graph of $f$

![Graph of first two derivatives](image)

(B) Graph of first two derivatives

**Solution** The slopes of the tangent lines to the graph of $f$ are increasing on the interval $[a, b]$. Therefore, $f'$ is an increasing function and its graph must be II. Since $f''(x)$ is the rate of change of $f'(x)$, and $f'(x)$ is increasing, $f''(x)$ is positive and its graph must be I.

**EXAMPLE 7** In a 1997 study, Boardman and Lave related the traffic speed $S$ on a two-lane road to traffic density $Q$ (number of cars per mile of road) by the formula

$$S = 2882Q^{-1} - 0.052Q + 31.73$$

for $60 \leq Q \leq 400$ (Figure 5).

Show that $dS/dQ < 0$ and $d^2S/dQ^2 > 0$ and interpret each in relation to the situation being modeled.
Figure 5 Speed as a function of traffic density.

Solution Taking the derivatives:

\[ \frac{dS}{dQ} = -2882Q^{-2} - 0.052 \]
\[ \frac{d^2S}{dQ^2} = 5764Q^{-3} \]

From these derivative formulas, it is clear that \( \frac{dS}{dQ} < 0 \) and \( \frac{d^2S}{dQ^2} > 0 \). Since \( \frac{dS}{dQ} \) is negative, it follows that as the traffic density increases, the traffic speed decreases, as we would expect.

Furthermore, since \( \frac{d^2S}{dQ^2} \) is positive, it follows that \( \frac{d^2S}{dQ^2} \) is increasing. Given \( \frac{dS}{dQ} \) is negative and increasing and therefore is getting closer to zero, we conclude that \( \frac{dS}{dQ} \) is getting smaller. This implies that the speed decrease associated with increasing traffic density is larger at low density than at high density. This is to be expected as well— with low density and a high speed, the speed is going to drop off quickly as the density increases, but with higher density and an already low speed the speed is not going to drop off very much more as the density increases further.

3.5 SUMMARY

- The higher derivatives \( f', f'', f''', \ldots \) are defined by successive differentiation:

\[
\frac{d^n f}{dx^n}(x) = \frac{d^n}{dx^n} f(x), \quad f''(x) = \frac{d}{dx} \left( \frac{d}{dx} f(x) \right) = \frac{d^3 f}{dx^3}, \ldots
\]

The \( n \)th derivative is denoted \( f^{(n)}(x) = \frac{d^n f}{dx^n} \).
- The second derivative plays an important role: It is the rate at which \( f'(x) \) changes. Graphically, \( f''(x) \) measures how fast the tangent lines change direction and thus measures the "bending" of the graph.
- If \( s(t) \) is the position of an object at time \( t \), then \( s'(t) \) is velocity and \( s''(t) \) is acceleration.

3.5 EXERCISES

Preliminary Questions
1. For each headline, rephrase as a statement about first and second derivatives and sketch a possible graph.
   - "Stocks Go Higher, Though the Pace of Their Gain Slows"
   - "Recent Rain Slow Roland Reservoir Water Level Drop"
   - "Asteroid Approaching Earth at Rapidly Increasing Rate!"
2. Sketch a graph of position as a function of time for an object that is slowing down and has positive acceleration.
3. Sketch a graph of position as a function of time for an object that is speeding up and has negative acceleration.
4. True or false? The third derivative of position with respect to time is zero for an object falling to Earth under the influence of gravity. Explain.
5. Which type of polynomial satisfies \( f''(x) = 0 \) for all \( x \)?
6. What is the millionth derivative of \( f(x) = e^x \)?
7. What are the seventh and eighth derivatives of \( f(x) = x^7 \)?
Exercises

In Exercises 1–16, calculate \( y'' \) and \( y''' \).

1. \( y = 14x^2 \)
2. \( y = 7 - 2x \)
3. \( y = x^4 - 25x^2 + 2x \)
4. \( y = 4x^3 - 9x^2 + 7 \)
5. \( y = \frac{4}{3}x^3 \)
6. \( y = \sqrt{x} \)
7. \( y = 20t^4/3 - 6t^{2/3} \)
8. \( y = x^{-9/3} \)
9. \( y = z - 4 \)
10. \( y = 5t^3 - 7t^{-8/3} \)
11. \( y = 6^2(20 + 7) \)
12. \( y = (x^2 + x)(x^3 + 1) \)
13. \( y = \frac{x-4}{x} \)
14. \( y = \frac{1}{1-x} \)
15. \( y = x^{-1/2} + (x + 1) \)
16. \( y = (r^{1/2} + r)(1 - r) \)

In Exercises 17–26, calculate the derivative indicated.

17. \( f^{(4)}(1), \quad f(x) = x^4 \)
18. \( g^{(n)}(-1), \quad g(t) = -4t^{-5} \)
19. \( \frac{d^2y}{dt^2} \bigg|_{t=1}, \quad y = 4t^3 + 3t^2 \)
20. \( \frac{d^4f}{dt^4} \bigg|_{t=1}, \quad f(t) = 6t^9 - 2t^5 \)
21. \( \frac{d^6x}{dt^6} \bigg|_{t=6}, \quad x = t^{-3/4} \)
22. \( f''(4), \quad f(t) = 2t^2 - t \)
23. \( f^{(n)}(-3), \quad f(x) = \frac{12}{x} - x^3 \)
24. \( f''(1), \quad f(t) = \frac{t}{t+1} \)
25. \( h^{(n)}(1), \quad h(x) = \frac{1}{\sqrt{x} + 1} \)
26. \( g^{(n)}(1), \quad g(x) = \frac{\sqrt{x}}{x + 1} \)

27. Calculate \( y^{(n)}(0) \) for \( 0 \leq k \leq 5 \), where \( y = x^4 + ax^3 + bx^2 + cx + d \) (with \( a, b, c, d \) the constants).

28. Which of the following satisfy \( f^{(k)}(x) = 0 \) for all \( k \geq 6 \)?
   (a) \( f(x) = 7x^4 + 4x^{-1} \)
   (b) \( f(x) = x^6 - 2 \)
   (c) \( f(x) = \sqrt{x} \)
   (d) \( f(x) = 1 - x^6 \)
   (e) \( f(x) = x^{9/5} \)
   (f) \( f(x) = 2x^3 + 3x^5 \)

29. Use the result in Example 3 to find \( \frac{d^6}{dx^6} x^{-1} \).

30. (a) Calculate the first five derivatives of \( f(x) = \sqrt{x} \).
   (b) Show that \( f^{(n)}(x) \) is a multiple of \( x^{-n+1/2} \).
   (c) Show that \( f^{(n)}(x) \) alternates in sign as \( (-1)^{n+1} \) for \( n \geq 1 \).
   (d) Find a formula for \( f^{(n)}(x) \) for \( n \geq 2 \). Hint: Verify that the coefficient is \( \pm 1 \cdot 3 \cdot 5 \cdots 2n - 3 \cdot 2n \).

31. \( f(x) = x^{-2} \)
32. \( f(x) = (x + 2)^{-1} \)
33. \( f(x) = x^{-1/2} \)
34. \( f(x) = x^{-3/2} \)
35. \( f(x) = \frac{x+1}{x^2} \)
36. \( f(x) = \frac{x-1}{\sqrt{x}} \)

37. (a) Find the acceleration at time \( t = 5 \) min of a helicopter whose height is \( s(t) = 300t - 4t^3 \) m.
   (b) Plot the acceleration \( x'' \) for \( 0 \leq t \leq 6 \). Is the helicopter speeding up or slowing down during this time interval? Explain.

38. Find an equation of the tangent line to the graph of \( y = f'(x) \) at \( x = 3 \), where \( f(x) = x^4 \).

39. Figure 6 shows \( f(x), f'(x), \) and \( f''(x) \). Determine which is which.

40. The second derivative \( f''(x) \) is shown in Figure 7. Which of (A) or (B) is the graph of \( f \) and which is \( f'' \)?

41. Figure 8 shows the graph of the position \( s \) of an object as a function of time \( t \). Determine the intervals on which the acceleration is positive.

42. Figure 9 shows the graph of the position \( s \) of an object as a function of time \( t \). For each interval \([0, 10], [10, 20]\), and so on, indicate whether the acceleration is negative, zero, or positive.
43. Find all values of \( n \) such that \( y = x^n \) satisfies
\[ x^2 y'' - 2xy' = 4y \]

44. Find all values of \( n \) such that \( y = x^n \) satisfies
\[ x^2 y'' - 12y = 0 \]

45. According to one model that takes into account air resistance, the acceleration \( a(t) \) (in m/s\(^2\)) of a skydiver of mass \( m \) in free-fall satisfies
\[ a(t) = -9.8 - \frac{k}{y(t)^2} \]
where \( v(t) \) is velocity (negative since the object is falling) and \( k \) is a constant. Suppose that \( m = 75 \) kg and \( k = 0.24 \) kg/m.
(a) What is the skydiver's velocity when \( a(t) = -4.9 \)?
(b) What is the skydiver's velocity when \( a(t) = 0 \)? (This velocity is the terminal velocity, the velocity attained when air resistance balances gravity and the skydiver falls at a constant speed.)

46. In contrast to Exercise 45, the size of a falling lightweight object may be more significant than its mass when taking into account air resistance. One model that takes such an approach for falling raindrops is
\[ \frac{d^2 x}{dt^2} = g - 0.0005 \left( \frac{dx}{dt} \right)^2 \]
where \( x(t) \) is the distance a raindrop has fallen (in meters), \( D \) is the raindrop diameter, and \( g = 9.8 \) m/s\(^2\). Terminal velocity \( v_{\text{term}} \) is defined as the velocity at which the drop has zero acceleration (one can show that velocity approaches \( v_{\text{term}} \) as time proceeds).
(a) Show that \( v_{\text{term}} = \sqrt{2mg/D} \).
(b) Find \( v_{\text{term}} \) for drops of diameter 10\(^{-3}\) and 10\(^{-4}\) m.
(c) In this model, do raindrops accelerate more rapidly at higher or lower velocities?

47. In a manufacturing process, a drill press automatically drills a hole into a sheet metal part on a conveyor. In the drilling operation the drill bit starts at rest directly above the part, descends quickly, drills a hole, and quickly returns to the start position. The maximum vertical speed of the drill bit is 4 in/s, and while drilling the hole, it must move no more than 2.6 in/s to avoid warping the metal. Let \( x(t) \) be the drill bit's height (in inches) above the part as a function of time \( t \) in seconds. Sketch possible graphs of the drill bit's velocity \( v(x(t)) \) and acceleration \( a(x(t)) \).

48. Use a computer algebra system to compute \( f^{(k)}(x) \) for \( k = 1, 2, 3 \) for the following functions:
(a) \( f(x) = (1 + x^3)^{3/5} \)
(b) \( f(x) = \frac{1 - x^4}{1 - 5x - 6x^2} \)

49. Use a computer algebra system to compute the \( f^{(k)}(x) \) for \( 1 \leq k \leq 4 \). Can you find a general formula for \( f^{(k)}(x) \)?

50. Find the 100th derivative of
\[ p(x) = (x + x^2 + x^3)^{10}(1 + x^2)^{11}(x^3 + x^2 + x^7) \]

51. What is \( p^{(100)}(x) \) for \( p(x) \) in Exercise 50?

52. Use the Product Rule twice to find a formula for \( (fg)^{10} \) in terms of \( f \) and \( g \) and their first and second derivatives.

53. Use the Product Rule to find a formula for \( (fg)^{10} \) and compare your result with the expansion of \( (a + b)^{10} \). Then try to guess the general formula for \( (fg)^{10} \).

54. Compute
\[ \lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} \]
for the following functions:
(a) \( f(x) = x \)
(b) \( f(x) = x^2 \)
(c) \( f(x) = x^3 \)
Based on these examples, what do you think the limit represents?

3.6 Trigonometric Functions

We can use the rules developed so far to differentiate functions involving powers of \( x \), but we cannot yet handle the trigonometric functions. What is missing are the formulas for the derivatives of \( \sin x \) and \( \cos x \). Fortunately, their derivatives are simple—each is the derivative of the other up to a sign.
Recall our convention: Angles are measured in radians, unless otherwise specified.

**Theorem 1: Derivative of Sine and Cosine**

The functions \( y = \sin x \) and \( y = \cos x \) are differentiable and
\[
\frac{d}{dx} \sin x = \cos x \quad \text{and} \quad \frac{d}{dx} \cos x = -\sin x
\]
Proof We must go back to the definition of the derivative:

\[
\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}
\]

We cannot cancel the \(h\) by rewriting the difference quotient, but we can use the addition formula (see marginal note) to write the numerator as a sum of two terms:

\[
\sin(x + h) - \sin x = \sin x \cos h + \cos x \sin h - \sin x = (\sin x \cos h - \sin x) + \cos x \sin h
\]

(\text{addition formula})

\[
= \sin x (\cos h - 1) + \cos x \sin h
\]

This gives us

\[
\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}
\]

\[
= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}
\]

We can take \(\sin x\) and \(\cos x\) outside the limits in Eq. (2) because they do not depend on the limiting variable \(h\). The two limits are determined by Theorem 2 in Section 2.6, which indicates that

\[
\lim_{h \to 0} \frac{1 - \cos h}{h} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{\sin h}{h} = 1
\]

It follows that \(\lim_{h \to 0} \frac{\sin h}{h} = 0\), and therefore, Eq. (2) reduces to \(\frac{d}{dx} \sin x = \cos x\), as desired. The formula \(\frac{d}{dx} \cos x = -\sin x\) is proved similarly (see Exercise 57).

\textbf{Conceptual Insight} One property of \(f(x) = \sin x\) that makes its derivative formula so simple is that \(\lim_{h \to 0} \frac{\sin h}{h} = 1\). The value of this limit depends on measuring angles in radians. The simplicity of this limit explains why we measure angles in radians when working with trigonometric functions in calculus. If instead we measure angles in degrees, then, as we pointed out in Section 2.2, \(\lim_{h \to 0} \frac{\sin h}{h} = \frac{\pi}{180}\). Nothing else in the previous proof would change, and we would end up with the unwieldy derivative formula \(\frac{d}{dx} \sin x = \frac{\sqrt{3}}{180} \cos x\).

\textbf{Example 1} For \(f(x) = \sin x\), compute \(f'\) at \(x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \text{and} \frac{5\pi}{6}\).

\textbf{Solution} We have \(f'(x) = \cos x\). Thus, \(f'(0) = \cos(0) = 1\), \(f'(\frac{\pi}{6}) = \cos \left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\), \(f'(\frac{\pi}{4}) = \cos \left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\), \(f'(\frac{5\pi}{6}) = \cos \left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}\).

Note, in Example 6 in Section 3.1, for \(f(x) = \sin x\) we estimated \(f' \left(\frac{\pi}{6}\right) \approx 0.8660\). Now we have the exact value \(f' \left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\).

\textbf{Graphical Insight} The formula \((\sin x)' = \cos x\) seems reasonable when we compare the graphs in Figure 1. The tangent lines to the graph of \(y = \sin x\) have positive slope on the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\), and on this interval, the derivative \(y' = \cos x\) is positive. The tangent lines have negative slope on the interval \((\frac{\pi}{2}, \frac{3\pi}{2})\), where \(y' = \cos x\) is negative. The tangent lines are horizontal at \(x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}\), where \(\cos x = 0\).
EXAMPLE 2 Calculate $f''(x)$, where $f(x) = x \cos x$.

Solution By the Product Rule,

$$f'(x) = x' \cos x + x(\cos x)' = \cos x - x \sin x$$

$$f''(x) = (\cos x - x \sin x)' = -\sin x - (x'(\sin x) + x(\sin x)')$$

$$= -\sin x - x \cos x$$

EXAMPLE 3 A projectile is shot from ground level at 100 ft/s at a launch angle of $\theta$ that is between 0 and $\pi/2$ (Figure 2). Assume that the projectile is acted on by gravity, but not air resistance. Then, in a simple projectile-motion model (see Section 14.5) it can be shown that the projectile lands at a distance $R(\theta) = 625 \sin \theta \cos \theta$ ft from the launch point. What is the rate of change of the range with respect to the launch angle? For what angles does the range increase/decrease with increasing launch angle? What angle provides the maximum range, and what is that maximum range?

Solution Using the Product Rule and the derivative rules for $\sin \theta$ and $\cos \theta$, we have

$$R'(\theta) = 625(\sin \theta)(-\sin \theta) + (\cos \theta)(\cos \theta)$$

$$= 625(\cos^2 \theta - \sin^2 \theta) = 625 \cos 2\theta$$

where the last equality results from the double angle formula for $\cos 2\theta$.

Since $\cos 2\theta$ is positive for $0 \leq \theta < \pi/4$ and is negative for $\pi/4 < \theta \leq \pi/2$, it follows that the range increases with increasing launch angle for angles between 0 and $\pi/4$ and decreases with increasing launch angle for angles between $\pi/4$ and $\pi/2$. The maximum range occurs at $\theta = \pi/4$, and that maximum range is $R(\pi/4) = 312.5$ ft.

The derivatives of the other standard trigonometric functions can be computed using the Quotient Rule. We derive the formula for $(\tan x)'$ in Example 4 and the remaining formulas in Exercises 35–37.

THEOREM 2 Derivatives of Standard Trigonometric Functions

$$\frac{d}{dx} \tan x = \sec^2 x,$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \cot x = -\csc^2 x,$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$
**EXAMPLE 4** Derive the formula \( \frac{d}{dx} \tan x = \sec^2 x \) (Figure 3).

**Solution** Use the Quotient Rule and the identity \( \cos^2 x + \sin^2 x = 1 \):

\[
\frac{d}{dx} \tan x = \left( \frac{\sin x}{\cos x} \right)' = \frac{\cos x \cdot (\sin x)' - \sin x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\]

**EXAMPLE 5** Determine \( y' \) for \( y = \tan \theta \sec \theta \), and find an equation of the tangent line to the graph at \( \theta = \frac{\pi}{4} \).

**Solution** By the Product Rule,

\[
y' = (\tan \theta)' \sec \theta + \tan \theta (\sec \theta)' = \sec^2 \theta \sec \theta + \tan \theta (\sec \theta \tan \theta) = \sec^3 \theta + \tan^2 \theta \sec \theta
\]

Now use the values \( \sec \frac{\pi}{4} = \sqrt{2} \) and \( \tan \frac{\pi}{4} = 1 \) to compute

\[
y' \left( \frac{\pi}{4} \right) = \sec^3 \left( \frac{\pi}{4} \right) + \tan^2 \left( \frac{\pi}{4} \right) = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}
\]

An equation of the tangent line (Figure 4) is \( y - \sqrt{2} = 3\sqrt{2} (\theta - \frac{\pi}{4}) \).

### 3.6 SUMMARY

- The derivatives of the trigonometric functions:

\[
\begin{align*}
\frac{d}{dx} \sin x &= \cos x \\
\frac{d}{dx} \cos x &= -\sin x \\
\frac{d}{dx} \tan x &= \sec^2 x \\
\frac{d}{dx} \sec x &= \sec x \tan x \\
\frac{d}{dx} \cot x &= -\csc^2 x \\
\frac{d}{dx} \csc x &= -\csc x \cot x
\end{align*}
\]

### 3.6 EXERCISES

**Preliminary Questions**

1. Determine the sign (+ or −) that yields the correct formula for the following:
   - (a) \( \frac{d}{dx} (\sin x + \cos x) = \pm \sin x \pm \cos x \)
   - (b) \( \frac{d}{dx} \sec x = \pm \sec x \tan x \)
   - (c) \( \frac{d}{dx} \cot x = \pm \csc^2 x \)

2. Which of the following functions can be differentiated using the rules we have covered so far?
   - (a) \( y = 3 \cos x \cot x \)
   - (b) \( y = \cos(x^2) \)
   - (c) \( y = 2x \sin x \)

3. For each, give an equation of the tangent line to the graph at \( x = 0 \):
   - (a) \( y = \sin x \)
   - (b) \( y = \cos x \)

4. How is the addition formula for sine used in deriving the formula \( (\sin x)' = \cos x \)?
Exercises

In Exercises 1–4, find an equation of the tangent line at the point indicated.

1. \( y = \sin x \), \( x = \frac{\pi}{4} \)
2. \( y = \cos x \), \( x = \frac{\pi}{3} \)
3. \( y = \tan x \), \( x = \frac{\pi}{3} \)
4. \( y = \sec x \), \( x = \frac{\pi}{6} \)

In Exercises 5–24, compute the derivative.

5. \( f(x) = \sin x \cos x \)
6. \( f(x) = x^2 \cos x \)
7. \( f(x) = x \sin x \)
8. \( f(x) = 9 \sec x + 12 \cot x \)
9. \( H(t) = \sin t \sec^2 t \)
10. \( h(t) = 9 \sec t + t \cot t \)
11. \( f(\theta) = \tan \theta \sec \theta \)
12. \( k(\theta) = \theta^2 \sin^2 \theta \)
13. \( f(x) = (2x^4 - 4x^{-1}) \sec x \)
14. \( f(x) = x \tan x \)
15. \( y = \frac{\sec \theta}{\theta} \)
16. \( G(z) = \frac{1}{\tan z - \cot z} \)
17. \( R(y) = \frac{3 \cos y - 4}{\sin y} \)
18. \( f(x) = x \frac{\sin x + 2}{\sin x + 2} \)
19. \( f(x) = \frac{1 + \tan x}{1 - \tan x} \)
20. \( f(\theta) = \theta \tan \theta \sec \theta \)
21. \( f(x) = \frac{\sin x + 1}{\sin x - 1} \)
22. \( h(t) = \frac{\sec^2 t}{t} \)
23. \( R(\theta) = \frac{4 + \cos \theta}{3 - 3 \sin \theta} \)
24. \( g(x) = \frac{\cos x}{4 + \cos \theta} \)

In Exercises 25–34, find an equation of the tangent line at the point specified.

25. \( y = x^3 + \cos x \), \( x = 0 \)
26. \( y = \tan \theta \), \( \theta = \frac{\pi}{6} \)
27. \( y = \frac{\sin x}{1 + \cos x} \), \( x = \frac{\pi}{3} \)
28. \( y = \sin x + \cos x \), \( x = 0 \)
29. \( y = 2(\sin \theta + \cos \theta) \), \( \theta = \frac{\pi}{3} \)
30. \( y = \cos x \cos x - \cot x \), \( x = \frac{\pi}{4} \)
31. \( y = (\cot x)(\cos x) \), \( t = \frac{\pi}{3} \)
32. \( y = x \cos^2 x \), \( x = \frac{\pi}{4} \)
33. \( y = x^2(1 - \sin x) \), \( x = \frac{3\pi}{2} \)
34. \( y = \frac{\sin \theta \cos \theta}{\theta} \), \( \theta = \frac{\pi}{4} \)

In Exercises 35–37, use Theorem 1 to derive the formula.

35. \( \frac{d}{dx} \cot x = -\csc^2 x \)
36. \( \frac{d}{dx} \sec x = \sec x \tan x \)
37. \( \frac{d}{dx} \csc x = -\csc x \cot x \)
38. Show that both \( y = \sin x \) and \( y = \cos x \) satisfy \( y'' = -y \).

In Exercises 39–42, calculate the higher derivative.

39. \( f''(\theta) \), \( f''(\theta) = \theta \tan \theta \)
40. \( \frac{d^2}{dt^2} \cos^2 t \)
41. \( y'' \), \( y'' = \tan x \)
42. \( y'' \), \( y'' = t^2 \sin t \)

Calculate the first five derivatives of \( f(x) = \cos x \). Then determine \( f^{(5)}(x) \) and \( f^{(10)}(x) \).

43. Calculate the first five derivatives of \( f(x) = \sin x \). Then determine \( f^{(5)}(x) \) and \( f^{(10)}(x) \).

44. Calculate the first five derivatives of \( f(x) = \sin x \). Then determine \( f^{(5)}(x) \) and \( f^{(10)}(x) \).

45. Let \( f(x) = \sin x \). We can compute \( f^{(n)}(x) \) as follows: First, express \( n = 4m + r \) where \( m \) is a whole number and \( r = 0, 1, 2, \) or \( 3 \). Then determine \( f^{(m)}(x) \) from \( f \). Explain how to do the latter step.

46. Let \( f(x) = \cos x \). We can compute \( f^{(n)}(x) \) as follows: First, express \( n = 4m + r \) where \( m \) is a whole number and \( r = 0, 1, 2, \) or \( 3 \). Then determine \( f^{(m)}(x) \) from \( f \). Explain how to do the latter step.

47. Let \( f(x) = \sin^2 x \) and \( g(x) = \cos^2 x \).
(a) Use an identity and prove \( f'(x) = -g'(x) \) without directly computing \( f'(x) \) and \( g'(x) \).
(b) Now verify the result in (a) by directly computing \( f'(x) \) and \( g'(x) \).
48. Let \( f(x) = \tan^2 x \) and \( g(x) = \sec^2 x \).
(a) Use an identity and prove \( f'(x) = g'(x) \) without directly computing \( f'(x) \) and \( g'(x) \).
(b) Now verify the result in (a) by directly computing \( f'(x) \) and \( g'(x) \).

49. Find the values of \( x \) between 0 and 2π where the tangent line to the graph of \( y = \sin x \cos x \) is horizontal.

50. [GU] Plot the graph \( f(\theta) = \sec \theta + \csc \theta \) over \( [0, 2\pi] \) and determine the number of solutions to \( f'(\theta) = 0 \) in this interval graphically. Then compute \( f'(\theta) \) and find the solutions.

51. [GU] Let \( g(t) = t - \sin t \).
(a) Plot the graph of \( g \) with a graphing utility for \( 0 \leq t \leq 4\pi \).
(b) Show that the slope of the tangent line is nonnegative. Verify this on your graph.
(c) For which values of \( t \) in the given range is the tangent line horizontal?

52. [CAS] Let \( f(x) = (\sin x)/x \) for \( x \neq 0 \) and \( f(0) = 1 \).
(a) Plot \( f \) on \([-3\pi, 3\pi] \).
(b) Show that \( f'(0) = 0 \) if \( c = \tan c \). Approximate the smallest positive value of \( c \) such that \( f''(c) = 0 \).
(c) Verify that the horizontal line \( y = f(c) = \tan c \) is tangent to the graph of \( y = f(x) \) at \( x = c \) by plotting them on the same set of axes.

53. [CAS] Show that no tangent line to the graph of \( f(x) = \tan x \) has zero slope. What is the least slope of a tangent line? Justify by sketching the graph of \( f'(x) = \sec^2 x \).

54. The height at time \( t \) (in seconds) of a mass, oscillating at the end of a spring, is \( s(t) = 300 + 40 \sin t \). Find the velocity and acceleration at \( t = \frac{\pi}{4} \).

55. A projectile is launched from ground level with an initial velocity \( v_0 \) at an angle \( \theta \), where \( 0 \leq \theta \leq \pi/2 \). Its horizontal range is
\[
R = \frac{v_0^2 \sin 2\theta}{g}
\]
where \( g \) is the acceleration due to gravity. Calculate \( dR/d\theta \). The maximum range occurs where \( dR/d\theta = 0 \). Show that that occurs at \( \theta = \pi/4 \) and that the maximum range is \( v_0^2 / g \).

56. The graph of \( y = \sin x \) is shown in Figure 5, along with a tangent line at \( x = \theta \). Show that if \( \frac{\pi}{2} < \theta < \pi \), then the distance along the \( x \)-axis between \( \theta \) and the point where the tangent line intersects the \( x \)-axis is equal to \( \tan \theta \).

![Figure 5](image-url)
Further Insights and Challenges

57. Use the limit definition of the derivative and the addition law for the
cosine function to prove that \( (\cos x)' = -\sin x \).

58. Use the addition formula for the tangent
\[
\tan(x + h) = \frac{\tan x + \tan h}{1 + \tan x \tan h}
\]
to compute \( (\tan x)' \) directly as a limit of the difference quotients. You will
also need to show that \( \lim_{h \to 0} \frac{\tan h}{h} = 1 \).

59. Verify the following identity and use it to give another proof of the
formula \((\sin x)' = \cos x\): \[
\sin(x + h) - \sin x = 2 \cos \left( x + \frac{h}{2} \right) \sin \left( \frac{h}{2} \right)
\]
*Hint*: Use the addition formula for sine to prove that
\[ \sin(a + b) - \sin(a - b) = 2 \cos a \sin b \]

60. Show that a nonzero polynomial function \( y = f(x) \) cannot satisfy
the equation \( y'' = -y \). Use this to prove that neither \( f(x) = \sin x \) nor
\( f(x) = \cos x \) is a polynomial. Can you think of another way to reach this
conclusion by considering limits as \( x \to \infty \)?

61. Let \( f(x) = x \sin x \) and \( g(x) = x \cos x \).
(a) Show that \( f'(x) = g(x) + x \sin x \) and \( g'(x) = -f(x) + \cos x \).
(b) Verify that \( f''(x) = -f'(x) + 2 \cos x \) and
\[ g''(x) = -g'(x) - 2 \sin x \]
62. Figure 6 shows the geometry behind the derivative formula
\( (\sin \theta)' = \cos \theta \).
(a) \( \Delta \sin \theta = BC \)
(b) \( \angle BDA = \theta \) \( \text{Hint: } \overline{OA} \perp \overline{AD} \).
(c) \( BD = (\cos \theta) AD \)
Now explain the following intuitive argument: If \( h \) is small, then
\( BC \approx BD \) and \( AD \approx h \), so \( \Delta \sin \theta \approx (\cos \theta)h \) and \( (\sin \theta)' = \cos \theta. \)

3.7 The Chain Rule

The Chain Rule is used to differentiate composite functions such as \( y = \cos(x^2) \) and
\( y = \sqrt{x^3 + 1} \).

Recall that a *composite function* is obtained by evaluating one function at the output
of another. The composite of \( f \) with \( g \), denoted \( f \circ g \), is defined by
\[
(f \circ g)(x) = f(g(x))
\]
For convenience, we call \( f \) the *outside* function and \( g \) the *inside* function. Often, we
write the composite function as \( f(u) \), where \( u = g(x) \). For example, \( y = \cos(x^2) \) is the
function \( y = \cos u \), where \( u = x^3 \).

**Theorem 1: Chain Rule** If \( f \) and \( g \) are differentiable, then the composite function
\( (f \circ g)(x) = f(g(x)) \) is differentiable and
\[
(f \circ g)'(x) = f'(g(x))g'(x)
\]

We will prove the Chain Rule at the end of the section.

**Example 1** Calculate the derivative of \( y = \cos(x^3) \).

**Solution** As noted above, \( y = \cos(x^3) \) is a composite \( f(g(x)) \), where
\[
\begin{align*}
  f(u) &= \cos u, & u &= g(x) = x^3 \\
  f'(u) &= -\sin u, & g'(x) &= 3x^2 
\end{align*}
\]

Since \( u = x^3 \), \( f'(g(x)) = f'(u) = f'(x^3) = -\sin(x^3) \). So, by the Chain Rule,
\[
\frac{d}{dx} \cos(x^3) = -\sin(x^3) \frac{(3x^2)}{f'(g(x))} = -3x^2 \sin(x^3).
\]
Alternatively, we can build this derivative using the verbal description. Here, the outside function is \( \cos \square \) and the inside function is \( x^3 \). We have

- The derivative of the outside: \(- \sin \square\)
- The derivative of the outside at the inside: \(-\sin(x^3)\)
- The derivative of the outside at the inside times the derivative of the inside: \((-\sin(x^3))(3x^2)\)

So \( \frac{d}{dx} \cos(x^3) = -3x^2 \sin(x^3) \).

**EXAMPLE 2** Calculate the derivative of \( y = \sqrt{x^4 + 1} \).

**Solution** The function \( y = \sqrt{x^4 + 1} \) is a composite \( f(g(x)) \), where

\[
f(u) = u^{1/2}, \quad u = g(x) = x^4 + 1 \]

\[
f'(u) = \frac{1}{2} u^{-1/2}, \quad g'(x) = 4x^3
\]

Note that \( f'(g(x)) = \frac{1}{2}(x^4 + 1)^{-1/2} \), so by the Chain Rule,

\[
\frac{dy}{dx} = \frac{d}{dx} \sqrt{x^4 + 1} = \frac{d}{du} \left[ \frac{1}{2}(u^{1/2}) \right] \cdot \frac{du}{dx} = \frac{4x^3}{2\sqrt{x^4 + 1}}
\]

Alternatively, the outside is \( \sqrt{\square} = \square^{1/2} \) and the inside is \( x^4 + 1 \). Therefore,

- The derivative of the outside: \( \frac{1}{2}\square^{-1/2} \)
- The derivative of the outside at the inside: \( \frac{1}{2}(x^4 + 1)^{-1/2} \)
- The derivative of the outside at the inside times the derivative of the inside: \( \frac{1}{2}(x^4 + 1)^{-1/2} \cdot 4x^3 \)

So \( \frac{dy}{dx} \sqrt{x^4 + 1} = \frac{4x^3}{2\sqrt{x^4 + 1}} \).

**EXAMPLE 3** Calculate \( \frac{dy}{dx} \) for \( y = \tan \left( \frac{x}{x+1} \right) \).

**Solution** The outside function is \( f(u) = \tan u \). Because \( f'(u) = \sec^2 u \), the Chain Rule gives us

\[
\frac{d}{dx} \tan \left( \frac{x}{x+1} \right) = \sec^2 \left( \frac{x}{x+1} \right) \frac{d}{dx} \left( \frac{x}{x+1} \right)
\]

Derivative of inside function

Now, by the Quotient Rule,

\[
\frac{d}{dx} \left( \frac{x}{x+1} \right) = \frac{(x+1) \frac{d}{dx} x - x \frac{d}{dx} (x+1)}{(x+1)^2} = \frac{1}{(x+1)^2}
\]

We obtain

\[
\frac{d}{dx} \tan \left( \frac{x}{x+1} \right) = \sec^2 \left( \frac{x}{x+1} \right) \frac{1}{(x+1)^2} = \frac{\sec^2 \left( \frac{x}{x+1} \right)}{(x+1)^2}
\]

**EXAMPLE 4** In Example 5 in Section 1.4 we introduced \( L(t) = 12 + 3.1 \sin \left( \frac{2\pi}{365} t \right) \) as a model for the length of a day in hours from sunrise to sunset in Orange City, Iowa, where \( t \) is the day in the year after the spring equinox on March 21. Determine \( L'(t) \) and
use it to calculate the rate that the length of the days are changing on July 1, August 9, September 15, and October 1.

Solution  By the Chain Rule,

\[ L'(t) = 3.1 \cos \left( \frac{2\pi}{365} t \right) \cdot \frac{2\pi}{365} = \frac{6.2\pi}{365} \cos \left( \frac{2\pi}{365} t \right) \]

We use this formula to compute \( L' \) for the given dates as follows:

July 1 corresponds to \( t = 102 \), and \( L'(102) \approx -0.01 \) h/day \( \approx -0.6 \) min/day.

August 9 corresponds to \( t = 142 \), and \( L'(142) \approx -0.04 \) h/day \( \approx -2.4 \) min/day.

September 15 corresponds to \( t = 179 \), and \( L'(179) \approx -0.05 \) h/day \( \approx -3.0 \) min/day.

October 1 corresponds to \( t = 195 \), and \( L'(195) \approx -0.05 \) h/day \( \approx -3.0 \) min/day.

These results may confirm your experience with the changing length of days in the summer. Once summer begins, the lengths of the days start to diminish (\( L'(t) < 0 \)). As we see here, this rate of decrease gets larger throughout the summer, and as summer is ending and fall arrives the days are shortening fastest.

It is instructive to write the Chain Rule in Leibniz notation. Let

\[ y = f(u) = f(g(x)) \]

Then, by the Chain Rule,

\[ \frac{dy}{dx} = f'(u)g'(x) = \frac{df}{du} \frac{du}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \]

CONCEPTUAL INSIGHT  In Leibniz notation, it appears as if we are multiplying fractions and the Chain Rule is simply a matter of "canceling the \( du \)." Since the expressions \( dy/du \) and \( du/dx \) are not fractions, this does not make sense literally, but it does suggest that derivatives behave as if they were fractions (this is reasonable because a derivative is a limit of fractions, namely of the difference quotients). Leibniz's form also emphasizes a key aspect of the Chain Rule: Rates of change multiply. To illustrate, suppose that (thanks to your knowledge of calculus) your salary increases twice as fast as your friend's. If your friend's salary increases $4000 per year, your salary will increase at the rate of 2 x $4000 or $8000/yr. In terms of derivatives,

\[ \frac{d(\text{your salary})}{dt} = \frac{d(\text{your salary})}{d(\text{friend's salary})} \times \frac{d(\text{friend's salary})}{dt} \]

\[ \$8000/yr = 2 \times \$4000/yr \]

EXAMPLE 5  A spherical balloon has a radius \( r \) that is increasing at a rate of 3 cm/s. At what rate is the volume \( V \) of the balloon increasing when \( r = 10 \) cm?

Solution  Because we are asked to determine the rate at which \( V \) is increasing, we must find \( dV/dt \). We are given that \( dr/dt = 3 \) cm/s. The Chain Rule allows us to express \( dV/dt \) in terms of \( dV/dr \) and \( dr/dt \):

\[ \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \]

Rate of change of volume with respect to time  Rate of change of volume with respect to radius  Rate of change of radius with respect to time
To compute \( \frac{dV}{dr} \), we use the formula for the volume of a sphere, \( V = \frac{4}{3} \pi r^3 \):

\[
\frac{dV}{dr} = \frac{d}{dr} \left( \frac{4}{3} \pi r^3 \right) = 4 \pi r^2
\]

Because \( \frac{dr}{dt} = 3 \), we obtain

\[
\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4 \pi r^2 (3) = 12 \pi r^2
\]

For \( r = 10 \),

\[
\left. \frac{dV}{dt} \right|_{r=10} = (12 \pi) 10^2 = 1200 \pi \approx 3770
\]

The volume of the balloon is increasing at a rate of approximately 3770 cm\(^3\)/s.

We now discuss an important special case of the Chain Rule.

**Theorem 2 General Power Rule**  If \( g \) is differentiable, then

\[
\frac{d}{dx} (g(x))^n = n(g(x))^{n-1} g'(x) \quad \text{(for any number } n)\]

**Proof**  Let \( f(u) = u^n \). Then \( (g(x))^n = f(g(x)) \), and the Chain Rule yields

\[
\frac{d}{dx} (g(x))^n = f'(g(x))g'(x) = n(g(x))^{n-1} g'(x)
\]

**Example 6 General Power Rule**  Find the derivatives of

(a) \( y = (x^2 + 7x + 2)^{-1/2} \) and  
(b) \( y = \sec^4 t \).

**Solution**  Apply \( \frac{d}{dx} g(x)^n = ng(x)^{n-1} g'(x) \):

(a)
\[
\frac{d}{dx} (x^2 + 7x + 2)^{-1/3} = -\frac{1}{3} (x^2 + 7x + 2)^{-4/3} \frac{d}{dx} (x^2 + 7x + 2) \\
= -\frac{1}{3} (x^2 + 7x + 2)^{-4/3} (2x + 7)
\]

(b)  \( \frac{d}{dt} \sec^4 t = 4 \sec^3 t \frac{d}{dt} \sec t = 4 \sec^3 t (\sec t \tan t) = 4 \sec^4 t \tan t \)

**Example 7 Using the Chain Rule Twice**  Calculate \( \frac{d}{dx} \sqrt{1 + \sqrt{x^2 + 1}} \).

**Solution**  In the computation that follows, we apply the Chain Rule, first to the square root of the inside function \( u = 1 + \sqrt{x^2 + 1} \) and then to the derivative of the inside function:

\[
\frac{d}{dx} \left( 1 + (x^2 + 1)^{1/2} \right)^{1/2} = \frac{1}{2} \left( 1 + (x^2 + 1)^{1/2} \right)^{-1/2} \frac{d}{dx} \left( 1 + (x^2 + 1)^{1/2} \right) \\
= \frac{1}{2} \left( 1 + (x^2 + 1)^{1/2} \right)^{-1/2} \left( \frac{1}{2} (x^2 + 1)^{-1/2} (2x) \right) \\
= \frac{1}{2} \frac{2x}{(x^2 + 1)^{1/2}} \left( 1 + (x^2 + 1)^{1/2} \right)^{-1/2}
\]
Proof of the Chain Rule  The difference quotient for the composite \( f \circ g \) is

\[
\frac{f(g(x + h)) - f(g(x))}{h} \quad (h \neq 0)
\]

We express the difference quotient in a more complicated form that, we will see, results in the appearance of the appropriate terms in the Chain Rule formula when we take the limit as \( h \to 0 \).

\[
\frac{f(g(x + h)) - f(g(x))}{h} = \frac{f(g(x + h)) - f(g(x))}{g(x + h) - g(x)} \times \frac{g(x + h) - g(x)}{h}
\]

This is legitimate only if the denominator \( g(x + h) - g(x) \) is nonzero. Therefore, to continue our proof, we make the extra assumption that \( g(x + h) - g(x) \neq 0 \) for all \( h \) close to but not equal to 0. This assumption is not necessary, but without it, the argument is more technical (see Exercise 109).

Under our assumption, we may use Eq. (1) to write \( (f \circ g)'(x) \) as a product of two limits:

\[
(f \circ g)'(x) = \lim_{h \to 0} \frac{f(g(x + h)) - f(g(x))}{g(x + h) - g(x)} \times \lim_{h \to 0} \frac{g(x + h) - g(x)}{h}
\]

We show that this equals \( f'(g(x)) \). This is \( g'(x) \).

The second limit on the right is \( g'(x) \). The Chain Rule will follow if we show that the first limit equals \( f'(g(x)) \). To verify this, set

\[ k = g(x + h) - g(x) \]

Then \( g(x + h) = g(x) + k \) and

\[
\frac{f(g(x + h)) - f(g(x))}{g(x + h) - g(x)} = \frac{f(g(x) + k) - f(g(x))}{k}
\]

The function \( g \) is continuous because it is differentiable. Therefore, \( g(x + h) \) tends to \( g(x) \) and \( k = g(x + h) - g(x) \) tends to zero as \( h \to 0 \). Thus, we may rewrite the limit in terms of \( k \) to obtain the desired result:

\[
\lim_{k \to 0} \frac{f(g(x + h)) - f(g(x))}{g(x + h) - g(x)} = \lim_{k \to 0} \frac{f(g(x) + k) - f(g(x))}{k} = f'(g(x))
\]

It now follows that \( (f \circ g)'(x) = f'(g(x))g'(x) \).

3.7 SUMMARY

- The Chain Rule expresses \( (f \circ g)' \) in terms of \( f' \) and \( g' \):
  \[
  (f(g(x)))' = f'(g(x))g'(x)
  \]

- In Leibniz notation:
  \[
  \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}, \text{ where } y = f(u) \text{ and } u = g(x)
  \]

- General Power Rule:
  \[
  \frac{d}{dx}(g(x))^n = n(g(x))^{n-1}g'(x)
  \]
3.7 EXERCISES

Preliminary Questions
1. Identify the outside and inside functions for each of these composite functions:
   (a) \( y = \sqrt{x^2 + 9x^2} \)
   (b) \( y = \tan(x^2 + 1) \)
   (c) \( y = \sec^5 x \)
   (d) \( y = (1 + x^2)^6 \)

2. Which of the following can be differentiated without using the Chain Rule?
   (a) \( y = \tan(7x^2 + 2) \)
   (b) \( y = \frac{x}{x+1} \)
   (c) \( y = \sqrt{x} \cdot \sec x \)
   (d) \( y = x \cdot \sqrt{\sec x} \)
   (e) \( y = \sqrt{x} \cdot \sec x \)
   (f) \( y = \tan(4x) \)

3. Which is the derivative of \( f(5x) \)?
   (a) \( 5f'(x) \)
   (b) \( 5f'(5x) \)
   (c) \( f'(5x) \)

4. Suppose that \( f'(4) = g'(4) = g(4) = 1 \). Do we have enough information to compute \( F'(4) \), where \( F(x) = f(g(x)) \)? If not, what is missing?

Exercises

In Exercises 1–4, fill in a table of the following type:

<table>
<thead>
<tr>
<th>( f(g(x)) )</th>
<th>( f'(x) )</th>
<th>( f'(g(x)) )</th>
<th>( g'(x) )</th>
<th>( (f \circ g)'(x) )</th>
</tr>
</thead>
</table>

1. \( f(u) = u^{3/2} \), \( g(x) = x^4 + 1 \)
2. \( f(u) = u^2 \), \( g(x) = 3x + 5 \)
3. \( f(u) = \tan u \), \( g(x) = x^4 \)
4. \( f(u) = u^4 + u \), \( g(x) = \cos x \)

In Exercises 5 and 6, write the function as a composite \( f(g(x)) \) and compute the derivative using the Chain Rule.
5. \( y = (x + \sin x)^4 \)
6. \( y = \cos(x^3) \)
7. Calculate \( \frac{d}{dx} \cos u \) for the following choices of \( u(x) \):
   (a) \( u(x) = 9 - x^2 \)
   (b) \( u(x) = x^{-1} \)
   (c) \( u(x) = \tan x \)
8. Calculate \( \frac{d}{dx} f(x^2 + 1) \) for the following choices of \( f(u) \):
   (a) \( f(u) = \sin u \)
   (b) \( f(u) = 3u^{3/2} \)
   (c) \( f(u) = u^2 - u \)
9. Compute \( \frac{df}{du} \) if \( f = 2 \) and \( \frac{du}{dx} = 6 \).
10. Compute \( \frac{df}{dx} \) if \( f(u) = u^2 \), \( u(2) = -5 \), and \( u'(2) = -5 \).
11. Let \( f(x) = (2x^2 - 3x) \). Compute \( f'(x) \) three different ways: 1) Multiplying out and then differentiating, 2) using the Product Rule, and 3) using the Chain Rule. Show that the results coincide.
12. Let \( f(x) = (x + \sin x)^{-1} \). Compute \( f'(x) \) separately using the Quotient Rule and the Chain Rule. Show that the results coincide.

In Exercises 13–24, compute the derivative using derivative rules that have been introduced so far.
13. \( y = (x^4 + 5)^3 \)
14. \( y = (8x^4 + 5)^3 \)
15. \( y = \sqrt{7x - 3} \)
16. \( y = (4x^4 - 7x^3)^3 \)
17. \( y = (x^2 + 9)^{1/2} \)
18. \( y = (x^3 + 3x + 9)^{1/2} \)
19. \( y = \cos^4 \theta \)
20. \( y = \cos(9\theta + 41) \)
21. \( y = (2 \cos \theta + 5 \sin \theta)^9 \)
22. \( y = \sqrt{9 + x + \sin x} \)
23. \( y = \sin (\sqrt{x^2 + 2x + 9}) \)
24. \( y = \tan(4 - 3x) \sec(3 - 4x) \)
25. \( f(u) = \sin u \), \( g(x) = 2x + 1 \)
26. \( f(u) = 2u + 1 \), \( g(x) = \sin x \)
27. \( f(u) = u^2 - u^{-1} \), \( g(x) = \tan x \)
28. \( f(u) = \frac{u}{u - 1} \), \( g(x) = \csc x \)
29. \( f(u) = \cos u \), \( g(x) = x^2 + 1 \)
30. \( f(u) = u^3 \), \( g(x) = \frac{1}{x+1} \)

In Exercises 31–44, use the Chain Rule to find the derivative.
31. \( y = \sin(x^3) \)
32. \( y = \sin^3 x \)
33. \( y = \sqrt{t^2 + 9} \)
34. \( y = (t^2 + 3t + 1)^{1/2} \)
35. \( y = (x^2 - x^3 - 1)^{1/3} \)
36. \( y = (x^2 + 1 - 1)^{1/2} \)
37. \( y = \frac{x + 1}{x - 1} \)
38. \( y = \cos^3(12\theta) \)
39. \( y = \sec \frac{1}{x} \)
40. \( y = \tan(0^2 - 4\theta) \)
41. \( y = \tan(\theta + \cos \theta) \)
42. \( y = \sqrt{\cos^2 \theta + 1} \)
43. \( y = \csc(9 - 2\theta) \)
44. \( y = \cos(\sqrt{\theta - 1}) \)

In Exercises 45–74, compute the derivative using derivative rules that have been introduced so far.
45. \( y = \tan(x^2 + 4x) \)
46. \( y = \sin(x^2 + 4x) \)
47. \( y = x \cos(1 - 3x) \)
48. \( y = \sin(x^2) \cos(x^2) \)
49. \( y = (4t + 9)^{1/2} \)
50. \( y = (x + 1)^{1/2}(2x - 1)^3 \)
51. \( y = (x^3 + \cos x)^{-4} \)
52. \( y = \sin(\cos x) \)
53. \( y = \sqrt{x} \cos x \)
54. \( y = (9 - x^2)^{1/2} \)
55. \( y = \cos(\theta + \sin x) \cos x \)
56. \( y = \frac{x + 1}{x + 2} \)
57. \( y = \tan^3 x + \tan(x^2) \)
58. \( y = \sqrt{x^3 - 3\cos x} \)
59. \( y = \sqrt{x + \frac{1}{x}} \)
60. \( y = (\cos^3 x + 3 \cos x + 7)^9 \)
61. \[ y = \frac{\cos(1+x)}{1 + \cos x} \]
62. \[ y = \sec(\sqrt{2} - 9) \]
63. \[ y = a \sec(Bx) \]
64. \[ y = \cos(\sqrt{3x^5}) \]
65. \[ y = \left(1 + \cos^2(x^4 + 1)^9\right)^9 \]
66. \[ y = \sqrt{\cos 2x + \sin 4x} \]
67. \[ y = (1 - \cos^2(1 - x^3))^{\frac{3}{2}} \]
68. \[ y = \sin(\sqrt{\sin \theta + 1}) \]
69. \[ y = \left(\frac{x + \frac{1}{3}}{x^2 + 9}\right)^{-1/2} \]
70. \[ y = \sec\left(1 + (4x - x^2)\right)^{-3/2} \]
71. \[ y = \sqrt{1 + \sqrt{1 + \sqrt{x}}} \]
72. \[ y = \sqrt{\sqrt{x} + 1 + 1} \]
73. \[ y = \frac{1}{\sqrt{kx} + b} \]
74. \[ y = \frac{1}{kx} + b \]

**In Exercises 75–78, compute the higher derivative.**

75. \[ \frac{d^2}{dx^2} \sin(2x) \]
76. \[ \frac{d^2}{dx^2} (x^2 + 9)^3 \]
77. \[ \frac{d^3}{dx^3} (9 - x)^8 \]
78. \[ \frac{d^3}{dx^3} \sin(2x) \]

79. Assume that the average molecular velocity \( v \) of a gas in a particular container is given by \( v(T) = 29 - \sqrt{T} \) m/s, where \( T \) is the temperature in kelvins. The temperature is related to the pressure (in atmospheres) by \( T = 200P \). Find \( \frac{d v}{dP} \) when \( r = 1.5 \).

80. The power \( P \) in a circuit is \( P = RI^2 \), where \( R \) is the resistance and \( i \) is the current. Find \( \frac{d P}{d R} \) at \( r = 1 \) if \( R = 1000 \Omega \) and \( i \) varies according to \( i = \sin(4\pi t) \) (time in seconds).

81. An expanding sphere has radius \( r = 0.4t \) cm at time \( t \) (in seconds). Let \( V \) be the sphere's volume. Find \( \frac{dV}{dt} \) when (a) \( r = 3 \) and (b) \( r = 3 \).

82. The function \( L(t) = 12 + 5.5 \sin(\frac{\pi}{12}) \) models the length of a day from sunrise to sunset in Moscow, Russia, where \( t \) is the day in the year after the spring equinox on March 21. Determine \( L'(t) \), and use it to calculate the rate that the length of the days are changing on March 25, April 30, and June 10. Discuss what the results say about the changing day length in the spring in Moscow.

83. The function \( L(t) = 12 + 3.4 \sin(\frac{3\pi}{12}) \) models the length of a day from sunrise to sunset in Sapporo, Japan, where \( t \) is the day in the year after the spring equinox on March 21. Determine \( L'(t) \), and use it to calculate the rate that the length of the days are changing on December 1, January 1, and February 1. Discuss what the results say about the changing day length in the late fall and winter in Sapporo.

84. From a 2005 study by the Fisheries Research Services in Aberdeen, Scotland, we infer that the average length in centimeters of the species *Clupea harengus* (Atlantic herring) as a function of age \( t \) (in years) can be modeled by the function

\[ L(t) = 32 \left(1 + 0.37t + 0.068t^2 + 0.0085t^3 + 0.00094t^4\right)^{-1} \]

for \( 0 \leq t \leq 13 \). See Figure 1.

**Figure 1** Average length of the species *Clupea harengus*.

(a) How fast is the average length changing at age \( t = 6 \) years?
(b) Use a plot of \( g'(t) \) to estimate the age \( t \) at which average length is changing at a rate of 5 cm/yr.

85. According to a 1999 study by Starkky and Scaranobbio, the average weight (in kilograms) at age \( t \) (in years) for channel catfish in the Lower Yellowstone River (Figure 2) is approximated by the function

\[ W(t) = \left(0.14 + 0.11t - 0.002t^2 + 0.000023t^3\right)^{0.4} \]

Find the rate at which the average weight is changing at \( t = 10 \) years.

**Figure 2** Average weight of channel catfish at age \( t \).

86. Calculate \( M'(0) \) in terms of the constants \( a, b, k, m, \) and \( n \), where

\[ M(t) = \left(a + b - b \left(1 + km t + \frac{1}{m} (kmt)^2\right)\right)^{1/n} \]

87. Assume that

\[ f(1) = 4, \quad f'(1) = -3, \quad g(2) = 1, \quad g'(2) = 3 \]

Calculate the derivatives of the following functions at \( x = 2 \):

(a) \( f(g(x)) \)
(b) \( f(x)/2 \)
(c) \( g(2g(x)) \)

88. Assume that

\[ f(0) = 2, \quad f'(0) = 3, \quad h(0) = -1, \quad h'(0) = 7 \]

Calculate the derivatives of the following functions at \( x = 0 \):

(a) \( f(x)^3 \)
(b) \( f(7x) \)
(c) \( f(4x)h(5x) \)

89. Compute the derivative of \( h(x) \) at \( x = \frac{\pi}{6} \), assuming that \( h'(0.5) = 10 \).

90. Let \( P(x) = f'(g(x)) \), where the graphs of \( f \) and \( g \) are shown in Figure 3. Estimate \( g'(2) \) and \( f'(g(2)) \) and compute \( P'(2) \).
Further Insights and Challenges

103. Show that if \( f \), \( g \), and \( h \) are differentiable, then

\[
[f(g(h(x)))]' = f'(g(h(x)))g'(h(x))h'(x)
\]

104. Show that differentiation reverses parity: If \( f \) is even, then \( f' \) is odd, and if \( f \) is odd, then \( f' \) is even. Hint: Differentiate \( f(-x) \).

105. (a) Sketch a graph of any even function \( f \) and explain graphically why \( f' \) is odd.
(b) Suppose that \( f' \) is even. Is \( f \) necessarily odd? Hint: Check whether this is true for linear functions.

106. Power Rule for Fractional Exponents

Consider the functions \( f(u) = u^p \) and \( g(x) = x^{p/q} \). Assume that \( g \) is differentiable.

(a) Show that \( f(g(x)) = x^{p/q} \) (recall the Laws of Exponents).
(b) Apply the Chain Rule and the Power Rule for whole-number exponents to show that \( f'(g(x))g'(x) = px^{q-1} \).
(c) Use the result of (b) to derive the Power Rule for \( x^{p/q} \).

107. Prove that for all whole numbers \( n \geq 1 \),

\[
\frac{d^n}{dx^n}\sin x = \sin \left( x + \frac{n\pi}{2} \right)
\]

Hint: Use the identity \( \cos x = \sin \left( x + \frac{\pi}{2} \right) \).

108. A Discontinuous Derivative

Use the limit definition to show that \( g(0) \) exists but \( g'(0) \neq \lim_{x \to 0} g'(x) \), where

\[
g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}
\]

109. Chain Rule

This exercise proves the Chain Rule without the special assumption made in the text. For any number \( a \), define a new function \( F(u) = \frac{f(u) - f(b)}{u - b} \) for all \( u \neq b \).

(a) Show that if we define \( F(b) = f'(b) \), then \( F \) is continuous at \( u = b \).
(b) Take \( b = g(a) \). Show that if \( x \neq a \), then for all \( u \),

\[
\frac{f(u) - f(g(a))}{x - a} = \frac{F(u) - g(a)}{u - a}
\]

Note that both sides are zero if \( u = g(a) \).
(c) Substitute \( u = g(x) \) in Eq. (2) to obtain

\[
\frac{f(g(x)) - f(g(a))}{x - a} = \frac{F(g(x)) - g(a)}{x - a}
\]

Derive the Chain Rule by computing the limit of both sides as \( x \to a \).

3.8 Implicit Differentiation

We have developed techniques for calculating a derivative \( dy/dx \) when \( y \) is given in terms of \( x \) by a formula—such as \( y = x^3 + 1 \); that is, when \( y \) is expressed explicitly as a function of \( x \). But suppose that \( y \) is instead related to \( x \) by an equation such as
In this case, we say that \( y \) is defined implicitly as a function of \( x \). How can we find the slope of the tangent line at a point, such as \((1,1)\), on the graph (Figure 1)? Although it may be difficult or even impossible to solve for \( y \) explicitly as a function of \( x \), we can find \( dy/dx \) using the method of implicit differentiation (see Example 2).

To illustrate, we first compute the slope of the tangent line to the unit circle at \( \left( \frac{3}{5}, \frac{4}{5} \right) \) (Figure 2). The equation of the unit circle is
\[
x^2 + y^2 = 1
\]
Compute \( dy/dx \) by taking the derivative of both sides of the equation:
\[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (1)
\]
\[
\frac{d}{dx} (x^2) + \frac{d}{dx} (y^2) = 0
\]
\[
2x + \frac{d}{dx} (y^2) = 0
\]
How do we handle the term \( \frac{d}{dx} (y^2) \)? We use the Chain Rule. Think of \( y \) as a function \( y = y(x) \). Then \( y^2 = (y(x))^2 \) and by the Chain Rule,
\[
\frac{d}{dx} y^2 = \frac{d}{dx} (y(x))^2 = 2y(x) \frac{dy}{dx} = 2y \frac{dy}{dx}
\]
Equation (2) becomes \( 2x + 2y \frac{dy}{dx} = 0 \), and we can solve for \( \frac{dy}{dx} \) if \( y \neq 0 \):

\[
\frac{dy}{dx} = \frac{-x}{y}
\]

**Example 1** Use Eq. (3) to find the slope of the tangent line at the point \( P = \left( \frac{3}{5}, \frac{4}{5} \right) \) on the unit circle.

**Solution** Set \( x = \frac{3}{5} \) and \( y = \frac{4}{5} \) in Eq. (3):
\[
\frac{dy}{dx} \bigg|_P = \frac{-x}{y} = \frac{-\frac{3}{5}}{\frac{4}{5}} = \frac{-3}{4}
\]

In this particular example, we could have computed \( dy/dx \) directly, without implicit differentiation. The upper semicircle is the graph of \( y = \sqrt{1-x^2} \) and
\[
\frac{dy}{dx} = \frac{d}{dx} \sqrt{1-x^2} = \frac{1}{2}(1-x^2)^{-1/2} \frac{d}{dx} (1-x^2) = -\frac{x}{\sqrt{1-x^2}}
\]
This formula expresses \( dy/dx \) in terms of \( x \) alone, whereas Eq. (3) expresses \( dy/dx \) in terms of both \( x \) and \( y \), as is typical when we use implicit differentiation. The two formulas agree because \( y = \sqrt{1-x^2} \).

Before presenting additional examples, let's examine again how the factor \( dy/dx \) arises when we differentiate an expression involving \( y \) with respect to \( x \). It would not appear if we were differentiating with respect to \( y \). Thus,
\[
\frac{d}{dy} \sin y = \cos y \quad \text{but} \quad \frac{d}{dx} \sin y = (\cos y) \frac{dy}{dx}
\]
\[
\frac{d}{dy} y^4 = 4y^3 \quad \text{but} \quad \frac{d}{dx} y^4 = 4y^3 \frac{dy}{dx}
\]
Similarly, the Product Rule applied to $xy$ yields
\[ \frac{d}{dx} (xy) = \frac{dx}{dx} y + x \frac{dy}{dx} = y + x \frac{dy}{dx} \]

The Quotient Rule applied to $\frac{t^2}{y}$ yields
\[ \frac{d}{dt} \left( \frac{t^2}{y} \right) = \frac{y \frac{dt}{dt} t^2 - \frac{t}{y} \frac{dt}{dt} }{y^2} = \frac{2ty - t^2 \frac{dy}{dt}}{y^2} \]

**EXAMPLE 2** Find an equation of the tangent line at the point $P = (1, 1)$ on the curve in Figure 1 with equation
\[ y^4 + xy = x^3 - x + 2 \]

**Solution** We break up the calculation into two steps.

**Step 1. Differentiate both sides of the equation with respect to $x$.**

Note that each occurrence of $y$ in the original equation generates an additional $\frac{dy}{dx}$ upon differentiation.

\[ \frac{d}{dx} y^4 + \frac{d}{dx} (xy) = \frac{d}{dx} (x^3 - x + 2) \]
\[ 4y^3 \frac{dy}{dx} + (y + x \frac{dy}{dx}) = 3x^2 - 1 \]

**Step 2. Solve for $\frac{dy}{dx}$.**

Move the terms involving $dy/dx$ in Eq. (4) to the left and place the remaining terms on the right:
\[ 4y^3 \frac{dy}{dx} + x \frac{dy}{dx} = 3x^2 - 1 - y \]
Then factor out $dy/dx$ and divide:
\[ (4y^3 + x) \frac{dy}{dx} = 3x^2 - 1 - y \]
\[ \frac{dy}{dx} = \frac{3x^2 - 1 - y}{4y^3 + x} \]

To find the derivative at $P = (1, 1)$, apply Eq. (5) with $x = 1$ and $y = 1$:
\[ \frac{dy}{dx} \bigg|_{(1,1)} = \frac{3 \cdot 1^2 - 1 - 1}{4 \cdot 1^3 + 1} = \frac{1}{5} \]

An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 1)$ or $y = \frac{1}{5}x + \frac{4}{5}$.

**CONCEPTUAL INSIGHT** The graph of an equation does not always define a function of $x$ because there may be more than one $y$-value for a given value of $x$. Implicit differentiation works because the graph is generally made up of several pieces called branches, each of which does define a function (a proof of this fact relies on the Implicit Function Theorem from advanced calculus). For example, the branches of the unit circle $x^2 + y^2 = 1$ are the graphs of the functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$. Similarly, the graph in Figure 3 has an upper and a lower branch. In most examples, the branches are differentiable except at certain exceptional points where the tangent line may be vertical.
FIGURE 3 Each branch of the graph of \( y^4 + xy = x^3 - x + 2 \) defines a function of \( x \).

EXAMPLE 3 Calculate \( dy/dx \) at the point \((\frac{\pi}{4}, \frac{\pi}{2})\) on the curve 
\[
\sqrt{2} \cos(x + y) = \cos x - \cos y
\]

Solution We follow the steps of the previous example, this time writing \( y' \) for \( dy/dx \):
\[
\frac{d}{dx} \left( \sqrt{2} \cos(x + y) \right) = \frac{d}{dx} \cos x - \frac{d}{dx} \cos y
\]
\[
-\sqrt{2} \sin(x + y) \cdot (1 + y') = -\sin x + (\sin y)y'
\]
(Chain Rule)
\[
-\sqrt{2} \sin(x + y) - \sqrt{2}y' \sin(x + y) = -\sin x + y' \sin y
\]
\[
- y' \left( \sin y + \sqrt{2} \sin(x + y) \right) = \sqrt{2} \sin(x + y) - \sin x
\]
(place \( y' \)-terms on left)
\[
y' = \frac{\sin x - \sqrt{2} \sin(x + y)}{\sin y + \sqrt{2} \sin(x + y)}
\]
The derivative at the point \((\frac{\pi}{4}, \frac{\pi}{2})\) is
\[
\frac{dy}{dx} \left( \frac{\pi}{4}, \frac{\pi}{2} \right) = \frac{\sin \frac{\pi}{4} - \sqrt{2} \sin \left( \frac{\pi}{4} + \frac{\pi}{2} \right)}{\sin \frac{\pi}{4} + \sqrt{2} \sin \left( \frac{\pi}{4} + \frac{\pi}{2} \right)} = \frac{\sqrt{2}/2 - \sqrt{2}}{\sqrt{2}/2 + \sqrt{2}} = -\frac{1}{3}
\]

EXAMPLE 4 Shortcut to Derivative at a Specific Point Calculate \( \frac{dy}{dt} \bigg|_P \) at the point \( P = (0, \frac{5\pi}{2}) \) on the curve (Figure 4):
\[
y \cos(y + t + t^2) = t^3
\]

Solution As before, differentiate both sides of the equation (we write \( y' \) for \( dy/dt \)):
\[
\frac{dt}{dt} y \cos(y + t + t^2) = \frac{dt}{dt} t^3
\]
\[
y' \cos(y + t + t^2) - y \sin(y + t + t^2)(y' + 1 + 2t) = 3t^2
\]
\[
6
\]
We could continue to solve for \( y' \) in terms of \( t \) and \( y \), but that is not necessary since we are only interested in \( dy/dt \) at the point \( P \). Instead, we can substitute \( t = 0, y = \frac{5\pi}{2} \) directly in Eq. (6) and then solve for \( y' \):
\[
y' \cos \left( \frac{5\pi}{2} + 0 + 0^2 \right) - \left( \frac{5\pi}{2} \right) \sin \left( \frac{5\pi}{2} + 0 + 0^2 \right) (y' + 1 + 0) = 0
\]
\[
0 - \left( \frac{5\pi}{2} \right) (1)(y' + 1) = 0
\]
This gives us \( y' + 1 = 0 \) or \( y' = -1 \).
Finding Higher Order Derivatives Implicitly

By using the implicit derivative process repeatedly, we can find higher order derivatives of a function that is defined implicitly. We do so in the next example.

**EXAMPLE 5** Find \( \frac{d^2y}{dx^2} \) for \( x^2 + 4y^2 = 7 \).

**Solution** We differentiate with respect to \( x \), writing \( y' \) for \( \frac{dy}{dx} \):

\[
2x + 8yy' = 0
\]

Solving for \( y' \), we obtain

\[
y' = \frac{-x}{4y}
\]

Differentiating again with respect to \( x \), we obtain

\[
y'' = \frac{4y(-1) - (-x)(4y')}{16y^2} = \frac{-y + xy'}{4y^2}
\]

Substituting in the fact that \( y' = \frac{-x}{4y} \), yields

\[
y'' = \frac{-y + x(-x/(4y))}{4y^2} = \frac{-4y^2 - x^2}{16y^3} = \frac{-7}{16y^3}
\]

The last equality holds since \( x^2 + 4y^2 = 7 \).

### 3.8 SUMMARY

- Implicit differentiation is used to compute \( \frac{dy}{dx} \) when \( x \) and \( y \) are related by an equation.

  * **Step 1.** Take the derivative of both sides of the equation with respect to \( x \), treating \( y \) as a function of \( x \).
  * **Step 2.** Solve for \( \frac{dy}{dx} \) by collecting the terms involving \( \frac{dy}{dx} \) on one side and the remaining terms on the other side of the equation.

- Remember to include the factor \( \frac{dy}{dx} \) when differentiating expressions involving \( y \) with respect to \( x \). For instance,

\[
\frac{d}{dx} \sin y = (\cos y) \frac{dy}{dx}
\]

### 3.8 EXERCISES

**Preliminary Questions**

1. Which differentiation rule is used to show \( \frac{d}{dx} \sin y = \cos y \frac{dy}{dx} \)?

2. One of (a)–(c) is incorrect. Find and correct the mistake.

   (a) \( \frac{d}{dy} \sin(y^2) = 2y \cos(y^2) \)
   
   (b) \( \frac{d}{dx} \sin(x^2) = 2x \cos(x^2) \)
   
   (c) \( \frac{d}{dx} \sin(y^2) = 2y \cos(y^2) \)

3. On an exam, Jason was asked to differentiate the equation

\[
x^2 + 2xy + y^2 = 7
\]

Find the errors in Jason’s answer: \( 2x + 2x + 3y^2 = 0 \).

4. Which of (a) or (b) is equal to \( \frac{d}{dx} \) ( \( x \sin t \))?

   (a) \( (x \cos t) \frac{dt}{dx} \)
   
   (b) \( (x \cos t) \frac{dt}{dx} + \sin t \)

5. Assume that \( a \) is a constant and that \( y \) is implicitly a function of \( x \). Compute the derivative with respect to \( x \) of each of \( a^2, x^2, \) and \( y^2 \).
Exercises

1. Show that if you differentiate both sides of $x^2 + 2y^3 = 6$, the result is $2x + 6y^2\frac{dy}{dx} = 0$. Then solve for $dy/dx$ and evaluate it at the point $(2, 1)$.

2. Show that if you differentiate both sides of $xy + 4x + 2y = 1$, the result is $(x + 2y)\frac{dy}{dx} + y + 4 = 0$. Then solve for $dy/dx$ and evaluate it at the point $(1, -1)$.

In Exercises 3–10, differentiate the expression with respect to $x$, assuming that $y$ is implicitly a function of $x$.

3. $x^3y^3$

4. $\frac{x^2}{y^2}$

5. $(x^2 + y^2)^{3/2}$

6. $\sqrt{x+y}$

7. $x\sqrt{y}$

8. $\tan(xy)$

9. $\frac{y}{y+1}$

10. $\sin\left(\frac{y}{x}\right)$

In Exercises 11–28, calculate the derivative with respect to $x$ of $y$ or the other variable appearing in the equation.

11. $3y^3 + x^2 = 5$

12. $y^4 - 2y = 4x^3 + x$

13. $x^2y + 2x^3y = x + y$

14. $xy^3 + x^2y^5 - x^3 = 3$

15. $x^3 + 2R^2 = 1$

16. $x^4 + z^2 = 1$

17. $\frac{y}{x} + \frac{x}{y} = 2y$

18. $\sqrt{x} + x = \frac{1}{x} + \frac{1}{y}$

19. $y^{-2/3} + x^{2/3} = 1$

20. $x^{1/2} + y^{2/3} = -4y$

21. $y + \frac{1}{y} = x^2 + x$

22. $\sin(xt) = t$

23. $\sin(x + y) = x + \cos y$

24. $\tan(x^2 y) = (x + y)^3$

25. $\tan(x + y) = \tan x + \tan y$

26. $x\sin y - y\cos x = 2$

27. $x + \cos(3x - y) = xy$

28. $2x^3 - x - y = \sqrt{x^2 + y^4}$

29. Show that $x + yx^{-1} = 1$ and $y = x - x^2$ define the same curve [except that $(0, 0)$ is not a solution of the first equation] and that implicit differentiation yields $y' = yx^{-1} - x$ and $y' = 1 - 2x$. Explain why these formulas produce the same values for the derivative.

30. Use the method of Example 4 to compute $\frac{dy}{dx}$ at $P = (2, 1)$ on the curve $y^2x^3 + y^3x^4 - 10x + y = 5$.

In Exercises 31 and 32, find $dy/dx$ at the given point.

31. $(x + 2)^3 - 6(2y + 3)^2 = 3$, (1, -1)

32. $\sin^2(3y) = x + y$, \(\left(\frac{2 - \pi}{4}, \frac{\pi}{4}\right)\)

In Exercises 33–40, find an equation of the tangent line at the given point.

33. $xy + x^2y^2 = 6$, (2, 1)

34. $x^{2/3} + y^{2/3} = 2$, (1, 1)

35. $x^2 + \sin y = xy^2 + 1$, (1, 0)

36. $\sin(x - y) = x\cos(y + \frac{\pi}{2})$, \(\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\)

37. $2x^{1/2} + 4y^{1/2} = xy$, (1, 4)

38. $\frac{x}{x+1} + \frac{y}{y+1} = 1$, (1, 1)

39. $\sin(2x - y) = \frac{x^2}{y}$, (0, $\pi$)

40. $x + \sqrt{x} = y^2 + y^4$, (1, 1)

41. Find the points on the graph of $y^2 = x^3 - 3x + 1$ (Figure 5) where the tangent line is horizontal, as follows:

(a) First show that $2yy' = 3x^2 - 3$, where $y' = dy/dx$.

(b) Do not solve for $y'$. Rather, set $y' = 0$ and solve for $x$. This yields two values of $x$ where the slope may be zero.

(c) Show that the positive value of $x$ does not correspond to a point on the graph.

(d) The negative value corresponds to the two points on the graph where the tangent line is horizontal. Find their coordinates.

42. Show, by differentiating the equation, that if the tangent line at a point $(x, y)$ on the curve $x^2y - 2x + 8y = 2$ is horizontal, then $x = 1$. Then substitute $y = x^{-1}$ in $x^2y - 2x + 8y = 2$ to show that the tangent line is horizontal at the points $(2, \frac{1}{2})$ and $(\frac{1}{2}, -4)$.

43. Find all points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal (Figure 6).

44. Show that no point on the graph of $x^2 - 3xy + y^2 = 1$ has a horizontal tangent line.

45. Figure 1 shows the graph of $y^4 + xy = x^2 - x + 2$. Find $dy/dx$ at the two points on the graph with $x$-coordinate 0 and find an equation of the tangent line at each of those points.

46. Folium of Descartes The curve $x^3 + y^3 = 3xy$ (Figure 7) was first discussed in 1638 by the French philosopher-mathematician René Descartes, who called it the folium (meaning "leaf"). Descartes's scientific colleague Gilles de Roberval called it the jasmine flower. Both men believed incorrectly that the leaf shape in the first quadrant was repeated in each quadrant, giving the appearance of petals of a flower. Find an equation of the tangent line at the point \(\left(\frac{3}{4}, \frac{3}{4}\right)\).
47. Find a point on the folium \( x^2 + y^3 = 3xy \) other than the origin at which the tangent line is horizontal.

48. Plot \( x^2 + y^2 = 3xy + b \) for several values of \( b \) and describe how the graph changes as \( b \to 0 \). Then compute \( dy/dx \) at the point \((b^{1/3}, 0)\). How does this value change as \( b \to \infty \)? Do your plots confirm this conclusion?

49. Find the \( x \)-coordinates of the points where the tangent line is horizontal on the trident curve \( xy = x^3 - 5x^2 + 2x - 1 \), so named by Isaac Newton in his treatise on curves published in 1710 (Figure 8). Hint: \( 2x^2 - 5x^2 + 1 = (2x - 1)(x^2 - 2x - 1) \).

\[ \text{Figure 8: Trident curve: } xy = x^3 - 5x^2 + 2x - 1. \]

50. Find an equation of the tangent line at each of the four points on the curve \( (x^2 + y^2 - 4x)^2 = 2(x^2 + y^2) \) where \( x = 1 \). This curve (Figure 9) is an example of a limaçon of Pascal, named after the father of the French philosopher Blaise Pascal, who first described it in 1650.

\[ \text{Figure 9: Limaçon: } (x^2 + y^2 - 4x)^2 = 2(x^2 + y^2). \]

51. Find the derivative, \( dy/dx \), at the points where \( x = 1 \) on the folium \( (x^2 + y^3)^2 = \frac{25}{4}xy^3 \). See Figure 10.

\[ \text{Figure 10: Folium curve: } (x^2 + y^3)^2 = \frac{25}{4}xy^3. \]

52. (CAS) Plot \( (x^2 + y^3)^2 = 12(x^2 - y^2) + 2 \) for \( x \) and \( y \) between \(-4\) and \( 4 \) using a computer algebra system. How many horizontal tangent lines does the curve appear to have? Find the points where these occur.

53. Calculate \( dx/dy \) for the equation \( y^4 + 1 = y^2 + x^2 \) and find the points on the graph where the tangent line is vertical.

54. Show that the tangent lines at \( x = 1 \pm \sqrt{2} \) to the conchoid with equation \( (x - 1)^2(x^2 + y^2) = 2x^2 \) are vertical (Figure 11).

\[ \text{Figure 11: Conchoid: } (x - 1)^2(x^2 + y^2) = 2x^2. \]

55. (CAS) Use a computer algebra system to plot \( y^2 = x^3 - 4x \) for \( x \) and \( y \) between \(-4\) and \( 4 \). Show that if \( dx/dy = 0 \), then \( y = 0 \). Conclude that the tangent line is vertical at the points where the curve intersects the \( x \)-axis. Does your plot confirm this conclusion?

56. Show that for all points \( P \) on the graph in Figure 12, the segments \( \overline{OP} \) and \( \overline{PR} \) have equal length.

\[ \text{Figure 12: Graph of } x^2 - y^2 = a^2. \]

In Exercises 57–58, first compute \( y' \) and \( y'' \) by implicit differentiation. Then solve the given equation for \( y \), and compute \( y' \) and \( y'' \) by direct differentiation. Finally, show that the results obtained by each approach are the same.

57. \( xy = y - 2 \)

58. \( xy^3 = 8 \)

In Exercises 59–62, use implicit differentiation to calculate higher derivatives.

59. Consider the equation \( y^3 - \frac{3}{2}x^2 = 1 \).

(a) Show that \( y' = x/y^3 \) and differentiate again to show that

\[ y'' = \frac{y^2 - 2xyy'}{y'^3}. \]

(b) Express \( y'' \) in terms of \( x \) and \( y \) using part (a).
60. Use the method of the previous exercise to show that \( y'' = -y^{-3} \) on the circle \( x^2 + y^2 = 1 \).

61. Calculate \( y'' \) at the point \((1, 1)\) on the curve \( xy^2 + y - 2 = 0 \) by the following steps:
(a) Find \( y' \) by implicit differentiation and calculate \( y' \) at the point \((1, 1)\).
(b) Differentiate the expression for \( y' \) found in (a). Then compute \( y'' \) at \((1, 1)\) by substituting \( x = 1 \), \( y = 1 \), and the value of \( y' \) found in (a).

62. Use the method of the previous exercise to compute \( y'' \) at the point \((1, 1)\) on the curve \( x^3 + y^3 = 3x + y - 2 \).

Exercises 63 and 64 explore the radius of curvature of curves. There can be many circles that are tangent to a curve at a particular point, but there is one that provides a "best fit" (Figure 13). This circle is called an osculating circle. We define it formally in Section 14.4. The radius of the osculating circle is called the radius of curvature of the curve and can be computed using either of the formulas:

\[
r = \frac{(1 + (dy/dx)^2)^{3/2}}{|d^2y/dx^2|}
\]

or

\[
r = \frac{(1 + (dx/dy)^2)^{3/2}}{|d^2x/dy^2|}
\]

**FIGURE 13** The osculating circle (solid red) at a point is the tangent circle that best fits the curve.

**Further Insights and Challenges**

69. Show that if \( P \) lies on the intersection of the two curves \( x^2 - y^2 = c \) and \( xy = d \) (\( c, d \) constants), then the tangents to the curves at \( P \) are perpendicular.

70. The lemniscate curve \( (x^2 + y^2)^2 = 4(x^2 - y^2) \) was discovered by Jacob Bernoulli in 1694, who noted that it is "shaped like a figure 8, or a knot, or the bow of a ribbon." Find the coordinates of the four points at which the tangent line is horizontal (Figure 14).

**FIGURE 14** Lemniscate curve: \( (x^2 + y^2)^2 = 4(x^2 - y^2) \).

71. Divide the curve in Figure 15 into five branches, each of which is the graph of a function. Sketch the branches.

73. Consider the ellipse \( x^2 + 4y^2 = 16 \).
(a) Compute the radius of curvature in terms of \( x \) and \( y \).
(b) Compute the radius of curvature at \((4, 0)\), \((2, \sqrt{3})\), and \((0, 2)\). Sketch the ellipse, plot these three points, and label them with the corresponding radius of curvature.

74. Consider the ellipse \( 9x^2 + y^2 = 36 \).
(a) Compute the radius of curvature in terms of \( x \) and \( y \).
(b) Compute the radius of curvature at \((-2, 0)\), \((1, -3\sqrt{3})\), and \((0, 6)\). Sketch the ellipse, plot these three points, and label them with the corresponding radius of curvature.

In Exercises 65–67, \( x \) and \( y \) are functions of a variable \( t \). Use implicit differentiation to express \( dy/dt \) in terms of \( dx/dt \), \( x \), and \( y \).

65. \( x^2y = 3 \)

66. \( x^3 - 6xy^2 = y \)

67. \( y^4 + 2x^2 = xy \)

68. The volume \( V \) and pressure \( P \) of gas in a piston (which vary in time \( t \)) satisfy \( PV^{3/2} = C \), where \( C \) is a constant. Prove that

\[
\frac{dP}{dt} = \frac{3}{2} \frac{P}{V}
\]

The ratio of the derivatives is negative. Could you have predicted this from the relation \( PV^{3/2} = C \)?

3.9 Related Rates

Suppose you are filling a bottle that has a cylindrical bottom and a top shaped like a cone. Water is flowing into the bottle from a faucet at a constant rate (Figure 1). Notice that the water level rises at a constant rate in the cylindrical part, but in the conical part, the level...
rises more rapidly. If you are not paying attention—expecting the water level to continue to rise at a constant rate—you might overfill the bottle.

To investigate this situation, we build a related rates model, a relationship between the rates of change of different variables that themselves are related. In this case, the rates that are related are the rate of increase of the volume of water in the bottle and the rate of increase of the height of the water. We investigate these rates of increase of volume and height for the cylindrical bottom in Example 1 and separately for the cone-shaped top in Example 2.

EXAMPLE 1 Filling a Cylindrical Container Water flows into a cylindrical container at a rate of 5 in.³/s. Assume that the container has a height of 6 in. and a base radius of 2 in. At what rate is the water level rising in the container?

Solution Let V represent the volume of the water in the container in in.³, and let h represent the height of the water in in. We draw a cylinder and label it with all the information we have (Figure 2). The goal is to find a relationship between the known rate, \( \frac{dV}{dt} \), and the desired rate, \( \frac{dh}{dt} \). That is,

\[
determine \frac{dh}{dt} \text{ given that } \frac{dV}{dt} = 5 \text{ in.}³/\text{s}.
\]

First, we find a relationship between V and h and then, because we want a relationship between \( \frac{dV}{dt} \) and \( \frac{dh}{dt} \), we differentiate the relationship between V and h with respect to t.

The geometry is simple: The volume of the water in the container is the volume of a cylinder with height h and base radius 2. So \( V = \pi 2^2 h = 4\pi h \). We differentiate with respect to t and obtain

\[
\frac{dV}{dt} = 4\pi \frac{dh}{dt}
\]

We are given that \( \frac{dV}{dt} = 5 \), so substituting that in the above equation and solving for \( \frac{dh}{dt} \), we find

\[
\frac{dh}{dt} = \frac{5}{4\pi} \approx 0.40
\]

Thus, the water level is rising in the container at a rate of approximately 0.40 in./s. Note that, as expected, the water level is changing at a constant rate.

We summarize the steps that we typically follow in developing a related rates model and solving an associated problem.

**Step 1.** Identify variables and the rates that are related.

**Step 2.** Find an equation relating the variables and differentiate it.

**Step 3.** Use given information to solve the problem.

EXAMPLE 2 Filling a Conical Container Water flows into a conical container at a rate of 5 in.³/s. Assume that the container has a height of 4 in. and a base radius of 2 in. Show that the rate that the water level is rising depends on the level of the water in the container, rising faster the higher the water level.
Solution

**Step 1. Identify variables and the rates that are related.**

Let $V$ represent the volume of the water in the container in $\text{in.}^3$, and let $h$ represent its height in in. Draw a cone with the given information (Figure 3). The goal:

\[
\text{Find a relationship between } \frac{dV}{dt} \text{ and } \frac{dh}{dt}.
\]

**Step 2. Find an equation relating the variables and differentiate it.**

We need to find a relationship between $V$ and $h$ that we can differentiate with respect to $t$. The geometry in this case is a little more involved than with the cylindrical container.

To express a relationship between the volume of the water and height, we regard the shaded volume in Figure 4 as the difference between the volume of the conical container and the volume of the conical space in the container above the water. The volume of the conical container is $\frac{1}{3} \pi (2^2)(4) = \frac{16\pi}{3}$, and the volume of the conical space is $\frac{1}{3} \pi r_s^2 h_s$, where $r_s$ and $h_s$ are the base radius and height, respectively, of the conical space. Note that $h_s = 4 - h$. Also note that there are similar triangles in Figure 4 from which we obtain $\frac{h_s}{r_s} = \frac{4}{2} = 2$. Thus, $r_s = \frac{1}{2} h_s = \frac{h - 4}{2}$. Now, putting together the terms, we have

\[
V = \frac{16\pi}{3} - \frac{1}{3} \pi \left(\frac{4 - h}{2}\right)^2 (4 - h) = \frac{16\pi}{3} - \frac{1}{12} \pi (4 - h)^3
\]

Differentiating with respect to $t$, and using the Chain Rule on the $(4 - h)^3$ term, we obtain

\[
\frac{dV}{dt} = 0 - \frac{1}{12} \pi (3(4 - h)^2) \left(\frac{dh}{dt}\right) = \frac{1}{4} \pi (4 - h)^2 \frac{dh}{dt}
\]

**Step 3. Use the given information to solve the problem.**

Substituting in 5 for $\frac{dV}{dt}$ and solving for $\frac{dh}{dt}$, the result is

\[
\frac{dh}{dt} = \frac{20}{\pi (4 - h)^2}
\]

As we expected, the rate of change $\frac{dh}{dt}$ of the water level depends on the level $h$ itself. In fact, it increases as the level increases. For example, at $h = 1, 2, 3$ in. we find $\frac{dh}{dt} \approx 0.71, 1.59, 6.37$ in/s, respectively.
Next, we examine the "sliding ladder problem" where a ladder that is leaning against a wall has its bottom pulled away at constant velocity. The question is, "How fast does the top of the ladder move?" What is interesting and perhaps surprising is that the top and bottom travel at different speeds. Figure 5 shows this clearly: The bottom travels the same distance over each time interval, but the top travels farther during the second time interval than the first. In other words, the top is speeding up while the bottom moves at a constant speed. We will explore this further through a related rates model.

EXAMPLE 3 Sliding Ladder Problem A 5-m ladder leans against a wall. The bottom of the ladder is 1.5 m from the wall at time $t = 0$ and slides away from the wall at a rate of 0.8 m/s. Find the velocity of the top of the ladder at time $t = 1$.

Solution

Step 1. Identify variables and the rates that are related.

Since we are considering how the top and bottom of the ladder change position, we use the variables $x$ for the distance from the bottom of the ladder to the wall and $h$ for the distance from the ground to the top of the ladder (Figure 6). The velocity of the bottom is $dx/dt = 0.8$ m/s. The unknown velocity of the top is $dh/dt$. Our goal is to compute $dh/dt$ at $t = 1$, given that $dx/dt = 0.8$ m/s and $x(0) = 1.5$ m.

Step 2. Find an equation relating the variables and differentiate it.

We wish to find a relationship between $dx/dt$ and $dh/dt$, and therefore we need an equation relating $x$ and $h$ (Figure 6). This is provided by the Pythagorean Theorem:

$$x^2 + h^2 = 5^2$$

To calculate $dh/dt$, we differentiate both sides of this equation with respect to $t$:

$$\frac{d}{dt} x^2 + \frac{d}{dt} h^2 = \frac{d}{dt} 25$$

$$2x \frac{dx}{dt} + 2h \frac{dh}{dt} = 0$$

Therefore, $\frac{dh}{dt} = -\frac{x}{h} \frac{dx}{dt}$.

Step 3. Use the given information to solve the problem.

Since $\frac{dx}{dt} = 0.8$, we have $\frac{dh}{dt} = -\frac{0.8x}{h}$.

To use this equation to determine $\frac{dh}{dt}$ at $t = 1$, we need to find $x$ and $h$ at that time. Since the bottom slides away from the wall at 0.8 m/s and we are given $x(0) = 1.5$, we have $x(1) = 2.3$ and $h(1) = \sqrt{5^2 - 2.3^2} \approx 4.44$. We obtain

$$\left.\frac{dh}{dt}\right|_{t=1} = -0.8 \frac{x(1)}{h(1)} \approx -0.8 \frac{2.3}{4.44} \approx -0.41 \text{ m/s}$$

The negative value for $\frac{dh}{dt}$ reflects the fact that the top of the ladder is sliding down the wall.

CONCEPTUAL INSIGHT A puzzling feature of Eq. (1) is that the velocity $dh/dt$, which is equal to $-0.8x/h$, becomes infinite as $h \to 0$ (as the top of the ladder gets close to the ground). Since this is impossible, our mathematical model must break down as $h \to 0$. In fact, using physics one can show that the ladder's top loses contact with the wall before reaching the bottom. From that moment on, the formula for $dh/dt$ is no longer valid.
EXAMPLE 4 Tracking a Rocket A spy uses a telescope to track a rocket launched vertically from a launch pad 6 km away, as in Figure 7. At the moment when the angle \( \theta \) between the telescope and the ground is \( \frac{\pi}{3} \), the angle is changing at a rate of 0.9 radians per minute. What is the rocket’s velocity at that moment?

Solution

Step 1. Identify variables and the rates that are related.

Let \( y \) be the height of the rocket at time \( t \). We wish to determine the rocket’s velocity \( \frac{dy}{dt} \) when \( \theta = \frac{\pi}{3} \). We are given that \( \frac{d\theta}{dt} = 0.9 \). Thus, our goal is to

\[
\text{compute } \frac{dy}{dt} \bigg|_{\theta=\frac{\pi}{3}}, \text{ given that } \frac{d\theta}{dt} = 0.9 \text{ rad/min when } \theta = \frac{\pi}{3}
\]

Step 2. Find an equation relating the variables and differentiate it.

We want a relationship between \( \frac{dy}{dt} \) and \( \frac{d\theta}{dt} \); therefore, we need to find a relation between \( \theta \) and \( y \). As we see in Figure 7,

\[
\tan \theta = \frac{y}{6}
\]

Now differentiate with respect to time:

\[
\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{6} \frac{dy}{dt}
\]

\[
\frac{dy}{dt} = \frac{6}{\cos^2 \theta} \frac{d\theta}{dt}
\]

Step 3. Use the given information to solve the problem.

At the given moment, \( \theta = \frac{\pi}{3} \) and \( \frac{d\theta}{dt} = 0.9 \), so Eq. (2) yields

\[
\frac{dy}{dt} = \frac{6}{\cos^2(\pi/3)} (0.9) = \frac{6}{(0.5)^2} (0.9) = 21.6 \text{ km/min}
\]

The rocket’s velocity at \( \theta = \frac{\pi}{3} \) is 21.6 km/min, or approximately 1296 km/h.

EXAMPLE 5 Farmer John’s tractor, traveling at 3 m/s, pulls a rope attached to a bale of hay through a pulley. With dimensions as indicated in Figure 8, how fast is the bale rising when \( x \), the horizontal distance from the tractor to the hay bale, is 5 meters?

Solution

Step 1. Identify variables and the rates that are related.

Let \( x \) be the horizontal distance from the tractor to the bale of hay, and let \( h \) be the height above ground of the top of the bale. We want to determine \( \frac{dh}{dt} \) when \( x = 5 \). We are given \( \frac{dx}{dt} = 3 \). Thus, our goal is to

\[
\text{compute } \frac{dh}{dt} \bigg|_{x=5}, \text{ given that } \frac{dx}{dt} = 3 \text{ m/s}
\]

Step 2. Find an equation relating the variables and differentiate it.

Let \( L \) be the total length of the rope. From Figure 8 (using the Pythagorean Theorem),

\[
L = \sqrt{x^2 + 4.5^2} + (6 - h)
\]

Differentiating with respect to \( t \), we obtain

\[
\frac{dL}{dt} = \frac{d}{dt} \left( \sqrt{x^2 + 4.5^2} + (6 - h) \right) = \frac{x}{\sqrt{x^2 + 4.5^2}} \frac{dx}{dt} - \frac{dh}{dt}
\]
Now $L$, the length of the rope, is constant. Therefore, $dL/dt = 0$. It follows from Eq. (3) that

$$
\frac{dh}{dt} = \frac{x \frac{dx}{dt}}{\sqrt{x^2 + 4.5^2}}
$$

**Step 3.** Use the given information to solve the problem.

Apply Eq. (4) with $x = 5$ and $dx/dt = 3$. The bale is rising at the rate

$$
\frac{dh}{dt} = \frac{(5)\cdot(3)}{\sqrt{5^2 + 4.5^2}} \approx 2.23 \text{ m/s}
$$

### 3.9 SUMMARY

- Related-rate models arise in situations where two or more variables are related and we wish to explore how the rate of change of one of the variables is related to the rates of change of the other variable(s).
- The following steps can be helpful in developing a related rates model and solving an associated problem:

  **Step 1.** Identify variables and the rates that are related.
  **Step 2.** Find an equation relating the variables and differentiate it. (A diagram is often helpful in identifying variables and relationships between them.)
  **Step 3.** Use the given information to solve the problem.

### 3.9 EXERCISES

#### Preliminary Questions

1. If $\frac{dx}{dt} = 3$ and $y = x^2$, what is $\frac{dy}{dt}$ when $x = -3, 2, 5$?

2. If $\frac{dy}{dt} = 2$ and $y = x^3$, what is $\frac{dx}{dt}$ when $x = -4, 2, 6$?

3. Assign variables and restate the following problem in terms of known and unknown derivatives (but do not solve it): How fast is the volume of a cube increasing if its side increases at a rate of 0.5 cm/min?

4. What is the relation between $dV/dt$ and $dr/dt$ if $V = \left(\frac{1}{3}\right)\pi r^3$?

#### Exercises

In Questions 5 and 6, water pours into a cylindrical glass of radius 4 cm. Let $V$ and $h$ denote the volume and water level, respectively, at time $t$.

5. Restate this question in terms of $dV/dt$ and $dh/dt$: How fast is the water level rising if water pours in at a rate of 2 cm$^3$/min?

6. Restate this question in terms of $dV/dt$ and $dh/dt$: At what rate is water pouring in if the water level rises at a rate of 1 cm/min?

In Exercises 5–8, assume that the radius $r$ of a sphere is expanding at a rate of 30 cm/min. The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and its surface area is $4\pi r^2$. Determine the given rate.

5. Volume with respect to time when $r = 15$ cm

6. Volume with respect to time at $t = 2$ min, assuming that $r = 0$ at $t = 0$

7. Surface area with respect to time when $r = 40$ cm

8. Surface area with respect to time at $t = 2$ min, assuming that $r = 10$ at $t = 0$
9. A conical tank (as in Example 2) has height 3 m and radius 2 m at the base. Water flows in at a rate of 2 m³/min. How fast is the water level rising when the level is 1 m and when the level is 2 m?

10. (Ge) A conical tank (as in Example 2) has height 8 m and radius 4 m at the base. Water flows in at a rate of 3 m³/min. Determine $\frac{dh}{dt}$ as a function of $h$, and provide a graph of this relationship.

In Exercises 11–14, refer to a 5-m ladder sliding down a wall, as in Figures 5 and 6. The variable $h$ is the height of the ladder's top at time $t$, and $x$ is the distance from the wall to the ladder's bottom.

11. Assume the bottom slides away from the wall at a rate of 0.8 m/s. Find the velocity of the top of the ladder at $t = 2$ s if the bottom is 1.5 m from the wall at $t = 0$ s.

12. Suppose that the top is sliding down the wall at a rate of 1.2 m/s. Calculate $\frac{dx}{dt}$ when $h = 3$ m.

13. Suppose that $h(t) = 4$ and the bottom slides down the wall at a rate of 1.2 m/s. Calculate $x$ and $\frac{dx}{dt}$ at $t = 2$ s.

14. What is the relation between $h$ and $x$ at the moment when the top and bottom of the ladder move at the same speed?

15. The radius $r$ and height $h$ of a circular cone change at a rate of 2 cm/s. How fast is the volume of the cone increasing when $r = 10$ and $h = 20$?

16. A road perpendicular to a highway leads to a farmhouse located 2 km away (Figure 9). An automobile travels past the farmhouse at a speed of 80 km/h. How fast is the distance between the automobile and the farmhouse increasing when the automobile is 6 km past the intersection of the highway and the road?

![Figure 9](image)

17. A man of height 1.8 m walks away from a 5-m lamppost at a speed of 1.2 m/s (Figure 10). Find the rate at which his shadow is increasing in length.

![Figure 10](image)

18. As Claudia walks away from a 264-cm lamppost, the tip of her shadow moves twice as fast as she does. What is Claudia's height?

19. At a given moment, a plane passes directly above a radar station at an altitude of 6 km.
(a) The plane's speed is 800 km/h. How fast is the distance between the plane and the station changing half a minute later?
(b) How fast is the distance between the plane and the station changing when the plane passes directly above the station?

20. In the setting of Exercise 19, let $\theta$ be the angle that the line through the radar station and the plane makes with the horizontal. How fast is $\theta$ changing 12 min after the plane passes over the radar station?

21. A hot air balloon rising vertically is tracked by an observer located 4 km from the liftoff point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{3}$, and it is changing at a rate of 0.2 rad/min. How fast is the balloon rising at this moment?

22. A laser pointer is placed on a platform that rotates at a rate of 20 revolutions per minute. The beam hits a wall 8 m away, producing a dot of light that moves horizontally along the wall. Let $\theta$ be the angle between the beam and the line through the searchlight perpendicular to the wall (Figure 11). How fast is this dot moving when $\theta = \frac{\pi}{6}$?

![Figure 11](image)

23. A rocket travels vertically at a speed of 1200 km/h. The rocket is tracked through a telescope by an observer located 16 km from the launching pad. Find the rate at which the angle between the telescope and the ground is increasing 3 min after liftoff.

24. Using a telescope, you track a rocket that was launched 4 km away, recording the angle $\theta$ between the telescope and the ground at half-second intervals. Estimate the velocity of the rocket if $\theta(10) = 0.205$ and $\theta(10.5) = 0.225$.

25. A police car traveling south toward Sioux Falls, Iowa, at 160 km/h pursues a truck traveling east away from Sioux Falls at 140 km/h (Figure 12). At time $t = 0$, the police car is 20 km north and the truck is 30 km east of Sioux Falls. Calculate the rate at which the distance between the vehicles is changing:
(a) At time $t = 0$
(b) 5 min later

![Figure 12](image)

26. A car travels down a highway at 25 m/s. An observer stands 150 m from the highway.
(a) How fast is the distance from the observer to the car increasing when the car passes in front of the observer? Explain your answer without making any calculations.
(b) How fast is the distance increasing 20 s later?
27. In the setting of Example 5, at a certain moment, the tractor’s speed is 3 m/s and the bale is rising at 2 m/s. How far is the tractor from the bale at this moment?

28. Placido pulls a rope attached to a wagon through a pulley at a rate of \( q \) m/s. With dimensions as in Figure 13:
(a) Find a formula for the speed of the wagon in terms of \( q \) and the variable \( x \) in the figure.
(b) Find the speed of the wagon when \( x = 0.6 \) if \( q = 0.5 \) m/s.

![Figure 13](image)

29. Julian is jogging around a circular track of radius 50 m. In a coordinate system with its origin at the center of the track, Julian’s \( x \)-coordinate is changing at a rate of \(-1.25 \) m/s when his coordinates are \((40, 30)\). Find \( dy/dt \) at this moment.

30. A particle moves counterclockwise around the ellipse with equation \( 9x^2 + 16y^2 = 25 \) (Figure 14).
(a) In which of the four quadrants is \( dx/dt > 0 \)? Explain.
(b) Find a relation between \( dx/dt \) and \( dy/dt \).
(c) At what rate is the \( x \)-coordinate changing when the particle passes the point \((1, 1)\) if its \( y \)-coordinate is increasing at a rate of \( 6 \) m/s?
(d) Find \( dy/dt \) when the particle is at the top and bottom of the ellipse.

![Figure 14](image)

34. Two parallel paths 15 m apart run east–west through the woods. Brooke jogs east on one path at 10 km/h, while Jamail walks west on the other path at 6 km/h. If they pass each other at time \( t = 0 \), how far apart are they 3 s later, and how fast is the distance between them changing at that moment?

35. A particle travels along a curve \( y = f(x) \) as in Figure 16. Let \( L(t) \) be the particle’s distance from the origin.
(a) Show that
\[
\frac{dL}{dt} = \frac{(x + f(x)f'(x))}{\sqrt{x^2 + f(x)^2}} \frac{dx}{dt}
\]
if the particle’s location at time \( t \) is \( P = (x, f(x)) \).
(b) Calculate \( L'(1) \) when \( x = 1 \) and \( x = 2 \) if \( f(x) = \sqrt{3x^2 - 8x + 9} \) and \( dx/dt = 4 \).

![Figure 16](image)

36. Let \( \theta \) be the angle in Figure 16, where \( P = (x, f(x)) \). In the setting of the previous exercise, show that
\[
\frac{d\theta}{dt} = \frac{\left( \frac{xf'(x) - f(x)}{x^2 + f(x)^2} \right)}{dx/dt}
\]

Hint: Differentiate \( \tan \theta = f(x)/x \) with respect to \( t \), and observe that \( \cos \theta = x / \sqrt{x^2 + f(x)^2} \).

Exercise 37 and 38 refer to the baseball diamond (a square of side 90 ft) in Figure 17.

37. A baseball player runs from home plate toward first base at 20 ft/s. How fast is the player’s distance from second base changing when the player is halfway to first base?

38. Player 1 runs to first base at a speed of 20 ft/s, while player 2 runs from second base to third base at a speed of 15 ft/s. Let \( s \) be the distance between the two players. How fast is \( s \) changing when player 1 is 30 ft from home plate and player 2 is 60 ft from second base?
39. The conical watering can in Figure 18 has a grid of holes. Water flows out through the holes at a rate of \( kA \) m\(^3\)/min, where \( k \) is a constant and \( A \) is the surface area of the part of the cone in contact with the water. This surface area is \( A = \pi r \sqrt{h^2 + r^2} \) and the volume is \( V = \frac{1}{3} \pi r^2 h \). Calculate the rate \( dh/dt \) at which the water level changes at \( h = 0.3 \) m, assuming that \( k = 0.25 \) m/min.

![Figure 18](image)

### Further Insights and Challenges

40. A bowl contains water that evaporates at a rate proportional to the surface area of water exposed to the air (Figure 19). Let \( A(h) \) be the cross-sectional area of the bowl at height \( h \).

(a) Explain why \( V(h + \Delta h) - V(h) \approx A(h) \Delta h \) if \( \Delta h \) is small.

(b) Use (a) to argue that \( \frac{dV}{dh} = A(h) \).

(c) Show that the water level \( h \) decreases at a constant rate.

![Figure 19](image)

41. A roller coaster has the shape of the graph in Figure 20. Show that when the roller coaster passes the point \((x, f(x))\), the vertical velocity of the roller coaster is equal to \( f'(x) \) times its horizontal velocity.

![Figure 20](image)

42. Two trains leave a station at \( t = 0 \) and travel with constant velocity \( v \) along straight tracks that make an angle \( \theta \).

(a) Show that the trains are separating from each other at a rate \( v \sqrt{2 - 2 \cos \theta} \).

43. As the wheel of radius \( r \) cm in Figure 21 rotates, the rod of length \( L \) attached at point \( P \) drives a piston back and forth in a straight line. Let \( x \) be the distance from the origin to point \( Q \) at the end of the rod, as shown in the figure.

(a) Use the Pythagorean Theorem to show that

\[
L^2 = (x - r \cos \theta)^2 + r^2 \sin^2 \theta
\]

(b) Differentiate Eq. (5) with respect to \( t \) to prove that

\[
2(x - r \cos \theta) \left( \frac{dx}{dt} + r \sin \theta \frac{d\theta}{dt} \right) + 2r^2 \sin \theta \cos \theta \frac{d\theta}{dt} = 0
\]

(c) Calculate the speed of the piston when \( \theta = \frac{\pi}{2} \), assuming that \( r = 10 \) cm, \( L = 50 \) cm, and the wheel rotates at 4 revolutions per minute.

![Figure 21](image)

44. A spectator seated 300 m away from the center of a circular track of radius 100 m watches an athlete run laps at a speed of 3 m/s. How fast is the distance between the spectator and athlete changing when the runner is approaching the spectator and the distance between them is 250 m? Hint: The diagram for this problem is similar to Figure 21, with \( r = 100 \) and \( x = 300 \).
CHAPTER REVIEW EXERCISES

In Exercises 1–4, refer to the function $f$ whose graph is shown in Figure 1.

1. Compute the average rate of change of $f(x)$ over $[0, 2]$. What is the graphical interpretation of this average rate?

2. For which value of $h$ is \( \frac{f(0.7 + h) - f(0.7)}{h} \) equal to the slope of the secant line between the points where $x = 0.7$ and $x = 1.1$?

3. Estimate \( \frac{f(0.7 + h) - f(0.7)}{h} \) for $h = 0.3$. Is the value of this difference quotient greater than or less than $f'(0.7)$?

4. Estimate $f'(0.7)$ and $f'(1.1)$.

In Exercises 5–8, compute $f'(a)$ using the limit definition and find an equation of the tangent line to the graph of $f$ at $x = a$.

5. $f(x) = x^2 - x$, $a = 1$

6. $f(x) = 5 - 3x$, $a = 2$

7. $f(x) = x^{-1}$, $a = 4$

8. $f(x) = x^3$, $a = -2$

In Exercises 9–12, compute $dy/dx$ using the limit definition.

9. $y = 4 - x^2$

10. $y = \sqrt{2x + 1}$

11. $y = \frac{1}{2-x}$

12. $y = \frac{1}{(x-1)^2}$

In Exercises 13–16, express the limit as a derivative.

13. $\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$

14. $\lim_{x \to 1} \frac{x^3 + 1}{x + 1}$

15. $\lim_{t \to \pi} \frac{\sin t \cos t}{t - \pi}$

16. $\lim_{\theta \to \pi} \frac{\cos \theta - \sin \theta + 1}{\theta - \pi}$

17. Find $f(4)$ and $f'(4)$ if the tangent line to the graph of $f$ at $x = 4$ has equation $y = 3x - 14$.

18. Each graph in Figure 2 shows the graph of a function $f$ and its derivative $f'$. Determine which is the function and which is the derivative.

19. Is (A), (B), or (C) the graph of the derivative of the function $f$ shown in Figure 3?

20. Sketch the graph of $f'$ if the graph of $f$ appears as in Figure 4.

21. Sketch the graph of a continuous function $f$ if the graph of $f'$ appears as in Figure 5 and $f(0) = 0$.  

![Figure 1](image1.png)

![Figure 2](image2.png)

![Figure 3](image3.png)

![Figure 4](image4.png)

![Figure 5](image5.png)
22. Let \( N(t) \) be the percentage of a state population infected with a flu virus on week \( t \) of an epidemic. What percentage is likely to be infected in week 4 if \( N(3) = 8 \) and \( N'(t) = 1.2t \)?
23. A girl's height \( h(t) \) (in centimeters) is measured at time \( t \) (in years) for \( 0 \leq t \leq 14 \):

\[
\begin{align*}
52 & , 75.1, 87.5, 96.7, 104.5, 111.8, 118.7, 125.2, 131.5, 137.5, 143.3, 149.2, 155.3, 160.8, 164.7
\end{align*}
\]

(a) What is the average growth rate over the 14-yr period?
(b) Is the average growth rate larger over the first half or the second half of this period?
(c) Estimate \( h'(t) \) (in cm/yr) for \( t = 3, 8 \).

24. A planet's period \( P \) (number of days to complete one revolution around the sun) is approximately \( 0.199A^{3/2} \), where \( A \) is the average distance (in millions of kilometers) from the planet to the sun. Calculate \( P \) and \( dP/dA \) for Earth using the value \( A = 150 \).

In Exercises 25 and 26, use the following table of values for the number \( A(t) \) of automobiles (in millions) manufactured in the United States in year \( t \).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(t) )</td>
<td>6.55</td>
<td>8.58</td>
<td>8.83</td>
<td>9.67</td>
<td>7.32</td>
<td>6.72</td>
<td>8.50</td>
</tr>
</tbody>
</table>

25. What is the interpretation of \( A'(t) \)? Estimate \( A'(1977) \). Does \( A'(1974) \) appear to be positive or negative?

26. Given the data, which of (A)–(C) in Figure 6 could be the graph of the derivative \( A' \)? Explain.

![Figure 6](image)

In Exercises 27–52, compute the derivative.

27. \( y = 3x^5 - 7x^2 + 4 \)
28. \( y = 4x^{-3/2} \)
29. \( y = t^{-3} \)
30. \( y = 4x^2 - x^{-2} \)
31. \( y = \frac{x + 1}{x^2 + 1} \)
32. \( y = \frac{3t^2 - 2}{4t - 9} \)
33. \( y = (4x^4 - 9x)^6 \)
34. \( y = (3t^2 + 20t - 3)^6 \)
35. \( y = (2 + 9t^2)^{3/2} \)
36. \( y = (x + 1)^3(x + 4)^4 \)
37. \( y = \frac{z}{\sqrt{1 - z}} \)
38. \( y = \left(1 + \frac{1}{x}\right)^3 \)
39. \( y = \frac{x^4 + \sqrt{x}}{x^2} \)
40. \( y = \frac{1}{(1 - x)\sqrt{2 - x}} \)
41. \( y = \sqrt{x + \sqrt{x + \sqrt{x}}} \)
42. \( h(z) = (z + (z + 1)^{1/2})^{-3/2} \)
43. \( y = \tan(t^3) \)
44. \( y = 4\cos(2 - 3x) \)
45. \( y = \sin(2x)\cos^2 x \)
46. \( y = \sin\left(\frac{4}{\theta}\right) \)
47. \( y = \frac{t}{1 + \sec t} \)
48. \( y = z\csc(9z + 1) \)
49. \( y = \frac{8}{1 + \cot \theta} \)
50. \( y = \sin^{100} x \)
51. \( y = \cos(x^{100}) \)
52. \( y = \cos(\cos(\cos(\theta))) \)

In Exercises 53–58, use the following table of values to calculate the derivative of the given function at \( x = 2 \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

53. \( S(x) = 3f(x) - 2g(x) \)
54. \( H(x) = f(x)g(x) \)
55. \( R(x) = \frac{f(x)}{g(x)} \)
56. \( G(x) = f(g(x)) \)
57. \( F(x) = f(g(2x)) \)
58. \( K(x) = f(x^2) \)

59. Find the points on the graph of \( f(x) = x^3 - 3x^2 + x + 4 \) where the tangent line has slope 10.
60. Find the points on the graph of \( x^{3/3} + y^{2/3} = 1 \) where the tangent line has slope 1.
61. Find \( a \) such that the tangent lines to \( y = x^3 - 2x^2 + x + 1 \) at \( x = a \) and \( x = a + 1 \) are parallel.
62. Use the table to compute the average rate of change of candidate A's percentage of votes over the intervals from day 20 to day 15, day 15 to day 10, and day 10 to day 5. If this trend continues over the last 5 days before the election, will candidate A win?

<table>
<thead>
<tr>
<th>Days Before Election</th>
<th>20</th>
<th>15</th>
<th>10</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Candidate A</td>
<td>44.8%</td>
<td>46.8%</td>
<td>48.3%</td>
<td>49.3%</td>
</tr>
<tr>
<td>Candidate B</td>
<td>55.2%</td>
<td>53.2%</td>
<td>51.7%</td>
<td>50.7%</td>
</tr>
</tbody>
</table>

In Exercises 63–68, calculate \( y' \).

63. \( y = 12x^3 - 5x^2 + 3x \)
64. \( y = x^{-2/5} \)
65. \( y = \sqrt{2x + 3} \)
66. \( y = \frac{4x}{x + 1} \)
67. \( y = \tan(x^2) \)
68. \( y = \sin^2(4x + 9) \)

In Exercises 69–74, compute \( \frac{dy}{dx} \).

69. \( x^3 - y^3 = 4 \)
70. \( 4x^2 - 9y^2 = 36 \)
71. \( y = xy^2 + 2x^2 \)
72. \( \frac{y}{x} = x + y \)
73. \( y = \sin(x + y) \)
74. \( \tan(x + y) = xy \)
In Exercises 75–76 compute \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \):

75. \( x^2 - 4y^2 = 8 \)

76. \( 6xy + y^2 = 10 \)

77. In Figure 7, for the three graphs on the left, identify \( f, f', \) and \( f'' \). Do the same for the three graphs on the right.

![Figure 7](image)

81. A bead slides down the curve \( xy = 10 \). Find the bead's horizontal velocity at time \( t = 2 \) s if the height of the bead at time \( t \) seconds is \( y = 400 - 16t^2 \) cm.

82. In Figure 9, \( x \) is increasing at 2 cm/s, \( y \) is increasing at 3 cm/s, and \( \theta \) is decreasing such that the area of the triangle has the constant value 4 cm².

(a) How fast is \( \theta \) decreasing when \( x = 4, y = 47 \)?

(b) How fast is the distance between \( P \) and \( Q \) changing when \( x = 4, y = 47 \)? Hint: Use the Law of Cosines.

![Figure 9](image)

83. A light moving at 0.8 m/s approaches a man standing 4 m from a wall (Figure 10). The light is 1 m above the ground. How fast is the tip \( P \) of the man’s shadow moving when the light is 7 m from the wall?

![Figure 10](image)
4 APPLICATIONS OF THE DERIVATIVE

This chapter puts the derivative to work. The first and second derivatives are used to analyze functions and their graphs and to solve optimization problems (finding minimum and maximum values of a function). Newton's Method in Section 4.7 employs the derivative to approximate solutions of equations.

4.1 Linear Approximation and Applications

In this section, we introduce the process of linear approximation that uses the tangent line to the graph of a function $f$ at $x = a$ to approximate $f(x)$ for $x$ near $a$. These approximation methods are desirable because linear functions are usually easier to use and compute with than nonlinear ones. We introduce a few different formulas involving linear approximation. There are different settings and situations where each is useful. Keep in mind that they all come from the same basic idea that the tangent line approximates the function close to the point of tangency (Figure 1).

**Linear Approximation**

In some situations, we are interested in the effect of a small change. For example,

- How does a small change in angle affect the distance of a basketball shot? (Exercise 45)
- How are revenues at the box office affected by a small change in ticket prices? (Exercise 35)
- The cube root of 27 is 3. How much greater is the cube root of 27.2? (Exercise 7)

In each case, we have a function $f$ and we're interested in the change

$$\Delta f = f(a + \Delta x) - f(a)$$

where $\Delta x$ is small. The Linear Approximation uses the slope of the tangent line (i.e., the derivative) to estimate $\Delta f$ without computing it exactly. By definition, the derivative is the limit

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

So when $\Delta x$ is small, we have $\Delta f/\Delta x \approx f'(a)$, and thus,

$$\Delta f \approx f'(a)\Delta x$$

**Linear Approximation of $\Delta f$**

If $f$ is differentiable at $x = a$ and $\Delta x$ is small, then

$$\Delta f \approx f'(a)\Delta x$$

It is important to understand the different roles played by $\Delta f$ and $f'(a)\Delta x$. The quantity of interest is the actual change $\Delta f$. We estimate it by $f'(a)\Delta x$, the change on the tangent line with slope $f'(a)$. The Linear Approximation tells us that up to a small error, $\Delta f$ is approximately equal to $f'(a)\Delta x$ when $\Delta x$ is small.
**Graphical Insight** As we indicated, the Linear Approximation is an approximation using a tangent line. In fact, it is sometimes called the tangent line approximation. Observe in Figure 2 that $\Delta f$ is the vertical change in the graph from $x = a$ to $x = a + \Delta x$. For a line, the vertical change is equal to the slope times the horizontal change $\Delta x$, and since the tangent line has slope $f'(a)$, its vertical change is $f'(a)\Delta x$. So the Linear Approximation approximates $\Delta f$ by the vertical change in the tangent line. When $\Delta x$ is small, the two quantities are nearly equal.

**Figure 2** Graphical meaning of the Linear Approximation $\Delta f \approx f'(a)\Delta x$.

**Example 1** Use the Linear Approximation to estimate the change in $f(x) = \frac{1}{x}$ as $x$ goes from 10 to 10.2; that is, to estimate $\Delta f = \frac{1}{10.2} - \frac{1}{10}$. How accurate is the estimate?

**Solution** We apply the Linear Approximation to $f(x) = \frac{1}{x}$ with $a = 10$ and $\Delta x = 0.2$.

We have $f'(x) = -x^{-2}$ and $f'(10) = -0.01$, so $\Delta f$ is approximated by

$$\Delta f \approx f'(10)\Delta x = (-0.01)(0.2) = -0.002$$

Since $\Delta f = \frac{1}{10.2} - \frac{1}{10}$, we have the approximation

$$\frac{1}{10.2} - \frac{1}{10} \approx -0.002$$

A calculator gives the value $\frac{1}{10.2} - \frac{1}{10} \approx -0.00196$, and thus, our error is less than $10^{-4}$:

$$\text{error} \approx \left| -0.00196 - (-0.002) \right| = 0.00004 < 10^{-4}$$

**Example 2** Approximate how much greater $\sqrt[8]{8.1}$ is than $\sqrt[8]{8} = 2$, and then use the result to approximate $\sqrt[6]{8.1}$.

**Solution** We are interested in $\sqrt[8]{8.1} - \sqrt[8]{8}$, so we apply the Linear Approximation to $f(x) = x^{1/8}$ with $a = 8$ and $\Delta x = 0.1$. We have

$$f'(x) = \frac{1}{8}x^{-7/8} \quad \text{and} \quad f'(8) = \left(\frac{1}{8}\right)8^{-7/8} = \left(\frac{1}{8}\right)\left(\frac{1}{8}\right) = \frac{1}{12}$$

Therefore, $\Delta f \approx f'(8)\Delta x = \frac{1}{12}(0.1) \approx 0.0083$, and since

$$\Delta f = f(a + \Delta x) - f(a) = \sqrt[8]{8 + 0.1} - \sqrt[8]{8} = \sqrt[8]{8.1} - 2$$

we have the approximation

$$\sqrt[6]{8.1} - 2 \approx 0.0083$$
Thus, $\sqrt{8.1}$ is greater than $\sqrt{8}$ by approximately 0.0083. It follows that
$$\sqrt{8.1} \approx 2 + 0.0083 = 2.0083$$

Suppose that we measure the diameter $D$ of a circle and use this result to compute the area of the circle. If our measurement of $D$ is inexact, the area computation will also be inexact. What is the effect of the measurement error on the resulting area computation? This can be estimated using the Linear Approximation, as in the next example.

**Example 3** Effect of an Inexact Measurement The Cheezy Pizza Parlor claims that its pizzas are circular with diameter 50 cm (Figure 3).

(a) What is the area of the pizza?
(b) Estimate the quantity of pizza lost or gained if the diameter is off by at most 1.2 cm.

**Solution** First, we need a formula for the area $A$ of a circle in terms of its diameter $D$. Since the radius is $r = D/2$, the area is
$$A(D) = \pi r^2 = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi}{4} D^2$$

(a) If $D = 50$ cm, then the pizza has area $A(50) = (\frac{\pi}{4})(50)^2 \approx 1963.5$ cm$^2$.

(b) If the actual diameter is equal to $50 + \Delta D$, then the loss or gain in pizza area is $A(50 + \Delta D) - A(50) = \Delta A$. We apply Linear Approximation to $A(D)$ with $D = 50$ and $\Delta D = \pm 1.2$. Observe that $A'(50) = \frac{\pi}{2}$ and $A'(50) = 25\pi \approx 78.5$ cm, so the Linear Approximation yields
$$\Delta A \approx A'(50) \Delta D \approx (78.5) \Delta D$$

Because $\Delta D$ is at most $\pm 1.2$ cm, the loss or gain in pizza is no more than around
$$\Delta A \approx \pm (78.5)(1.2) \approx \pm 94.2 \text{ cm}^2$$

This is a loss or gain of approximately 4.8% of the area of 1963.5 cm$^2$.

**Linearization**

To approximate the function $f$ itself rather than the change $\Delta f$, we use the linearization $L(x)$ centered at $x = a$, defined by
$$L(x) = f'(a)(x - a) + f(a)$$

Notice that $y = L(x)$ is the equation of the tangent line at $x = a$. For values of $x$ close to $a$, $L(x)$ provides a good approximation to $f(x)$ (Figure 4).
Approximating \( f \) by Its Linearization  If \( f \) is differentiable at \( a \) and \( x \) is close to \( a \), then \( f(x) \approx L(x) \), so

\[
f(x) \approx f(a) + f'(a)(x - a)
\]

Note that, by rearranging the terms in linearization formula, we obtain the Linear Approximation formula, \( \Delta f \approx f'(a) \Delta x \), that we introduced previously. Specifically, with \( \Delta x = x - a \) and \( \Delta f = f(x) - f(a) \), we have

\[
f(x) \approx f(a) + f'(a)(x - a) \\
\Delta f = f(x) - f(a) \approx f'(a) \Delta x \quad \text{(since } \Delta x = x - a) \\
\Delta f \approx f'(a) \Delta x
\]

**EXAMPLE 4** Compute the linearization of \( f(x) = \sqrt{x} \) at \( a = 1 \) (Figure 5).

**Solution** We evaluate \( f(x) \) and its derivative \( f'(x) = \frac{1}{2}x^{-1/2} \) at \( a = 1 \) to obtain \( f(1) = \sqrt{1} = 1 \) and \( f'(1) = \frac{1}{2} \). The linearization at \( a = 1 \) is

\[
L(x) = f(1) + f'(1)(x - 1) = 1 + \frac{1}{2}(x - 1) = \frac{1}{2}x + \frac{1}{2}
\]

The linearization can be used to approximate function values. The following table compares values of the linearization to values obtained from a calculator for the function \( f(x) = \sqrt{x} \) of the previous example. Note that the error is large for \( x = 9 \), as expected, because 9 is not close to the center \( a = 1 \) (Figure 5).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \sqrt{x} )</th>
<th>Linearization at ( a = 1 )</th>
<th>Calculator</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>√1.1</td>
<td>( L(1.1) = \frac{1}{2}(1.1) + \frac{1}{2} = 1.05 )</td>
<td>1.0488</td>
<td>0.0012</td>
</tr>
<tr>
<td>0.98</td>
<td>√0.98</td>
<td>( L(0.98) = \frac{1}{2}(0.98) + \frac{1}{2} = 0.99 )</td>
<td>0.98995</td>
<td>5 \cdot 10^{-5}</td>
</tr>
<tr>
<td>9</td>
<td>√9</td>
<td>( L(9) = \frac{1}{2}(9) + \frac{1}{2} = 5 )</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

In the next example, we compute the percentage error, which is often more important than the error itself because it gives us a measure of how large the error is in relation to the actual value. An error of 0.1 is more significant when the actual value is 3 than when the actual value is 333. By definition,

\[
\text{percentage error} = \left| \frac{\text{error}}{\text{actual value}} \right| \times 100\% 
\]

**EXAMPLE 5** Estimate \( \tan \left( \frac{\pi}{4} + 0.02 \right) \) and compute the percentage error.

**Solution** We use the linearization of \( f(x) = \tan x \) at \( a = \frac{\pi}{4} \) for our approximation. So we need to calculate the terms in \( f(\pi/4) + f'(\pi/4)(x - \pi/4) \):

\[
f \left( \frac{\pi}{4} \right) = \tan \left( \frac{\pi}{4} \right) = 1, \quad f' \left( \frac{\pi}{4} \right) = \sec^2 \left( \frac{\pi}{4} \right) = (\sqrt{2})^2 = 2
\]

\[
f \left( \frac{\pi}{4} \right) + f' \left( \frac{\pi}{4} \right) \left( x - \frac{\pi}{4} \right) = 1 + 2 \left( x - \frac{\pi}{4} \right)
\]

So, for \( x \) near \( \pi/4 \), we have the approximation formula

\[
\tan(x) \approx 1 + 2 \left( x - \frac{\pi}{4} \right)
\]
At \( x = \frac{\pi}{4} + 0.02 \), this approximation yields the estimate

\[
\tan\left(\frac{\pi}{4} + 0.02\right) \approx 1 + 2\left(\frac{\pi}{4} + 0.02 - \frac{\pi}{4}\right) = 1.04
\]

A calculator gives \( \tan\left(\frac{\pi}{4} + 0.02\right) \approx 1.0408 \), so

\[
\text{percentage error} \approx \left| \frac{1.0408 - 1.04}{1.0408} \right| \times 100 \approx 0.08\%
\]

\[\boxed{\text{Differential Form of Linear Approximation}}\]

Another way of expressing the Linear Approximation is via the differentials \( dx \) and \( dy \) that represent the change in \( x \) and \( y \), respectively, on the tangent line to \( f(x) \) at \( x = a \). Since these differentials represent change on the tangent line, we have

\[
dy = f'(a)dx
\]

As before, we let \( \Delta y \) represent the change in \( y \) on the graph of \( f \). It follows—as in the previous approximations in this section—that with a small change in \( x \), the change in \( y \) on the graph is approximately the change in \( y \) on the tangent line (Figure 6). Thus, \( \Delta y \approx dy \), yielding the following:

\[\boxed{\text{Differential Form of Linear Approximation}} \text{ If } f \text{ is differentiable at } a \text{ and } dx \text{ is small, then}
\]

\[
\Delta y \approx dy = f'(a)dx
\]

\[\text{CONCEPTUAL INSIGHT} \text{ At the start of the section, we observed that all of the approximation relationships presented here are based on the idea that the tangent line is a good approximation to the graph of the function near the point of tangency. The Linear Approximation, the linearization, and the Differential Form of Linear Approximation are illustrated in Figures 2, 4, and 6, respectively. Note that these figures all depict the graph of } f \text{ and the tangent line at } x = a. \text{ From figure to figure, various features are described or labeled differently in order to illustrate the important aspects of each approximation relationship.}

\[\text{You might wonder why we bother with the Differential Form of Linear Approximation. At this point, it just appears to be another way of expressing a relationship that we already had a perfectly good way of expressing. The intent here is to provide an initial}\]
glimpse into a tool, the differential, that is employed often by mathematicians, scientists, and engineers to express or approximate a small change involving related variables. A differential corresponds to the change on the tangent line (as we see here), on the tangent plane (see Section 15.4), or in the tangent space (in higher dimensions). Differentials provide a straightforward linear means for approximating and working with complicated relationships.

EXAMPLE 6 Thermal Expansion Changes in temperature can have subtle effects on physical properties of objects that we might think normally are constant. A thin metal cable has length $L = 12$ cm when the temperature is $T = 21^\circ$C. Estimate the change in length when $T$ rises to $24^\circ$C, assuming that

$$\frac{dL}{dT} = kL$$

where $k = 1.7 \times 10^{-5}$ C$^{-1}$ ($k$ is called the coefficient of thermal expansion).

Solution How does the Linear Approximation apply here? We will use the differential $dL$ to estimate the actual change in length $\Delta L$ when $T$ increases from $21^\circ$C to $24^\circ$C—that is, when $dT = 3^\circ$C. By Eq. (2), the differential $dL$ is

$$dL = \left(\frac{dL}{dT}\right) dT$$

By Eq. (4), since $L = 12$,

$$\frac{dL}{dT} \bigg|_{L=12} = kL = (1.7 \times 10^{-5})(12) \approx 2 \times 10^{-4} \text{ cm}^2\text{C}$$

Therefore, with $dT = 3$, we have

$$dL = \left(\frac{dL}{dT}\right) dT \approx (2 \times 10^{-4})(3) = 6 \times 10^{-4} \text{ cm}$$

Thus, $\Delta L \approx dL$ tells us that when the temperature increases from $21^\circ$C to $24^\circ$C, we can expect the cable to lengthen by approximately 0.0006 cm.

The Size of the Error

The examples in this section may have convinced you that the Linear Approximation yields a good approximation to $\Delta f$ when $\Delta x$ is small, but if we want to rely on the Linear Approximation, we need to know more about the size of the error:

$$E = \text{error} = |\Delta f - f'(a)\Delta x|$$

Graphically the error $E$ is the vertical gap between the graph of $f$ and the tangent line (Figure 7). In Section 11.7, we will prove the following Error Bound:

$$E \leq \frac{1}{2}K(\Delta x)^2$$

where $K$ is the maximum value of $|f''(x)|$ on the interval from $a$ to $a + \Delta x$.

The Error Bound tells us two important things. First, it says that the error is small when the second derivative (and hence $K$) is small. This makes sense, because $f''(x)$ measures how quickly the tangent lines change direction. When $|f''(x)|$ is smaller, the graph is flatter and the Linear Approximation is more accurate over a larger interval around $x = a$ (compare the graphs in Figure 8).
Second, the Error Bound tells us that the error is of order 2 in $\Delta x$, meaning that $E$ is no larger than a constant times $(\Delta x)^2$. So if $\Delta x$ is small, say, $\Delta x = 10^{-6}$, then $E$ has a substantially smaller order of magnitude, since $(\Delta x)^2 = 10^{-12}$. In particular, $E/\Delta x$ tends to zero (because $E/\Delta x \ll K\Delta x$), so the Error Bound tells us that the graph becomes nearly indistinguishable from its tangent line as we zoom in on the graph around $x = a$. This is a precise version of the "local linearity" property discussed in Section 3.2.

### 4.1 SUMMARY

The approximation formulas in this section are all based on the idea that the tangent line to the graph of a function $f$ at $x = a$ can be used to approximate $f(x)$ for $x$ near $a$.

- Let $\Delta f = f(a + \Delta x) - f(a)$. The Linear Approximation is the estimate
  \[ \Delta f \approx f'(a)\Delta x \quad \text{(for } \Delta x \text{ small)} \]

- The linearization of $f(x)$ centered at $x = a$ is the function for the tangent line
  \[ L(x) = f(a) + f'(a)(x - a) \]

- The approximation based on linearization is
  \[ f(x) \approx f(a) + f'(a)(x - a) \quad \text{(for } x \text{ close to } a) \]

- Differential notation: $dx = \Delta x$ is the change in $x$, $dy = f'(a)dx$ is the change on the tangent line, and $\Delta y = f(a + \Delta x) - f(a)$ is the change in $f$. In this notation, the Differential Form of Linear Approximation is
  \[ \Delta y \approx dy = f'(a)dx \quad \text{(for } dx \text{ small)} \]

- The error in the Linear Approximation is the quantity
  \[ \text{error} = |\Delta f - f'(a)\Delta x| \]

  The percentage error is often more significant because it is a measure of how large the error is in relation to the actual value:
  \[ \text{percentage error} = \left| \frac{\text{error}}{\text{actual value}} \right| \times 100\% \]

- The error $E$ in the Linear Approximation is bounded as follows:
  \[ E \leq \frac{1}{2} K (\Delta x)^2 \]

  where $K$ is the maximum value of $|f''(x)|$ on the interval from $a$ to $a + \Delta x$. 

```
4.1 EXERCISES

Preliminary Questions

1. True or False? The Linear Approximation says that the vertical change in the graph is approximately equal to the vertical change in the tangent line.

2. Estimate \( g(1.2) - g(1) \) if \( g'(1) = 4 \).

3. Estimate \( f(2.1) \) if \( f(2) = 1 \) and \( f'(2) = 3 \).

4. Complete the following sentence: The Linear Approximation shows that up to a small error, the change in output \( \Delta f \) is directly proportional to ________.

Exercises

In Exercises 1–6, use Eq. (1) to estimate \( \Delta f = f(3.02) - f(3) \).

1. \( f(x) = x^2 \)

2. \( f(x) = x^4 \)

3. \( f(x) = x^{-1} \)

4. \( f(x) = \frac{1}{x + 1} \)

5. \( f(x) = x^6 + 6 \)

6. \( f(x) = \tan \frac{\pi x}{3} \)

7. The cube root of 27 is 3. How much larger is the cube root of 27.2? Estimate using the Linear Approximation.

8. The cube root of 64 is 4. How much smaller is the cube root of 63.6? Estimate using the Linear Approximation.

In Exercises 9–12, use Eq. (1) to estimate \( \Delta f \). Use a calculator to compute both the error and the percentage error.

9. \( f(x) = \sqrt{1 + x}, \quad a = 3, \quad \Delta x = 0.2 \)

10. \( f(x) = 2x^2 - x, \quad a = 5, \quad \Delta x = -0.4 \)

11. \( f(x) = \frac{1}{1 + x^2}, \quad a = 3, \quad \Delta x = 0.5 \)

12. \( f(x) = \tan \left( x + \frac{\pi}{4} \right), \quad a = 0, \quad \Delta x = 0.01 \)

In Exercises 13–18, using Linear Approximation, estimate \( \Delta f \) for a change in \( x \) from \( a \) to \( b \). Use the estimate to approximate \( f(b) \), and find the error using a calculator.

13. \( f(x) = \sqrt{x}, \quad a = 25, \quad b = 26 \)

14. \( f(x) = x^{1/4}, \quad a = 16, \quad b = 16.5 \)

15. \( f(x) = \sqrt[3]{x}, \quad a = 100, \quad b = 101 \)

16. \( f(x) = \frac{1}{\sqrt{x}}, \quad a = 100, \quad b = 98 \)

17. \( f(x) = x^{1/3}, \quad a = 8, \quad b = 9 \)

18. \( f(x) = x^{2/3}, \quad a = 27, \quad b = 30 \)

In Exercises 19–26, find the linearization at \( x = a \) and then use it to approximate \( f(b) \).

19. \( f(x) = x^4, \quad a = 1, \quad b = 0.96 \)

20. \( f(x) = \frac{1}{x}, \quad a = 2, \quad b = 2.02 \)

21. \( f(x) = \sin^2 x, \quad a = \frac{\pi}{2}, \quad b = 1.10 \)

22. \( f(x) = \frac{x^2}{x - 3}, \quad a = 4, \quad b = 4.1 \)

23. \( f(x) = (1 + x)^{-1/2}, \quad a = 0, \quad b = 0.08 \)

24. \( f(x) = (1 + x)^{-1/2}, \quad a = 3, \quad b = 2.88 \)

25. \( y = \sin x, \quad a = \frac{\pi}{2} \)

26. \( y = \sin x, \quad a = \frac{\pi}{4} \)

In Exercises 27–30, estimate \( \Delta y \) using differentials [Eq. (3)].

27. \( y = \cos x, \quad a = \frac{\pi}{6}, \quad dx = 0.014 \)

28. \( y = \tan^2 x, \quad a = \frac{\pi}{4}, \quad dx = -0.02 \)

29. \( y = \frac{10 - x^2}{2 + x^2}, \quad a = 1, \quad dx = 0.01 \)

30. \( y = \frac{3 - \sqrt{x}}{\sqrt{x} + 3}, \quad a = 1, \quad dx = -0.1 \)

31. Estimate \( f(4, 0.3) \) for \( f(x) \) as in Figure 9.

**Figure 9**

At a certain moment, an object in linear motion has velocity 100 m/s. Estimate the distance traveled over the next quarter-second, and explain how this is an application of the Linear Approximation.

33. Which is larger: \( \sqrt{21} - \sqrt{2} \) or \( \sqrt{5} - \sqrt{2} \)? Explain using the Linear Approximation.

34. Estimate \( \sin 61^\circ - \sin 60^\circ \) using the Linear Approximation. Hint: Express \( \Delta \theta \) in radians.

35. Box office revenue at a cinema in Paris is \( R(p) = 3600p - 10p^3 \) euros per showing when the ticket price is \( p \) euros. Calculate \( R(p) \) for \( p = 9 \) and use the Linear Approximation to estimate \( \Delta R \) if \( p \) is raised or lowered by 0.5 euro.

36. The stopping distance for an automobile is \( D(v) = 1.14 + 0.054v^2 \) ft, where \( v \) is the speed in mph. Use the Linear Approximation to estimate the change in stopping distance per additional mph when \( v = 35 \) and when \( v = 55 \).

37. A thin silver wire has length \( L = 18 \) cm when the temperature is \( T = 30^\circ C \). Estimate \( \Delta L \) when \( T \) decreases to \( 25^\circ C \) if the coefficient of thermal expansion is \( \alpha = 1.9 \times 10^{-5}^\circ C^{-1} \) (see Example 6).

38. At a certain moment, the temperature in a snake cage satisfies \( \frac{dT}{dt} = 0.008^\circ C/s \). Estimate the rise in temperature over the next 10 s.

39. The atmospheric pressure \( P \) at altitude \( h = 20 \) km is \( P = 5.5 \) kilopascals. Estimate \( P \) at altitude \( h = 20.5 \) km assuming that

\[
\frac{dP}{dh} = -0.87
\]

Estimate \( \Delta P \) at \( h = 20 \) when \( \Delta h = 0.5 \).
40. The resistance $R$ of a copper wire at temperature $T = 20^\circ C$ is $R = 15 \ \Omega$. Estimate the resistance at $T = 22^\circ C$, assuming that $dR/dT \big|_{T=20} = 0.05 \ \Omega/\text{C}$. 

41. Newton's Law of Gravitation shows that if a person weighs $w$ pounds on the surface of the earth, then his or her weight at distance $x$ from the center of the earth is 

$$W(x) = \frac{wR^2}{x^2} \quad \text{for } x \geq R$$

where $R = 3960$ miles is the radius of the earth (Figure 10).

(a) Show that the weight lost at altitude $h$ miles above the earth's surface is approximately $\Delta W = -(0.0005w)h$. Hint: Use the Linear Approximation with $dx = h$.

(b) Estimate the weight lost by a 200-lb football player flying in a jet at an altitude of 7 miles.

![Figure 10](image)

**Figure 10** The distance to the center of the earth is $3960 + h$ miles.

42. Using Exercise 41(a), estimate the altitude at which a 130-lb pilot would weigh 129.5 lb.

43. A stone tossed vertically into the air with initial velocity $v$ cm/s reaches a maximum height of $h = v^2/1960$ cm.

(a) Estimate $\Delta h$ if $v = 700$ cm/s and $\Delta v = 1$ cm/s.

(b) Estimate $\Delta h$ if $v = 1000$ cm/s and $\Delta v = 1$ cm/s.

(c) In general, does a 1-cm/s increase in $v$ lead to a greater change in $h$ at low or high initial velocities? Explain.

44. The side $s$ of a square carpet is measured at 6 ft. Estimate the maximum error in the area $A$ of the carpet if $s$ is accurate to within 2 cm.

In Exercises 45 and 46, use the following fact derived from Newton's Laws: An object released at an angle $\theta$ with initial velocity $v$ ft/s travels a horizontal distance

$$x = \frac{1}{2}v^2 \sin \theta \text{ ft}$$

(11)

45. A player located 18.1 ft from the basket launches a successful jump shot from a height of 10 ft (level with the rim of the basket), at an angle $\theta = 34^\circ$ and initial velocity $v = 25$ ft/s.

(a) Show that $\Delta x = 0.255\Delta \theta$ for a small change of $\Delta \theta$.

(b) Is it likely that the shot would have been successful if the angle had been off by 2°?

(c) Estimate $\Delta x$ if $\theta = 34^\circ$, $v = 25$ ft/s, and $\Delta \theta = 2^\circ$.

![Figure 11](image)

**Figure 11** Trajectory of an object released at an angle $\theta$.

46. A golfer hits a golf ball at an angle of $\theta = 23^\circ$ with initial velocity $v = 120$ ft/s.

(a) Estimate $\Delta x$ if the ball is hit at the same velocity but the angle is increased by $3^\circ$.

(b) Estimate $\Delta x$ if the ball is hit at the same angle but the velocity is increased by 3 ft/s.

47. The radius of a spherical ball is measured at $r = 25$ cm. Estimate the maximum error in the volume and surface area if $r$ is accurate to within 0.5 cm.

48. The dosage $D$ of diphenhydramine for a dog of body mass $w$ kg is $D = 4.7w^{3/2}$ mg. Estimate the maximum allowable error in $w$ for a cocker spaniel of mass $w = 10$ kg if the percentage error in $D$ must be less than 3%.

49. The volume (in liters) and pressure $P$ (in atmospheres) of a certain gas satisfy $PV = 24$. A measurement yields $V = 4$ with a possible error of $\pm 0.3$ L. Compute $P$ and estimate the maximum error in this computation.

50. In the notation of Exercise 49, assume that a measurement yields $V = 4$. Estimate the maximum allowable error in $V$ if $P$ must have an error of less than 0.2 atm.

51. Approximate $f(2)$ if the linearization of $f(x)$ at $a = 2$ is $L(x) = 2x + 4$.

52. Compute the linearization of $f(x) = 3x - 4$ at $a = 0$ and $a = 2$. Prove more generally that a linear function coincides with its linearization at $x = a$ for all $a$.

53. Estimate $\sqrt{16.2}$ using the linearization $L(x)$ of $f(x) = \sqrt{x}$ at $a = 16$. Plot $f$ and $L$ on the same set of axes and from the plot indicate whether the estimate is greater than or less than the actual value.

54. Estimate $1/\sqrt{13}$ using a suitable linearization of $f(x) = 1/\sqrt{x}$. Plot $f$ and $L$ on the same set of axes and from the plot indicate whether the estimate is greater than or less than the actual value. Use a calculator to compute the percentage error.

In Exercises 55–56, approximate using linearization and use a calculator to compute the percentage error.

55. $\frac{1}{\sqrt{17}}$

56. $\frac{1}{101}$

57. $(10.03)^{1/2}$

58. $(17)^{1/4}$

59. $(64.1)^{1/3}$

60. $(1.2)^{1/3}$

61. $\tan(0.04)$

62. $\cos\left(\frac{3.1}{4}\right)$

63. $\frac{(3.1/2)^2}{\sin(3.1/2)}$

64. The linearization $L(x)$ of $f(x) = x^2 - x^{3/2}$ at $a = 4$. Then plot $f - L$ and identify an interval $I$ around $a = 4$ such that $|f(x) - L(x)| \leq 0.1$ for $x \in I$.

65. Show that the Linear Approximation to $f(x) = \sqrt{x}$ at $x = 9$ yields the estimate $\sqrt{9 + h} = 3 + \frac{h}{6}$. Set $K = 0.01$ and show that $|f''(x)| \leq K$ for $x \geq 9$. Then verify numerically that the error $E$ satisfies Eq. (5) for $h = 10^{-n}$, for $1 \leq n \leq 4$.

66. The Linear Approximation to $f(x) = \tan x$ at $x = \frac{\pi}{2}$ yields the estimate $\tan\left(\frac{\pi}{2} + h\right) = \tan \frac{\pi}{2} + h$ of $\tan\left(\frac{\pi}{2} + h\right)$. Set $K = 6.3$ and show, using a plot, that $|f''(x)| \leq K$ for $x \in (\pi/2, \pi/2 + 0.1)$. Then verify numerically that the error $E$ satisfies Eq. (5) for $h = 10^{-n}$, for $1 \leq n \leq 4$. 

Scanned with CamScanner
Further Insights and Challenges

67. Compute \(dy/dx\) at the point \(P = (2, 1)\) on the curve \(y^3 + 3xy = 7\) and show that the linearization at \(P\) is \(L(x) = -4x + 1\). Use \(L(x)\) to estimate the \(y\)-coordinate of the point on the curve where \(x = 2.1\).

68. Apply the method of Exercise 67 to \(P = (0.5, 1)\) on \(y^2 + y - 2x = 1\) to estimate the \(y\)-coordinate of the point on the curve where \(x = 0.55\).

69. Apply the method of Exercise 67 to \(P = (-1, 2)\) on \(y^2 + 7xy = 2\) to estimate the solution of \(y^2 - 7.7y = 2\) near \(y = 2\).

70. Show that for any real number \(k\), \((1 + \Delta x)^k \approx 1 + k\Delta x\) for small \(\Delta x\). Estimate \((1.02)^{.7}\) and \((1.02)^{-.3}\).

71. Let \(\Delta f = f(5 + h) - f(5)\), where \(f(x) = x^3\). Verify directly that \(E = |\Delta f - f'(5)h|\) satisfies (5) with \(K = 2\).

72. Let \(\Delta f = f(1 + h) - f(1)\), where \(f(x) = x^{-1}\). Show directly that \(E = |\Delta f - f'(1)h|\) is equal to \(h^2/(1 + h)\). Then prove that \(E \leq 2h^2\) if \(-\frac{1}{2} \leq h \leq \frac{1}{2}\). \(\text{Hint:}\) In this case, \(\frac{1}{2} \leq 1 + h \leq \frac{3}{2}\).

4.2 Extreme Values

In many applications, it is important to find the minimum or maximum value of a function \(f\). For example, a physician needs to know the maximum drug concentration in a patient’s bloodstream when a drug is administered. This amounts to finding the highest point on the graph of \(C\), the concentration at time \(t\) (Figure 1).

We refer to the maximum and minimum values (max and min for short) as extreme values or extrema (singular: extremum) and to the process of finding them as optimization. Sometimes, we are interested in finding the min or max for \(x\) in a particular interval \(I\), rather than on the entire domain of \(f\).

**Definition Extreme Values on an Interval** Let \(f\) be a function on an interval \(I\) and let \(a \in I\). We say that \(f(a)\) is the

- **Absolute minimum** of \(f\) on \(I\) if \(f(a) \leq f(x)\) for all \(x \in I\).
- **Absolute maximum** of \(f\) on \(I\) if \(f(a) \geq f(x)\) for all \(x \in I\).

Does every function have a minimum or maximum value? Clearly not, as we see by taking \(f(x) = x\). Indeed, \(f(x) = x\) increases without bound as \(x \to \infty\) and decreases without bound as \(x \to -\infty\). In fact, extreme values do not always exist even if we restrict ourselves to an interval \(I\). Figure 2 illustrates what can go wrong if \(I\) is open or \(f\) has a discontinuity.

- **Discontinuity:** (A) shows a discontinuous function with no maximum value. The values of \(f(x)\) get arbitrarily close to 3 from below, but 3 is not the maximum value because \(f(x)\) never actually takes on the value 3.
- **Open Interval:** In (B), \(g(x)\) is defined on the open interval \((a, b)\). It has no max because it tends to \(-\infty\) on the right, and it has no min because it tends to 10 on the left without ever reaching this value.

Fortunately, our next theorem guarantees that extreme values exist when the function is continuous and \(I\) is closed (Figure 2(C)).

![Figure 2](image-url)
Theorem 1: Existence of Extrema on a Closed Interval
A continuous function \( f \) on a closed (bounded) interval \( I = [a, b] \) takes on both a minimum and a maximum value on \( I \).

Conceptual Insight
Why does Theorem 1 require a closed interval? Think of the graph of a continuous function as a string. If the interval is closed, the string is pinned down at the two endpoints and cannot fly off to infinity or approach a \( \min/\max \) without reaching it as in Figure 2(B). Intuitively, therefore, it must have a highest and lowest point. As with the Intermediate Value Theorem in Section 2.8, a rigorous proof of Theorem 1 relies on the completeness property of the real numbers (see Appendix B).

Local Extrema and Critical Points
We focus now on the problem of finding extreme values. A key concept is that of a local minimum or maximum.

Definition: Local Extrema
We say that \( f(c) \) is a
- Local minimum occurring at \( x = c \) if \( f(c) \) is the minimum value of \( f \) on some open interval (in the domain of \( f \)) containing \( c \).
- Local maximum occurring at \( x = c \) if \( f(c) \) is the maximum value of \( f \) on some open interval (in the domain of \( f \)) containing \( c \).

A local max occurs at \( x = c \) if \((c, f(c))\) is the highest point on the graph within some small box [Figure 4(A)]. Thus, \( f(c) \) is greater than or equal to all other nearby values, but it does not have to be the absolute maximum value of \( f \) [Figure 3]. Local minima are similar. On the other hand, as Figure 4(B) illustrates, an absolute maximum of \( f \) on an interval \([a, b]\) need not be a local maximum of \( f \) in open intervals containing the point. In the figure, \( f(a) \) is the absolute max on \([a, b]\) but is not a local max on open intervals containing \( a \) because \( f(x) \) takes on greater values to the left of \( x = a \).

How do we find the local extrema? The crucial observation is that the tangent line at a local min or max is horizontal [Figure 5(A)]. In other words, if \( f(c) \) is a local min or max, then \( f'(c) = 0 \). However, this assumes that \( f \) is differentiable. Otherwise, the tangent line may not exist, as in Figure 5(B). To take both possibilities into account, we define the notion of a critical point.
Chapter 4: Applications of the Derivative

**Definition** Critical Points A number \( c \) in the domain of \( f \) is called a critical point if either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

**Example 1** Find the critical points of \( f(x) = x^3 - 9x^2 + 24x - 10 \).

Solution The function \( f \) is differentiable everywhere (Figure 6). Therefore, the critical points are the solutions of \( f'(x) = 0 \):

\[
f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x - 2)(x - 4)
\]

To find the critical points, we solve \( 3(x - 2)(x - 4) = 0 \). Thus, they are \( x = 2 \) and \( x = 4 \).

**Example 2** Nondifferentiable Function Find the critical points of \( f(x) = |x| \).

Solution As we see in Figure 7, \( f'(x) = -1 \) for \( x < 0 \) and \( f'(x) = 1 \) for \( x > 0 \). Therefore, \( f'(x) = 0 \) has no solutions with \( x \neq 0 \). However, \( f'(0) \) does not exist. Thus, \( c = 0 \) is a critical point.

The next theorem tells us that we can find local extrema by solving for the critical points. It is one of the most important results in calculus.

**Theorem 2** Fermat's Theorem on Local Extrema If \( f(c) \) is a local min or max, then \( c \) is a critical point of \( f \).

Proof Suppose that \( f(c) \) is a local minimum (the case of a local maximum is similar). If \( f'(c) \) does not exist, then \( c \) is a critical point and there is nothing more to prove. So, assume that \( f'(c) \) exists. We must then prove that \( f'(c) = 0 \).

Because \( f(c) \) is a local minimum, we have \( f(c + h) \geq f(c) \) for all sufficiently small \( h \neq 0 \). Equivalently, \( f(c + h) - f(c) \geq 0 \). Now divide this inequality by \( h \). Two possibilities occur depending on whether we are dividing by a positive value or a negative one:

\[
\begin{align*}
\frac{f(c + h) - f(c)}{h} & \geq 0 & \text{if } h > 0 \\
\frac{f(c + h) - f(c)}{h} & \leq 0 & \text{if } h < 0
\end{align*}
\]

Figure 8 shows the graphical interpretation of these inequalities. Taking the one-sided limits of both sides of (1) and (2), we obtain

\[
\begin{align*}
f'(c) &= \lim_{h \to 0^+} \frac{f(c + h) - f(c)}{h} \quad \lim_{h \to 0^+} 0 = 0 \\
&= \lim_{h \to 0^-} \frac{f(c + h) - f(c)}{h} \quad \lim_{h \to 0^-} 0 = 0
\end{align*}
\]

Thus, \( f'(c) \geq 0 \) and \( f'(c) \leq 0 \). The only possibility is \( f'(c) = 0 \) as claimed.
CONCEPTUAL INSIGHT Theorem 2 indicates that a local max or min must be a critical point. However, the “converse” need not be true. That is, having a critical point c does not guarantee a local min or max occurs at c. For example, \( f(x) = x^3 \) has derivative \( f'(x) = 3x^2 \) and \( f''(0) = 0 \). So, 0 is a critical point, but \( f(0) \) is neither a local min nor max (Figure 9). The origin is a point of inflection (studied in Section 4.4), where the tangent line crosses the graph.

**Optimizing on a Closed Interval**

Finally, we have all the tools needed for optimizing a continuous function on a closed interval. Theorem 1 guarantees that the extreme values exist, and the next theorem tells us where to find them, namely among the critical points or endpoints of the interval.

**THEOREM 3 Extreme Values on a Closed Interval** Assume that \( f \) is continuous on \([a, b]\) and let \( f(c) \) be the minimum or maximum value on \([a, b]\). Then \( c \) is either a critical point or one of the endpoints \( a \) or \( b \).

**Proof** If \( c \) is one of the endpoints \( a \) or \( b \), there is nothing to prove. If not, then \( c \) belongs to the open interval \((a, b)\). In this case, \( f(c) \) is also a local min or max because it is the min or max on \((a, b)\). By Fermat's Theorem, \( c \) is a critical point.

**EXAMPLE 3** Find the extrema of \( f(x) = 2x^3 - 15x^2 + 24x + 7 \) on \([0, 6]\).

**Solution** The extreme values occur at critical points or endpoints by Theorem 3, so we can break up the problem neatly into two steps.

**Step 1. Find the critical points.**

The function \( f \) is differentiable, so the critical points are solutions to \( f'(x) = 0 \).

\[ f'(x) = 6x^2 - 30x + 24 = 6(x - 1)(x - 4) \]

The critical points satisfy \( 6(x - 1)(x - 4) = 0 \), and therefore are \( x = 1 \) and \( 4 \).

**Step 2. Compare values of \( f(x) \) at the critical points and endpoints.**

<table>
<thead>
<tr>
<th>( x )-value</th>
<th>Value of ( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (critical point)</td>
<td>18</td>
</tr>
<tr>
<td>4 (critical point)</td>
<td>-9 min</td>
</tr>
<tr>
<td>0 (endpoint)</td>
<td>7</td>
</tr>
<tr>
<td>6 (endpoint)</td>
<td>43 max</td>
</tr>
</tbody>
</table>

The maximum value of \( f(x) \) on \([0, 6]\) is the greatest of the values in this table, namely \( f(6) = 43 \). Similarly, the minimum is \( f(4) = -9 \). See Figure 10.

**EXAMPLE 4 Function with a Cusp** Find the extrema of \( f(x) = 1 - (x - 1)^{2/3} \) on \([-1, 2]\).

**Solution** First, find the critical points:

\[ f'(x) = -\frac{2}{3}(x - 1)^{-1/3} = -\frac{2}{3(x - 1)^{1/3}} \]

The equation \( f'(x) = 0 \) has no solutions because \( f'(x) \) is never zero. However, \( f'(x) \) does not exist at \( x = 1 \), so there is a critical point there (Figure 11).
Next, compare values of \( f(x) \) at the critical points and endpoints:

<table>
<thead>
<tr>
<th>( x )-value</th>
<th>Value of ( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (critical point)</td>
<td>( f(1) = 1 ) max</td>
</tr>
<tr>
<td>-1 (endpoint)</td>
<td>( f(-1) \approx -0.59 ) min</td>
</tr>
<tr>
<td>2 (endpoint)</td>
<td>( f(2) = 0 )</td>
</tr>
</tbody>
</table>

So on \([-1, 2]\), the maximum of \( f \) is \( f(1) = 1 \) and the minimum is \( f(-1) \approx -0.59 \).

In the next example, the critical points lie outside the interval \([a, b]\), so they are not relevant to the problem.

**EXAMPLE 5  A Critical Point Lying Outside the Interval**

Find the extreme values of \( f(x) = \frac{x}{x^2 + 5} \) on \([4, 8]\) (Figure 12).

**Solution** Compute \( f'(x) \) using the Quotient Rule and solve for the critical points:

\[
f'(x) = \frac{(x^2 + 5)(x) - x(x^2 + 5)'}{(x^2 + 5)^2} = \frac{(x^2 + 5) - x(2x)}{(x^2 + 5)^2} = \frac{5 - x^2}{(x^2 + 5)^2} = 0
\]

We see that \( f'(x) = 0 \) if \( 5 - x^2 = 0 \). Hence, the critical points are \( x = \pm \sqrt{5} \approx \pm 2.2 \), both of which lie outside the interval \([4, 8]\). Therefore, the min and max of \( f(x) \) on \([4, 8]\) occur at the endpoints. In fact, \( f(4) = \frac{4}{21} \approx 0.19 \) and \( f(8) = \frac{8}{69} \approx 0.116 \). Therefore \( f(4) \) is the max and \( f(8) \) is the min on the interval \([4, 8]\).

**EXAMPLE 6  An Open-Interval Example**

The function \( S(\theta) = 240 + 24 \left( \frac{\sqrt{3} - \cos \theta}{\sin \theta} \right) \)

arises in a model—that we describe after this example—of the geometry of a honeycomb cell. Figure 13 shows the graph of \( S \) for \( 0 < \theta < \pi \). As \( \theta \) approaches 0 and \( \pi \) from inside the interval, \( S(\theta) \to \infty \). Therefore, there is no absolute maximum of \( S \) on \((0, \pi)\), but the graph suggests that there is an absolute minimum. Find it.

**Solution** Computing \( S'(\theta) \), we have

\[
S'(\theta) = 24 \left( \frac{(\sin \theta)(\sin \theta) - (\sqrt{3} - \cos \theta)(\cos \theta)}{\sin^2 \theta} \right) = 24 \left( \frac{1 - \sqrt{3} \cos \theta}{\sin^2 \theta} \right)
\]

The derivative is defined for all \( \theta \) in the interval and is zero where \( 1 - \sqrt{3} \cos \theta = 0 \). Therefore the absolute minimum occurs at \( \theta_m \) where \( \cos \theta_m = 1/\sqrt{3} \approx 0.5773 \). Solving numerically, we find \( \theta_m \approx 0.96 \) radians \( \approx 54.7^\circ \). Computing \( S(0.96) \), we find that the absolute minimum of \( S \) over \((0, \pi)\) is approximately 273.94.

**Honeycomb Geometry**

The honeycomb of bees has long been of scientific and mathematical interest (Figure 14). Some believe that, for a fixed cell volume, the specific shape of the cell minimizes the cell's surface area and thus the amount of wax needed to construct it. Each cell has an open hexagonal top, six quadrilateral sides, and three rhombi on the bottom. Imagine that (as in Figure 15) the sides on the hexagonal top are 4 mm, three of the vertical sides are 10 mm, and the remaining dimensions can vary. Let \( \theta \) be...
the angle between a vertical axis through the center of the cell and the bottom rhombi faces. Via geometry, two important facts about the shapes in the figure can be shown:

- The volume of the cell is independent of the angle $\theta$.
- The surface area $S$ of the cell (the total area of the six quadrilaterals and three rhombii) depends on $\theta$ according to

$S(\theta) = 240 + 24 \left( \frac{\sqrt{3} - \cos \theta}{\sin \theta} \right)$

The previous example indicates that for the cells we are considering, the minimum surface area occurs at $\theta \approx 54.7^\circ$. How does this minimum compare with the actual cells that the bees construct?

In the early eighteenth century, astronomer Giacomo Maraldi made extensive measurements of bees’ honeycomb and observed typical angle measurements consistent with the optimal angle we found. Subsequently, mathematicians Samuel Koenig and Colin Maclaurin performed a calculus-based analysis of the honeycomb geometry (as we have done here) supporting the idea that the bees are economical in their honeycomb construction. The question of why bees construct the honeycomb as they do is still unsettled, but calculus provides an interesting glimpse at the possibilities.

**Rolle’s Theorem**

As an application of our optimization methods, we prove Rolle’s Theorem: If $f$ is differentiable and takes on the same value at two different points $a$ and $b$, then somewhere between these two points the derivative is zero. Graphically, if the secant line between $x = a$ and $x = b$ is horizontal, then at least one tangent line between $a$ and $b$ is also horizontal (Figure 16).

**THEOREM 4 Rolle’s Theorem** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. If $f(a) = f(b)$, then there exists a number $c$ between $a$ and $b$ such that $f'(c) = 0$.

**Proof** Since $f$ is continuous and $[a, b]$ is closed, $f$ has a min and a max in $[a, b]$. Where do they occur? If either the min or the max occurs at a point $c$ in the open interval $(a, b)$, then $f(c)$ is a local extreme value and $f'(c) = 0$ by Fermat’s Theorem (Theorem 2). Otherwise, both the min and the max occur at the endpoints. However, $f(a) = f(b)$, so in this case, the min and max coincide and $f$ is a constant function with zero derivative. Then, $f'(c) = 0$ for all $c$ in $(a, b)$.

**EXAMPLE 7 Illustrating Rolle’s Theorem** Verify Rolle’s Theorem for

$$f(x) = x^4 - x^2 \quad \text{on} \quad [-2, 2]$$

**Solution** The hypotheses of Rolle’s Theorem are satisfied because $f$ is differentiable (and therefore continuous) everywhere, and $f(2) = f(-2)$:

$$f(2) = 2^4 - 2^2 = 12, \quad f(-2) = (-2)^4 - (-2)^2 = 12$$

We must verify that $f'(c) = 0$ has a solution in $(-2, 2)$. Since

$$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$$

we need to solve $2x(2x^2 - 1) = 0$. The solutions are $c = 0$ and $c = \pm \sqrt[3]{1} \approx \pm 0.707$. They all lie in $(-2, 2)$, so Rolle’s Theorem is satisfied with three values of $c$.

**EXAMPLE 8 Using Rolle’s Theorem** Show that $f(x) = x^3 + 9x - 4$ has precisely one real root.
Solution First, we note that \( f(0) = -4 \) is negative and \( f(1) = 6 \) is positive. By the Intermediate Value Theorem (Section 2.8), \( f \) has at least one root in \([0,1]\). If \( f \) had a second root \( b \), then we would have \( f(a) = f(b) = 0 \). Rolle's Theorem would then imply that \( f'(c) = 0 \) for some \( c \in (a,b) \). This is not possible because \( f'(x) = 3x^2 + 9 \geq 9 > 0 \), so \( f'(c) = 0 \) has no solutions. We conclude that \( a \) is the only real root of \( f \) (Figure 17).

**HISTORICAL PERSPECTIVE**

![Pierre de Fermat](1601-1665)  
![René Descartes](1596-1650)

Sometime in the 1630s, in the decade before Isaac Newton was born, the French mathematician Pierre de Fermat invented a general method for finding extreme values. Fermat said, in essence, that if you want to find extrema, you must set the derivative equal to zero and solve for the critical points, just as we have done in this section. He also described a general method for finding tangent lines that is not essentially different from our method of derivatives. For this reason, Fermat is often regarded as an inventor of calculus, together with Newton and Leibniz.

At around the same time, René Descartes (1596-1650) developed a different but less effective approach to finding tangent lines. Descartes, after whom Cartesian coordinates are named, was a profound thinker—the leading philosopher and scientist of his time in Europe. He is regarded today as the father of modern philosophy and the founder (along with Fermat) of analytic geometry. A dispute developed when Descartes learned through an intermediary that Fermat had criticized his work on optics. Sensitive and stubborn, Descartes retaliated by attacking Fermat's method of finding tangents, and only after some third-party refereeing did he admit that Fermat was correct. He wrote:

...Seeing the last method that you use for finding tangents to curved lines, I can reply to it in no other way than to say that it is very good and that, if you had explained it in this manner at the outset, I would have not contradicted it at all.

However, in subsequent private correspondence, Descartes was less generous, referring at one point to some of Fermat's work as "le galimatias le plus ridicule"—meaning the most ridiculous gibberish. Today, Fermat is recognized as one of the greatest mathematicians of his age, who made far-reaching contributions in several areas of mathematics.

### 4.2 SUMMARY

- The extreme values of \( f \) on an interval \( I \) are the minimum and maximum values of \( f \) for \( x \in I \) (also called absolute extrema on \( I \)).
- Basic Theorem: If \( f \) is continuous on a closed interval \( [a,b] \), then \( f \) has both a min and a max on \([a,b]\).
- \( f(c) \) is a local minimum if \( f(x) \geq f(c) \) for all \( x \) in some open interval around \( c \). Local maxima are defined similarly.
- \( x = c \) is a critical point of \( f \) if either \( f'(c) = 0 \) or \( f'(c) \) does not exist.
- Fermat's Theorem on Local Extreme: If \( f(c) \) is a local min or max, then \( c \) is a critical point.
- To find the extreme values of a continuous function \( f \) on a closed interval \( [a,b] \):
  1. **Step 1.** Find the critical points of \( f \) in \([a,b]\).
  2. **Step 2.** Calculate \( f(x) \) at the critical points in \([a,b]\) and at the endpoints. The min and max on \([a,b]\) are the least and greatest among the values computed in Step 2.
- Rolle's Theorem: If \( f \) is continuous on \([a,b]\) and differentiable on \((a,b)\), and if \( f(a) = f(b) \), then there exists \( c \) between \( a \) and \( b \) such that \( f'(c) = 0 \).
4.2 EXERCISES

Preliminary Questions

1. What is the definition of a critical point?

In Questions 2 and 3, which is the correct conclusion, (a) or (b)?

2. If \( f \) is not continuous on \([0, 1]\), then
   (a) \( f \) has no extreme values on \([0, 1]\).
   (b) \( f \) might not have any extreme values on \([0, 1]\).

3. If \( f \) is continuous but has no critical points in \([0, 1]\), then
   (a) \( f \) has no min or max on \([0, 1]\).
   (b) Either \( f(0) \) or \( f(1) \) is the minimum value on \([0, 1]\).

Exercises

1. The following refer to Figure 18.
   (a) What are the critical points of \( f \) on \([0, 8]\)?
   (b) What are the maximum and minimum values of \( f \) on \([0, 8]\)?
   (c) What are the local maximum and minimum values of \( f \), and where do they occur?
   (d) Find a closed interval on which both the minimum and maximum values of \( f \) occur at critical points.
   (e) Find an open interval on which \( f \) has neither a minimum nor a maximum value.
   (f) Find an open interval on which \( f \) has a maximum value but no minimum value.

![Figure 18](image)

2. State whether \( f(x) = x^{-1} \) (Figure 19) has a minimum or maximum value on the following intervals:
   (a) \((0, 2)\)
   (b) \((1, 2)\)
   (c) \([1, 2]\)

![Figure 19](image)

In Exercises 3–20, find all critical points of the function.

3. \( f(x) = x^2 - 2x + 4 \)
4. \( f(x) = 7x - 2 \)
5. \( f(x) = x^3 - \frac{3}{2}x^2 - 54x + 2 \)
6. \( f(t) = 8t^3 - t^5 \)
7. \( f(x) = x^{-1} - x^{-2} \)
8. \( g(z) = \frac{1}{z-1} - \frac{1}{z} \)
9. \( f(x) = \frac{x}{x^2 + 1} \)
10. \( f(x) = \frac{x^2}{x^2 - 4x + 8} \)
11. \( f(t) = t - 4\sqrt{t} + 1 \)
12. \( f(t) = 4t - \sqrt{t^2 + 1} \)
13. \( f(x) = x^3 - x^2 \)
14. \( f(x) = x + [2x + 1] \)
15. \( g(\theta) = \sin^2 \theta \)
16. \( R(\theta) = \cos \theta + \sin^2 \theta \)
17. Let \( f(x) = 2x^2 - 8x + 7 \).
   (a) Find the critical point \( c \) of \( f \) and compute \( f(c) \).
   (b) Find the extreme values of \( f \) on \([0, 5]\).
   (c) Find the extreme values of \( f \) on \([-4, 1]\).

18. Find the extreme values of \( f(x) = 2x^3 - 9x^2 + 12x \) on \([0, 3]\) and \([0, 2]\).

19. Find the critical points of \( f(x) = \sin x + \cos x \) and determine the extreme values on \([0, \frac{\pi}{2}]\).

20. Compute the critical points of \( h(t) = (t^2 - 1)^{1/3} \). Check that your answer is consistent with Figure 20. Then find the extreme values of \( h \) on \([0, 1]\) and on \([0, 2]\).

![Figure 20](image)

21. **(GU)** Plot \( f(x) = 4\sqrt{x} - 2x + 3 \) on \([0, 3]\) and indicate where it appears that the minimum and maximum occur. Then determine the minimum and maximum using calculus.

22. **(GU)** Plot \( f(x) = 2x^3 - 9x^2 + 12x \) on \([0, 3]\) and locate the extreme values graphically. Then verify your answer using calculus.
In Exercises 23–52, find the minimum and maximum values of the function on the given interval by comparing values at the critical points and endpoints.

23. \( y = 2x^2 + 4x + 5 \), \([-2, 2]\)
24. \( y = 2x^3 + 4x + 5 \), \([0, 2]\)
25. \( y = 6x - x^2 \), \([0, 5]\)
26. \( y = 6x - x^2 \), \([4, 6]\)
27. \( y = x^3 - 6x^2 + 8 \), \([-1, 6]\)
28. \( y = x^3 - 6x^2 + 8 \), \([-1, 6]\)
29. \( y = x^2 - 6x^2 + 8 \), \([-1, 1]\)
30. \( y = x^2 - 6x^2 + 8 \), \([-1, 1]\)
31. \( y = 2x^2 + 3x^2 \), \([1, 2]\)
32. \( y = x^2 - 12x^2 + 21x \), \([0, 2]\)
33. \( y = x^3 - 80x^3 \), \([-3, 3]\)
34. \( y = 2x^2 + 5x^2 \), \([-2, 2]\)
35. \( y = \frac{x^2 + 1}{x - 4} \), \([5, 6]\)
36. \( y = \frac{1 - x}{x^2 + 3x} \), \([1, 4]\)
37. \( y = -\frac{4x}{x + 1} \), \([0, 3]\)
38. \( y = 2\sqrt{x^2 + 1} - x \), \([0, 2]\)
39. \( y = (2 + x^2\sqrt{2 + (2 - x)^2}) \), \([0, 2]\)
40. \( y = \sqrt{1 + x^3 - 2x} \), \([0, 1]\)
41. \( y = \sqrt{1 + x^3 - 2x^2} \), \([0, 4]\)
42. \( y = (r - x^2)^3 \), \([-1, 2]\)
43. \( y = \sin x \cos x \), \([0, 4]\)
44. \( y = x + \sin x \), \([0, 2\pi]\)
45. \( y = \sqrt{2} \theta - \sec \theta \), \([0, 3\pi]\)
46. \( x^4 - 2x^2 + 1 \), \([-3, 3]\)
47. \( y = x^3 + x^2 - x \), \([-2, 2]\)
48. \( y = \cos \theta + \sin \theta \), \([0, 2\pi]\)
49. \( y = \theta - 2 \sin \theta \), \([0, 2\pi]\)
50. \( y = 4 \sin^3 \theta - 3 \cos^2 \theta \), \([0, 2\pi]\)
51. \( y = \tan x - 2x \), \([0, 1]\)
52. \( y = \sec x - 2 \tan x \), \([-\pi/6, \pi/3]\]

53. **GU** Plot \( f(x) = 2x^2 + 3x + 1 \) on \([0, 5]\) and use the graph to explain why there is a minimum value, but no maximum value, of \( f \) on \([0, 5]\). Use calculus to find the minimum value.

54. **GU** Plot \( f(x) = 2x^2 + 3x + 1 \) on \([0, 3]\) and use the graph to explain why there is a minimum value, but no maximum value, of \( f \) on \([0, 3]\). Use calculus to find the maximum value.

55. Let \( f(\theta) = 2 \sin 2\theta + \sin 4\theta \).
   (a) Show that \( \theta \) is a critical point if \( \cos 4\theta = -\cos 2\theta \).
   (b) Show, using a unit circle, that \( \cos 2\theta = -\cos 2\theta \) if and only if \( \theta = \pi + \theta_2 + 2k\pi \) for an integer \( k \).
   (c) Show that \( \cos 4\theta = -\cos 2\theta \) if and only if \( \theta = \pi + \theta_2 + \pi k \) or \( \theta = \pi + \theta_2 + \pi k \).
   (d) Find the six critical points of \( f \) on \([0, 2\pi]\) and find the extreme values of \( f \) on this interval.
   (e) **GU** Check your results against a graph of \( f \).

56. **GU** Find the critical points of \( f(x) = 2 \cos 3x + 3 \cos 2x \) in \([0, 2\pi]\). Check your answer against a graph of \( f \).

In Exercises 57–60, find the critical points and the extreme values on \([0, 4]\).

57. \( y = |x - 2| \)
58. \( y = |3x - 9| \)
59. \( y = |x^2 + 4x - 12| \)
60. \( y = |\cos x| \)

**FIGURE 21**

In Exercises 61–64, verify Rolle's Theorem for the given interval by checking \( f(a) = f(b) \) and then finding a value \( c \) in \((a, b)\) such that \( f'(c) = 0 \).

61. \( f(x) = x - x^{-1} \), \([-1, 2]\)
62. \( f(x) = \sin x \), \([\pi, 2\pi]\)
63. \( f(x) = \frac{x^2}{8x - 15} \), \([3, 5]\)
64. \( f(x) = \sin 3x - \cos^2 x \), \([\pi/6, \pi/3]\)

65. Prove that \( f(x) = x^3 + 2x^3 + 3x - 12 \) has precisely one real root.
66. Prove that \( f(x) = x^3 + 3x^3 + 6x \) has precisely one real root.
67. Prove that \( f(x) = x^3 + 3x^3 + 4x \) has no root \( c \) satisfying \( c > 0 \). *Hint:* Note that \( x = 0 \) is a root and apply Rolle's Theorem.
68. Prove that \( x = 4 \) is the greatest root of \( f(x) = x^4 - 8x^2 - 128 \).

69. The position of a mass oscillating at the end of a spring is \( s(t) = 2 \sin \omega t \), where \( \omega \) is the angular frequency. Show that the speed \( |s'(t)| \) is at a maximum when the acceleration \( s''(t) \) is zero and that \( |s'(t)| \) is at a maximum when \( s(t) \) is zero.

70. The concentration \( C(t) \) (in milligrams per cubic centimeter) of a drug in a patient's bloodstream after \( t \) hours is

\[
C(t) = 0.016t - \frac{1}{2} t^2 + 4t + 4
\]

Find the maximum concentration in the time interval \([0, 8]\) and the time at which it occurs.

71. In 1919, physicist Alfred Betz argued that the maximum efficiency of a wind turbine is around 59%. If wind enters a turbine with speed \( v_1 \) and exits with speed \( v_2 \), then the power extracted is the difference in kinetic energy per unit time:

\[
P = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_2^2 \quad \text{watts}
\]

where \( m \) is the mass of wind flowing through the rotor per unit time (Figure 22). Betz assumed that \( m = \rho A (v_1 + v_2)/2 \), where \( \rho \) is the density of air and \( A \) is the area swept out by the rotor. Wind flowing undisturbed through the same area \( A \) would have mass per unit time \( \rho A v_1 \) and power \( P_0 = \frac{1}{2} \rho A v_1^2 \). The fraction of power extracted by the turbine is \( F = P/P_0 \).

(a) Show that \( F \) depends only on the ratio \( r = v_2/v_1 \) and is equal to \( F(r) = \frac{r^2}{4(1 - r^2)(1 + r)} \), where \( 0 \leq r \leq 1 \).

(b) Show that the maximum value of \( F \), called the Betz Limit, is \( 16/27 \approx 0.59 \).

(c) **GU** Explain why Betz's formula for \( F \) is not meaningful for \( r \) close to zero. *Hint:* How much wind would pass through the turbine if \( v_2 \) were zero? Is this realistic?
72. **GU**  The Bohr radius $a_0$ of the hydrogen atom is the value of $r$ that minimizes the energy 
\[
E(r) = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}
\]
where $\hbar$, $m$, $e$, and $\epsilon_0$ are physical constants. Show that $a_0 = 4\pi\epsilon_0\hbar^2/(n^2e^2)$. Assume that the minimum occurs at a critical point, as suggested by Figure 23.

73. The response of a circuit or other oscillatory system to an input of frequency $\omega$ ("omega") is described by the function 
\[
\phi(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4D^2\omega^2}}
\]
Both $\omega_0$ (the natural frequency of the system) and $D$ (the damping factor) are positive constants. The graph of $\phi$ is called a resonance curve, and the positive frequency $\omega_0 > 0$, where $\phi$ takes its maximum value. If it exists, is called the resonant frequency. Show that $\omega_0 = \sqrt{(2D^2)}$ if $0 < D < \omega_0/\sqrt{2}$ and that no resonant frequency exists otherwise (Figure 24).

74. Find the maximum of $y = x - x^4$ on $[0, 1]$, where $n > 1$.
75. Find the maximum of $y = x^a - x^b$ on $[0, 1]$, where $0 < a < b$. In particular, find the maximum of $y = x^3 - x^2$ on $[0, 1]$.

In exercises 76-78, plot the function using a graphing utility and find its critical points and extreme values on $[-5, 5]$.

76. **GU**  $y = \frac{1}{1 + |x - 1|}$
77. **GU**  $y = \frac{1}{1 + |x - 1|} + \frac{1}{1 + |x - 4|}$
78. **GU**  $y = \frac{x}{|x^2 - 1| + |x^2 - 4|}$

79. (a) Use implicit differentiation to find the critical points on the curve $27x^2 = (x^2 + y^2)^3$.
(b) **GU**  Plot the curve and the horizontal tangent lines on the same set of axes.

80. Sketch the graph of a continuous function on $[0, 4]$ with a minimum value but no maximum value.
81. Sketch the graph of a continuous function on $[0, 4]$ having a local minimum but no absolute minimum.
82. Sketch the graph of a function on $[0, 4]$ having
   (a) Two local maxima and one local minimum
   (b) An absolute minimum that occurs at an endpoint, and an absolute maximum that occurs at a critical point
83. Sketch the graph of a function $f$ on $[0, 4]$ with a discontinuity such that $f$ has an absolute minimum but no absolute maximum.
84. A rainbow is produced by light rays that enter a raindrop (assumed spherical) and exit after being reflected internally as in Figure 25. The angle between the incoming and reflected rays is $\theta = 4r - 2l$, where the angle of incidence $i$ and the angle of refraction $r$ are related by Snell’s Law $\sin i = n \sin r$ with $n \approx 1.33$ (the index of refraction for air and water).
   (a) Use Snell’s Law to show that $\frac{dr}{di} = \frac{\cos i}{n \cos r}$.
   (b) Show that the maximum value $\theta_{max}$ of $\theta$ occurs when $i$ satisfies $\cos i = \frac{n^2 - 1}{3}$. Hint: Show that $\frac{d\theta}{d\theta} = 0$ if $\cos i = \frac{3}{n}$. Then use Snell’s Law to eliminate $r$.
   (c) Show that $\theta_{max} \approx 42.53^\circ$.

85. Find the maximum of $y = x - x^4$ on $[0, 1]$, where $n > 1$.
86. Find the maximum of $y = x^a - x^b$ on $[0, 1]$, where $0 < a < b$. In particular, find the maximum of $y = x^3 - x^2$ on $[0, 1]$.

In exercises 76-78, plot the function using a graphing utility and find its critical points and extreme values on $[-5, 5]$.

76. **GU**  $y = \frac{1}{1 + |x - 1|}$
77. **GU**  $y = \frac{1}{1 + |x - 1|} + \frac{1}{1 + |x - 4|}$
78. **GU**  $y = \frac{x}{|x^2 - 1| + |x^2 - 4|}$

79. (a) Use implicit differentiation to find the critical points on the curve $27x^2 = (x^2 + y^2)^3$.
(b) **GU**  Plot the curve and the horizontal tangent lines on the same set of axes.

80. Sketch the graph of a continuous function on $[0, 4]$ with a minimum value but no maximum value.
81. Sketch the graph of a continuous function on $[0, 4]$ having a local minimum but no absolute minimum.
82. Sketch the graph of a function on $[0, 4]$ having
   (a) Two local maxima and one local minimum
   (b) An absolute minimum that occurs at an endpoint, and an absolute maximum that occurs at a critical point
83. Sketch the graph of a function $f$ on $[0, 4]$ with a discontinuity such that $f$ has an absolute minimum but no absolute maximum.
84. A rainbow is produced by light rays that enter a raindrop (assumed spherical) and exit after being reflected internally as in Figure 25. The angle between the incoming and reflected rays is $\theta = 4r - 2l$, where the angle of incidence $i$ and the angle of refraction $r$ are related by Snell’s Law $\sin i = n \sin r$ with $n \approx 1.33$ (the index of refraction for air and water).
   (a) Use Snell’s Law to show that $\frac{dr}{di} = \frac{\cos i}{n \cos r}$.
   (b) Show that the maximum value $\theta_{max}$ of $\theta$ occurs when $i$ satisfies $\cos i = \frac{n^2 - 1}{3}$. Hint: Show that $\frac{d\theta}{d\theta} = 0$ if $\cos i = \frac{3}{n}$. Then use Snell’s Law to eliminate $r$.
   (c) Show that $\theta_{max} \approx 42.53^\circ$.
Further Insights and Challenges

85. Show that the extreme values of \( f(x) = a \sin x + b \cos x \) are \( \pm \sqrt{a^2 + b^2} \).

86. Show, by considering its minimum, that \( f(x) = x^2 - 2x + 3 \) takes on only positive values. More generally, find the conditions on \( r \) and \( s \) under which the quadratic function \( f(x) = x^2 + rx + s \) takes on only positive values. Give examples of \( r \) and \( s \) for which \( f \) takes on both positive and negative values.

87. Show that if the quadratic polynomial \( f(x) = x^2 + rx + s \) takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

88. Generalize Exercise 87: Show that if the horizontal line \( y = c \) intersects the graph of \( f(x) = x^2 + rx + s \) at two points \( (x_1, f(x_1)) \) and \( (x_2, f(x_2)) \), then \( f \) takes its minimum value at the midpoint \( M = \frac{x_1 + x_2}{2} \) (Figure 26).

89. A cubic polynomial may have a local min and max, or it may have neither (Figure 27). Find conditions on the coefficients \( a \) and \( b \) of

\[
f(x) = \frac{1}{3}x^3 + \frac{1}{2}ax^2 + bx + c
\]

that ensure \( f \) has neither a local min nor a local max. Hint: Apply Exercise 86 to \( f(x) \).

90. Find the min and max of

\[
f(x) = x^p(1-x)^q
\]

on \([0,1]\)

where \( p, q > 0 \).

91. \( \square \) Prove that if \( f \) is continuous and \( f(a) \) and \( f(b) \) are local minima where \( a < b \), then there exists a value \( c \) between \( a \) and \( b \) such that \( f(c) \) is a local maximum. (Hint: Apply Theorem 1 to the interval \([a,b]\).) Show that continuity is a necessary hypothesis by sketching the graph of a function (necessarily discontinuous) with two local minima but no local maximum.

4.3 The Mean Value Theorem and Monotonicity

We have taken for granted that if \( f'(x) \) is positive, the function \( f \) is increasing, and if \( f'(x) \) is negative, \( f \) is decreasing. In this section, we prove this rigorously using an important result called the Mean Value Theorem (MVT). Then we develop a method for "testing" critical points—that is, for determining whether they correspond to local minima, local maxima, or neither.

The MVT says that a secant line between two points \((a, f(a))\) and \((b, f(b))\) on a graph is parallel to at least one tangent line in the interval \((a,b)\) (Figure 1). Since the secant line between \((a, f(a))\) and \((b, f(b))\) has slope \( \frac{f(b) - f(a)}{b-a} \) and since two lines are parallel if they have the same slope, the MVT is claiming that there exists a point \( c \) between \( a \) and \( b \) such that

\[
\frac{f'(c)}{\text{Slope of tangent line}} = \frac{f(b) - f(a)}{\text{Slope of secant line}}
\]

**Theorem 1** The Mean Value Theorem Assume that \( f \) is continuous on the closed interval \([a,b]\) and differentiable on \((a,b)\). Then there exists at least one value \( c \) in \((a,b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b-a}
\]
Rolle's Theorem (Section 4.2) is the special case of the MVT in which \( f(a) = f(b) \). In this case, the conclusion is that \( f'(c) = 0 \).

**Graphical Insight** Imagine what happens when a secant line is moved parallel to itself. Eventually, it becomes a tangent line, as shown in Figure 2. This is the idea behind the MVT. We present a formal proof at the end of this section.

**Conceptual Insight** The conclusion of the MVT can be rewritten as

\[
\frac{f(b) - f(a)}{b-a} = f'(c) = \frac{3-1}{9-1} = \frac{2}{4} = \frac{1}{2}
\]

We can think of this as a variation on the Linear Approximation, which says

\[
f(b) - f(a) \approx f'(a)(b-a)
\]

The MVT turns this approximation into an equality by replacing \( f'(a) \) with \( f'(c) \) for a suitable choice of \( c \) in \((a, b)\).

**Example 1** Verify the MVT with \( f(x) = \sqrt{x}, a = 1, \) and \( b = 9 \).

**Solution** First, compute the slope of the secant line (Figure 3):

\[
\frac{f(b) - f(a)}{b-a} = \frac{\sqrt{9} - \sqrt{1}}{9-1} = \frac{3-1}{9-1} = \frac{1}{2}
\]

We must find \( c \) such that \( f'(c) = 1/4 \). The derivative is \( f'(x) = \frac{1}{2}x^{-1/2} \), and

\[
f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{4} \quad \Rightarrow \quad 2\sqrt{c} = 4 \quad \Rightarrow \quad c = 4
\]

The value \( c = 4 \) lies in \((1, 9)\) and satisfies \( f'(4) = \frac{1}{4} \). This verifies the MVT.

As a first application, we prove that a function with zero derivative is constant.

**Corollary** If \( f \) is differentiable and \( f'(x) = 0 \) for all \( x \in (a, b) \), then \( f \) is constant on \((a, b)\). In other words, \( f(x) = C \) for some constant \( C \).

**Proof** If \( a_1 \) and \( b_1 \) are any two distinct points in \((a, b)\), then, by the MVT, there exists \( c \) between \( a_1 \) and \( b_1 \) such that

\[
f(b_1) - f(a_1) = f'(c)(b_1 - a_1) = 0 \quad \text{(since } f'(c) = 0 \text{)}
\]

Thus, \( f(b_1) = f(a_1) \). This says that \( f(x) \) is constant on \((a, b)\).

**Increasing/Decreasing Behavior of Functions**

We prove now that the sign of the derivative determines whether a function \( f \) is increasing or decreasing. Recall that \( f \) is

- Increasing on \((a, b)\) if \( f(x_1) < f(x_2) \) for all \( x_1, x_2 \in (a, b) \) such that \( x_1 < x_2 \).
- Decreasing on \((a, b)\) if \( f(x_1) > f(x_2) \) for all \( x_1, x_2 \in (a, b) \) such that \( x_1 < x_2 \).

We say that \( f \) is monotonic on \((a, b)\) if it is either increasing or decreasing on \((a, b)\).

**Theorem 2** The Sign of the Derivative Let \( f \) be a differentiable function on an open interval \((a, b)\).

- If \( f'(x) > 0 \) for \( x \in (a, b) \), then \( f \) is increasing on \((a, b)\).
- If \( f'(x) < 0 \) for \( x \in (a, b) \), then \( f \) is decreasing on \((a, b)\).
Proof  Suppose first that \( f'(x) > 0 \) for all \( x \in (a, b) \). The MVT tells us that for any two points \( x_1 < x_2 \) in \( (a, b) \), there exists \( c \) between \( x_1 \) and \( x_2 \) such that
\[
f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0
\]
The inequality holds because \( f'(c) \) and \( (x_2 - x_1) \) are both positive. Thus, \( f(x_2) > f(x_1) \), as required. The case \( f'(x) < 0 \) is similar.

**Example 2**  Show that the function \( f(x) = 3x - \cos 2x \) is increasing.

**Solution**  The derivative \( f'(x) = 3 + 2\sin 2x \) satisfies \( f'(x) > 0 \) for all \( x \). Indeed, \( \sin 2x \geq -1 \), and thus \( 3 + 2\sin 2x \geq 3 - 2 = 1 \). Therefore, \( f \) is an increasing function on the entire real line \((-\infty, \infty)\) [Figure 5].

**Example 3**  Find the intervals on which \( f(x) = x^2 - 2x - 3 \) is monotonic.

**Solution**  The derivative \( f'(x) = 2x - 2 = 2(x - 1) \) is positive for \( x > 1 \) and negative for \( x < 1 \). By Theorem 2, \( f \) is decreasing on the interval \((-\infty, 1)\) and increasing on the interval \((1, \infty)\), as confirmed in Figure 6.

### Testing Critical Points

There is a useful test for determining whether a critical point yields a min or max (or neither) based on the sign change of the derivative \( f'(x) \).

To explain the term "sign change," suppose that a function \( g \) satisfies \( g(c) = 0 \). We say that \( g(x) \) changes from positive to negative at \( x = c \) if \( g(x) > 0 \) to the left of \( c \) and \( g(x) < 0 \) to the right of \( c \) for \( x \) within a small open interval around \( c \) (Figure 7).

A sign change from negative to positive is defined similarly. Observe in Figure 7 that \( g(5) = 0 \) but \( g(x) \) does not change sign at \( x = 5 \).

Now suppose that \( f'(c) = 0 \) and that \( f'(x) \) changes sign at \( x = c \), say, from to +. Then \( f \) is increasing to the left of \( c \) and decreasing to the right, so \( f(c) \) is a local maximum. Similarly, if \( f'(x) \) changes sign from to +, then \( f(c) \) is a local minimum. See Figure 8(A). Figure 8(B) illustrates a case where \( f'(c) = 0 \) but \( f'(x) \) does not change sign. In this case, \( f'(x) > 0 \) for all \( x \) near but not equal to \( c \), so \( f \) is increasing and has neither a local min nor a local max at \( c \).

A similar analysis holds when \( f'(c) \) does not exist and the possibilities for the sign of \( f' \) on either side of \( c \) are considered. As a result, we have the following theorem:

**Theorem 3**  First Derivative Test for Critical Points  Let \( c \) be a critical point of \( f \). Then

- \( f'(x) \) changes from + to - at \( c \) \( \Rightarrow \) \( f(c) \) is a local maximum.
- \( f'(x) \) changes from - to + at \( c \) \( \Rightarrow \) \( f(c) \) is a local minimum.
To carry out the First Derivative Test, we make a useful observation: \( f'(x) \) can change sign at a critical point, but it cannot change sign on the interval between two consecutive critical points as long as the function is defined over the whole interval. In such a case, we can determine the sign of \( f'(x) \) on an interval between consecutive critical points by evaluating \( f'(x) \) at any test point \( x_0 \) inside the interval. The sign of \( f'(x_0) \) is the sign of \( f'(x) \) on the entire interval. In a case where a function's domain is made up of separate intervals, this analysis of the sign of \( f' \) needs to be carried out individually on each of the intervals.

**Example 4** Analyze the critical points of \( f(x) = x^3 - 27x - 20 \).

**Solution** Our analysis will confirm the picture in Figure 8(A).

**Step 1. Find the critical points.**
We have \( f'(x) = 3x^2 - 27 = 3(x^2 - 9) \). The critical points satisfy \( f'(c) = 0 \) and therefore are \( c = \pm 3 \).

**Step 2. Find the sign of \( f'(x) \) on the intervals between the critical points.**
The critical points \( c = \pm 3 \) divide the real line into three intervals:

\[
(-\infty, -3), \quad (-3, 3), \quad (3, \infty)
\]

To determine the sign of \( f'(x) \) on these intervals, we choose a test point inside each interval and evaluate. For example, in \((-\infty, -3)\) we choose \( x = -4 \). Because \( f'(-4) = 21 > 0 \), \( f'(x) \) is positive on the entire interval \((-3, \infty)\). Taking this result, along with the results from test points at 0 and 4, we have

\[
\begin{align*}
  f'(-4) &= 21 > 0 & \Rightarrow & & f'(x) > 0 \quad \text{for all } x \in (-\infty, -3) \\
  f'(0) &= -27 < 0 & \Rightarrow & & f'(x) < 0 \quad \text{for all } x \in (-3, 3) \\
  f'(4) &= 21 > 0 & \Rightarrow & & f'(x) > 0 \quad \text{for all } x \in (3, \infty)
\end{align*}
\]

This information is displayed in the following sign diagram:

<table>
<thead>
<tr>
<th>Behavior of ( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of ( f'(x) )</td>
</tr>
<tr>
<td>( + )</td>
</tr>
<tr>
<td>(-3)</td>
</tr>
</tbody>
</table>
Step 3. Use the First Derivative Test.

- $c = 3$: $f'(x)$ changes from $+$ to $-$ $\Rightarrow$ $f(-3) = 34$ is a local maximum value.
- $c = -3$: $f'(x)$ changes from $-$ to $+$ $\Rightarrow$ $f(3) = -74$ is a local minimum value.

**EXAMPLE 5** Analyze the critical points and the increase/decrease behavior of $f(x) = \cos^2 x + \sin x$ in $(0, \pi)$.

**Solution** First, find the critical points:

$$f'(x) = -2\cos x \sin x + \cos x = (\cos x)(1 - 2\sin x)$$

Therefore, the critical points are solutions to $\cos x = 0$ or $\sin x = \frac{1}{2}$. Since we are just examining the interval $(0, \pi)$, the critical points of interest are $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{5\pi}{6}$. They divide $(0, \pi)$ into four intervals:

$$\left(0, \frac{\pi}{6}\right), \left(\frac{\pi}{6}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{5\pi}{6}\right), \left(\frac{5\pi}{6}, \pi\right)$$

We determine the sign of $f'(x)$ by evaluating $f'(x)$ at a test point inside each interval. Since $\frac{\pi}{6} \approx 0.52$, $\frac{\pi}{3} \approx 1.05$, $\frac{5\pi}{6} \approx 2.62$, and $\pi \approx 3.14$, we can use the following test points:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>Sign of $f'(x)$</th>
<th>Behavior of $f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, \frac{\pi}{6})$</td>
<td>$f'(0.5) \approx 0.04$</td>
<td>$+$</td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$(\frac{\pi}{6}, \frac{\pi}{2})$</td>
<td>$f'(1.0) \approx -0.37$</td>
<td>$-$</td>
<td>$\downarrow$</td>
</tr>
<tr>
<td>$(\frac{\pi}{2}, \frac{5\pi}{6})$</td>
<td>$f'(2.0) \approx 0.34$</td>
<td>$+$</td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$(\frac{5\pi}{6}, \pi)$</td>
<td>$f'(3.0) \approx -0.71$</td>
<td>$-$</td>
<td>$\downarrow$</td>
</tr>
</tbody>
</table>

Now apply the First Derivative Test:

- Local max at $c = \frac{\pi}{6}$ and $c = \frac{5\pi}{6}$ because $f'(x)$ changes from $+$ to $-$.
- Local min at $c = \frac{\pi}{3}$ because $f'(x)$ changes from $-$ to $+$.

The behavior of $f(x)$ and $f'(x)$ is reflected in the graphs in Figure 9.

**EXAMPLE 6** Analyze the critical points and the increase/decrease behavior of $f(x) = x^2 + \frac{1}{x^2}$.

**Solution** Note that $f$ is undefined at $x = 0$, so we need to analyze $f$ separately on $(-\infty, 0)$ and $(0, \infty)$. We have

$$f'(x) = 2x - \frac{2}{x^3}$$

The critical points are solutions to $x - \frac{2}{x^3} = 0$; that is, to $x^4 - 1 = 0$. They are $c = \pm 1$. Since we need to consider $f$ separately on $(-\infty, 0)$ and $(0, \infty)$, there are four intervals on which we need to examine the sign of $f'(x)$: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. We determine the sign of $f'(x)$ by evaluating $f'(x)$ at a test point inside each interval.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>Sign of $f'(x)$</th>
<th>Behavior of $f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -1)$</td>
<td>$f'(-2) = -3.75$</td>
<td>$-$</td>
<td>$\downarrow$</td>
</tr>
<tr>
<td>$(-1, 0)$</td>
<td>$f'(-0.5) = 15$</td>
<td>$+$</td>
<td>$\uparrow$</td>
</tr>
<tr>
<td>$(0, 1)$</td>
<td>$f'(0.5) = -15$</td>
<td>$-$</td>
<td>$\downarrow$</td>
</tr>
<tr>
<td>$(1, \infty)$</td>
<td>$f'(2) = 3.75$</td>
<td>$+$</td>
<td>$\uparrow$</td>
</tr>
</tbody>
</table>

Applying the First Derivative Test, we see that both critical points are local minima. This is verified in the graph in Figure 10.
EXAMPLE 7  A Critical Point Where $f'(x)$ Is Undefined
Analyze the critical points of $f(x) = (1 - x)^{2/3}$.

Solution  The derivative is $f'(x) = -\frac{2}{3}(1-x)^{-1/3} = -\frac{2}{3(1-x)^{1/3}}$. The only critical point occurs at $c = 1$, when $f'(x)$ is undefined. For $x < 1$, $f'(x)$ is negative. For $x > 1$, $f'(x)$ is positive. So $f'(x)$ changes sign as we pass through $c = 1$, and by the First Derivative Test, $f(c)$ is a local minimum. See Figure 11.

EXAMPLE 8  Infinitely Many Critical Points, No Local Extrema
Analyze the critical points of $f(x) = x - \sin x$.

Solution  We have $f'(x) = 1 - \cos x$, and therefore critical points occur at solutions to $\cos x = 1$; that is, at $n\pi$ for all even integers $n$. At none of the critical points does the sign of $f'$ change since $f'(x) \geq 0$ for all $x$. Therefore, none of the critical points are local extrema (Figure 12).

Proof of the MVT  Let $m = \frac{f(b) - f(a)}{b - a}$ be the slope of the secant line joining $(a, f(a))$ and $(b, f(b))$. The secant line has equation $y = mx + r$ for some $r$ (Figure 13). Now consider the function

$$G(x) = f(x) - (mx + r)$$

As indicated in Figure 13, $G(x)$ is the vertical distance between the graph and the secant line at $x$ (it is negative at points where the graph of $f$ lies below the secant line). This distance is zero at the endpoints, and therefore, $G(a) = G(b) = 0$. By Rolle’s Theorem (Section 4.2), there exists a point $c$ in $(a, b)$ such that $G'(c) = 0$. But $G'(x) = f'(x) - m$, so $G'(c) = f'(c) - m = 0$, and $f'(c) = m$ as desired.

4.3 SUMMARY

- The Mean Value Theorem (MVT): If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists at least one value $c$ in $(a, b)$ such that

  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  This conclusion can also be written

  $$f(b) - f(a) = f'(c)(b - a)$$

- Important corollary of the MVT: If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant on $(a, b)$.

- The sign of $f'(x)$ determines whether $f$ is increasing or decreasing:

  $$f'(x) > 0 \text{ for } x \in (a, b) \implies f \text{ is increasing on } (a, b)$$

  $$f'(x) < 0 \text{ for } x \in (a, b) \implies f \text{ is decreasing on } (a, b)$$

- On an interval over which $f$ is defined, the sign of $f'(x)$ can change only at the critical points, so $f$ is monotonic (increasing or decreasing) on the intervals between the critical points.

- On an interval over which $f$ is defined, to find the sign of $f'(x)$ on an interval between two critical points, calculate the sign of $f'(x_0)$ at any test point $x_0$ in that interval.

- First Derivative Test: If $f$ is differentiable and $c$ is a critical point, then

<table>
<thead>
<tr>
<th>Sign change of $f'(x)$ at $c$</th>
<th>Type of critical point</th>
</tr>
</thead>
<tbody>
<tr>
<td>From $+ \to -$</td>
<td>Local maximum</td>
</tr>
<tr>
<td>From $- \to +$</td>
<td>Local minimum</td>
</tr>
</tbody>
</table>
4.3 EXERCISES

Preliminary Questions
1. For which value of $m$ is the following statement correct? If $f(2) = 3$ and $f(4) = 9$, and $f'$ is differentiable, then $f$ has a tangent line of slope $m$.
2. Assume $f$ is differentiable. Which of the following statements does not follow from the MVT?
   (a) If $f$ has a secant line of slope 0, then $f$ has a tangent line of slope 0.
   (b) If $f(5) < f(9)$, then $f'(c) > 0$ for some $c \in (5, 9)$.
   (c) If $f$ has a tangent line of slope 0, then $f$ has a secant line of slope 0.
   (d) If $f'(x) > 0$ for all $x$, then every secant line has positive slope.
3. Can a function with the real numbers as its domain that takes on only negative values have a positive derivative? If so, sketch an example.
4. For $f$ with derivative as in Figure 14:
   (a) Is $f(c)$ a local minimum or maximum?
   (b) Is $f$ a decreasing function?

Exercises

In Exercises 1–8, find a point $c$ satisfying the conclusion of the MVT for the given function and interval.
1. $y = x^{-1}$, $[2, 8]$  
2. $y = x^2$, $[9, 25]$  
3. $y = \cos x - \sin x$, $[0, 2\pi]$  
4. $y = \frac{x}{x + 2}$, $[1, 4]$  
5. $y = x^3$, $[-4, 5]$  
6. $y = (x - 1)(x - 3)$, $[1, 3]$  
7. $y = x \sin x$, $[-\frac{\pi}{2}, \frac{\pi}{2}]$  
8. $y = x - \sin(\pi x)$, $[-1, 1]$

In Exercises 9–12, find a point $c$ satisfying the conclusion of the MVT for the given function and interval. Then draw the graph of the function, the secant line between the endpoints of the graph and the tangent line at $(c, f(c))$, to see that the secant and tangent lines are, in fact, parallel.
9. $y = x^2$, $[0, 1]$  
10. $y = x^2/3$, $[0, 8]$  
11. $f(x) = \frac{1}{1 + x}$, $[0, 1]$  
12. $y = \sqrt{x}$, $[0, 3]$

13. Let $f(x) = x^3 + x^2$. The secant line between $(0, 0)$ and $(1, 2)$ has slope 2 (check this), so by the MVT, $f'(c) = 2$ for some $c \in (0, 1)$. Plot $f$ and the secant line on the same axes. Then plot $y = 2x + b$ for different values of $b$ until the line becomes tangent to the graph of $f$. Zoom in on the point of tangency to estimate the $x$-coordinate $c$ of the point of tangency.

14. Plot the derivative of $f(x) = 3x^3 - 5x^2$. Describe its sign changes and use this to determine the local extreme values of $f$. Then graph $f'$ to confirm your conclusions.

15. Determine the intervals on which $f'(x)$ is positive and negative, assuming that Figure 15 is the graph of $f$.

16. Determine the intervals on which $f$ is increasing or decreasing, assuming that Figure 15 is the graph of $f$.

17. State whether $f(2)$ and $f(4)$ are local minima or local maxima, assuming that Figure 15 is the graph of $f$.

5. Which of the six standard trigonometric functions have infinitely many local minima and infinitely many local maxima but no absolute maximum and no absolute minimum over their whole domain?

6. Compose the absolute value with a familiar function to define a function $f$ that has infinitely many local maxima, all of which occur where $f' = 0$, and has infinitely many local minima, all of which occur where $f'$ is undefined.

18. Figure 16 shows the graph of the derivative $f'$ of a function $f$. Find the critical points of $f$ and determine whether they are local minima, local maxima, or neither.

In Exercises 19–22, sketch the graph of a function $f$ whose derivative $f'$ has the given description.
19. $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < 3$
20. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$
21. $f'(x)$ is negative on $(1, 3)$ and positive everywhere else.
22. $f'(x)$ makes the sign transitions $+$, $-$, $+$, $-$.

In Exercises 23–26, find all critical points of $f$ and use the First Derivative Test to determine whether they are local minima or local maxima.
23. $f(x) = 4 + 6x - x^2$
24. $f(x) = x^3 - 12x - 4$
25. $f(x) = \frac{x^3}{x + 1}$
26. $f(x) = x^3 + x^{-3}$
In Exercises 27–50, find the critical points and the intervals on which the function is increasing or decreasing. Use the First Derivative Test to determine whether the critical point yields a local min or max (or neither).

27. \( y = -x^2 + 7x - 17 \)
28. \( y = 5x^2 + 6x - 4 \)
29. \( y = x^3 - 12x \)
30. \( y = \ln(x - 2) \)
31. \( y = 3x^3 + 5x^2 - 6x - 24x \)
32. \( y = x^2 + (10 - x)^2 \)
33. \( y = \frac{1}{2}x^2 + \frac{3}{2}x + 2x + 4 \)
34. \( y = x^4 + x^3 \)
35. \( y = x^5 + x^3 + x \)
36. \( y = x^2 + x^3 + x \)
37. \( y = x^4 - 4x^2 + 2x \) \( (x > 0) \)
38. \( y = \frac{x^1}{2} - x^2 \) \( (x > 0) \)
39. \( y = x + x^{-1} \)
40. \( y = x^2 - 4x^{-1} \)
41. \( y = 1 + \frac{x^3}{x^2 + 1} \)
42. \( y = \frac{2x + 1}{x^2 + 1} \)
43. \( y = \frac{x^3}{x^2 + 1} \)
44. \( y = \frac{x^3}{x^2 - 3} \)
45. \( y = \theta + \sin \theta + \cos \theta \) \( [0, 2\pi] \)
46. \( y = \sin \theta + \sqrt{3} \cos \theta \) \( [0, 2\pi] \)
47. \( y = \sin^2 \theta + \sin \theta \) \( [0, 2\pi] \)
48. \( y = \theta - 2 \cos \theta \) \( [0, 2\pi] \)
49. \( y = x^{1/3} \)
50. \( y = x^{1/3} - x \)
51. Show that \( f(x) = x^2 + bx + c \) is decreasing on \((-\infty, -\frac{b}{2})\) and increasing on \((-\frac{b}{2}, \infty)\).
52. Show that \( f(x) = x^3 - 2x^2 + 2x \) is an increasing function. Hint: Find the minimum value of \( f' \).
53. Find conditions on \( a \) and \( b \) that ensure \( f(x) = x^3 + ax + b \) is increasing on \((\infty, \infty)\).

Further Insights and Challenges

63. Show that a cubic function \( f(x) = x^3 + ax^2 + bx + c \) is increasing on \((-\infty, \infty)\) if \( b > a^2/3 \).
64. Prove that if \( f(0) = g(0) \) and \( f'(x) \leq g'(x) \) for \( x \geq 0 \), then \( f(x) \leq g(x) \) for all \( x \geq 0 \). Hint: Show that the function given by \( y = f(x) - g(x) \) is nonincreasing.
65. Use Exercise 64 to prove that \( x \leq \tan x \) for \( 0 \leq x < \frac{\pi}{2} \) and \( \sin x \leq x \) for \( x \geq 0 \).
66. Use Exercises 64 and 65 to prove the following assertions for all \( x \geq 0 \) (each assertion follows from the previous one):
   (a) \( \cos x \geq 1 - \frac{x^2}{2} \)
   (b) \( \sin x \geq x - \frac{x^3}{6} \)
   (c) \( \cos x \geq 1 - \frac{x^2}{2} + \frac{x^4}{24} \)
   Can you guess the next inequality in the series?
67. Suppose that \( f(x) \) is a function such that \( f(0) = 1 \) and for all \( x \), \( f'(x) = f(x) \) and \( f(x) > 0 \) (in Chapter 7, we will see that \( f(x) \) is the exponential function \( e^x \)). Prove that for all \( x \geq 0 \), each assertion follows from the previous one.
   (a) \( f(x) \geq 1 \)
   (b) \( f(x) \geq 1 + x \)
   (c) \( f(x) \geq 1 + x + \frac{x^2}{2} \)
   Then prove by induction that for every whole number \( n \) and all \( x \geq 0 \),
   \[ f(x) \geq 1 + x + \frac{x^2}{2} + \cdots + \frac{1}{n!}x^n \]
68. Assume that \( f'' \) exists and \( f''(x) = 0 \) for all \( x \). Prove that \( f(x) = mx + b \), where \( m = f'(0) \) and \( b = f(0) \).
69. Define \( f(x) = x^2 \sin \left( \frac{1}{x} \right) \) for \( x \neq 0 \) and \( f(0) = 0 \).
   (a) Show that \( f' \) is continuous at \( x = 0 \) and that \( x = 0 \) is a critical point of \( f \).
   (b) Examine the graphs of \( f \) and \( f' \). Can the First Derivative Test be applied?
   (c) Show that \( f(0) \) is neither a local min nor a local max.
70. Suppose that \( f(x) \) satisfies the following equation (an example of a differential equation):
   \[ f''(x) = -f(x) \]
   (a) Show that \( f(x)^2 + f'(x)^2 = f(0)^2 + f'(0)^2 \) for all \( x \). Hint: Show that the function on the left has zero derivative.
   (b) Verify that \( f(x) = \sin x \) and \( f(x) = \cos x \) satisfy Eq. (1), and deduce that \( \sin x + \cos x = 1 \).
71. Suppose that functions \( f \) and \( g \) satisfy Eq. (1) and have the same initial values—that is, \( f(0) = g(0) \) and \( f'(0) = g'(0) \). Prove that \( f(x) = g(x) \) for all \( x \). Hint: Apply Exercise 70(a) to \( f - g \).
72. Use Exercise 71 to prove that \( f(x) = \sin x \) is the unique solution of Eq. (1) such that \( f(0) = 0 \) and \( f'(0) = 1 \); and \( g(x) = \cos x \) is the unique solution such that \( g(0) = 1 \) and \( g'(0) = 0 \). This result can be used to develop all the properties of the trigonometric functions "analytically"—that is, without reference to triangles.
4.4 The Second Derivative and Concavity

In the previous section, we studied the increasing/decreasing behavior of a function, as determined by the sign of the derivative. Another important property is concavity, which refers to the way the graph bends. Informally, a curve is concave up if it bends up and concave down if it bends down (Figure 1).

![Concave up and Concave down](image)

**Figure 1**

To analyze concavity in a precise fashion, let's examine how concavity is related to tangent lines and derivatives. Observe in Figure 2 that when \( f \) is concave up, \( f' \) is increasing (the slopes of the tangent lines increase as we move to the right). Similarly, when \( f \) is concave down, \( f' \) is decreasing. This suggests the following definition.

![Concave up and Concave down](image)

**Figure 2**

**Definition** Concavity

Let \( f \) be a differentiable function on an open interval \((a, b)\).

Then

- \( f \) is concave up on \((a, b)\) if \( f' \) is increasing on \((a, b)\).
- \( f \) is concave down on \((a, b)\) if \( f' \) is decreasing on \((a, b)\).

**Example 1** Concavity and Stock Prices

The stocks of two companies, Arenot Industries (AI) and Blubenthal Business Associates (BBA), went up in value, and both currently sell for $75 (Figure 3). However, one is clearly a better investment than the other, assuming these trends continue in the same manner. Explain in terms of concavity.

![Stock price graphs](image)

**Figure 3**

**Solution** The graph of Stock AI is concave down, so its growth rate (first derivative) is declining as time goes on. The graph of Stock BBA is concave up, so its growth rate is increasing. If these trends continue, Stock BBA is the better investment.

The concavity of a function is determined by the sign of its second derivative. Indeed, if \( f''(x) > 0 \), then \( f' \) is increasing and hence \( f \) is concave up. Similarly, if \( f''(x) < 0 \), then \( f' \) is decreasing and \( f \) is concave down.
**THEOREM 1 Test for Concavity**
Assume that $f''(x)$ exists for all $x \in (a, b)$.

- If $f''(x) > 0$ for all $x \in (a, b)$, then $f$ is concave up on $(a, b)$.
- If $f''(x) < 0$ for all $x \in (a, b)$, then $f$ is concave down on $(a, b)$.

Of special interest are the points on the graph where the concavity changes. We say that $P = (c, f(c))$ is a **point of inflection** of $f$ if the concavity changes from up to down or from down to up at $x = c$. Figure 4 shows a curve made up of two arcs—one is concave down and one is concave up (the word “arc” refers to a piece of a curve). The point $P$ where the arcs are joined is a point of inflection. We will denote points of inflection in graphs by a solid square □.

![Figure 4](image)

According to Theorem 1, the concavity of $f$ is determined by the sign of $f''(x)$. Therefore, a point of inflection is a point where $f''(x)$ changes sign.

**THEOREM 2 Test for Inflection Points**
If $f''(c) = 0$ or $f''(c)$ does not exist and $f''(x)$ changes sign at $x = c$, then $f$ has a point of inflection at $x = c$.

**EXAMPLE 2** Find the points of inflection of $f(x) = \cos x$ on $[0, 2\pi]$.

**Solution** We have

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad \text{and} \quad f''(x) = 0 \text{ for } x = \frac{\pi}{2}, \frac{3\pi}{2}.$$  

Figure 5 shows that $f''(x)$ changes sign at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, so $f$ has a point of inflection at both points.

**EXAMPLE 3** Points of Inflection and Intervals of Concavity

Find the points of inflection and the intervals on which $f(x) = 3x^3 - 5x^4 + 1$ is concave up and concave down.

**Solution** The first derivative is $f'(x) = 9x^2 - 20x^3$ and

$$f''(x) = 60x^3 - 60x^2 = 60x^2(x - 1).$$

The zeros of $f''(x) = 60x^2(x - 1)$ are $x = 0$ and $x = 1$. They divide the $x$-axis into three intervals: $(-\infty, 0), (0, 1),$ and $(1, \infty)$. We determine the sign of $f''(x)$ and the concavity of $f$ by computing test values within each interval (Figure 6):

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>Sign of $f''(x)$</th>
<th>Behavior of $f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>$f''(-1) = -120$</td>
<td>$-$</td>
<td>Concave down</td>
</tr>
<tr>
<td>$(0, 1)$</td>
<td>$f''(\frac{1}{2}) = -\frac{15}{2}$</td>
<td>$-$</td>
<td>Concave down</td>
</tr>
<tr>
<td>$(1, \infty)$</td>
<td>$f''(2) = 240$</td>
<td>$+$</td>
<td>Concave up</td>
</tr>
</tbody>
</table>

Since the concavity changes at $x = 1$ there is an inflection point there. The inflection point is $(1, -1)$. Note that, even though $f'(0) = 0$, there is not an inflection point at $x = 0$ because the concavity does not change at $x = 0$. □

**CAUTION** A critical point $c$ is just a single number, whereas a point of inflection $(c, f(c))$ is a point in the $xy$-plane.
Usually, we find the inflection points by solving \( f''(x) = 0 \). However, an inflection point can also occur at a point \((c, f(c))\), where \( f''(c) \) does not exist.

**Example 4** A Case Where the Second Derivative Does Not Exist

Find the points of inflection of \( f(x) = x^{2/3} \).

**Solution** In this case, \( f'(x) = \frac{2}{3}x^{-1/3} \) and \( f''(x) = \frac{10}{9x^{4/3}} \). Although \( f''(0) \) does not exist, \( f''(x) \) does change sign at \( x = 0 \):

\[
f''(x) = \frac{10}{9x^{4/3}} \begin{cases} > 0 & \text{for } x > 0 \\ < 0 & \text{for } x < 0 \end{cases}
\]

Therefore, the concavity of \( f \) changes at \( x = 0 \), and \((0,0)\) is a point of inflection (Figure 7).

**Graphical Insight** Points of inflection are easy to spot on the graph of the first derivative \( f' \). If \( f''(x) = 0 \) and \( f''(x) \) changes sign at \( x = c \), then the increasing/decreasing behavior of \( f' \) changes at \( x = c \):

- If \( f''(x) \) goes from positive to negative at \( x = c \), then \( f' \) has a local max at \( x = c \).
- If \( f''(x) \) goes from negative to positive at \( x = c \), then \( f' \) has a local min at \( x = c \).

Thus, inflection points of \( f \) occur where \( f' \) has a local min or max (Figure 8).

**Second Derivative Test for Critical Points**

There is a simple test for critical points based on concavity. Suppose that \( f'(c) = 0 \). As we see in Figure 9, \( f(c) \) is a local max if \( f \) is concave down, and it is a local min if \( f \) is concave up. Concavity is determined by the sign of \( f''(x) \), so we obtain the Second Derivative Test in Theorem 3. (See Exercise 61 for a detailed proof.)

**Theorem 3** Second Derivative Test

Let \( c \) be a critical point of \( f(x) \). If \( f''(c) \) exists, then

- \( f''(c) > 0 \) \( \Rightarrow \) \( f(c) \) is a local minimum.
- \( f''(c) < 0 \) \( \Rightarrow \) \( f(c) \) is a local maximum.
- \( f''(c) = 0 \) \( \Rightarrow \) inconclusive: \( f(c) \) may be a local min, a local max, or neither.

The mnemonic device appearing in Figure 10 provides an easy way to remember the test.
EXAMPLE 5  Analyze the critical points of \( f(x) = 2x^3 + 3x^2 - 12x \).

Solution  First, we find the critical points by solving

\[
f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1) = 0
\]

The critical points are \( c = -2, 1 \) (Figure 11) and \( f''(x) = 12x + 6 \), so by the Second Derivative Test,

\[
f''(-2) = -24 + 6 = -18 < 0 \quad \Rightarrow \quad f(-2) \text{ is a local max}
\]
\[
f''(1) = 12 + 6 = 18 > 0 \quad \Rightarrow \quad f(1) \text{ is a local max}
\]

EXAMPLE 6  Second Derivative Test Inconclusive  Analyze the critical points of \( f(x) = x^5 - 5x^4 \).

Solution  The first two derivatives are

\[
f'(x) = 5x^4 - 20x^3 = 5x^3(x - 4)
\]
\[
f''(x) = 20x^3 - 60x^2
\]

The critical points are \( c = 0, 4 \), and the Second Derivative Test yields

\[
f''(0) = 0 \quad \Rightarrow \quad \text{Second Derivative Test fails}
\]
\[
f''(4) = 320 > 0 \quad \Rightarrow \quad f(4) \text{ is a local min}
\]

The Second Derivative Test fails at \( x = 0 \), so we fall back on the First Derivative Test. Choosing test points to the left and right of \( x = 0 \), we find

\[
f'(-1) = 5 + 20 = 25 > 0 \quad \Rightarrow \quad f'(x) \text{ is positive on } (-\infty, 0)
\]
\[
f'(1) = 5 - 20 = -15 < 0 \quad \Rightarrow \quad f'(x) \text{ is negative on } (0, 4)
\]

Since \( f'(x) \) changes from + to − at \( x = 0 \), \( f(0) \) is a local max (Figure 12).

4.4 SUMMARY

- A differentiable function \( f \) is concave up on \((a, b)\) if \( f' \) is increasing and concave down if \( f' \) is decreasing on \((a, b)\).
- The signs of the first two derivatives provide the following information:

<table>
<thead>
<tr>
<th>First derivative</th>
<th>Second derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f' &gt; 0 )</td>
<td>( f'' &gt; 0 )</td>
</tr>
<tr>
<td>( f' &lt; 0 )</td>
<td>( f'' &lt; 0 )</td>
</tr>
</tbody>
</table>

- A point of inflection is a point \((c, f(c))\) where the concavity changes from concave up to concave down, or vice versa.
- Second Derivative Test: If \( f'(c) = 0 \) and \( f''(c) \) exists, then
  - \( f(c) \) is a local maximum value if \( f''(c) < 0 \)
  - \( f(c) \) is a local minimum value if \( f''(c) > 0 \)
  - The test fails if \( f''(c) = 0 \)
- If this test fails, use the First Derivative Test.
4.4 EXERCISES

Preliminary Questions

1. If \( f \) is concave up, then \( f' \) is (choose one)
   (a) increasing (b) decreasing

2. What conclusion can you draw if \( f'(c) = 0 \) and \( f''(c) < 0 \)?

3. True or false? If \( f'(c) \) is a local min, then \( f''(c) \) must be positive.

Exercises

1. Match the graphs in Figure 13 with the description:
   (a) \( f''(x) < 0 \) for all \( x \).
   (b) \( f''(x) \) goes from + to -.
   (c) \( f''(x) > 0 \) for all \( x \).
   (d) \( f''(x) \) goes from - to +.

   ![Figure 13]

2. Match each statement with a graph in Figure 14 that represents company profits as a function of time.
   (a) The outlook is great: The growth rate keeps increasing.
   (b) We’re losing money, but not as quickly as before.
   (c) We’re losing money, and it’s getting worse as time goes on.
   (d) We’re doing well, but our growth rate is leveling off.
   (e) Business had been cooling off, but now it’s picking up.
   (f) Business had been picking up, but now it’s cooling off.

   ![Figure 14]

3. **GU** Plot \( f(x) = x^3 + 4x^2 - x - 4 \) and indicate on the graph where it appears that inflection points occur. Then find the inflection points using calculus.

4. **GU** Plot \( f(x) = x(x - 4)^2 \) and indicate on the graph where it appears that inflection points occur. Then find the inflection points using calculus.

In Exercises 5–18, determine the intervals on which the function is concave up or down and find the points of inflection:

5. \( y = x^2 - 4x + 3 \)

6. \( y = x^3 - 6x^2 + 4 \)

7. \( y = 10x^3 - x^2 \)

8. \( y = 5x^2 + x^4 \)

9. \( y = \theta - 2\sin \theta \), \([0, 2\pi]\)

10. \( y = \theta + \sin^2 \theta \), \([0, \pi]\)

11. \( y = x(x - 8\sqrt{2}) \) (\( x \geq 0 \))

12. \( y = x^{3/2} - 35x^2 \)

13. \( y = (x - 2)(1 - x)^2 \)

14. \( y = x^{7/5} \)

15. \( y = \frac{1}{x^2 + 3} \)

16. \( y = \frac{x}{x^2 + 9} \)

17. \( f(x) = \frac{x^3}{1 + x} \)

18. \( w(t) = \frac{t^4 - 1}{t} \)

19. The position of an ambulance in kilometers on a straight road over a period of 4 hours is given by the graph in Figure 15.
   (a) Describe the motion of the ambulance.
   (b) Explain what the fact that this graph is concave up tells us about the speed of the ambulance.

   ![Figure 15]

20. The position of a bicyclist on a straight road in kilometers over a period of 4 h is given by the graph in Figure 16, where inflection points occur when \( t = 0.5 \) and \( t = 2 \).
   (a) Describe the motion of the bicyclist.
   (b) Explain what the concavity of the graph over various intervals tells us about the speed of the bicyclist.

   ![Figure 16]

21. **□** The growth of a sunflower during the first 100 days after sprouting is modeled well by the logistic curve \( y = h(t) \) shown in Figure 17. Estimate the growth rate at the point of inflection and explain its significance. Then make a rough sketch of the first and second derivatives of \( h \).

   ![Figure 17]
22. Assume that Figure 18 is the graph of \( f \). Where do the points of inflection of \( f \) occur, and on which interval is \( f \) concave down?

![Figure 18](image)

23. Repeat Exercise 22 but assume that Figure 18 is the graph of the derivative \( f' \).

24. Repeat Exercise 22 but assume that Figure 18 is the graph of the second derivative \( f'' \).

25. Figure 19 shows the derivative \( f' \) on \([0, 1.2]\). Locate the points of inflection of \( f \) and the points where the local minimum and maximum occur. Determine the intervals on which \( f \) has the following properties:

   (a) Increasing  
   (b) Decreasing  
   (c) Concave up  
   (d) Concave down

![Figure 19](image)

26. Letícia has been selling solar-powered laptop chargers through her Web site, with monthly sales as recorded below. In a report to investors, she states, "Sales reached a point of inflection when I started using payper-click advertising." In which month did that occur? Explain.

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales</td>
<td>2</td>
<td>30</td>
<td>50</td>
<td>60</td>
<td>90</td>
<td>150</td>
<td>230</td>
<td>340</td>
</tr>
</tbody>
</table>

In Exercises 27–38, find the critical points and apply the Second Derivative Test (or state that it fails).

27. \( f(x) = x^3 - 12x^2 + 45x \)
28. \( f(x) = x^4 - 8x^2 + 1 \)
29. \( f(x) = 3x^4 - 8x^3 + 6x^2 \)
30. \( f(x) = x^3 - x \)
31. \( f(x) = \frac{x^2 - 8x}{x + 1} \)
32. \( f(x) = \frac{1}{x^2 - x + 2} \)
33. \( y = 6x^{3/2} - 4x^{1/2} \)
34. \( y = 9x^{7/3} - 21x^{1/2} \)
35. \( f(x) = \sin x \cos x, \quad [0, \pi] \)
36. \( y = \frac{1}{\sin x + 4}, \quad [0, 2\pi] \)
37. \( 2 + \tan^2 x, \quad (-\pi/2, \pi/2) \)
38. \( \sin x \cos^3 x, \quad [0, \pi] \)

In Exercises 39–52, find the intervals on which \( f \) is concave up or down, the points of inflection, the critical points, and the local minima and maxima.

39. \( f(x) = x^2 - 2x^2 + x \)
40. \( f(x) = x^2(x - 4) \)
41. \( f(t) = t^3 - t^3 \)
42. \( f(x) = 2x^4 - 3x^2 + 2 \)
43. \( f(x) = x^2 - 8x^{1/2} \quad (x \geq 0) \)
44. \( f(x) = x^{3/2} - 4x^{-1/2} \quad (x > 0) \)
45. \( f(x) = \frac{x}{x^2 + 27} \)
46. \( f(x) = \frac{1}{x^3 + 1} \)
47. \( f(x) = x^{4/3} - x \)
48. \( f(x) = (x - 1)^{3/5} \)
49. \( f(t) = \cos^2 t, \quad [0, 2\pi] \)
50. \( f(x) = \cos^2 x, \quad [0, \pi] \)
51. \( f(x) = \tan x, \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \)
52. \( f(x) = \frac{x}{(x^2 + 3)^2} \)

53. Sketch the graph of an increasing function such that \( f''(x) \) changes from + to − at \( x = 2 \) and from − to + at \( x = 4 \). Do the same for a decreasing function.

In Exercises 54–56, sketch the graph of a function \( f \) satisfying all of the given conditions.

54. \( f''(x) > 0 \) and \( f''(x) < 0 \) for all \( x \)
55. (i) \( f''(x) > 0 \) for all \( x \), and  
   (ii) \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \)
56. (i) \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \), and  
   (ii) \( f''(x) < 0 \) for \( |x| > 2 \), and \( f''(x) > 0 \) for \( |x| < 2 \)

57. An infectious flu spreads slowly at the beginning of an epidemic. The infection process accelerates until a majority of the susceptible individuals are infected, at which point the process slows down.

(a) If \( R(t) \) is the number of individuals infected at time \( t \), describe the concavity of the graph of \( R \) near the beginning and end of the epidemic.

(b) Describe the status of the epidemic on the day that \( R \) has a point of inflection.

58. Water is pumped into a sphere at a constant rate (Figure 20). Let \( h(t) \) be the water level at time \( t \). Sketch the graph of \( h \) (approximately, but with the correct concavity). Where does the point of inflection occur?

59. Water is pumped into a sphere of radius \( R \) at a variable rate in such a way that the water level rises at a constant rate (Figure 20). Let \( V(t) \) be the volume of water in the tank at time \( t \). Sketch the graph \( V \) (approximately, but with the correct concavity). Where does the point of inflection occur?

60. (Continuation of Exercise 59) If the sphere has radius \( R \), the volume of water is

\[ V = \pi R^3 \left( \frac{1}{3} \right) \]

where \( h \) is the water level. Assume the level rises at a constant rate of 1 (i.e., \( h = 1 \)).

(a) Find the inflection point of \( V \). Does this agree with your conclusion in Exercise 59?

(b) Plot \( V \) for \( R = 1 \).
Further Insights and Challenges

In Exercises 61–63, assume that \( f \) is differentiable.

61. Proof of the Second Derivative Test Let \( c \) be a critical point such that \( f''(c) > 0 \) [the case \( f''(c) < 0 \) is similar].

(a) Show that \( f''(c) = \lim_{h \to 0} \frac{f'(c + h) - f'(c)}{h} \).
(b) Use (a) to show that there exists an open interval \((a, b)\) containing \( c \) such that \( f'(x) < 0 \) if \( a < x < c \) and \( f'(x) > 0 \) if \( c < x < b \). Conclude that \( f(c) \) is a local minimum.

62. Prove that if \( f'' \) exists and \( f''(x) > 0 \) for all \( x \), then the graph of \( f \) "sits above" its tangent lines.

(a) For any \( c \), set \( G(x) = f(x) - f'(c)(x - c) - f(c) \). It is sufficient to prove that \( G(x) \geq 0 \) for all \( c \). Explain why with a sketch.
(b) Show that \( G(c) = G'(c) = 0 \) and \( G''(x) > 0 \) for all \( x \). Conclude that \( G(x) < 0 \) for \( x < c \) and \( G(x) > 0 \) for \( x > c \). Then deduce, using the Mean Value Theorem (MVT), that \( G(x) > G(c) \) for \( x \neq c \).

63. Assume that \( f'' \) exists and let \( c \) be a point of inflection of \( f \).

(a) Use the method of Exercise 62 to prove that the tangent line at \( x = c \) crosses the graph (Figure 21). Hint: Show that \( G(x) \) changes sign at \( x = c \).
(b) Verify this conclusion for \( f(x) = \frac{x^2}{3x^2 + 1} \) by graphing \( f \) and the tangent line at each inflection point on the same set of axes.

64. Let \( C(x) \) be the cost of producing \( x \) units of a certain good. Assume that the graph of \( C \) is concave up.

(a) Show that the average cost \( A(x) = C(x)/x \) is minimized at the production level \( x_0 \) such that average cost equals marginal cost—that is, \( A(x_0) = C'(x_0) \).
(b) Show that the line through \((0, 0)\) and \((x_0, C(x_0))\) is tangent to the graph of \( C \).

4.5 Analyzing and Sketching Graphs of Functions

In this section, our goal is to study graphs of functions \( f \) using the information provided by the first two derivatives \( f' \) and \( f'' \). You will see that you can acquire a good understanding of the properties of a graph without plotting a large number of points. Even though almost all graphs you may see are produced by computer (including, of course, the graphs in this textbook), the tools of calculus provide information beyond the image displayed on a computer. This information includes the exact locations of critical points and inflection points, the rates of increase and decrease over the function's domain, and the concavity of the function.

Most graphs are made up of smaller arcs that have one of the four basic shapes, corresponding to the four possible sign combinations of \( f' \) and \( f'' \) (Figure 1). Since \( f' \) and \( f'' \) can each have sign + or −, the sign combinations are

\[ ++ \quad +- \quad -+ \quad -- \]

In this notation, the first sign refers to \( f' \) and the second sign to \( f'' \). For instance, \(-+\) indicates that \( f'(x) < 0 \) and \( f''(x) > 0 \). We use a slanted arrow over the first sign to indicate whether the function is increasing or decreasing, and an upturned or downturned \( \cup \) over the second sign to indicate the concavity.
In analyzing a graph, we focus on the transition points, where the basic shape changes due to a sign change in either $f'$ (local min or max) or $f''$ (point of inflection). In this section, local extrema are indicated by solid dots, and points of inflection are indicated by green solid squares (Figure 2).

In examining the properties of a function, it is often useful to investigate the asymptotic behavior—that is, the behavior of $f(x)$ as $x$ approaches either $\pm\infty$ or a vertical asymptote.

In the examples that follow, we use calculus to investigate the behavior of specific functions, and then we use the information we gather to construct a picture of the function's graph—that is, to "sketch the graph." The first three examples treat polynomials. Recall from Section 2.7 that the limits at infinity of a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

(assuming that $a_n \neq 0$) are determined by

$$\lim_{x \to \infty} f(x) = a_n \lim_{x \to \infty} x^n$$

In general, the graph of a polynomial oscillates up and down a finite number of times and tends to positive or negative infinity as $x$ tends to positive or negative infinity. Typical examples appear in Figure 3.

![Figure 3 Graphs of polynomials.](image)

### Example 1: Quadratic Polynomial

Investigate the behavior of $f(x) = x^2 - 4x + 3$ and sketch its graph.

**Solution**

Note that $f(x) = (x - 1)(x - 3)$ so the graph intersects the $x$-axis at $x = 1$ and $x = 3$. We have $f'(x) = 2x - 4 = 2(x - 2)$. We can see directly that $f'(x)$ is negative for $x < 2$ and positive for $x > 2$, but let's confirm this using test values, as in previous sections:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>Sign of $f'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 2)$</td>
<td>$f'(1) = -2$</td>
<td>$-$</td>
</tr>
<tr>
<td>$(2, \infty)$</td>
<td>$f'(3) = 2$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Furthermore, $f''(x) = 2$ is positive, so the graph is everywhere concave up. To sketch the graph, plot the local minimum $(2, -1)$, the $y$-intercept, and the roots $x = 1, 3$. Since the leading term of $f$ is $x^2$, $f(x)$ tends to $\infty$ as $x \to \pm\infty$. This asymptotic behavior is noted by the arrows in Figure 4.

### Example 2: Cubic Polynomial

Investigate the behavior of the cubic function $f(x) = \frac{1}{3} x^3 - \frac{1}{2} x^2 - 2x + 3$ and sketch the graph.

**Solution**

**Step 1. Determine the signs of $f'$ and $f''$.**

First, solve for the critical points:

$$f'(x) = x^2 - x - 2 = (x + 1)(x - 2)$$
The critical points are $c = -1, 2$, and they divide the $x$-axis into three intervals $(-\infty, -1), (-1, 2)$, and $(2, \infty)$, on which we determine the sign of $f'$ by computing test values:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>Sign of $f'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -1)$</td>
<td>$f'(-2) = -4$</td>
<td>$+$</td>
</tr>
<tr>
<td>$(-\infty, -1)$</td>
<td>$f'(0) = -2$</td>
<td>$-$</td>
</tr>
<tr>
<td>$(2, \infty)$</td>
<td>$f'(3) = 4$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Next, $f''(x) = 2x - 1$, and therefore $x = \frac{1}{2}$ is the only solution to $f''(x) = 0$. We have

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>Sign of $f''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, \frac{1}{2})$</td>
<td>$f''(0) = -1$</td>
<td>$-$</td>
</tr>
<tr>
<td>$(\frac{1}{2}, \infty)$</td>
<td>$f''(2) = 1$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

**Step 2.** Note transition points and sign combinations.
This step merges the information about $f'$ and $f''$ in a sign diagram (Figure 5). There are three transition points:

- $c = -1$: local max since $f'$ changes from + to -.
- $c = \frac{1}{2}$: corresponds to a point of inflection since $f''$ changes sign.
- $c = 2$: local min since $f'$ changes from - to +.

In Figure 6(A), we plot the transition points and, for added accuracy, the $y$-intercept, the $y$-intercept $f(0)$, using the values

$$f(-1) = \frac{25}{6}, \quad f\left(\frac{1}{2}\right) = \frac{23}{12}, \quad f(0) = 3, \quad f(2) = -\frac{1}{3}$$

**Step 3.** Draw arcs of appropriate shape and asymptotic behavior.
The leading term of $f(x)$ is $\frac{1}{3}x^3$. Therefore, $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

To create the sketch, it remains only to connect the transition points by arcs of the appropriate concavity and asymptotic behavior, as in Figure 6(B) and (C).

**EXAMPLE 3** Investigate the behavior of $f(x) = 3x^4 - 8x^3 + 6x^2 + 1$ and sketch its graph.

**Solution**

**Step 1.** Determine the signs of $f'$ and $f''$.
First, solve for the transition points:

$$f'(x) = 12x^3 - 24x^2 + 12x = 12x(x - 1)^2, \quad \text{so } f' = 0 \implies x = 0, 1$$

$$f''(x) = 36x^2 - 48x + 12 = 12(x - 1)(3x - 1), \quad \text{so } f'' = 0 \implies x = \frac{1}{3}, 1$$
The signs of \( f' \) and \( f'' \) are recorded in the following tables:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>Sign of ( f' )</th>
<th>Interval</th>
<th>Test value</th>
<th>Sign of ( f'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (-\infty, 0) )</td>
<td>( f'(-1) = -48 )</td>
<td>-</td>
<td>( (-\infty, \frac{1}{4}) )</td>
<td>( f''(0) = 12 )</td>
<td>+</td>
</tr>
<tr>
<td>( (0, 1) )</td>
<td>( f'(\frac{1}{3}) = \frac{3}{2} )</td>
<td>+</td>
<td>( (\frac{1}{4}, 1) )</td>
<td>( f''(\frac{1}{2}) = -3 )</td>
<td>-</td>
</tr>
<tr>
<td>( (1, \infty) )</td>
<td>( f'(2) = 24 )</td>
<td>+</td>
<td>( (1, \infty) )</td>
<td>( f''(2) = 60 )</td>
<td>+</td>
</tr>
</tbody>
</table>

**Step 2.** Note transition points and sign combinations.

The transition points \( c = 0, \frac{1}{4}, 1 \) divide the \( x \)-axis into four intervals (Figure 7). The type of sign change determines the nature of the transition point:

- \( c = 0 \): local min since \( f' \) changes from \(-\) to \(+\).
- \( c = \frac{1}{4} \): corresponds to a point of inflection since \( f'' \) changes sign.
- \( c = 1 \): neither a local min nor a local max since \( f' \) does not change sign, but it is a point of inflection since \( f''(x) \) changes sign.

We plot the transition points \( c = 0, \frac{1}{4}, 1 \) in Figure 8(A) using function values \( f(0) = 1, f(\frac{1}{4}) = \frac{33}{32}, \) and \( f(1) = 2 \).

**Step 3.** Draw arcs of appropriate shape and asymptotic behavior.

Before drawing the arcs, we note that \( f(x) \) has leading term \( 3x^3 \), so \( f(x) \) tends to \( \infty \) as \( x \to \infty \) and as \( x \to -\infty \). We obtain Figure 8(B).

**EXAMPLE 4** Investigate the behavior of \( f(x) = \cos x + \frac{1}{2}x \) over \([0, \pi]\), and sketch its graph.

**Solution** First, we find the transition points for \( x \) in \([0, \pi]\):

\[
f'(x) = -\sin x + \frac{1}{2}, \quad \text{so} \quad f'(x) = 0 \Rightarrow x = \frac{\pi}{6}, \quad \frac{5\pi}{6}
\]

\[
f''(x) = -\cos x, \quad \text{so} \quad f''(x) = 0 \Rightarrow x = \frac{\pi}{2}
\]

The sign combinations are shown in the following tables:

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test value</th>
<th>Sign of ( f' )</th>
<th>Interval</th>
<th>Test value</th>
<th>Sign of ( f'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0, \frac{\pi}{6}) )</td>
<td>( f'(\frac{\pi}{12}) \approx 0.24 )</td>
<td>+</td>
<td>( (0, \frac{\pi}{4}) )</td>
<td>( f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} )</td>
<td>-</td>
</tr>
<tr>
<td>( (\frac{\pi}{6}, \frac{5\pi}{6}) )</td>
<td>( f'(\frac{\pi}{4}) = -\frac{1}{2} )</td>
<td>-</td>
<td>( (\frac{\pi}{4}, \pi) )</td>
<td>( f''(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2} )</td>
<td>+</td>
</tr>
<tr>
<td>( (\frac{5\pi}{6}, \pi) )</td>
<td>( f'(\frac{11\pi}{12}) \approx 0.24 )</td>
<td>+</td>
<td>( (\frac{\pi}{6}, \frac{\pi}{4}) )</td>
<td>( f''(\frac{\pi}{12}) \approx 0.24 )</td>
<td>+</td>
</tr>
</tbody>
</table>

We record the sign changes and transition points in Figure 9 and sketch the graph using the values

\( f(0) = 1, \quad f\left(\frac{\pi}{6}\right) \approx 1.13, \quad f\left(\frac{\pi}{2}\right) \approx 0.79, \quad f\left(\frac{5\pi}{6}\right) \approx 0.44, \quad f(\pi) \approx 0.57 \)
EXAMPLE 5 Investigate the behavior of \( f(x) = \frac{3x + 2}{2x - 4} \) and sketch its graph.

Solution The function \( f \) is not defined for all \( x \). This plays a role in our analysis so we add a Step 0 to our procedure.

Step 0. Determine the domain of \( f \).
Since \( f(x) \) is not defined for \( x = 2 \), the domain of \( f \) consists of the two intervals \((\infty, 2)\) and \((2, \infty)\). We must analyze \( f \) on these intervals separately.

Step 1. Determine the signs of \( f' \) and \( f'' \).
Calculation shows that
\[
f'(x) = -\frac{4}{(x-2)^2}, \quad f''(x) = \frac{8}{(x-2)^3}
\]
Although \( f'(x) \) is not defined at \( x = 2 \), it is not a critical point because \( x = 2 \) is not in the domain of \( f \). In fact, \( f'(x) \) is negative for \( x \neq 2 \), so \( f \) is decreasing and has no critical points.

On the other hand, \( f''(x) > 0 \) for \( x > 2 \) and \( f''(x) < 0 \) for \( x < 2 \), so the concavity of \( f \) changes at \( x = 2 \). However, there is not an inflection point at \( x = 2 \) because—as was the case above—\( x = 2 \) is not in the domain of \( f \).

Step 2. Note transition points and sign combinations.
There are no transition points in the domain of \( f \).
\[
\begin{align*}
(-\infty, 2) & \quad f'(x) < 0 \text{ and } f''(x) < 0 \\
(2, \infty) & \quad f'(x) < 0 \text{ and } f''(x) > 0
\end{align*}
\]

Step 3. Draw arcs of appropriate shape and asymptotic behavior.
The following limits show that \( y = \frac{3}{2} \) is a horizontal asymptote:
\[
\lim_{x \to \infty} \frac{3x + 2}{2x - 4} = \lim_{x \to \infty} \frac{3 + 2x^{-1}}{2 - 4x^{-1}} = \frac{3}{2}
\]
The line \( x = 2 \) is a vertical asymptote because \( f(x) \) has infinite one-sided limits
\[
\lim_{x \to 2^-} \frac{3x + 2}{2x - 4} = -\infty, \quad \lim_{x \to 2^+} \frac{3x + 2}{2x - 4} = \infty
\]
To verify this, note that for \( x \) near 2, the numerator \( 3x + 2 \) is positive while the denominator \( 2x - 4 \) is small and negative for \( x < 2 \) and is small and positive for \( x > 2 \). Figure 10(A) summarizes the asymptotic behavior.

Now, to the left of \( x = 2 \), the graph is decreasing \([f'(x) < 0]\), is concave down \([f''(x) < 0]\), and approaches the asymptotes. The \( x \)-intercept is \( x = -\frac{2}{3} \) because \( f(-\frac{2}{3}) = 0 \), and the \( y \)-intercept is \( y = f(0) = -\frac{1}{2} \). We obtain the left part of the graph as shown in Figure 10(B). To the right of \( x = 2 \), the graph is decreasing \([f'(x) < 0]\), is concave up \([f''(x) > 0]\), and approaches the asymptotes as shown.

![Graph of y = \( \frac{3x + 2}{2x - 4} \).](attachment:image.png)
EXAMPLE 6 Sketch the graph of \( f(x) = \frac{1}{x^2 - 1} \).

Solution The function \( f \) is defined for \( x \neq \pm 1 \). By calculation,

\[
f'(x) = -\frac{2x}{(x^2 - 1)^2}, \quad f''(x) = \frac{6x^2 + 2}{(x^2 - 1)^3}
\]

The sign of \( f''(x) \) is equal to the sign of \( x^2 - 1 \) because \( 6x^2 + 2 \) is positive:

- \( f''(x) > 0 \) for \( x < -1 \) or \( x > 1 \) and \( f''(x) < 0 \) for \(-1 < x < 1\)

Figure 11 summarizes the sign information.

The \( x \)-axis, \( y = 0 \), is a horizontal asymptote because

\[
\lim_{x \to \pm\infty} \frac{1}{x^2 - 1} = 0 \quad \text{and} \quad \lim_{x \to \pm\infty} \frac{1}{x^2 - 1} = 0
\]

The lines \( x = \pm 1 \) are vertical asymptotes. To determine the one-sided limits, note that \( f(x) < 0 \) for \(-1 < x < 1\) and \( f(x) > 0 \) for \(|x| > 1\). Therefore, as \( x \to \pm 1 \), \( f(x) \) approaches \(-\infty\) from within the interval \((-1, 1)\), and it approaches \(\infty\) from outside \((-1, 1)\) (Figure 12). We display the sketch in Figure 13.

<table>
<thead>
<tr>
<th>Vertical Asymptote</th>
<th>Left-Hand Limit</th>
<th>Right-Hand Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = -1 )</td>
<td>( \lim_{x \to -1^-} \frac{1}{x^2 - 1} = \infty )</td>
<td>( \lim_{x \to -1^+} \frac{1}{x^2 - 1} = -\infty )</td>
</tr>
<tr>
<td>( x = 1 )</td>
<td>( \lim_{x \to 1^-} \frac{1}{x^2 - 1} = -\infty )</td>
<td>( \lim_{x \to 1^+} \frac{1}{x^2 - 1} = \infty )</td>
</tr>
</tbody>
</table>

FIGURE 12 Behavior at vertical asymptotes.

FIGURE 13 Graph of \( y = \frac{1}{x^2 - 1} \).

4.5 SUMMARY

- Most graphs are made up of arcs that have one of the four basic shapes (Figure 14):

<table>
<thead>
<tr>
<th>Sign combination</th>
<th>Curve type</th>
</tr>
</thead>
<tbody>
<tr>
<td>++</td>
<td>( f' &gt; 0, f'' &gt; 0 )  Increasing and concave up</td>
</tr>
<tr>
<td>+-</td>
<td>( f' &gt; 0, f'' &lt; 0 )  Increasing and concave down</td>
</tr>
<tr>
<td>-+</td>
<td>( f' &lt; 0, f'' &gt; 0 )  Decreasing and concave up</td>
</tr>
<tr>
<td>--</td>
<td>( f' &lt; 0, f'' &lt; 0 )  Decreasing and concave down</td>
</tr>
</tbody>
</table>

- A transition point is a point in the domain of \( f \) at which either \( f' \) changes sign (local min or max) or \( f'' \) changes sign (point of inflection).
- It is convenient to break up the curve-sketching process into steps:
  Step 0. Determine the domain of \( f \).
  Step 1. Determine the signs of \( f' \) and \( f'' \).
  Step 2. Note transition points and sign combinations.
  Step 3. Determine the asymptotic behavior of \( f(x) \).
  Step 4. Draw arcs of appropriate shape and asymptotic behavior.
4.5 Exercises

Preliminary Questions
1. Sketch an arc where \( f' \) and \( f'' \) have the sign combination \( ++ \). Do the same for \( -- \).

2. If the sign combination of \( f' \) and \( f'' \) changes from \( ++ \) to \( -- \) at \( x = c \), then (choose the correct answer)
   (a) \( f(c) \) is a local min.
   (b) \( f(c) \) is a local max.
   (c) \( f(c) \) is a point of inflection.

3. The second derivative of the function \( f(x) = (x - 4)^{-1} \) is \( f''(x) = 2(x - 4)^{-3} \). Although \( f''(x) \) changes sign at \( x = 4 \), \( f \) does not have a point of inflection at \( x = 4 \). Why not?

Exercises
1. Determine the sign combinations of \( f' \) and \( f'' \) for each interval \( A \rightarrow G \) in Figure 15.

2. State the sign change at each transition point \( A \rightarrow G \) in Figure 16. Example: \( f'(x) \) goes from \( + \) to \( - \) at \( A \).

3. \( ++, +-, -- \)
4. \( +-, --, + \)
5. \( --, +, + \)
6. \( +, ++, + \)

7. Sketch the graph of a function that could have the graphs of \( f' \) and \( f'' \) appearing in Figure 17.

8. Sketch the graph of a function that could have the graphs of \( f' \) and \( f'' \) appearing in Figure 18.

9. Investigate the behavior and sketch the graph of \( y = x^2 - 5x + 4 \).
10. Investigate the behavior and sketch the graph of \( y = 12 - 5x - 2x^2 \).
11. Investigate the behavior and sketch the graph of \( f(x) = x^3 - 3x^2 + 2 \). Include the zeros of \( f \), which are \( x = 1 \) and \( 1 \pm \sqrt{3} \) (approximately \(-0.73, 2.73\)).
12. Show that \( f(x) = x^3 - 2x^2 + 6x \) has a point of inflection but no local extreme values. Sketch the graph.
13. Extend the sketch of the graph of \( f(x) = \cos x + \frac{1}{4}x \) in Example 4 to the interval \([0, 5\pi]\).
14. Investigate the behavior and sketch the graphs of \( y = x^{2/3} \) and \( y = x^{4/3} \).

In Exercises 15–36, find the transition points, intervals of increase/decrease, concavity, and asymptotic behavior. Then sketch the graph, with this information indicated.

15. \( y = x^3 + 24x^2 \)
16. \( y = x^3 - 3x + 5 \)
17. \( y = x^2 - 4x^3 \)
18. \( y = \frac{1}{4}x^3 + 2x^2 + 2x \)
19. \( y = 4 - 2x^2 + \frac{1}{2}x \)
20. \( y = 7x^2 - 6x^2 + 1 \)
21. \( y = x^4 + 5x \)
22. \( y = x^2 - 15x \)
23. \( y = x^2 - 3x^3 + 4x \)
24. \( y = x^2(x - 3)^2 \)
25. \( y = x^7 - 14x^6 \)
26. \( y = x^6 - 9x^4 \)
27. \( y = x - 4\sqrt{x} \)
28. \( y = \sqrt{x} + \sqrt{15 - x} \)
29. \( y = x(8 - x)^{1/3} \)
30. \( y = (x^2 - 4x)^{1/3} \)
31. \( y = (2x - x^2)^{1/3} \)
32. \( y = (x^3 - 3x)^{1/3} \)
33. \( y = x - x^{-1} \)
34. \( y = x^2 - x^{-2} \)
35. \( y = x^3 - 48/x^2 \)
36. \( y = x^2 - x + x^{-1} \)
37. Investigate the behavior and sketch the graph of the function \( f(x) = \frac{(x-3)(x-1)^{2/3}}{18} \) using the formulas
   \[ f'(x) = \frac{50(x - \frac{3}{5})}{(x - 1)^{4/3}}, \quad f''(x) = \frac{20(x - \frac{3}{5})}{(x - 1)^{5/3}} \]
38. Investigate the behavior and sketch the graph of \( f(x) = \frac{x}{x^2 + 1} \) using the formulas
   \[ f'(x) = \frac{1 - x^2}{(1 + x^2)^2}, \quad f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3} \]

[CAS] In Exercises 39–42, sketch the graph of the function, indicating all transition points. If necessary, use a graphing utility or computer algebra system to locate the transition points numerically.

39. \( y = x^3 - \frac{4}{x^2 + 1} \)
40. \( y = 12\sqrt{x^2 + 2x + 4} - x^2 \)
41. \( y = x^4 - 4x^2 + x + 1 \)
42. \( y = 2\sqrt{x} - \sin x, \quad 0 \leq x \leq 2\pi \)

In Exercises 43–48, sketch the graph over the given interval, with all transition points indicated.
43. \( y = x + \sin x, \quad [0, 2\pi] \)
44. \( y = \sin x + \cos x, \quad [0, 2\pi] \)
45. \( y = 2\sin x - \cos^2 x, \quad [0, 2\pi] \)
46. \( y = \sin x + \frac{1}{2}x, \quad [0, 2\pi] \)
47. \( y = \sin x + \sqrt{3}\cos x, \quad [0, \pi] \)
48. \( y = \sin x - \frac{1}{2}\sin 2x, \quad [0, \pi] \)

49. Are all sign transitions possible? Explain with a sketch why the transitions \(++ \rightarrow ++ \) and \(-- \rightarrow -- \) do not occur if the function is differentiable. (See Exercise 78 for a proof.)

50. Suppose that \( f \) is twice differentiable satisfying (i) \( f(0) = 1 \), (ii) \( f'(x) > 0 \) for all \( x \neq 0 \), and (iii) \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \). Let \( g(x) = f(x^2) \).
   (a) Sketch a possible graph of \( f \).
   (b) Prove that \( g \) has no points of inflection and a unique local extreme value at \( x = 0 \). Sketch a possible graph of \( g \).

In Exercises 51–52, draw the graph of a function \( f \) having the given limits at \( \pm \infty \) and for which \( f'(x) \) and \( f''(x) \) take on the given sign combinations in order.
51. \( \lim_{x \to \infty} f(x) = -\infty, \lim_{x \to \infty} f'(x) = 0, \quad ++, --, --, ++, -- \)
52. \( \lim_{x \to \infty} f(x) = -1, \lim_{x \to \infty} f'(x) = 1, \quad ++, --, --, ++ \)

53. Match the graphs in Figure 19 with the two functions \( y = \frac{3x}{x^2 - 1} \) and \( y = \frac{x^2}{x^2 - 1} \). Explain.

54. Match the functions below with their graphs in Figure 20.
   (a) \( y = \frac{1}{x^2 - 1} \)
   (b) \( y = \frac{x^2}{x^2 + 1} \)
   (c) \( y = \frac{1}{x^2 + 1} \)
   (d) \( y = \frac{x}{x^2 - 1} \)

55. \( y = \frac{x - 2}{x^2 - 3} \)
56. \( y = \frac{x}{x - 3} \)
57. \( y = \frac{x + 3}{x^2 - 2} \)
58. \( y = \frac{x}{x^2 - 1} \)
59. \( y = \frac{1}{x^2 - 1} \)
60. \( y = \frac{1}{x^2 - 1} \)
61. \( y = \frac{1}{x(x - 2)} \)
62. \( y = \frac{1}{x^2 - 9} \)
63. \( y = \frac{1}{x^2 - 6x + 8} \)
64. \( y = \frac{x^2 + 1}{x} \)
65. \( y = \frac{1}{x^2 + 4x + 3} \)
66. \( y = \frac{1}{x^2 + 1} \)
67. \( y = \frac{1}{x^2 - (x - 2)^2} \)
68. \( y = \frac{1}{x^2 - 9} \)
69. \( y = \frac{1}{(x^2 + 1)^2} \)
70. \( y = \frac{1}{(x^2 - 1)(x^2 + 1)} \)
71. \( y = \frac{1}{x^2 + 1} \)
72. \( y = \frac{1}{x^2 + 1} \)
Further Insights and Challenges

In Exercises 73–77, we explore functions whose graphs approach a non-horizontal line as $x \to \infty$. A line $y = ax + b$ is called a slant asymptote if

$$
\lim_{x \to \infty} (f(x) - (ax + b)) = 0
$$
or

$$
\lim_{x \to -\infty} (f(x) - (ax + b)) = 0
$$

73. Let $f(x) = \frac{x^2}{x - 1}$ (Figure 21). Verify the following:

(a) $f(0)$ is a local max and $f(2)$ a local min.

(b) $f$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

(c) $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to 1^+} f(x) = \infty$.

(d) $y = x + 1$ is a slant asymptote of $f$ as $x \to -\infty$.

(e) The slant asymptote lies above the graph of $f$ for $x < 1$ and below the graph for $x > 1$.

74. If $f(x) = P(x)/Q(x)$, where $P$ and $Q$ are polynomials of degrees $m + 1$ and $n$, then by long division, we can write

$$
f(x) = (ax + b) + P_1(x)/Q(x)
$$

where $P_1$ is a polynomial of degree $< m$. Show that $y = ax + b$ is the slant asymptote of $f(x)$. Use this procedure to find the slant asymptotes of the following functions:

(a) $y = \frac{x^2}{x + 2}$

(b) $y = \frac{x^3 + x}{x^2 + x + 1}$

75. Sketch the graph of

$$
f(x) = \frac{x^2}{x + 1}
$$

Proceed as in the previous exercise to find the slant asymptote.

76. Show that $y = 3x$ is a slant asymptote for $f(x) = 3x + x^{-2}$. Determine whether $f(x)$ approaches the slant asymptote from above or below, and make a sketch of the graph.

77. Sketch the graph of $f(x) = \frac{1 - x^2}{2 - x}$.

78. Assume that $f'$ and $f''$ exist for all $x$ and let $c$ be a critical point of $f$. Show that $f(x)$ cannot make a transition from $++$ to $-+$ at $x = c$. Hint: Apply the MVT to $f'(x)$.

79. Assume that $f''$ exists and $f''(x) > 0$ for all $x$. Show that $f(x)$ cannot be negative for all $x$. Hint: Show that $f'(b) \neq 0$ for some $b$ and use the result of Exercise 62 in Section 4.4.

4.6 Applied Optimization

Optimization plays a role in a wide range of disciplines, including the physical sciences, economics, and biology. For example, scientists have studied how migrating birds choose an optimal velocity $v$ that maximizes the distance $D$ they can travel without stopping, given the energy that can be stored as body fat (Figure 1).

In many optimization problems, the first step is to write down the **objective function**. This is the function whose minimum or maximum we seek. Once we find the objective function, we can apply the techniques developed in this chapter. Our first examples require optimization on a closed interval $[a, b]$. Let's recall the steps for finding extrema developed in Section 4.2:

(i) Find the critical points of $f$ in $[a, b]$.

(ii) Evaluate $f(x)$ at the critical points and the endpoints $a$ and $b$.

(iii) The least and greatest values are the extreme values of $f$ on $[a, b]$.

**EXAMPLE 1** A piece of wire of length $L$ is bent into the shape of a rectangle (Figure 2). Which dimensions produce the rectangle of maximum area?
Solution The rectangle has area \( A = xy \), where \( x \) and \( y \) are the lengths of the sides. Since \( A \) depends on two variables \( x \) and \( y \), we cannot find the maximum until we eliminate one of the variables. We can do this because the variables are related: The rectangle has perimeter \( L = 2x + 2y \), so \( y = \frac{1}{2}L - x \). This allows us to rewrite the area in terms of \( x \) alone to obtain the objective function

\[
A(x) = x \left( \frac{1}{2}L - x \right) = \frac{1}{2}Lx - x^2
\]

On which interval does the optimization take place? The sides of the rectangle are non-negative, so we require both \( x \geq 0 \) and \( \frac{1}{2}L - x \geq 0 \). Thus, \( 0 \leq x \leq \frac{1}{2}L \). Our problem is to maximize \( A(x) \) on the closed interval \([0, \frac{1}{2}L]\).

We have \( A'(x) = \frac{1}{2}L - 2x \). Solving \( A'(x) = 0 \), we obtain just a single critical point, \( x = \frac{1}{4}L \). Comparing values of \( A \), we find:

Endpoints:

\[
A(0) = 0
\]

\[
A \left( \frac{1}{2}L \right) = \frac{1}{2}L \left( \frac{1}{2}L - \frac{1}{2}L \right) = 0
\]

Critical point:

\[
A \left( \frac{1}{4}L \right) = \left( \frac{1}{4}L \right) \left( \frac{1}{2}L - \frac{1}{4}L \right) = \frac{1}{16}L^2
\]

The greatest value occurs for \( x = \frac{1}{4}L \), and in this case, \( y = \frac{1}{2}L - \frac{1}{4}L = \frac{1}{4}L \). The rectangle of maximum area is the square of sides \( x = y = \frac{1}{4}L \).

**Example 2** Minimizing Travel Time Your task is to build a road joining the small town of Calverton to Route 1 to enable drivers to reach Capital City in the shortest time (Figure 3). How should this be done if the speed limit is 60 km/hour on the road and 110 km/h on Route 1? The perpendicular distance from Calverton to Route 1 is 30 km, and Capital City is 50 km down Route 1.

Solution We will solve this problem in three steps. These steps can be helpful when solving other optimization problems.

**Step 1. Choose variables.**

We need to determine the point \( Q \) where the road will join the Route 1. So let \( x \) be the distance from \( Q \) to the point \( P \) where the perpendicular joins Route 1.

**Step 2. Find the objective function and the interval.**

Our objective function is the time \( T(x) \) of the trip as a function of \( x \). To find a formula for \( T(x) \), recall that distance traveled at constant velocity \( v \) is \( d = vt \), and the time required to travel a distance \( d \) is \( t = d/v \). The road has length \( \sqrt{30^2 + x^2} \) by the Pythagorean Theorem, so at velocity \( v = 60 \) km/h, it takes

\[
\frac{\sqrt{30^2 + x^2}}{60} \text{ hours to travel from Calverton to } Q
\]

The segment of Route 1 from \( Q \) to Capital City has length \( 50 - x \). At velocity \( v = 110 \) km/h, it takes

\[
\frac{50 - x}{110} \text{ hours to travel from } Q \text{ to the city}
\]

The total number of hours for the trip is

\[
T(x) = \frac{\sqrt{30^2 + x^2}}{60} + \frac{50 - x}{110}
\]

Our interval is \( 0 \leq x \leq 50 \) because the road joins Route 1 somewhere between \( P \) and Capital City. So our task is to minimize \( T \) on \([0, 50]\) (Figure 4).
Step 2. Find the objective function and the interval.
Since one unit becomes vacant with each $100 increase in rent above $2000, we find that \((r - 2000)/100\) units are vacant when \(r > 2000\). Therefore,
\[
N(r) = 30 - \frac{1}{100}(r - 2000) = 50 - \frac{1}{100}r
\]
Total monthly profit is equal to the number of occupied units times the profit per unit, which is \(r - 200\) (because each unit costs $200 in maintenance), so
\[
P(r) = N(r)(r - 200) = (50 - \frac{1}{100}r)(r - 200) = -10,000 + 52r - \frac{1}{100}r^2
\]
Which interval of \(r\)-values should we consider? There is no reason to lower the rent below \(r = 2000\) because all units are already occupied when \(r = 2000\). On the other hand, for the upper limit of \(r\) we take the rent at which no units are occupied; that is, the \(r\) for which \(N(r) = 0\). That occurs at \(r = 100 \cdot 50 = 5000\). Therefore, we consider \(P(r)\) over the interval \(2000 \leq r \leq 5000\).

Step 3. Optimize.
Solve for the critical points:
\[
P'(r) = 52 - \frac{1}{50}r \quad \text{so} \quad P'(r) = 0 \quad \Rightarrow \quad r = 2600
\]
and compare values at the critical point and the endpoints:
\[
P(2000) = 54,000, \quad P(2600) = 62,400, \quad P(5000) = 0
\]
We conclude that the profit is maximized when the rent is set at \(r = 2600\). In this case, 24 units are occupied. Note that if the maximum profit had occurred at a price that gave us a fractional number of units occupied, we could not have achieved that maximum. Instead, we would have taken the price corresponding to rounding the fractional number up or down to the integer number of units that maximized our profit.

Open Versus Closed Intervals
In contrast to the case of a closed interval, when optimizing a function over an open interval, there is no guarantee that a min or max exists. For example, in Figure 6, a minimum exists at \(x = c\) but there is no maximum value. As we approach the endpoint at \(b\), the function values increase, but there is no maximum because \(b\) is not included in the interval (and furthermore the function is not defined there).

If a min or max does exist on an open interval, then it must occur at a critical point (because it is also a local min or max).

With a closed interval, to search for a min and max, we need to evaluate the function at the endpoints of the interval. With an open interval, we need to examine the behavior of the function as \(x\) approaches the endpoints of the interval in order to make conclusions about the existence (or lack thereof) of max values and min values. For example, if \(f(x)\) tends to infinity at the endpoints, then there is no maximum, and a minimum must occur at a critical point somewhere in the interval. We consider such a situation in the next example.

EXAMPLE 5 Design a cylindrical can of volume 900 cm³ so that it uses the least amount of metal (Figure 7). In other words, minimize the surface area of the can (including its top and bottom).

Solution

Step 1. Choose variables.
We want to find the radius and the height of the can with minimum surface area. Therefore, we let \(r\) be the radius and \(h\) the height. Furthermore, we denote the surface area of the can by \(A\).
Step 2. Find the objective function and the interval.
We express $A$ as a function of $r$ and $h$:

$$A = \pi r^2 + \pi r^2 + 2 \pi rh = 2 \pi r^2 + 2 \pi rh$$

The can's volume is $V = \pi r^2 h$. Since we require that $V = 900$ cm$^3$, we have the constraint equation $\pi r^2 h = 900$. Thus, $h = \left(\frac{900}{\pi r^2}\right)$ and

$$A(r) = 2 \pi r^2 + 2 \pi r \left(\frac{900}{\pi^2 r^2}\right) = 2 \pi r^2 + \frac{1800}{r}$$

The radius $r$ can take on any positive value, so we minimize $A(r)$ on $(0, \infty)$.

Step 3. Optimize the function.
Observe that $A(r)$ tends to infinity as $r$ approaches the endpoints of $(0, \infty)$:

- $A(r) \to \infty$ as $r \to 0$ (because of the $r^2$ term).
- $A(r) \to \infty$ as $r \to 0$ (because of the $1/r$ term).

Therefore, $A(r)$ must take on a minimum value at a critical point in $(0, \infty)$ (Figure 8).

We solve in the usual way:

$$\frac{dA}{dr} = 4 \pi r - \frac{1800}{r^2} = 0 \quad \Rightarrow \quad r^3 = \frac{450}{\pi} \quad \Rightarrow \quad r = \left(\frac{450}{\pi}\right)^{1/3} \approx 5.23 \text{ cm}$$

We also need to calculate the height:

$$h = \frac{900}{\pi r^2} = 2 \left(\frac{450}{\pi}\right)^{1/3} = 2 \left(\frac{450}{\pi}\right)^{2/3} = 2 \left(\frac{450}{\pi}\right)^{1/3} \approx 10.46 \text{ cm}$$

Since we have a single critical point in our interval, it follows that we obtain the minimum of $A$ there. Thus, the minimum surface area occurs when a can has radius approximately 5.23 cm and height approximately 10.46 cm. Notice that the optimal dimensions satisfy $h = 2r$. In other words, the optimal can is as tall as it is wide.

**EXAMPLE 6 Optimization Problem with No Solution**

Is it possible to design a cylinder of volume 900 cm$^3$ with the largest possible surface area?

Solution

The answer is no. In the previous example, we showed that a cylinder of volume 900 cm$^3$ and radius $r$ has surface area

$$A(r) = 2 \pi r^2 + \frac{1800}{r}$$

This function has no maximum value because it tends to infinity as $r \to 0$ or $r \to \infty$ (Figure 8). This means that a cylinder of fixed volume has a large surface area if it is either very fat and short ($r$ large) or very tall and skinny ($r$ small).
The Principle of Least Distance states that a light beam reflected in a mirror travels along the shortest path. More precisely, a beam traveling from A to B, as in Figure 9, is reflected at the point P for which the path APB has minimum length. In the next example, we show that this minimum occurs when the angle of incidence is equal to the angle of reflection, that is, θ₁ = θ₂.

**Example 7** Show that if P is the point for which the path APB in Figure 9 has minimal length, then θ₁ = θ₂.

**Solution** By the Pythagorean Theorem, the path APB has length

\[ f(x) = AP + PB = \sqrt{x^2 + h_1^2} + \sqrt{(L-x)^2 + h_2^2} \]

with x, h₁, and h₂ as in the figure. The function f is defined for all x and tends to infinity as x approaches ±oo (i.e., as P moves arbitrarily far to the right or left). It follows that f has an absolute minimum value, and it must occur at a critical point (see Figure 10). Taking the derivative:

\[ f'(x) = \frac{x}{\sqrt{x^2 + h_1^2}} - \frac{L-x}{\sqrt{(L-x)^2 + h_2^2}} \]

Since f'(x) is defined for all x, critical points occur where f'(x) = 0. It is not necessary to solve for x because our goal is not to find critical points, but rather to show that θ₁ = θ₂ at the minimum. To do this, we set the derivative equal to 0 in Eq. (1) and rewrite as

\[ \frac{x}{\sqrt{x^2 + h_1^2}} = \frac{L-x}{\sqrt{(L-x)^2 + h_2^2}} \]

Note that the critical point x that satisfies Eq. (2) must lie between 0 and L because no x < 0 can satisfy this equation (otherwise, we would have a negative value on the left and a positive on the right) and no x > L can satisfy this equation (for similar reasons). Since the critical point x lies in [0, L] we can associate angles θ₁ and θ₂ with x as in Figure 9. We claim that θ₁ = θ₂. To see this, observe that with θ₁ and θ₂ as pictured, we have

\[ \cos θ₁ = \frac{x}{\sqrt{x^2 + h_1^2}} \quad \text{and} \quad \cos θ₂ = \frac{L-x}{\sqrt{(L-x)^2 + h_2^2}} \]

Therefore, Eq. (2) implies that \( \cos θ₁ = \cos θ₂ \), and since θ₁ and θ₂ lie between 0 and \( \frac{π}{2} \), we conclude that θ₁ = θ₂ as claimed.

**Conceptual Insight** Often, a maximum or minimum at a critical point represents the best compromise between "competing factors." In Example 4, we maximized profit by finding the best compromise between raising the rent and keeping the apartment units occupied. In Example 5, our solution minimizes surface area by finding the best compromise between height and radius. In Example 2, the solution represents a compromise between the slower speed on the road that leads to Route 1 and the faster speed along Route 1. On the other hand, in Example 3, since there is no compromise, a solution occurs at an endpoint of the interval rather than at a critical point. The faster speed along the road yields a road straight to the city, avoiding Route 1 altogether.
4.6 SUMMARY

- There are usually three main steps in solving an applied optimization problem:
  
  Step 1. Choose variables.
  
  Determine which quantities are relevant, often by drawing a diagram, and assign appropriate variables.
  
  Step 2. Find the objective function and the interval.
  
  Restate as an optimization problem for a function \( f \) over an interval. If \( f \) depends on more than one variable, use a constraint equation to write \( f \) as a function of just one variable.
  
  Step 3. Optimize the objective function.
  
- If the interval is open, \( f \) does not necessarily take on a minimum or maximum value. But if it does, these must occur at critical points within the interval. To determine if a min or max exists, analyze the behavior of \( f \) as \( x \) approaches the endpoints of the interval.

4.6 EXERCISES

Preliminary Questions

1. The problem is to find the right triangle of perimeter 10 whose area is as large as possible. What is the constraint equation relating the base \( b \) and height \( h \) of the triangle?

2. Describe a way of showing that a continuous function on an open interval \((a, b)\) has a minimum value.

3. Is there a rectangle of area 100 of largest perimeter? Explain.

Exercises

1. Find the dimensions \( x \) and \( y \) of the rectangle of maximum area that can be formed using 3 m of wire.
   
   (a) What is the constraint equation relating \( x \) and \( y \)?
   
   (b) Find a formula for the area in terms of \( x \) alone.
   
   (c) What is the interval of optimization? Is it open or closed?
   
   (d) Solve the optimization problem.

2. Wire of length 12 m is divided into two pieces and each piece is bent into a square. How should this be done in order to minimize the sum of the areas of the two squares?
   
   (a) Express the sum of the areas of the squares in terms of the lengths \( x \) and \( y \) of the two pieces.
   
   (b) What is the constraint equation relating \( x \) and \( y \)?
   
   (c) What is the interval of optimization? Is it open or closed?
   
   (d) Solve the optimization problem.

3. A rectangular bird sanctuary is being created on one side along a straight riverbank. The remaining three sides are to be enclosed with a protective fence. If there are 12 km of fence available, find the dimension of the rectangle to maximize the area of the sanctuary.

4. The rectangular bird sanctuary with one side along a straight river is to be constructed so that it contains 8 km² of area. Find the dimensions of the rectangle to minimize the amount of fence necessary to enclose the remaining three sides.

5. Find two positive real numbers such that the sum of the first number squared and the second number is 48 and their product is a maximum.

6. Find two positive real numbers such that they sum to 108 and the product of the first times the square of the second is a maximum.

7. A wire of length 12 m is divided into two pieces and the pieces are bent into a square and a circle. How should this be done in order to minimize the sum of their areas?

8. Find the positive number \( x \) such that the sum of \( x \) and its reciprocal is as small as possible. Does this problem require optimization over an open interval or a closed interval?

9. Find two positive real numbers such that they add to 40 and their product is as large as possible.

10. Find two positive real numbers \( x \) and \( y \) such that they add to 120 and \( x^2 y \) is as large as possible.

11. Find two positive real numbers \( x \) and \( y \) such that their product is 800 and \( x + 2y \) is as small as possible.

12. A flexible tube of length 4 m is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

13. Find the dimensions of the box with square base with
   
   (a) Volume 12 and the minimal surface area.
   
   (b) Surface area 20 and maximal volume.

14. A jewelry box with a square base is to be built with copper-plated sides, nickel-plated bottom and top, and a volume of 40 cm³. If nickel plating costs $2 per cm² and copper plating costs $1 per cm², find the dimensions of the box to minimize the cost of the materials.

15. A rancher will use 600 m of fencing to build a corral in the shape of a semicircle on top of a rectangle (Figure 11). Find the dimensions that maximize the area of the corral.

---

**FIGURE 11**
16. What is the maximum area of a rectangle inscribed in a right triangle with legs of length 3 and 4 as in Figure 12? The sides of the rectangle are parallel to the legs of the triangle.

![Figure 12](image)

17. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius \( r = 4 \) (Figure 13).

![Figure 13](image)

18. Find the dimensions \( x \) and \( y \) of the rectangle inscribed in a circle of radius \( r \) that maximizes the quantity \( xy^2 \).

19. In the setting of Examples 2 and 3, let \( r \) denote the speed along the road, and \( h \) denote the speed along the highway.

(a) Show that the travel-time function \( T(x) \) has a critical point at

\[
x = \frac{30}{\sqrt{(h/r)^2 - 1}}
\]

and explain why this indicates that if \( r \geq h \) there is no critical point.

(b) Explain why there cannot be a critical point at \( x = 0 \), but depending on the speeds, the critical point can be arbitrarily close to 0.

20. In the setting of Examples 2 and 3, replace 30 and 50 with general distances \( D \) and \( L \), respectively. Also, let \( r \) denote the speed along the road, and \( h \) denote the speed along the highway. Show that the travel-time function \( T(x) \) has a critical point at

\[
x = \frac{D}{\sqrt{(h/r)^2 - 1}}
\]

21. In the article "Do Dogs Know Calculus?" the author Timothy Penning explained how he noticed that when he threw a ball diagonally into Lake Michigan along a straight shoreline, his dog Elvis seemed to pick the optimal point in which to enter the water so as to minimize his time to reach the ball, as in Figure 14. He timed the dog and found Elvis could run at 6.4 m/s on the sand and swim at 0.91 m/s. If Tim stood at point \( A \) and threw the ball to a point \( B \) in the water, which was a perpendicular distance 10 m from point \( C \) on the shore, where \( C \) is a distance 15 m from where he stood, at what distance \( x \) from point \( C \) did Elvis enter the water if the dog effectively minimized his time to reach the ball?

![Figure 14](image)

22. A four-wheel-drive vehicle is transporting an injured hiker to the hospital from a point that is 30 km from the nearest point on a straight road. The hospital is 50 km down that road from that nearest point. If the vehicle can drive at 30 kph over the terrain and at 120 kph on the road, how far down the road should the vehicle aim to reach the road to minimize the time it takes to reach the hospital?

23. Find the point on the line \( y = x \) closest to the point \((1, 0)\). Hint: It is equivalent and easier to minimize the square of the distance.

24. Find the point \( P \) on the parabola \( y = x^2 \) closest to the point \((3, 0)\) (Figure 15).

![Figure 15](image)

25. Find the coordinates of the point on the graph of \( y = x + 2x^{-1} \) closest to the origin in the region \( x > 0 \) (Figure 16).

![Figure 16](image)

26. Problem of Tartaglia (1500–1557) Among all positive numbers \( a, b \) whose sum is 8, find those for which the product of the two numbers and their difference is largest.
27. Find the angle \( \theta \) that maximizes the area of the isosceles triangle whose legs have length \( l \) (Figure 17), using the fact the area is given by \( A = \frac{1}{2} l^2 \sin \theta \).

![Figure 17](image)

28. A right circular cone (Figure 18) has volume

\[
V = \frac{\pi}{3} r^2 h
\]

and surface area \( S = \pi r \sqrt{r^2 + h^2} \). Find the dimensions of the cone with surface area 1 and maximal volume.

![Figure 18](image)

29. Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 1 (Figure 19).

![Figure 19](image)

30. Find the radius and height of a cylindrical can of total surface area \( A \) whose volume is as large as possible. Does there exist a cylinder of surface area \( A \) and minimal total volume?

31. A poster of area 6000 cm\(^2\) has blank margins of width 10 cm on the top and bottom and 6 cm on the sides. Find the dimensions that maximize the printed area.

32. According to postal regulations, a carton is classified as "oversized" if the sum of its height and girth (perimeter of its base) exceeds 108 in. Find the dimensions of a carton with a square base that is not oversized and has maximum volume.

33. **Kepler's Wine Barrel Problem** In his work *Nova stereometria doliorum vinariaeum* (New Solid Geometry of a Wine Barrel), published in 1615, astronomer Johannes Kepler stated and solved the following problem: Find the dimensions of the cylinder of largest volume that can be inscribed in a sphere of radius \( R \). *Hint:* Show that an inscribed cylinder has volume \( 2\pi x (R^2 - x^2) \), where \( x \) is one-half the height of the cylinder.

34. Find the angle \( \theta \) that maximizes the area of the trapezoid with a base of length 4 and sides of length 2, as in Figure 20.

![Figure 20](image)

35. A landscape architect wishes to enclose a rectangular garden of area 1000 m\(^2\) on one side by a brick wall costing $90/m and on the other three sides by a metal fence costing $30/m. Which dimensions minimize the total cost?

36. The amount of light reaching a point at a distance \( r \) from a light source \( A \) of intensity \( I_A \) is \( I_A/r^2 \). Suppose that a second light source \( B \) of intensity \( I_B = 4I_A \) is located 10 m from \( A \). Find the point on the segment joining \( A \) and \( B \) where the total amount of light is at a minimum.

37. Find the maximum area of a rectangle inscribed in the region bounded by the graph of \( y = \frac{4-x}{2+x} \) and the axes (Figure 21).

![Figure 21](image)

38. Find the maximum area of a triangle formed by the axes and a tangent line to the graph of \( y = (x + 1)^{-2} \) with \( x > 0 \).

39. Find the maximum area of a rectangle circumscribed around a rectangle of sides \( L \) and \( H \). *Hint:* Express the area in terms of the angle \( \theta \) (Figure 22).

![Figure 22](image)

40. A contractor is engaged to build steps up the slope of a hill that has the shape of the graph of \( y = \frac{x^2(120-x)}{6400} \) for \( 0 \leq x \leq 80 \) with \( x \) in meters
41. Find the equation of the line through $P = (4, 12)$ such that the triangle bounded by this line and the axes in the first quadrant has minimal area.

42. Let $P = (a, b)$ lie in the first quadrant. Find the slope of the line through $P$ such that the triangle bounded by this line and the axes in the first quadrant has minimal area. Then show that $P$ is the midpoint of the hypotenuse of this triangle.

43. Archimedes's Problem A spherical cap (Figure 24) of radius $r$ and height $h$ has volume $V = \frac{\pi}{6} r^2 (3h - \frac{1}{4}h^2)$ and surface area $S = 2\pi rh$. Prove that the hemisphere encloses the largest volume among all spherical caps of fixed surface area $S$.

44. Find the isosceles triangle of smallest area (Figure 25) that circumscribes a circle of radius 1 (from Thomas Simpson's The Doctrine and Application of Fluxions, a calculus text that appeared in 1750).

45. A box of volume 72 m$^3$ with a square bottom and no top is constructed out of two different materials. The cost of the bottom is $40/m^2$ and the cost of the sides is $30/m^2$. Find the dimensions of the box that minimize total cost.

46. Find the dimensions of a cylinder of volume 1 m$^3$ of minimal cost if the top and bottom are made of material that costs twice as much as the material for the side.

47. Your task is to design a rectangular industrial warehouse consisting of three separate spaces of equal size as in Figure 26. The wall materials cost $500 per linear meter and your company allocates $2,400,000 for that part of the project involving the walls.

(a) Which dimensions maximize the area of the warehouse?

(b) What is the area of each compartment in this case?

48. Suppose, in the previous exercise, that the warehouse consists of $n$ separate spaces of equal size. Find a formula in terms of $n$ for the maximum possible area of the warehouse.

49. According to a model developed by economists E. Headley and J. Pesek, if fertilizer made from $N$ pounds of nitrogen and $P$ lb of phosphate is used on an acre of farmland, then the yield of corn (in bushels per acre) is

$$Y = 7.5 + 0.6N + 0.7P - 0.001N^2 - 0.002P^2 + 0.001NP$$

A farmer intends to spend $30/acre on fertilizer. If nitrogen costs 25 cents/lb and phosphate costs 20 cents/lb, which combination of $N$ and $P$ produces the highest yield of corn?

50. Experiments show that the quantities $x$ of corn and $y$ of soybean required to produce a hog of weight $Q$ satisfy $Q = 0.5x^{1/4}y^{1/4}$. The unit of $x$, $y$, and $Q$ is the cwt, an agricultural unit equal to 100 lb. Find the values of $x$ and $y$ that minimize the cost of a hog of weight $Q = 2.5$ cwt if corn costs $3/cwt and soy costs $7/cwt.

51. All units in a 100-unit apartment building are rented out when the monthly rent is set at $r = 800/month. Suppose that one unit becomes vacant with each $10 increase in rent and that each occupied unit incurs $60/month in maintenance. Which rent $r$ maximizes monthly profit?

52. An 8-billion-bushel corn crop brings a price of $2.40/bushel. A commodity broker uses the rule of thumb: If the crop is reduced by $x$ percent, then the price increases by $10x$ cents. Which crop size results in maximum revenue and what is the price per bushel? Hint: Revenue is equal to price times crop size.

53. The monthly output of a Spanish light bulb factory is $P = 2L^2K^2$ (in millions), where $L$ is the cost of labor and $K$ is the cost of equipment (in millions of euros). The company needs to produce 1.7 million units per month. Which values of $L$ and $K$ would minimize the total cost $L + K$?

54. The rectangular plot in Figure 27 has size 100 m $\times$ 200 m. Pipe is to be laid from $A$ to a point $P$ on side $BC$ and from there to $C$. The cost of laying pipe along the side of the plot is $45/m and the cost through the plot is $80/m (since it is underground). (a) Let $f(x)$ be the total cost, where $x$ is the distance from $P$ to $B$. Determine $f(x)$, but note that $f$ is discontinuous at $x = 0$ (when $x = 0$, the cost of the entire pipe is $45/m).

(b) What is the most economical way to lay the pipe? What if the cost along the sides is $65/m?"
55. Brandon is on one side of a river that is 30 m wide and wants to reach a point 200 m downstream on the opposite side as quickly as possible by swimming diagonally across the river and then running the rest of the way. Find the best route if Brandon can swim at 1.5 m/s and run at 4 m/s.

56. Snell's Law
When a light beam travels from a point \( A \) above a swimming pool to a point \( B \) below the water (Figure 28), it chooses the path that takes the least time. Let \( v_1 \) be the velocity of light in air and \( v_2 \) the velocity in water (it is known that \( v_1 > v_2 \)). Prove Snell's Law of Refraction:

\[
\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}
\]

![Figure 28](image)

57. Vascular Branching
A small blood vessel of radius \( r \) branches off at an angle \( \theta \) from a larger vessel of radius \( R \) to supply blood along a path from \( A \) to \( B \). According to Poiseuille's Law, the total resistance to blood flow is proportional to

\[
T = \left( \frac{a - b \cot \theta}{R^4} \right) + \frac{b \csc \theta}{R^4}
\]

where \( a \) and \( b \) are as in Figure 29. Show that the total resistance is minimized when \( \cos \theta = (r/R)^2 \).

![Figure 29](image)

In Exercises 58–59, a box (with no top) is to be constructed from a piece of cardboard with sides of lengths \( A \) and \( B \) by cutting out squares of length \( h \) from the corners and folding up the sides (Figure 30).

58. Find the value of \( h \) that maximizes the volume of the box if \( A = 15 \) and \( B = 24 \). What are the dimensions of this box?

59. Which values of \( A \) and \( B \) maximize the volume of the box if \( h = 10 \) cm and \( AB = 900 \) cm²?

![Figure 30](image)

60. Which value of \( h \) maximizes the volume of the box if \( A = B \)?

61. Given \( n \) numbers \( x_1, \ldots, x_n \), find the value of \( x \) minimizing the sum of the squares:

\[
(x - x_1)^2 + (x - x_2)^2 + \cdots + (x - x_n)^2
\]

First, solve for \( n = 2, 3 \) and then try it for arbitrary \( n \).

62. A billboard of height \( h \) is mounted on the side of a building with its bottom edge at a distance \( x \) from the street as in Figure 31. At what distance \( x \) should an observer stand from the wall to maximize the angle of observation \( \theta \)?

63. Solve Exercise 62 again using geometry rather than calculus. There is a unique circle passing through points \( B \) and \( C \) that is tangent to the street. Let \( R \) be the point of tangency. Note that the two angles labeled \( \psi \) in Figure 31 are equal because they subtend equal arcs on the circle.

(a) Show that the maximum value of \( \theta \) is \( \psi \). Hint: Show that \( \psi = \theta + \angle PBA \), where \( A \) is the intersection of the circle with \( PC \).

(b) Prove that this agrees with the answer to Exercise 62.

(c) Show that \( LQRB = LRCQ \) for the maximal angle \( \psi \).

![Figure 31](image)

64. Optimal Delivery Schedule
A gas station sells \( Q \) gallons of gasoline per year, which is delivered \( N \) times per year in equal shipments of \( Q/N \) gallons. The cost of each delivery is \( d \) dollars and the yearly storage costs are \( aT \), where \( T \) is the length of time (a fraction of a year) between shipments and \( s \) is a constant. Show that costs are minimized for \( N = \sqrt{aT/d} \). (Hint: \( T = 1/N \).) Find the optimal number of deliveries if \( Q = 2 \) million gal, \( d = \$8000 \), and \( s = 30 \) cents/gal-year. Your answer should be a whole number, so compare costs for the two integer values of \( N \) nearest the optimal value.

65. Victor Klee's Endpoint Maximum Problem
Given 40 m of straight fence, your goal is to build a rectangular enclosure using 80 additional meters of fence that encompasses the greatest area. Let \( A(x) \) be the area of the enclosure, with \( x \) as in Figure 32.

(a) Find the maximum value of \( A(x) \).

(b) Which interval of \( x \) values is relevant to our problem? Find the maximum value of \( A(x) \) on this interval.

![Figure 32](image)
66. Let \((a, b)\) be a fixed point in the first quadrant and let \(S(d)\) be the sum of the distances from \((d, 0)\) to the points \((0, 0)\), \((a, b)\), and \((a, -b)\).
   (a) Find the value of \(d\) for which \(S(d)\) is minimal. The answer depends on whether \(b < \sqrt{3}a\) or \(b \geq \sqrt{3}a\). Hint: Show that \(d = 0\) when \(b \geq \sqrt{3}a\).
   (b) \(\text{GU}\) Let \(a = 1\). Plot \(S\) for \(b = 0.5, \sqrt{3}, 3\) and describe the position of the minimum.

67. The force \(F\) (in Newtons) required to move a box of mass \(m\) kg in motion by pulling on an attached rope (Figure 33) is
   \[
   F(\theta) = \frac{fmg}{\cos \theta + f \sin \theta}
   \]
   where \(\theta\) is the angle between the rope and the horizontal, \(f\) is the coefficient of static friction, and \(g = 9.8\) m/s\(^2\). Find the angle \(\theta\) that minimizes the required force \(F\), assuming \(f = 0.4\). Hint: Find the maximum value of \(\cos \theta + f \sin \theta\).

70. The problem is to put a "roof" of side \(s\) on an attic room of height \(h\) and width \(b\). Find the smallest length \(x\) for which this is possible if \(b = 27\) and \(h = 8\) (Figure 35).

71. Redo Exercise 70 for arbitrary \(b\) and \(h\).

72. Find the maximum length of a pole that can be carried horizontally around a corner joining corridors of widths \(a = 24\) and \(b = 3\) (Figure 36).

73. Redo Exercise 72 for arbitrary widths \(a\) and \(b\).

74. Find the minimum length \(d\) of a beam that can clear a fence of height \(h\) and touch a wall located \(b\) ft behind the fence (Figure 37).

68. In the setting of Exercise 67, show that for any \(f\) the minimal force required is proportional to \(1/\sqrt{1 + f^2}\).

69. Bird Migration Ornithologists have found that the power (in joules per second) consumed by a certain pigeon flying at velocity \(v\) m/s is described well by the function \(P(v) = 170v^{-1} + 10^3v^2\) joules/s. Assume that the pigeon can store \(5 \times 10^3\) joules of usable energy as body fat.
   (a) Show that at velocity \(v\), a pigeon can fly a total distance of \(D(u) = (5 \times 10^3)/P(v)\) if it uses all of its stored energy.
   (b) Find the velocity \(u\) that minimizes \(P\).
   (c) Migrating birds are smart enough to fly at the velocity that maximizes distance traveled rather than minimizes power consumption. Show that the velocity \(u\) which maximizes \(D(u)\) satisfies \(P'(u)u = P(u)/u\). Show that \(u\) is obtained graphically as the velocity coordinate of the point where a line through the origin is tangent to the graph of \(P\) (Figure 34).
   (d) Find \(u\) and the maximum distance \(D(u)\).

75. A basketball player stands \(d\) feet from the basket. Let \(h\) and \(\alpha\) be as in Figure 38. Using physics, one can show that if the player releases the ball at an angle \(\theta\), then the initial velocity required to make the ball go through the basket satisfies
   \[
   v^2 = \frac{16d}{\cos^2 \theta (\tan \theta - \tan \alpha)}
   \]
   (a) Explain why this formula is meaningful only for \(\alpha < \theta < \frac{\pi}{2}\). Why does \(v\) approach infinity at the endpoints of this interval?
   (b) \(\text{GU}\) Take \(\alpha = \frac{\pi}{6}\) and plot \(v^2\) as a function of \(\theta\) for \(\frac{\pi}{6} < \theta < \frac{\pi}{2}\). Verify that the minimum occurs at \(\theta = \frac{\pi}{4}\).
   (c) Set \(F(\theta) = \cos^2 \theta (\tan \theta - \tan \alpha)\). Explain why \(v\) is minimized for \(\theta\) such that \(F(\theta)\) is maximized.
   (d) Verify that \(F'(\theta) = \cos(\alpha - 2\theta) \sec \alpha\) (you will need to use the addition formula for cosine) and show that the maximum value of \(F\) on \([\alpha, \frac{\pi}{2}]\) occurs at \(\theta_0 = \frac{\pi}{4} + \frac{\alpha}{2}\).
   (e) For a given \(u\), the optimal angle for shooting the basket is \(\theta_0\) because it minimizes \(v^2\) and therefore minimizes the energy required to make the shot (energy is proportional to \(v^2\)). Show that the velocity \(v_{opt}\) at the optimal angle \(\theta_0\) satisfies
   \[
   v_{opt}^2 = \frac{32d \cos \alpha}{1 - \sin \alpha} = \frac{32d^2}{-h + \sqrt{h^2 + k^2}}
   \]
76. Three towns A, B, and C are to be joined by an underground fiber cable as illustrated in Figure 39(A). Assume that C is located directly below the midpoint of AB. Find the junction point P that minimizes the total amount of cable used.

Further Insights and Challenges

77. Tom and Ali drive along a highway represented by the graph of f in Figure 40. During the trip, Ali views a billboard represented by the segment BC along the y-axis. Let Q be the y-intercept of the tangent line to \( y = f(x) \). Show that \( \theta \) is maximized at the value of x for which the angles \( \angle QPB \) and \( \angle QCP \) are equal. This generalizes Exercise 63(c) which corresponds to the case \( f(x) = 0 \). Hints:
(a) Show that \( \theta = \tan^{-1} \left( \frac{x^2 + (x f'(x))^2}{(b - c) x^2 + c f'(x) x - b f(x) + c f'(x)^2} \right) \)
(b) Show that the y-coordinate of Q is \( f(x) - x f'(x) \).
(c) Show that the condition \( \theta = \tan^{-1} \left( \frac{x^2 + (x f'(x))^2}{(b - c) x^2 + c f'(x) x - b f(x) + c f'(x)^2} \right) = \theta \)
(d) Conclude that \( \triangle QPB \) and \( \triangle QCP \) are similar triangles.

78. (a) Show that the time required for the first pulse to travel from A to D is \( t_1 = \frac{s}{v_1} \).
(b) Show that the time required for the second pulse is
\[
t_2 = \frac{2d}{v_1} \sec \theta + \frac{s - 2d \tan \theta}{v_2}
\]
provided that
\[
\tan \theta \leq \frac{s}{2d}
\]
(Note: If this inequality is not satisfied, then point B does not lie to the left of C.)
(c) Show that \( t_2 \) is minimized when \( \sin \theta = \frac{v_1}{v_2} \).

79. In this exercise, assume that \( v_2 / v_1 \geq \sqrt{1 + 4d(x)} \).
(a) Show that inequality (3) holds if \( v_1 = v_2 \).
(b) Show that the minimal time for the second pulse is
\[
t_2 = \frac{2d}{v_1} \left( 1 - k^2 \right)^{1/2} + \frac{s}{v_2}
\]
where \( k = \frac{v_1}{v_2} \).
(c) Conclude that
\[
\frac{t_2}{t_1} = \frac{2d(1 - k^2)^{1/2} + s}{s} + k.
\]

80. Continue with the assumption of the previous exercise.
(a) Find the thickness of the soil layer, assuming that \( v_1 = 0.7v_2 \), \( t_2 / t_1 = 1.3 \), and \( s = 400 \) m.
(b) The times \( t_1 \) and \( t_2 \) are measured experimentally. The equation in Exercise 79(c) shows that \( t_2 / t_1 = 1 \) is a linear function of \( 1/k \). What might you conclude if experiments were formed for several values of \( s \) and the points \( (1/s, t_2 / t_1) \) did not lie on a straight line?
81. In this exercise, we use Figure 42 to prove Heron's principle of Example 7 without calculus. By definition, $C$ is the reflection of $B$ across the line $MN$ (so that $BC$ is perpendicular to $MN$ and $BN = CN$). Let $P$ be the intersection of $AC$ and $MN$. Use geometry to justify the following:

(a) $\triangle PNB$ and $\triangle PNC$ are congruent and $\theta_1 = \theta_2$.
(b) The paths $APB$ and $APC$ have equal length.
(c) Similarly, $AQB$ and $AQC$ have equal length.
(d) The path $APC$ is shorter than $AQC$ for all $Q \neq P$.

Conclude that the shortest path $AQB$ occurs for $Q = P$.

82. A jewelry designer plans to incorporate a component made of gold in the shape of a frustum of a cone of height 1 cm and fixed lower radius $r$ (Figure 43). The upper radius $x$ can take on any value between 0 and $r$. Note that $x = 0$ and $x = r$ correspond to a cone and cylinder, respectively. As a function of $x$, the surface area (not including the top and bottom) is $S(x) = \pi s^2 (x + r)$, where $s$ is the slant height as indicated in the figure. Which value of $x$ yields the least expensive design [the minimum value of $S(x)$ for $0 \leq x \leq r$]?

(a) Show that $S(x) = \pi s^2 (x + r)^2$.
(b) Show that if $r > \sqrt{2}$, then $S$ is an increasing function. Conclude that the cone ($x = 0$) has minimal area in this case.
(c) Assume that $r > \sqrt{2}$. Show that $S$ has two critical points $x_1 < x_2$ in $(0, r)$, and that $S(x_1)$ is a local maximum, and $S(x_2)$ is a local minimum.
(d) Conclude that the minimum occurs at $x = 0$ or $x_2$.
(e) Find the minimum in the cases $r = 1.5$ and $r = 2$.

(f) Challenge: Let $c = \sqrt{\left(\frac{5 + 3\sqrt{3}}{4}\right)} \approx 1.597$. Prove that the minimum occurs at $x = 0$ (conter) if $\sqrt{2} < r < c$, but the minimum occurs at $x = x_2$ if $r > c$.

![Figure 43 Frustum of height 1 cm.](image)

### 4.7 Newton's Method

Newton's Method is a procedure for finding numerical approximations to zeros of functions. Numerical approximations are important because it is often impossible to find the zeros exactly. For example, the polynomial $f(x) = x^2 - x - 1$ has one real root $c$ (see Figure 1), but we can prove, using an advanced branch of mathematics called Galois Theory, that there is no algebraic formula for this root. In this section, using Newton's Method, we show that $c \approx 1.6173$, and we show that we can compute $c$ to any desired degree of accuracy with enough computation.

In Newton's Method, we begin by choosing a number $x_0$, which we believe is close to a root of the equation $f(x) = 0$. This starting value $x_0$ is called the **initial guess**. Newton's Method then produces a sequence $x_0, x_1, x_2, x_3, \ldots$ of successive approximations that, in favorable situations, converge to a root.

Figure 2 illustrates the procedure. Given an initial guess $x_0$, we draw the tangent line to the graph at $(x_0, f(x_0))$. The approximation $x_1$ is defined as the $x$-coordinate of the point where the tangent line intersects the $x$-axis. To produce the second approximation $x_2$ (also called the second iterate), we apply this procedure to $x_1$. Then, repeatedly applying this procedure, we produce the sequence of approximations $x_0, x_1, x_2, x_3, \ldots$.

Let's derive a formula for $x_1$. The tangent line at $(x_0, f(x_0))$ has equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

The tangent line crosses the $x$-axis at $x_1$, where $y = 0$, that is, where

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0$$

To solve for $x_1$, we first divide by $f'(x_0)$ (as long as it is not zero) to obtain

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

and therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

![Figure 1 Graph of $y = x^5 - x - 1$. With Newton's Method, we can approximate the root $c$ as accurately as we like.](image)
The second iterate $x_2$ is obtained by applying this formula to $x_1$ instead of $x_0$:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and so on. Notice in Figure 2 that $x_1$ is closer to the root than $x_0$ is and that $x_3$ is closer still. This is typical: The successive approximations usually converge to the actual root. However, there are cases where Newton's Method fails (see Figure 4).

**Newton's Method** To approximate a root of $f(x) = 0$:

**Step 1.** Choose an initial guess $x_0$ (close to the desired root if possible).

**Step 2.** Generate successive approximations $x_1, x_2, \ldots$, where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**EXAMPLE 1** Calculate the first five approximations $x_1, \ldots, x_5$ to a root of $f(x) = x^3 - x - 1$ using the initial guess $x_0 = 1$.

**Solution** We have $f'(x) = 5x^4 - 1$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^5 - x_0 - 1}{5x_0^4 - 1}$$

We compute the first two approximations as follows:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^3 - 1 - 1}{5(1)^4 - 1} = 1.25$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.25 - \frac{1.25^5 - 1.25 - 1}{5(1.25)^4 - 1} \approx 1.178459$$

Continuing, rounding to six decimal places at each stage, we obtain $x_3 \approx 1.167547$, $x_4 \approx 1.167304$, and $x_5 \approx 1.167304$. This suggests that, accurate to six decimal places, $1.167304$ is a root of $f(x) = x^3 - x - 1$.

We can check our approximation; evaluating $x^3 - x - 1$ at $x = 1.167304$, we obtain 0.00000018 (to eight decimal places), verifying that we have a good approximation to a root of $f(x) = x^3 - x - 1$.

**How Many Iterations Are Required?**

How many iterations of Newton's Method are required to approximate a root to within a given accuracy? There is no definitive answer, but in practice, it is usually safe to assume that if $x_n$ and $x_{n+1}$ agree to $m$ decimal places, then the approximation $x_n$ is correct to these $m$ places.
Example 2: Let $c$ be the smallest positive solution of $\sin 3x = \cos x$.

(a) Use a computer-generated graph to choose an initial guess $x_0$ for $c$.

(b) Use Newton's Method to approximate $c$ to within an error of at most $10^{-6}$.

Solution

(a) A solution of $\sin 3x = \cos x$ is a zero of the function $f(x) = \sin 3x - \cos x$. Figure 3 shows that the smallest positive zero is approximately halfway between 0 and $\frac{\pi}{2}$. Because $\frac{\pi}{2} \approx 0.785$, a good initial guess is $x_0 = 0.4$.

(b) Since $f'(x) = 3\cos 3x + \sin x$, Eq. (1) yields the formula

$$x_{n+1} = x_n - \frac{\sin 3x_n - \cos x_n}{3\cos 3x_n + \sin x_n}$$

With $x_0 = 0.4$ as the initial guess, the first four iterates are:

$$x_1 \approx 0.3926547447$$
$$x_2 \approx 0.3926900816$$
$$x_3 \approx 0.3926990816$$
$$x_4 \approx 0.3926990816$$

Stopping here, we can be fairly confident that $x_4$ approximates the smallest positive root $c$ to at least 12 places. In fact, $c = \frac{\pi}{2}$ and $x_4$ is accurate to 16 places.

Which Root Does Newton's Method Compute?

Sometimes, Newton's Method computes no root at all. In Figure 4, the iterates diverge to infinity. In practice, however, Newton's Method usually converges quickly, and if a particular choice of $x_0$ does not lead to a root, the best strategy is to try a different initial guess, consulting a graph if possible. If $f(x) = 0$ has more than one root, different initial guesses $x_0$ may lead to different roots.

Example 3: Figure 5 shows that $f(x) = x^4 - 6x^2 + x + 5$ has four real roots.

(a) Show that with $x_0 = 0$, Newton's Method converges to the root near $-2$.

(b) Show that with $x_0 = -1$, Newton's Method converges to the root near $-1$.

Solution: We have $f'(x) = 4x^3 - 12x + 1$ and

$$x_{n+1} = x_n - \frac{x_n^4 - 6x_n^2 + x_n + 5}{4x_n^3 - 12x_n + 1}$$

(a) On the basis of Table 1, we can be confident that when $x_0 = 0$, Newton's Method converges to a root near $-2.3$. Notice in Figure 5 that this is not the closest root to $x_0$.

(b) Table 2 suggests that with $x_0 = -1$, Newton's Method converges to the root near $-0.9$.

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
</tr>
<tr>
<td>$x_1$</td>
</tr>
<tr>
<td>$x_2$</td>
</tr>
<tr>
<td>$x_3$</td>
</tr>
<tr>
<td>$x_4$</td>
</tr>
<tr>
<td>$x_5$</td>
</tr>
<tr>
<td>$x_6$</td>
</tr>
<tr>
<td>$x_7$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
</tr>
<tr>
<td>$x_1$</td>
</tr>
<tr>
<td>$x_2$</td>
</tr>
<tr>
<td>$x_3$</td>
</tr>
<tr>
<td>$x_4$</td>
</tr>
</tbody>
</table>
EXAMPLE 4 Approximating \( \sqrt{5} \) We know that the solutions to \( x^2 - 5 = 0 \) are \( x = \pm \sqrt{5} \). We can use Newton’s method to obtain approximations to these values. Approximate \( \sqrt{5} \) using an initial guess \( x_0 = 2 \).

Solution We have \( f'(x) = 2x \). Therefore,

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 5}{2x_0}
\]

We compute the successive approximations as follows:

\[
x_1 = 2 - \frac{2 - 5}{2 \cdot 2} = 2.25
\]

\[
x_2 = 2.25 - \frac{2.25^2 - 5}{2 \cdot 2.25} \approx 2.23611
\]

\[
x_3 = 2.23611 - \frac{2.23611^2 - 5}{2 \cdot 2.23611} \approx 2.23606797789
\]

Therefore, \( \sqrt{5} \approx 2.23606797789 \).

A calculator computation of \( \sqrt{5} \) yields

\[
\sqrt{5} = 2.23606797750\ldots
\]

Observe that \( x_3 \) is accurate to within an error of less than \( 10^{-9} \). This is impressive accuracy for just three iterations of Newton’s Method.

4.7 SUMMARY

- Newton’s Method: To find a sequence of numerical approximations to a root of \( f \), begin with an initial guess \( x_0 \). Then construct the sequence \( x_0, x_1, x_2, \ldots \) using the formula

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

You should choose the initial guess \( x_0 \) as close as possible to a root, possibly by referring to a graph. In favorable cases, the sequence converges rapidly to a root.

- If \( x_n \) and \( x_{n+1} \) agree to \( m \) decimal places, it is usually safe to assume that \( x_n \) agrees with a root to \( m \) decimal places.

4.7 EXERCISES

Preliminary Questions

1. How many iterations of Newton’s Method are required to compute a root if \( f \) is a linear function?

2. What happens in Newton’s Method if your initial guess happens to be a zero of \( f' \)?

3. What happens in Newton’s Method if your initial guess happens to be a local min or max of \( f \)?

4. Is the following a reasonable description of Newton’s Method: “A root of the equation of the tangent line to the graph of \( f \) is used as an approximation to a root of \( f \) itself”? Explain.

Exercises

In this exercise set, all approximations should be carried out using Newton’s Method.

In Exercises 1-6, apply Newton’s Method to \( f \) and initial guess \( x_0 \) to calculate \( x_1, x_2, x_3 \).

1. \( f(x) = x^2 - 6 \), \( x_0 = 2 \)

2. \( f(x) = x^3 - 3x + 1 \), \( x_0 = 3 \)

3. \( f(x) = x^3 - 10 \), \( x_0 = 2 \)

4. \( f(x) = x^3 + x + 1 \), \( x_0 = -1 \)

5. \( f(x) = \cos x - 4x \), \( x_0 = 1 \)
6. \( f(x) = 1 - x \sin x \), \( x_0 = 7 \).
7. Use Figure 6 to choose an initial guess \( x_0 \) to the unique real root of \( x^3 + 2x + 5 = 0 \) and compute the first three Newton iterates.

**Figure 6** Graph of \( y = x^3 + 2x + 5 \).

8. Approximate a solution of \( \sin x = \cos 2x \) in the interval \([0, \frac{\pi}{2}]\) to three decimal places. Then find the exact solution and compare with your approximation.
9. Approximate the point of intersection of the graphs \( y = x^2 + 4 + \frac{1}{x} \) and \( y = 2\sqrt{x} \) to three decimal places (Figure 7).

**Figure 7**

10. The first positive solution of \( \sin x = 0 \) is \( x = \pi \). Use Newton's Method to calculate \( \pi \) to four decimal places.

In Exercises 11–14, approximate to three decimal places using Newton's Method and compare with the value from a calculator.
11. \( \sqrt{11} \)
12. \( 5^{1/3} \)
13. \( 2^{1/3} \)
14. \( 3^{-1/4} \)
15. Approximate the largest positive root of \( f(x) = x^4 - 6x^2 + x + 5 \) to within an error of at most \( 10^{-4} \). Refer to Figure 5.

**GU** In Exercises 16–19, approximate the value specified to three decimal places using Newton's Method. Use a plot to choose an initial guess.
16. Largest positive root of \( f(x) = x^3 - 5x + 1 \)
17. Negative root of \( f(x) = x^3 - 20x + 10 \)
18. Positive solution of \( \sin \theta = 0.89 \)
19. Positive solution of \( 4 \cos x = x^2 \)
20. Let \( x_1, x_2 \) be the estimates to a root obtained by applying Newton's Method with \( x_0 = 1 \) to the function graphed in Figure 8. Estimate the numerical values of \( x_1 \) and \( x_2 \), and draw the tangent lines used to obtain them.

**Figure 8**

21. **GU** Find the smallest positive value of \( x \) at which \( y = x \) and \( y = \tan x \) intersect. *Hint:* Draw a plot.

22. In 1535, the mathematician Antonio Fior challenged his rival Niccolò Tartaglia to solve this problem: A tree stands 12 feet high; it is broken into two parts at such a point that the height of the part left standing is the cube root of the length of the part cut away. What is the height of the part left standing? Show that this is equivalent to solving \( x^3 + x = 12 \) and finding the height to three decimal places. Tartaglia, who had discovered the secret of solving the cubic equation, was able to determine the exact answer:

\[
x = \frac{\sqrt[3]{2919} + 54 - \sqrt[3]{2919} - 54}{2}
\]

23. Find (to two decimal places) the coordinates of the point \( P \) in Figure 9 where the tangent line to \( y = \cos x \) passes through the origin.

**Figure 9**

Newton's Method is often used to determine interest rates in financial calculations. In Exercises 24–26, \( r \) denotes a yearly interest rate expressed as a decimal (rather than as a percent).

24. If \( P \) dollars are deposited every month in an account earning interest at the yearly rate \( r \), then the value \( S \) of the account after \( N \) years is

\[
S = P \left( \frac{b^{12N+1} - b}{b - 1} \right), \quad \text{where} \quad b = 1 + \frac{r}{12}
\]

You have decided to deposit \( P = \$100 \) per month.
(a) Determine \( S \) after 5 years if \( r = 0.07 \) (i.e., 7%).
(b) Show that to save \( \$10,000 \) after 5 years, you must earn interest at a rate \( r \) determined by the equation \( b^{60} - 101b + 100 = 0 \). Use Newton's Method to solve for \( b \). Then find \( r \). Note that \( b = 1 \) is a root, but you want the root satisfying \( b > 1 \).

25. If you borrow \( L \) dollars for \( N \) years at a yearly interest rate \( r \), your monthly payment of \( P \) dollars is calculated using the equation

\[
L = P \left( \frac{1 - b^{-12N}}{b - 1} \right), \quad \text{where} \quad b = 1 + \frac{r}{12}
\]

(a) Find \( P \) if \( L = \$5000 \), \( N = 3 \), and \( r = 0.08 \) (8%).
(b) You are offered a loan of \( L = \$5000 \) to be paid back over 3 years with monthly payments of \( P = \$200 \). Use Newton's Method to compute \( b \) and find the implied interest rate \( r \) of this loan. *Hint:* Show that

\[
(L/P)b^{12N+1} - (1 + L/P)b^{12N} + 1 = 0
\]

26. If you deposit \( P \) dollars in a retirement fund every year for \( N \) years with the intention of then withdrawing \( Q \) dollars per year for \( M \) years, you must earn interest at a rate \( r \) satisfying

\[
P(b^{-M} - 1) = Q(1 - b^{-M}), \quad \text{where} \quad b = 1 + r
\]

Assume that \( \$2000 \) is deposited each year for 30 years and the goal is to withdraw \( \$10,000 \) per year for 25 years. Use Newton's Method to compute \( b \) and then find \( r \). Note that \( b = 1 \) is a root, but you want the root satisfying \( b > 1 \).

27. There is no simple formula for the position at time \( t \) of a planet \( P \) in its orbit (an ellipse) around the sun. Introduce the auxiliary circle and angle \( \theta \) in Figure 10 (note that \( P \) determines \( \theta \) because it is the central angle of point \( B \) on the circle). Let \( a = OA \) and \( e = OS/OA \) (the eccentricity of the orbit).
(a) Show that sector \( BSA \) has area \( \frac{a^2}{2} \sin \theta \).

(b) By Kepler's Second Law, the area of sector \( BSA \) is proportional to the time \( t \) elapsed since the planet passed point \( A \), and because the circle has area \( \pi a^2 \), \( BSA \) has area \( \pi a^2 \theta / T \), where \( T \) is the period of the orbit. Deduce Kepler's Equation:

\[
\frac{2\pi t}{T} = \theta - \frac{e}{2} \sin \theta
\]

(c) The eccentricity of Mercury's orbit is approximately \( e = 0.2 \). Use Newton's Method to find \( \theta \) after a quarter of Mercury's year has elapsed (\( t = T/4 \)). Convert \( \theta \) to degrees. Has Mercury covered more than a quarter of its orbit at \( t = T/4 \)?

**FIGURE 10**

**FIGURE 11** Graph of \( f(x) = \frac{1}{3}x^3 - 4x + 1 \).

28. The roots of \( f(x) = \frac{1}{3}x^3 - 4x + 1 \) to three decimal places are \(-3.583, 0.251, \text{ and } 3.332 \) (Figure 11). Determine the root to which Newton's Method converges for the initial choices \( x_0 = 1.85, 1.7, \text{ and } 1.55 \). The answer shows that a small change in \( x_0 \) can have a significant effect on the outcome of Newton's Method.

29. What happens when you apply Newton's Method to find a zero of \( f(x) = x^{1/3} \)? Note that \( x = 0 \) is the only zero.

30. What happens when you apply Newton's Method to the equation \( x^3 - 20x = 0 \) with the unlucky initial guess \( x_0 = 27 \).

**Further Insights and Challenges**

31. Newton's Method can be used to compute reciprocals without performing division. Let \( c > 0 \) and set \( f(x) = x^{-1} - c \).

(a) Show that \( x = (f(x)/f'(x)) = 2x - cx^2 \).

(b) Calculate the first three iterates of Newton's Method with \( c = 10.3 \) and the two initial guesses \( x_0 = 0.1 \) and \( x_0 = 0.5 \).

(c) Explain graphically why \( x_0 = 0.5 \) does not yield a sequence converging to \( 1/10.3 \).

In Exercises 32 and 33, consider a metal rod of length \( L \), fastened at both ends. If you cut the rod and weld an additional segment of length \( m \), leaving the ends fixed, the rod will bow up into a circular arc of radius \( R \) (unknown), as indicated in Figure 12.

**FIGURE 12** The bold circular arc has length \( L + m \).

32. Let \( h \) be the maximum vertical displacement of the rod.

(a) Show that \( L = 2R \sin \theta \) and conclude that

\[
\frac{h}{L} = \frac{\sin \theta}{2} = \frac{L}{L + m}.
\]

(b) Show that \( L + m = 2R \sin \theta \) and then prove

\[
\frac{\sin \theta}{\theta} = \frac{L}{L + m}.
\]

33. Let \( L = 3 \) and \( m = 1 \). Apply Newton's Method to Eq. (2) to estimate \( \theta \), and use this to estimate \( h \).

34. Quadratic Convergence to Square Roots

Let \( f(x) = x^2 - c \) and let \( e_n = x_n - \sqrt{c} \) be the error in \( x_n \).

(a) Show that \( x_{n+1} = \frac{1}{2}(x_n + c/x_n) \) and \( e_{n+1} = e_n^2 / 2x_n^2 \).

(b) Show that if \( x_0 > \sqrt{c} \), then \( x_n > \sqrt{c} \) for all \( n \). Explain graphically.

(c) Show that if \( x_0 > \sqrt{c} \), then \( e_{n+1} \leq e_n^2 / (2\sqrt{c}) \).

**CHAPTER REVIEW EXERCISES**

In Exercises 1–6, estimate using the Linear Approximation or Linearization, and use a calculator to estimate the error.

1. \( 8.1^{1/3} - 2 \)

2. \( \frac{1}{\sqrt[4]{1.1}} - \frac{1}{2} \)

3. \( 625^{1/4} - 624^{1/4} \)

4. \( \sqrt{101} \)

5. \( \frac{1}{1.02} \)

6. \( \frac{1}{\sqrt{33}} \)

In Exercises 7–12, find the linearization at the point indicated.

7. \( y = \sqrt{x}, \quad a = 25 \)

8. \( y(t) = 32t - 4t^2, \quad a = 2 \)

9. \( A(r) = \frac{4}{3} \pi r^3, \quad a = 3 \)
10. \( V(h) = 4h(2 - h)(4 - 2h) \). \( a = 1 \)

11. \( P(t) = \sin(39 + \pi) \). \( a = \frac{\pi}{3} \)

12. \( R(t) = \tan \left( \frac{t}{1} \right) \). \( a = \frac{1}{2} \)

In Exercises 13–16, use the Linear Approximation.

13. The position of an object in linear motion at time \( t \) is \( s(t) = 0.4t^2 + (t + 1)^{-1} \). Estimate the distance traveled over the time interval \([4,4.2]\).

14. A bond that pays $10,000 in 6 years is offered for sale at a price \( P \). The percentage yield \( Y \) of the bond is:

\[ Y = 100 \left( \left( \frac{10,000}{P} \right)^{1/6} - 1 \right) \]

Verify that if \( P = 8500 \), then \( Y = 4.91\% \). Estimate the drop in yield if the price rises to $7700.

15. When a bus pass from Albuquerque to Los Alamos is priced at \( p \) dollars, a bus company takes in a monthly revenue of \( R(p) = 1.5p - 0.01p^2 \) (in thousands of dollars). (a) Estimate \( \Delta R \) if the price rises from $50 to $53. (b) If \( p = 80 \), how will revenue be affected by a small increase in price? Explain using the Linear Approximation.

16. Show that \( \sqrt{a^2 + b} \approx a + \frac{b}{2a} \) if \( b \) is small. Use this to estimate \( \sqrt{26} \) and find the error using a calculator.

17. Use the Intermediate Value Theorem to show that \( \sin x - \cos x = 3x \) has a solution, and use Rolle's Theorem to show that this solution is unique.

18. Show that \( f(x) = 2x^3 + 2x + \sin x + 1 \) has precisely one real root.

19. Verify the MVT for \( f(x) = x + \frac{1}{2} \) on \([2,5]\).

20. Suppose that \( f(1) = 5 \) and \( f'(x) \geq 2 \) for \( x \geq 1 \). Use the MVT to show that \( f(8) \geq 19 \).

21. Use the MVT to prove that if \( f'(x) \leq 2 \) for \( x > 0 \) and \( f(0) = 4 \), then \( f(x) \leq 2x + 4 \) for all \( x \geq 0 \).

22. A function \( f \) has derivative \( f'(x) = \frac{1}{x^4 + 1} \). Where on the interval \([1,4]\) does \( f \) take on its maximum value?

In Exercises 23–28, find the critical points and determine whether they are minima, maxima, or neither.

23. \( f(x) = x^3 - 4x^2 + 4x \)

24. \( a(t) = t^4 - 8t^2 \)

25. \( f(x) = x^2(x + 2)^3 \)

26. \( f(x) = x^{2/3}(1-x) \)

27. \( g(\theta) = \sin^2 \theta + \theta \)

28. \( h(\theta) = 2\cos 2\theta + \cos 4\theta \)

In Exercises 29–36, find the extreme values on the interval.

29. \( f(x) = x(10-x) \), \([-1,3]\)

30. \( f(x) = 6x^4 - 4x^6 \), \([-2,2]\)

31. \( g(\theta) = \sin^2 \theta - \cos \theta \), \([0,2\pi]\)

32. \( R(t) = \frac{t}{t^2 + t + 1} \), \([0,3]\)

33. \( f(x) = x^{2/3} - 2x^{1/3} \), \([-1,3]\)

34. \( f(x) = 4x - \tan^2 x \), \([-\frac{\pi}{4}, \frac{\pi}{4}]\)

35. \( f(x) = x - x^{3/2} \), \([0,2]\)

36. \( f(x) = \sec x - \cos x \), \([-\frac{\pi}{4}, \frac{\pi}{4}]\)

37. Find the critical points and extreme values of \( f(x) = |x - 1| + |x - 6| \) in \([0,8]\).

38. Match the description of \( f \) with the graph of its derivative \( f' \) in Figure 1.

(a) \( f \) is increasing and concave up.

(b) \( f \) is decreasing and concave up.

(c) \( f \) is increasing and concave down.

![Figure 1: Graphs of the derivative.](image)

In Exercises 39–44, find the points of inflection.

39. \( y = x^3 - 4x^2 + 4x \)

40. \( y = x - 2\cos x \)

41. \( y = \frac{x^2}{x^2 + 4} \)

42. \( y = \frac{x}{(x^3 - 4)^{1/3}} \)

43. \( f(x) = x^3 - x \)

44. \( f(x) = \sin 2x - 4\cos x \)

In Exercises 45–54, sketch the graph, noting the transition points and asymptotic behavior.

45. \( y = 12x - 3x^3 \)

46. \( y = 8x^2 - x^4 \)

47. \( y = x^3 - 2x^2 + 3 \)

48. \( y = 4x - x^{3/2} \)

49. \( y = \frac{x}{x^3 + 1} \)

50. \( y = \frac{x}{(x^3 - 4)^{1/3}} \)

51. \( y = \frac{1}{|x + 2| + 1} \)

52. \( y = \sqrt{2 - x^3} \)

53. \( y = \sqrt[3]{\sin x - \cos x} \) on \([0, 2\pi]\)

54. \( y = 2x - \tan x \) on \([0, 2\pi]\)

55. Draw a curve \( y = f(x) \) for which \( f' \) and \( f'' \) have signs as indicated in Figure 2.

![Figure 2](image)

56. Find the dimensions of a cylindrical can with a bottom but no top of volume \( 4 \text{ m}^3 \) that uses the least amount of metal.
57. A rectangular open-topped box of height $h$ with a square base of side $b$ has volume $V = 4 \text{ m}^3$. Two of the side faces are made of material costing $40/\text{m}^2$. The remaining sides cost $20/\text{m}^2$. Which values of $b$ and $h$ minimize the cost of the box?

58. The corn yield on a certain farm is

$$Y = -0.118x^2 + 8.5x + 12.9 \quad \text{(bushels per acre)}$$

where $x$ is the number of corn plants per acre (in thousands). Assume that corn seed costs $1.25 \text{ (per thousand seeds)}$ and that corn can be sold for $1.50/\text{bushel}$. Let $P(x)$ be the profit (revenue minus the cost of seeds) at planting level $x$.

(a) Compute $P(x_0)$ for the value $x_0$ that maximizes yield $Y$.

(b) Find the maximum value of $P(x)$. Does maximum yield lead to maximum profit?

59. A quantity $N(t)$ satisfies $dN/dt = 2/t - 5/t^2$ for $t \geq 4$ (in days). At which time is $N$ increasing most rapidly?

60. A truck gets 10 miles per gallon (mpg) of diesel fuel traveling along an interstate highway at 50 mph. This mileage decreases by $0.15 \text{ mpg}$ for each mile per hour increase above 50 mph.

(a) If the truck driver is paid $30/h and diesel fuel costs $3/gal, which speed $v$ between 50 and 70 mph will minimize the cost of a trip along the highway? Notice that the actual cost depends on the length of the trip, but the optimal speed does not.

(b) [GU] Plot cost as a function of $v$ (choose the length arbitrarily) and verify your answer to part (a).

(c) [GU] Do you expect the optimal speed $v$ to increase or decrease if fuel costs go down to $P = 2/gal$? Plot the graphs of cost as a function of $v$ for $P = 2$ and $P = 3$ on the same axis and verify your conclusion.

61. Find the maximum volume of a right-circular cone placed upside-down in a right-circular cone of radius $R = 3$ and height $H = 4$ as in Figure 3. A cone of radius $r$ and height $h$ has volume $\frac{1}{3}\pi r^2h$.

62. Redo Exercise 61 for arbitrary $R$ and $H$.

63. Show that the maximum area of a parallelogram $ADEF$ that is inscribed in a triangle $ABC$, as in Figure 4, is equal to one-half the area of $\triangle ABC$.

64. A box of volume 8 $\text{m}^3$ with a square top and bottom is constructed out of two types of metal. The metal for the top and bottom costs $50/\text{m}^2$ and the metal for the sides costs $30/\text{m}^2$. Find the dimensions of the box that minimize total cost.

65. Let $f$ be a function whose graph does not pass through the $x$-axis and let $Q = (a, 0)$. Let $P = (x_0, f(x_0))$ be the point on the graph closest to $Q$ (Figure 5). Prove that $PQ$ is perpendicular to the tangent line to the graph of $x_0$. Hint: Find the minimum value of the square of the distance from $(x, f(x))$ to $(a, 0)$.

66. Take a circular piece of paper of radius $R$, remove a sector of angle $\theta$ (Figure 6), and fold the remaining piece into a cone-shaped cup. Which angle $\theta$ produces the cup of largest volume?

67. Use Newton’s Method to estimate $\sqrt{23}$ to four decimal places.

68. Use Newton’s Method to find a root of $f(x) = x^2 - x - 1$ to four decimal places.
5 INTEGRATION

The last two chapters developed the derivative, one of the primary topics in calculus. The derivative was motivated by the basic problem of finding a line tangent to a curve at a given point. In addition, we saw that the derivative is much more significant than just a means for finding tangent lines.

In this chapter, we introduce the next primary topic, the definite integral. It too is motivated by a basic problem: finding the area under a curve. And in this case as well, once you become familiar with the definite integral you will realize that its importance goes well beyond computing area.

Computing tangent lines and areas may seem completely unrelated, and therefore the derivative and the definite integral may appear to be completely separate topics. They aren't. There is a deep connection between them that is revealed by the Fundamental Theorem of Calculus, discussed in Sections 5.4 and 5.5. This theorem expresses the "inverse" relationship between integration and differentiation. It plays a truly fundamental role in nearly all applications of calculus, both theoretical and practical.

5.1 Approximating and Computing Area

Why might we be interested in the area under a graph? Consider an object moving in a straight line with constant positive velocity \( v \). The distance traveled over a time interval \([t_1, t_2]\) is equal to \( v \Delta t \), where \( \Delta t = (t_2 - t_1) \) is the time elapsed. This is the well-known formula

\[
\text{distance traveled} = \frac{\text{velocity} \times \text{time elapsed}}{v \Delta t}
\]

Because \( v \) is constant, the graph of velocity is a horizontal line (Figure 1) and \( v \Delta t \) is equal to the area of the rectangular region under the graph of velocity over \([t_1, t_2]\). So we can write Eq. (1) as

\[
\text{distance traveled} = \text{area under the graph of velocity over } [t_1, t_2]
\]

There is, however, an important difference between these two equations: Eq. (1) makes sense only if velocity \( v \) is constant, whereas Eq. (2) is correct even if the velocity changes with time. We examine the relationship in Eq. (2) further in Section 5.6. Thus, the advantage of expressing distance traveled as an area is that it enables us to deal with much more general types of motion.

To see why Eq. (2) might be true in general, let's consider the case where velocity changes over time but is constant on intervals. In other words, we assume that the object's velocity changes abruptly from one interval to the next as in Figure 2. The distance traveled over each time interval is equal to the area of the rectangle above that interval, so the total distance traveled is the sum of the areas of the rectangles. In Figure 2,

\[
\text{distance traveled over } [0, 8] = 10 + 15 + 30 + 10 = 65 \text{ m}
\]

**Sum of areas of rectangles**

![Figure 1](image1.png)

**Figure 1** The rectangle has area \( v \Delta t \), which is equal to the distance traveled.

![Figure 2](image2.png)

**Figure 2** Distance traveled equals the sum of the areas of the rectangles.
Our strategy when velocity changes continuously (Figure 3) is to approximate the area under the graph by a sum of areas of rectangles. We can continually improve the approximation by using thinner and thinner rectangles, and then take a limit to obtain an exact value of distance traveled. This idea leads to the concept of an integral.

**Approximating Area by Rectangles**

Our goal is to compute the area under the graph of a function $f$. In this section, we assume that $f$ is continuous and positive, so that the graph of $f$ lies above the $x$-axis (Figure 4). The first step is to approximate the area using rectangles.

To begin, choose a whole number $N$ and divide $[a, b]$ into $N$ subintervals of equal width, as in Figure 4(A). The full interval $[a, b]$ has width $b - a$, so each subinterval has width $\Delta x = (b - a)/N$. The right endpoints of the subintervals are

$$x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ldots, \quad x_{N-1} = a + (N - 1)\Delta x, \quad x_N = a + N\Delta x$$

Note that the last right endpoint is $x_N = b$ because $a + N\Delta x = a + N((b - a)/N) = b$.

The general term $x_j$ can be expressed as $x_j = a + j\Delta x$, indicating that it is obtained by adding $j$ intervals of width $\Delta x$ to $a$. Next, as in Figure 4(B), above each subinterval construct a rectangle whose height is the value of $f(x)$ at the right endpoint of the subinterval.

The sum of the areas of these rectangles provides an approximation to the area under the graph. The first rectangle has base $\Delta x$ and height $f(x_1)$, so its area is $f(x_1)\Delta x$. Similarly, the second rectangle has height $f(x_2)$ and area $f(x_2)\Delta x$, and so on. The sum of the areas of the rectangles is denoted $R_N$ and is called the $N$th right-endpoint approximation:

$$R_N = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_N)\Delta x$$

Factoring out $\Delta x$, we obtain the formula

$$R_N = \Delta x \left( f(x_1) + f(x_2) + \ldots + f(x_N) \right)$$
In words: \( R_N \) is equal to \( \Delta x \) times the sum of the function values at the right endpoints of the subintervals.

**EXAMPLE 1** Calculate \( R_4 \) and \( R_6 \) for \( f(x) = x^2 \) on the interval \([1, 3]\).

**Solution**

**Step 1.** Determine \( \Delta x \) and the right endpoints.

To calculate \( R_4 \), divide \([1, 3]\) into four subintervals of width \( \Delta x = \frac{3-1}{4} = \frac{1}{2} \). The right endpoints are the numbers \( x_j = a + j\Delta x = 1 + j\left(\frac{1}{2}\right) \) for \( j = 1, 2, 3, 4 \). They are spaced at intervals of \( \frac{1}{2} \) beginning at \( \frac{3}{2} \), so, as we see in Figure 5(A), the right endpoints are \( \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, 3 \).

**Step 2.** Calculate \( \Delta x \) times the sum of function values.

\( R_4 \) is \( \Delta x \) times the sum of the function values at the right endpoints:

\[
R_4 = \frac{1}{2} \left( f\left(\frac{3}{2}\right) + f\left(\frac{4}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{6}{2}\right) \right)
\]

\[
= \frac{1}{2} \left( \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(\frac{6}{2}\right)^2 \right) = \frac{43}{4} = 10.75
\]

\( R_6 \) is similar; \( \Delta x = \frac{3-1}{6} = \frac{1}{3} \), and the right endpoints are spaced at intervals of \( \frac{1}{3} \) beginning at \( \frac{3}{2} \) and ending at 3, as in Figure 5(B). Thus,

\[
R_6 = \frac{1}{3} \left( f\left(\frac{4}{3}\right) + f\left(\frac{5}{3}\right) + f\left(\frac{6}{3}\right) + f\left(\frac{7}{3}\right) + f\left(\frac{8}{3}\right) + f\left(\frac{9}{3}\right) \right)
\]

\[
= \frac{1}{3} \left( \frac{16}{9} + \frac{25}{9} + \frac{36}{9} + \frac{49}{9} + \frac{64}{9} + \frac{81}{9} \right) = \frac{271}{27} \approx 10.037
\]

**FIGURE 5**

The \( R_4 \) and \( R_6 \) values in the previous example are clearly overestimates of the area under \( y = x^2 \) between 1 and 3, although the approximation improves going from \( R_4 \) to \( R_6 \). Later in the section, we show how to obtain a general expression for \( R_N \). With that, we can then take the limit as \( N \to \infty \) to obtain an exact value of \( 8\frac{2}{3} \) as the area (see Exercise 55).

**Summation Notation**

**Summation notation** is a standard notation for writing sums in compact form. The sum of numbers \( a_m, \ldots, a_n \) \((m \leq n)\) is denoted

\[
\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n
\]
The Greek letter $\sum$ (capital sigma) stands for "sum," and the notation $\sum_{j=m}^{n}$ tells us to start the summation at $j = m$ and end it at $j = n$. For example,

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

In this summation, the $j$th term is $a_j = j^2$. We refer to $j^2$ as the general term. The letter $j$ is called the summation index. It is also referred to as a dummy variable because any other letter can be used instead. For example,

$$\sum_{k=4}^{6} (k^2 - 2k) = (4^2 - 2(4)) + (5^2 - 2(5)) + (6^2 - 2(6)) = 47$$

$$\sum_{m=7}^{9} 1 = 1 + 1 + 1 = 3 \quad \text{(because } a_7 = a_8 = a_9 = 1)$$

The usual commutative, associative, and distributive laws of addition give us the following rules for manipulating summations.

**Linearity of Summations**

- $\sum_{j=m}^{n} (a_j + b_j) = \sum_{j=m}^{n} a_j + \sum_{j=m}^{n} b_j$
- $\sum_{j=m}^{n} Ca_j = C \sum_{j=m}^{n} a_j \quad \text{(C any constant)}$
- $\sum_{j=1}^{n} C = nC \quad \text{(C any constant and } n \geq 1)$

For example,

$$\sum_{j=3}^{5} (j^2 + j) = (3^2 + 3) + (4^2 + 4) + (5^2 + 5) = 62$$

can also be expressed as

$$\sum_{j=3}^{5} j^2 + \sum_{j=3}^{5} j = (3^2 + 4^2 + 5^2) + (3 + 4 + 5) = 50 + 12 = 62$$

The linearity properties can be used to write a single summation as a linear combination of several summations. For example,

$$\sum_{k=0}^{100} (7k^2 - 4k + 9) = \sum_{k=0}^{100} 7k^2 + \sum_{k=0}^{100} (-4k) + \sum_{k=0}^{100} 9$$

$$= 7 \sum_{k=0}^{100} k^2 - 4 \sum_{k=0}^{100} k + 9 \sum_{k=0}^{100} 1$$

It is convenient to use summation notation when working with area approximations. For example, $R_N$ is a sum with general term $f(x_j)$:

$$R_N = \Delta x [f(x_1) + f(x_2) + \cdots + f(x_N)]$$
The summation extends from $j = 1$ to $j = N$, so we can write $R_N$ concisely as

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j)$$

We shall make use of two other rectangular approximations to area: the left-endpoint and the midpoint approximations. Divide $[a, b]$ into $N$ subintervals as before. In the left-endpoint approximation $L_N$, the heights of the rectangles are the values of $f(x)$ at the left endpoints [Figure 6(A)]. These left endpoints are

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \ldots, \quad x_{N-1} = a + (N-1)\Delta x$$

and the sum of the areas of the left-endpoint rectangles is

$$L_N = \Delta x \left(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{N-1})\right)$$

Note that both $R_N$ and $L_N$ have general term $f(x_j)$, but the sum for $L_N$ runs from $j = 0$ to $j = N - 1$ rather than from $j = 1$ to $j = N$:

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j)$$

In the midpoint approximation $M_N$, the heights of the rectangles are the values of $f(x)$ at the midpoints of the subintervals rather than at the endpoints. As we see in Figure 6(B), the midpoints are

$$\frac{x_0 + x_1}{2}, \quad \frac{x_1 + x_2}{2}, \quad \ldots, \quad \frac{x_{N-1} + x_N}{2}$$

The sum of the areas of the midpoint rectangles is

$$M_N = \Delta x \left(f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{N-1} + x_N}{2}\right)\right)$$

In summation notation,

$$M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$$

**Figure 6**

(A) Left-endpoint rectangles

(B) Midpoint rectangles

**Example 2** Calculate $R_6$, $L_6$, and $M_6$ for $f(x) = x^{-1}$ on $[2, 4]$.

**Solution** In this case, $\Delta x = (b - a)/N = (4 - 2)/6 = \frac{1}{3}$. For the six intervals, the right endpoints are $\frac{7}{3}, \frac{5}{3}, \frac{10}{3}, \frac{11}{3}$, and $4$, and the left endpoints are $2, \frac{7}{3}, \frac{3}{2}, \frac{10}{3}$, and $\frac{11}{3}$. 
Therefore (Figure 7),
\[
R_6 = \frac{1}{3} \left( f \left( \frac{7}{3} \right) + f \left( \frac{8}{3} \right) + f \left( \frac{10}{3} \right) + f \left( \frac{11}{3} \right) + f(4) \right) \\
= \frac{1}{3} \left( \frac{3}{7} + \frac{3}{8} + \frac{3}{10} + \frac{3}{11} + \frac{1}{4} \right) \approx 0.653
\]
\[
L_6 = \frac{1}{3} \left( f \left( \frac{8}{3} \right) + f \left( \frac{10}{3} \right) + f \left( \frac{11}{3} \right) \right) \\
= \frac{1}{3} \left( \frac{1}{2} + \frac{1}{3} + \frac{3}{10} + \frac{3}{11} \right) \approx 0.737
\]
The general term in \( M_b \) is
\[
f \left( \frac{x_j + x_{j+1}}{2} \right)
\]
In this case, the midpoints are \( \frac{13}{6}, \frac{15}{6}, \frac{17}{6}, \frac{19}{6}, \frac{21}{6} \) and \( \frac{23}{6} \). Summing from \( j = 0 \) to 5, we obtain (Figure 8)
\[
M_6 = \frac{1}{3} \left( f \left( \frac{13}{6} \right) + f \left( \frac{15}{6} \right) + f \left( \frac{17}{6} \right) + f \left( \frac{19}{6} \right) + f \left( \frac{21}{6} \right) + f \left( \frac{23}{6} \right) \right) \\
= \frac{1}{3} \left( \frac{6}{13} + \frac{6}{15} + \frac{6}{17} + \frac{6}{19} + \frac{6}{21} + \frac{6}{23} \right) \approx 0.692
\]

**GRAPHICAL INSIGHT Monotonic Functions** Observe in Figure 7 that the left-endpoint rectangles for \( f(x) = x^{-1} \) extend above the graph and the right-endpoint rectangles lie below it. The exact area \( A \) under the graph of \( f \) from 2 to 4 must lie between \( R_6 \) and \( L_6 \), and so, according to the previous example, \( 0.65 < A < 0.74 \). More generally, when \( f \) is monotonic (increasing or decreasing), the exact area lies between \( R_N \) and \( L_N \) (Figure 9):

- \( f \) increasing \( \Rightarrow L_N \leq \text{area under graph} \leq R_N \)
- \( f \) decreasing \( \Rightarrow R_N \leq \text{area under graph} \leq L_N \)

Notice that \( M_6 \) lies between \( R_6 \) and \( L_6 \). This is always the case for a monotonic function (see Exercise 91). In the upcoming sections, we will see how to determine the exact area in the example. It turns out to be \( \ln 2 \approx 0.693 \), and thus, \( M_6 \) provides the best estimate, which is understandable considering these observations.

**Computing Area as the Limit of Approximations**

Figure 10 shows several right-endpoint approximations. Notice that the error in computing the area, corresponding to the yellow region above the graph, gets smaller as the number of rectangles increases. In fact, it appears that we can make the error as small as we please by taking the number \( N \) of rectangles large enough. If so, it makes sense to consider the limit as \( N \to \infty \), as this should give us the exact area under the curve. The next theorem guarantees that the limit exists (see Theorem 7 in Appendix D for a proof and Exercise 91 for a special case).
In Theorem 1, it is not assumed that \( f(x) \geq 0 \). If \( f(x) \) takes on negative values, the limit \( L \) no longer represents area under the graph, but we can interpret it as a "signed area," discussed in the next section.

**THEOREM 1** If \( f \) is continuous on \([a, b]\), then the endpoint and midpoint approximations approach one and the same limit as \( N \to \infty \). In other words, there is a value \( L \) such that

\[
\lim_{N \to \infty} R_N = \lim_{N \to \infty} L_N = \lim_{N \to \infty} M_N = L
\]

If \( f(x) \geq 0 \) on \([a, b]\), we define the area under the graph over \([a, b]\) to be \( L \).

**CONCEPTUAL INSIGHT** In calculus, limits are used to define basic quantities that otherwise would not have a precise meaning. Theorem 1 allows us to define area as a limit \( L \) in much the same way that we define the slope of a tangent line as the limit of slopes of secant lines.

The next three examples illustrate Theorem 1 using formulas for power sums. The \( k \)th power sum is defined as the sum of the \( k \)th powers of the first \( N \) integers. We shall use the power sum formulas for \( k = 1, 2, 3 \).

**Power Sums**

\[
\sum_{j=1}^{N} j = 1 + 2 + \cdots + N = \frac{N(N + 1)}{2} = \frac{N^2 + N}{2}
\]

\[
\sum_{j=1}^{N} j^2 = 1^2 + 2^2 + \cdots + N^2 = \frac{N(N + 1)(2N + 1)}{6} = \frac{N^3 + N^2}{3} + \frac{N}{6}
\]

\[
\sum_{j=1}^{N} j^3 = 1^3 + 2^3 + \cdots + N^3 = \frac{N^2(N + 1)^2}{4} = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}
\]

For example, by Eq. (4),

\[
\sum_{j=1}^{6} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = \frac{6^3 + 6^2}{3} + \frac{6}{6} = 91
\]

As a first illustration, we verify this limit approach to area by computing the area of a right triangle, a figure whose area we can also compute geometrically.

**EXAMPLE 3** Find the area \( A \) under the graph of \( f(x) = x \) over \([0, 4]\) in three ways:

(a) Using geometry

(b) \( \lim_{N \to \infty} R_N \)

(c) \( \lim_{N \to \infty} L_N \)

Solution The region under the graph is a right triangle with base \( b = 4 \) and height \( h = 4 \) (Figure 11).

(a) By geometry, \( A = \frac{1}{2}bh = \frac{1}{2}(4)(4) = 8 \).

(b) We compute this area again as a limit. Since \( \Delta x = (b - a)/N = 4/N \) and \( f(x) = x \),

\[
f(x_j) = f(a + j \Delta x) = f \left( 0 + j \left( \frac{4}{N} \right) \right) = \frac{4j}{N}
\]

\[
R_N = \Delta x \sum_{j=1}^{N} f(x_j) = 4 \sum_{j=1}^{N} \frac{4j}{N} = \frac{16}{N^2} \sum_{j=1}^{N} j
\]

(continued)
In the last equality, we factored out $4/N$ from the sum. This is valid because $4/N$ is a constant that does not depend on $j$. Now use formula (3):

$$R_N = \frac{16}{N^2} \sum_{j=1}^{N} j = \frac{16}{N^2} \left( \frac{N(N+1)}{2} \right) = \frac{8}{N^2} \left( N^2 + N \right) = 8 + \frac{8}{N}$$

The second term $8/N$ tends to zero as $N$ approaches $\infty$, so

$$A = \lim_{N \to \infty} R_N = \lim_{N \to \infty} \left( 8 + \frac{8}{N} \right) = 8$$

As expected, this limit yields the same value as the formula $\frac{1}{2}bh$.

(c) The right-endpoint approximation is similar, but the sum begins at $j = 0$ and ends at $j = N - 1$:

$$L_N = \frac{16}{N^2} \sum_{j=0}^{N-1} j = \frac{16}{N^2} \sum_{j=1}^{N-1} j = \frac{16}{N^2} \left( \frac{(N-1)N}{2} \right) = 8 - \frac{8}{N}$$

Note in the second step that we replaced the sum beginning at $j = 0$ with a sum beginning at $j = 1$. This is valid because the term for $j = 0$ is zero and may be dropped. Again, we find that $A = \lim_{N \to \infty} L_N = \lim_{N \to \infty} \left( 8 - \frac{8}{N} \right) = 8$.

In the next example, we compute the area under a curved graph. Unlike the previous example, it is not possible to compute this area directly using geometry.

**EXAMPLE 4** Let $A$ be the area under the graph of $f(x) = 2x^2 - x + 3$ over $[2, 4]$ (Figure 12). Compute $A$ as the limit $\lim_{N \to \infty} R_N$.

**Solution**

**Step 1.** Express $R_N$ in terms of power sums.

In this case, $\Delta x = (4 - 2)/N = 2/N$ and

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \Delta x \sum_{j=1}^{N} f(a + j\Delta x) = \frac{2}{N} \sum_{j=1}^{N} \left( 2 + \frac{2j}{N} \right)$$

Let's use algebra to simplify the general term. Since $f(x) = 2x^2 - x + 3$,

$$f \left( 2 + \frac{2j}{N} \right) = \left( 2 + \frac{2j}{N} \right)^2 - \left( 2 + \frac{2j}{N} \right) + 3$$

$$= 2 \left( 4 + \frac{8j}{N} + \frac{4j^2}{N^2} \right) - \left( 2 + \frac{2j}{N} \right) + 3 = \frac{8j^2}{N^2} + \frac{14j}{N} + 9$$
Now we can express $R_N$ in terms of power sums:

$$R_N = \frac{2}{N} \sum_{j=1}^{N} \left( \frac{8}{N^3} j^2 + \frac{14}{N} j + 9 \right) = \frac{2}{N} \sum_{j=1}^{N} \frac{8}{N^3} j^2 + \frac{2}{N} \sum_{j=1}^{N} \frac{14}{N} j + \frac{2}{N} \sum_{j=1}^{N} 9$$

$$= \frac{16}{N^3} \sum_{j=1}^{N} j^2 + \frac{28}{N^2} \sum_{j=1}^{N} j + \frac{18}{N} \sum_{j=1}^{N} 1$$

**Step 2. Use the formulas for the power sums.**

Using formulas (3) and (4) for the power sums in Eq. (7), we obtain

$$R_N = \frac{16}{N^3} \left( \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \right) + \frac{28}{N^2} \left( \frac{N^2}{2} + \frac{N}{2} \right) + \frac{18}{N} \left( N \right)$$

$$= \left( \frac{16}{3} + \frac{8}{N} + \frac{8}{3N^2} \right) + \left( 14 + \frac{14}{N} \right) + 18$$

$$= \frac{112}{3} + \frac{22}{N} + \frac{8}{3N^2}$$

**Step 3. Calculate the limit.**

$$A = \lim_{N \to \infty} R_N = \lim_{N \to \infty} \left( \frac{112}{3} + \frac{22}{N} + \frac{8}{3N^2} \right) = \frac{112}{3}$$

The area under the graph of any polynomial can be calculated using power sum formulas as in the examples above. For other functions, such as in the next example, the limit defining the area may be difficult or impossible to evaluate directly.

**EXAMPLE 5** Let $A$ be the area under the graph of $f(x) = \sin x$ on the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$ (Figure 13). Set up (but do not compute) the expression $A = \lim_{N \to \infty} R_N$ for determining the area.

**Solution** In this case, $\Delta x = (3\pi/4 - \pi/4)/N = \pi/(2N)$ and the area $A$ is

$$A = \lim_{N \to \infty} R_N = \lim_{N \to \infty} \Delta x \sum_{j=1}^{N} f(a + j\Delta x) = \lim_{N \to \infty} \frac{\pi}{2N} \sum_{j=1}^{N} \sin \left( \frac{\pi}{4} + \frac{\pi j}{2N} \right)$$

The limit in the previous example is difficult to compute, however, it can be approximated by computing $R_N$ for large $N$ (see Exercise 83). In Section 5.4, we will see that this area is straightforward to compute (and is $\sqrt{2}$) using the definite integral and the Fundamental Theorem of Calculus.

**HISTORICAL PERSPECTIVE**

We used the formulas for the $k$th power sums for $k = 1, 2, 3$. Do similar formulas exist for all powers $k$? This problem was studied in the seventeenth century and eventually solved around 1690 by the great Swiss mathematician Jacob Bernoulli. Of this discovery, he wrote...
5.1 SUMMARY

- Approximations to the area under the graph of $f$ over the interval $[a, b]$ 
  \[ \Delta x = \frac{b - a}{N}, x_j = a + j \Delta x \]:

  \[ R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \Delta x (f(x_1) + f(x_2) + \cdots + f(x_N)) \]

  \[ L_N = \Delta x \sum_{j=0}^{N-1} f(x_j) = \Delta x (f(x_0) + f(x_1) + \cdots + f(x_{N-1})) \]

  \[ M_N = \Delta x \sum_{j=0}^{N-1} f \left( \frac{x_j + x_{j+1}}{2} \right) \]

  \[ = \Delta x \left( f \left( \frac{x_0 + x_1}{2} \right) + \cdots + f \left( \frac{x_{N-1} + x_N}{2} \right) \right) \]

- If $f$ is continuous on $[a, b]$, then the endpoint and midpoint approximations approach one and the same limit $L$:
  \[ \lim_{N \to \infty} R_N = \lim_{N \to \infty} L_N = \lim_{N \to \infty} M_N = L \]

- If $f(x) \geq 0$ on $[a, b]$, we take $L$ as the definition of the area under the graph of $y = f(x)$ over $[a, b]$.

5.1 EXERCISES

Preliminary Questions

1. What are the right and left endpoints if $[2, 5]$ is divided into six subintervals?

2. The interval $[1, 5]$ is divided into eight subintervals.
   (a) What is the left endpoint of the last subinterval?
   (b) What are the right endpoints of the first two subintervals?

3. Which of the following pairs of sums are not equal?

   (a) \[ \sum_{i=1}^{4} i, \sum_{i=1}^{4} (i - 1) \]
   (b) \[ \sum_{j=1}^{4} j^2, \sum_{k=2}^{5} k^2 \]
   (c) \[ \sum_{i=1}^{4} j, \sum_{i=2}^{4} (i - 1) \]
   (d) \[ \sum_{i=1}^{4} i(i + 1), \sum_{j=2}^{4} (j - 1)j \]

4. Explain: \[ \sum_{j=1}^{100} j = \sum_{j=1}^{100} 1 \text{ but } \sum_{j=1}^{100} 1 \text{ is not equal to } \sum_{j=1}^{100} 1. \]

5. Explain why $L_{100} \geq R_{100}$ for $f(x) = x^{-2}$ on $[3, 7]$.

Exercises

1. Figure 14 shows the velocity of an object over a 3-minute interval. Determine the distance traveled over the intervals $[0, 3]$ and $[1, 2.5]$ (remember to convert from kilometers per hour to kilometers per minute).

2. An ostrich (Figure 15) runs with velocity 20 km/hour for 2 minutes (min), 12 km/h for 3 min, and 40 km/h for another minute. Compute the total distance traveled and indicate with a graph how this quantity can be interpreted as an area.

![Figure 14](image1.png)

![Figure 15](image2.png)
3. A rainstorm hit Portland, Maine, in October 1996, resulting in record rainfall. The rainfall rate \( R(t) \) on October 21 is recorded, in centimeters per hour, in the following table, where \( t \) is the number of hours since midnight. Compute the total rainfall during this 24-hour period and indicate on a graph how this quantity can be interpreted as an area.

<table>
<thead>
<tr>
<th>( t ) (h)</th>
<th>0-2</th>
<th>2-4</th>
<th>4-9</th>
<th>9-12</th>
<th>12-20</th>
<th>20-24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(t) ) (cm/h)</td>
<td>0.5</td>
<td>0.3</td>
<td>1.0</td>
<td>2.5</td>
<td>1.5</td>
<td>0.6</td>
</tr>
</tbody>
</table>

4. The velocity of an object is \( v(t) = 12t \) m/s. Use Eq. (2) and geometry to find the distance traveled over the time intervals \([0, 2]\) and \([2, 5]\).

5. Compute \( R_5 \) and \( L_5 \) over \([0, 1]\) using the following values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>50</td>
<td>48</td>
<td>46</td>
<td>44</td>
<td>42</td>
<td>40</td>
</tr>
</tbody>
</table>

6. Compute \( R_6 \), \( L_6 \), and \( M_3 \) to estimate the distance traveled over \([0, 3]\) if the velocity at half-second intervals is as follows:

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ) (m/s)</td>
<td>0</td>
<td>12</td>
<td>18</td>
<td>25</td>
<td>20</td>
<td>14</td>
<td>20</td>
</tr>
</tbody>
</table>

7. Let \( f(x) = 2x + 3 \).
   (a) Compute \( R_5 \) and \( L_5 \) over \([0, 3]\).
   (b) Use geometry to find the exact value of the area \( A \), and compute the errors \( |A - R_5| \) and \( |A - L_5| \) in the approximations.

8. Repeat Exercise 7 for \( f(x) = 20 - 3x \) over \([2, 4]\).

9. Calculate \( R_5 \) and \( L_5 \) for \( f(x) = x^2 - x + 4 \) over \([1, 4]\). Then sketch the graph of \( f \) and the rectangles that make up each approximation. Is the area under the graph larger or smaller than \( R_5 \)? Is it larger or smaller than \( L_5 \)?

10. Let \( f(x) = \sqrt{x^2 + 1} \) and \( \Delta x = \frac{1}{4} \). Sketch the graph of \( f \) and draw the right-endpoint rectangles whose area is represented by the sum \( \sum_{i=1}^{4} f(1 + i\Delta x)\Delta x \).

11. Estimate \( R_3 \), \( M_3 \), and \( L_5 \) over \([0, 1.5]\) for the function in Figure 16.

12. Calculate the area of the shaded rectangles in Figure 17. Which approximation do these rectangles represent?

13. Let \( f(x) = x^2 \).
   (a) Sketch the function over the interval \([0, 2]\) and the rectangles corresponding to \( L_4 \). Calculate the area contained within them.
   (b) Sketch the function over the interval \([0, 2]\) again but with the rectangles corresponding to \( R_4 \). Calculate the area contained within them.
   (c) Make a conclusion about the area under the curve \( f(x) = x^2 \) over the interval \([0, 2]\).

14. Let \( f(x) = \sqrt{x} \).
   (a) Sketch the function over the interval \([0, 4]\) and the rectangles corresponding to \( L_4 \). Calculate the area contained within them.
   (b) Sketch the function over the interval \([0, 4]\) again but with the rectangles corresponding to \( R_4 \). Calculate the area contained within them.
   (c) Make a conclusion about the area under the curve \( f(x) = \sqrt{x} \) over the interval \([0, 4]\).

In Exercises 15–22, calculate the approximation for the given function and interval.

15. \( L_4 \), \( f(x) = \sqrt{2-x} \), \([0, 2]\)
16. \( L_4 \), \( f(x) = \sqrt{6x + 2} \), \([1, 3]\)
17. \( R_4 \), \( f(x) = 2x - x^2 \), \([0, 2]\)
18. \( R_3 \), \( f(x) = x^2 + x \), \([-1, 1]\)
19. \( M_3 \), \( f(x) = \frac{1}{x^2 + 1} \), \([1, 5]\)
20. \( M_4 \), \( f(x) = \sqrt{x} \), \([3, 5]\)
21. \( L_4 \), \( f(x) = \cos^2 x \), \([\frac{\pi}{6}, \frac{\pi}{2}]\)
22. \( L_6 \), \( f(x) = x^2 + 3x \), \([-2, 1]\)

In Exercises 23–28, write the sum in summation notation.

23. \( 4^3 + 5^3 + 6^3 + 7^3 \)
24. \( (2^2 + 2) + (3^2 + 3) + (4^2 + 4) + (5^2 + 5) \)
25. \( (2^2 + 2) + (3^2 + 2) + (4^2 + 2) + (5^2 + 2) \)
26. \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \)
27. \( \frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{(n+1)(n+2)} \)
28. \( \sin(n\pi) + \sin(n\pi/2) + \sin(n\pi/3) + \cdots + \sin(n\pi/(n+1)) \)

29. Calculate the sums:
   (a) \( \sum_{i=1}^{9} i \)
   (b) \( \sum_{i=1}^{4} i \)
   (c) \( \sum_{i=1}^{3} i^2 \)

30. Calculate the sums:
   (a) \( \sum_{j=3}^{n} \sin\left(\frac{j\pi}{2}\right) \)
   (b) \( \sum_{k=1}^{5} \frac{1}{k-1} \)
   (c) \( \sum_{j=1}^{2} 2^{3-j} \)
31. Let \( b_1 = 4, b_2 = 1, b_3 = 2, \) and \( b_4 = -4 \). Calculate:

(a) \( \sum_{j=2}^{5} b_j \)

(b) \( \sum_{j=1}^{3} (2b_j - b_j) \)

(c) \( \sum_{k=1}^{3} k b_k \)

32. Assume that \( a_1 = -5, a_2 = 20, \) and \( b_1 = 7 \). Calculate:

(a) \( \sum_{i=1}^{10} (4a_i + 3) \)

(b) \( \sum_{i=2}^{10} a_i \)

(c) \( \sum_{i=1}^{10} (2a_i - 3b_i) \)

33. Calculate \( \sum_{j=10}^{200} j \). Hint: Write as a difference of two sums and use formula (3).

34. Calculate \( \sum_{j=1}^{200} (2j + 1)^2 \). Hint: Expand and use formulas (3)-(4).

In Exercises 35–42, use linearity and formulas (3)-(5) to rewrite and evaluate the sums.

35. \( \sum_{j=1}^{20} 8j^3 \)

36. \( \sum_{k=1}^{20} (4k - 3) \)

37. \( \sum_{n=1}^{150} n^2 \)

38. \( \sum_{k=1}^{200} k^3 \)

39. \( \sum_{j=1}^{30} j(j - 1) \)

40. \( \sum_{j=0}^{30} (3j + 4) \)

41. \( \sum_{m=1}^{40} (4 - m)^2 \)

42. \( \sum_{m=1}^{300} (4 - m)^2 \)

In Exercises 43–46, use formulas (3)-(5) to evaluate the limit.

43. \( \lim_{N \to \infty} \sum_{i=1}^{N} \frac{1}{N^2} \)

44. \( \lim_{N \to \infty} \sum_{j=1}^{N} \frac{j^3}{N^4} \)

45. \( \lim_{N \to \infty} \sum_{i=1}^{N} \left( \frac{j^2 - i + 1}{N^3} \right) \)

46. \( \lim_{N \to \infty} \sum_{i=1}^{N} \left( \frac{j^3}{N^4} - \frac{20}{N} \right) \)

In Exercises 47–52, calculate the limit for the given function and interval. Verify your answer by using geometry.

47. \( \lim_{N \to \infty} R_N, \ f(x) = 9x, \ [0, 2] \)

48. \( \lim_{N \to \infty} R_N, \ f(x) = 3x + 6, \ [1, 4] \)

49. \( \lim_{N \to \infty} L_N, \ f(x) = \frac{1}{2} x + 2, \ [0, 4] \)

50. \( \lim_{N \to \infty} L_N, \ f(x) = 4x - 2, \ [1, 3] \)

51. \( \lim_{N \to \infty} M_N, \ f(x) = x, \ [0, 2] \)

52. \( \lim_{N \to \infty} M_N, \ f(x) = 12 - 4x, \ [2, 6] \)

53. Show, for \( f(x) = 3x^2 + 4x \) over \([0, 2] \), that

\[ R_N = \frac{2}{N} \sum_{j=1}^{N} \left( \frac{12j^2 + 8j}{N^2} \right) \]

Then evaluate \( \lim_{N \to \infty} R_N \).

54. Show, for \( f(x) = 3x^2 - x^2 \) over \([1, 5] \), that

\[ R_N = \frac{4}{N} \sum_{j=1}^{N} \left( \frac{192j^2}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2 \right) \]

Then evaluate \( \lim_{N \to \infty} R_N \).

In Exercises 55–62, find a formula for \( R_N \) and compute the area under the graph as a limit.

55. \( f(x) = x^2, \ [1, 3] \)

56. \( f(x) = x^2, \ [-1, 5] \)

57. \( f(x) = 6x^2 - 4, \ [2, 5] \)

58. \( f(x) = x^2 + 7x, \ [6, 11] \)

59. \( f(x) = x^3 - x, \ [0, 2] \)

60. \( f(x) = 2x^3 + x^2, \ [-2, 2] \)

61. \( f(x) = 2x + 1, \ [a, b] \) \( (a, b \) constants with \( a < b \))

62. \( f(x) = x^2, \ [a, b] \) \( (a, b \) constants with \( a < b \))

In Exercises 63–66, describe the area represented by the limits.

63. \( \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \left( \frac{j}{N} \right)^4 \)

64. \( \lim_{N \to \infty} \frac{3}{N} \sum_{j=1}^{N} \left( \frac{2j}{N} \right)^4 \)

65. \( \lim_{N \to \infty} \frac{5}{N} \sum_{j=0}^{N-1} \left( -2 + \frac{5j}{N} \right)^4 \)

66. \( \lim_{N \to \infty} \frac{\pi}{2N} \sum_{j=1}^{N} \sin \left( \frac{\pi j}{2N} \right) \)

In Exercises 67–72, express the area under the graph as a limit using the approximation indicated (in summation notation), but do not evaluate.

67. \( R_N, \ f(x) = \sin x \) over \([0, \pi] \)

68. \( R_N, \ f(x) = x^{-1} \) over \([1, 7] \)

69. \( L_N, \ f(x) = \sqrt{2x} + 1 \) over \([7, 11] \)

70. \( L_N, \ f(x) = \cos x \) over \([\frac{\pi}{2}, \frac{3\pi}{2}] \)

71. \( M_N, \ f(x) = \tan x \) over \([\frac{\pi}{4}, \frac{\pi}{2}] \)

72. \( M_N, \ f(x) = x^2 \) over \([3, 5] \)

73. Evaluate \( \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \left( 1 - \frac{j}{N} \right)^2 \) by interpreting it as the area of part of a familiar geometric figure.

In Exercises 74–76, let \( f(x) = x^2 \) and let \( R_N, L_N, \) and \( M_N \) be the approximations for the interval \([0, 1] \).

74. Show that \( R_N = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2} \). Interpret the quantity \( \frac{1}{2N} + \frac{1}{6N^2} \) as the area of a region.

75. Show that

\[ L_N = \frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2}, \quad M_N = \frac{1}{3} - \frac{1}{12N^2} \]

Then, given that the area under the graph of \( y = x^2 \) over \([0, 1] \) is \( \frac{1}{3} \), rank the three approximations \( R_N, L_N, \) and \( M_N \) in order of increasing accuracy (use Exercise 74).

76. For each of \( R_N, L_N, \) and \( M_N \), find the smallest integer \( N \) for which the error is less than 0.001.
In Exercises 77-82, use the Graphical Insight on page 244 to obtain bounds on the area.

77. Let \( A \) be the area under \( f(x) = \sqrt{x} \) over \([0, 1]\). By computing \( R_4 \) and \( L_4 \), prove that \( 0.51 \leq A \leq 0.77 \). Explain your reasoning.

78. Use \( R_3 \) and \( L_5 \) to show that the area \( A \) under \( y = \sqrt{x} \) over \([10, 13]\) satisfies \( 0.0218 \leq A \leq 0.0244 \).

79. Use \( R_4 \) and \( L_4 \) to show that the area \( A \) under the graph of \( y = \sin x \) over \([0, \frac{\pi}{2}]\) satisfies \( 0.79 \leq A \leq 1.19 \).

80. Show that the area \( A \) under \( f(x) = x^{-1} \) over \([1, 8]\) satisfies
\[
\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \leq A \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}
\]

81. **CAS** Show that the area \( A \) under \( y = x^{1/4} \) over \([0, 1]\) satisfies \( L_N \leq A \leq R_N \) for all \( N \). Use a computer algebra system to calculate \( L_N \) and \( R_N \) for \( N = 100 \) and \( 200 \), and determine \( A \) to two decimal places.

Further Insights and Challenges

87. Although the accuracy of \( R_N \) generally improves as \( N \) increases, this need not be true for small values of \( N \). Draw the graph of a positive continuous function \( f \) on an interval such that \( R_1 \) is closer than \( R_2 \) to the exact area under the graph. Can such a function be monotonic?

88. Draw the graph of a positive continuous function on an interval such that \( R_2 \) and \( L_2 \) are both smaller than the exact area under the graph. Can such a function be monotonic?

89. **Check** Explain graphically: The endpoint approximations are less accurate when \( f'(x) \) is large.

90. Prove that for any function \( f \) on \([a, b]\),
\[
R_N - L_N = \frac{b-a}{N} (f(b) - f(a))
\]

91. **CAS** In this exercise, we prove that \( \lim_{N \to \infty} R_N \) and \( \lim_{N \to \infty} L_N \) exist and are equal if \( f \) is increasing (the case of \( f \) decreasing is similar). We use the concept of a least upper bound discussed in Appendix B.

(a) Explain with a graph why \( L_N \leq R_N \) for all \( N, M \geq 1 \).

(b) By (a), the sequence \( \{L_N\} \) is bounded, so it has a least upper bound \( L \).

By definition, \( L \) is the smallest number such that \( L_N \leq L \) for all \( N \). Show that \( L = \lim_{N \to \infty} R_N \) for all \( M \).

(c) According to (b), \( L_N \leq L \leq R_N \) for all \( N \). Use Eq. (8) to show that \( \lim_{N \to \infty} L_N = L \) and \( \lim_{N \to \infty} R_N = L \).

92. **CAS** Use Eq. (8) to show that if \( f \) is positive and monotonic, then the area \( A \) under its graph over \([a, b]\) satisfies
\[
|R_N - A| \leq \frac{b-a}{N} |f(b) - f(a)|
\]

In Exercises 93-94, use Eq. (9) to find \( N \) such that \( |R_N - A| < 10^{-3} \) for the given function and interval.

93. \( f(x) = \sqrt{x} \), \([1, 4]\)

94. \( f(x) = \sqrt{9-x^2} \), \([0, 3]\)

95. **CAS** Prove that if \( f \) is positive and monotonic, then \( M_N \) lies between \( R_N \) and \( L_N \) and is closer to the actual area under the graph than both \( R_N \) and \( L_N \). Hint: In the case that \( f \) is increasing, Figure 18 shows that the part of the error in \( R_N \) due to the \( i \)th rectangle is the sum of the areas \( A + B + D \), and for \( M_N \) it is \( |B - E| \). On the other hand, \( A \geq E \).

5.2 The Definite Integral

In the previous section, we saw that if \( f \) is continuous on an interval \([a, b]\), then the endpoint and midpoint approximations approach a common limit \( L \) as \( N \to \infty \):

\[
L = \lim_{N \to \infty} R_N = \lim_{N \to \infty} L_N = \lim_{N \to \infty} M_N
\]

In a moment, we will state formally that \( L \) is the definite integral of \( f \) over \([a, b]\). Before doing so, we introduce Riemann sums, sums that are structured similar to the approximating sums \( L_N, R_N, \) and \( M_N \), and that generalize them.
Recall that \( R_N, L_N, \) and \( M_N \) use rectangles of equal width \( \Delta x \), whose heights are the values of \( f(x) \) at the endpoints or midpoints of the subintervals. In Riemann sums, we relax these requirements: The rectangles need not have equal width, \( f(x) \) can have any real-number value, and the height (which could be negative) may be any value of \( f(x) \) within the subinterval.

To specify a Riemann sum, we choose a partition and a set of sample points:

- **Partition** \( P \) of size \( N \): a choice of points that divides \([a, b]\) into \( N \) subintervals (not necessarily of equal width):
  \[
P : a = x_0 < x_1 < x_2 < \cdots < x_N = b
  \]
- **Sample points** \( C = \{c_1, \ldots, c_N\} \): \( c_i \) belongs to the subinterval \([x_{i-1}, x_i]\) for all \( i = 1, \ldots, N \), (and could be any point in the subinterval).

See Figures 1 and 2(A). The length of the \( i \)th subinterval \([x_{i-1}, x_i]\) is

\[
\Delta x_i = x_i - x_{i-1}
\]

The norm of the partition \( P \), denoted \( \| P \| \), is the maximum of the lengths \( \Delta x_i \).

Given \( P \) and \( C \), as Figure 2(B) illustrates, on each subinterval \([x_{i-1}, x_i]\) we have a rectangle whose height is \( f(c_i) \). We allow the possibility that \( f(c_i) < 0 \). For each rectangle, we refer to \( f(c_i) \Delta x_i \) as its signed area. Note that

- If \( f(c_i) > 0 \), the signed area is positive and is equal to the area of the rectangle
- If \( f(c_i) < 0 \) then the signed area is negative and is equal to the negative of the area of the rectangle.

The Riemann sum is the sum of the \( f(c_i) \Delta x_i \) terms that are determined by \( P \) and \( C \), and it is expressed as

\[
R(f, P, C) = \sum_{i=1}^{N} f(c_i) \Delta x_i = f(c_1) \Delta x_1 + f(c_2) \Delta x_2 + \cdots + f(c_N) \Delta x_N
\]

**Example 1** Calculate the Riemann sum \( R(f, P, C) \), where \( f(x) = 8 + 12 \sin x - 4x \) on \([0, 4]\).

\[
P : x_0 = 0, x_1 = 1, x_2 = 1.8, x_3 = 2.9, x_4 = 4
\]

\[
C : c_1 = 0.4, c_2 = 1.2, c_3 = 2, c_4 = 3.5
\]

What is the norm \( \| P \| \)?
Solution  The widths of the subintervals in the partition (Figure 3) are
\[ \Delta x_1 = x_1 - x_0 = 1 - 0 = 1, \quad \Delta x_2 = x_2 - x_1 = 1.8 - 1 = 0.8 \]
\[ \Delta x_3 = x_3 - x_2 = 2.9 - 1.8 = 1.1, \quad \Delta x_4 = x_4 - x_3 = 4 - 2.9 = 1.1 \]
The norm of the partition is \( \|P\| = 1.1 \) since the two longest subintervals have width 1.1. Using a calculator, we obtain
\[ R(f, P, C) = f(0.4)\Delta x_1 + f(1.2)\Delta x_2 + f(2)\Delta x_3 + f(3.5)\Delta x_4 \approx 11.071(0.4) + 14.38(0.8) + 10.91(1.1) - 10.2(1.1) \approx 23.35 \]

Note in Figure 2(C) that as the norm \( \|P\| \) tends to zero (meaning that the rectangles get thinner), the number of rectangles \( N \) tends to \( \infty \) and the sums \( R(f, P, C) \) approach a limiting value.

This leads to the following definition: \( f \) is integrable over \([a, b]\) if all of the Riemann sums (not just the endpoint and midpoint approximations) approach one and the same limit \( L \) as \( \|P\| \) tends to zero. Formally, we write
\[
L = \lim_{\|P\| \to 0} R(f, P, C) = \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i
\]

if \( |R(f, P, C) - L| \) gets arbitrarily small as the norm \( \|P\| \) tends to zero, no matter how we choose the partition and sample points. The limit \( L \) is called the definite integral of \( f \) over \([a, b]\).

**DEFINITION** Definite Integral  The definite integral of \( f \) over \([a, b]\), denoted by the integral sign, is the limit of Riemann sums:
\[
\int_{a}^{b} f(x) \, dx = \lim_{\|P\| \to 0} R(f, P, C) = \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i
\]

When this limit exists, we say that \( f \) is integrable over \([a, b]\).

The definite integral is often called, more simply, the integral of \( f \) over \([a, b]\). The process of computing integrals is called integration and \( f(x) \) is called the integrand. The endpoints \( a \) and \( b \) of \([a, b]\) are called the limits of integration. Finally, we remark that any variable may be used as a variable of integration. That is, that variable is a dummy variable. Thus, the following three integrals all denote the same quantity:
\[
\int_{a}^{b} \sin x \, dx, \quad \int_{t}^{b} \sin t \, dt, \quad \int_{a}^{b} \sin u \, du
\]

The next theorem assures us that continuous functions (and even functions with finitely many jump discontinuities) are integrable (see Appendix D for a proof). In practice, we rely on this theorem rather than attempting to prove directly that a given function is integrable.

**THEOREM 1** If \( f \) is continuous on \([a, b]\), or if \( f \) is continuous except at finitely many jump discontinuities in \([a, b]\), then \( f \) is integrable over \([a, b]\).

A constant function \( f(x) = K \) is integrable over every interval \([a, b]\). The following theorem provides the value of the integral and indicates that when \( K \) is positive, the integral is the area of the rectangle in Figure 4.
THEOREM 2 Integral of a Constant

For any constant function $f(x) = K$, 
$$\int_a^b f(x) \, dx = K(b - a)$$

Proof Let $R(f, P, C) = \sum_{i=1}^N f(c_i)\Delta x_i$ be a Riemann sum for $f$ over $[a, b]$. Then, $f(c_i) = K$ for each $c_i$, and therefore,
$$R(f, P, C) = \sum_{i=1}^N f(c_i)\Delta x_i = \sum_{i=1}^N K\Delta x_i = K \sum_{i=1}^N \Delta x_i = K(b - a)$$
The latter equality holds because the sum of the $\Delta x_i$ is the overall length $b - a$ of the interval $[a, b]$. Since every Riemann sum has the value $K(b - a)$, it follows that the integral does as well. Thus, 
$$\int_a^b f(x) \, dx = K(b - a).$$

The Definite Integral and Signed Area

We motivated the development of approximating sums, Riemann sums, and the definite integral with the problem of determining the area under the graph of a function. The definite integral provides an exact value for such areas. Specifically, if $f$ is integrable and $f(x) \geq 0$ over $[a, b]$, then the area under the graph of $f$ over $[a, b]$ is 
$$\int_a^b f(x) \, dx.$$ 

When the geometry of a region is simple (e.g., rectangles, triangles, circles) the definite integral corresponds to the area obtained by a geometric formula, as demonstrated for a rectangle in Theorem 2. In some instances, this correspondence enables us to compute a definite integral using geometric formulas (as in Example 2 below).

Allowing the possibility that $f(x)$ takes on both positive and negative values, we define the notion of signed area, where regions below the $x$-axis provide a negative contribution (Figure 5). Intuitively, the signed area of a region is the area above the $x$-axis minus the area below.

The $f(c_i)\Delta x_i$ terms in a Riemann sum are signed areas of rectangles. Therefore, a signed area such as in Figure 5 can be approximated by Riemann sums and is given by a definite integral. Thus, for all integrable $f$ over $[a, b]$,

The **signed area** between the graph of $f$ and the $x$-axis over $[a, b]$ is 
$$\int_a^b f(x) \, dx.$$ 

EXAMPLE 2 Definite Integrals Via Simple Geometry

Calculate 
$$\int_0^5 (3 - x) \, dx \quad \text{and} \quad \int_0^5 |3 - x| \, dx.$$ 

Solution The region between $y = 3 - x$ and the $x$-axis consists of two triangles of areas $\frac{9}{2}$ and $2$ (Figure 6(A)). The triangle with area $\frac{9}{2}$ lies above the $x$-axis and therefore has signed area $\frac{9}{2}$. The second triangle lies below the $x$-axis, so it has signed area $-2$. In the graph of $y = |3 - x|$, both triangles lie above the $x$-axis (Figure 6(B)). It follows that 
$$\int_0^5 (3 - x) \, dx = \frac{9}{2} - 2 = \frac{5}{2} \quad \text{and} \quad \int_0^5 |3 - x| \, dx = \frac{9}{2} + 2 = \frac{13}{2}.$$ 

In the next example, the first integral is geometrically simple but the second is not. For the second, we need to take a limit of Riemann sums to arrive at a value. In Section 5.4, we will start to develop tools that make the computation of integrals such as the latter much simpler.

Scanned with CamScanner
EXAMPLE 3 For $b > 0$, calculate
\[
(a) \quad \int_0^b x \, dx \quad \text{and} \quad (b) \quad \int_0^b x^2 \, dx
\]
Solution The integrals represent the shaded areas in Figure 7.

(a) This integral represents the area of a triangle with base and height equal to $b$. Therefore,
\[
\int_0^b x \, dx = \frac{1}{2} b \cdot b = \frac{b^2}{2}
\]

(b) We evaluate by taking a limit of Riemann sums. Because $f(x) = x^2$ is continuous, it is also integrable by Theorem 1. It follows that the right-endpoint approximations $R_N$ converge to the integral. Therefore, we compute $\int_0^b x^2 \, dx$ by computing $\lim_{N \to \infty} R_N$.

We divide the interval $[0, b]$ into $N$ subintervals of width $\Delta x = \frac{b-0}{N} = \frac{b}{N}$.

\[
R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{b}{N} \sum_{j=1}^{N} \left( \frac{j}{N} \right) = \frac{b}{N} \sum_{j=1}^{N} \left( \frac{j}{N} \right)^2
\]
\[
= \frac{b^3}{N^3} \sum_{j=1}^{N} j^2 = \frac{b^3}{N^3} \left( \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \right) = \frac{b^3}{3} + \frac{b^3}{2N} + \frac{b^3}{6N^2}
\]

Taking the limit of $R_N$ as $N \to \infty$, we obtain
\[
\int_0^b x^2 \, dx = \frac{b^3}{3}.
\]

The results of the previous example will be helpful in other definite integral computations in this section. Summarizing, if $b > 0$, then
\[
\int_0^b x \, dx = \frac{b^2}{2} \quad \text{and} \quad \int_0^b x^2 \, dx = \frac{b^3}{3}
\]

The integrals in Eq. (5) suggest a pattern. What do you think the value of $\int_0^b x^n \, dx$ is?

The $n = 3$ case is considered in Exercise 50. We will compute this integral for general $n$ in Section 5.4.

In cases where we do not have an exact representation of a function, such as in the following example, the best we can do to compute a definite integral is to approximate it using a Riemann sum.

EXAMPLE 4 A Grid-Connected Energy System At their home, Malina and Hors have solar panels and an energy system that is grid connected. They use energy from their panels and from the Apollo Power Company (APC) for their household needs. When the
panels are supplying more energy than they need, the excess is fed into the grid and they receive energy credit from APC.

We can view their daily APC energy use $E$ (in kilowatts) as a function of time $t$ (in hours). When $E < 0$, they are supplying energy to APC.

Their net APC energy use over a period of time $a < t < b$ is given by $\int_a^b E(t) \, dt$ (in kilowatt-hours). For the function $E$ shown in the graph in Figure 8, approximate the net APC energy use from 6:00 AM ($t = 6$) to 6:00 PM ($t = 18$).

![Graph of energy use $E$ in kilowatts as a function of time $t$ in hours since midnight.](image)

**Figure 8** Energy use $E$ in kilowatts as a function of time $t$ in hours since midnight.

**Solution** We approximate $\int_6^{18} E(t) \, dt$ with a left-endpoint approximation and $\Delta t = 1$.

That sum is expressed as

$$L = E(6)\Delta t + E(7)\Delta t + \cdots + E(17)\Delta t$$

$$= (2.5)(1) + (2)(1) + (0.3)(1) + (0.5)(1) + (0.4)(1) + (-0.8)(1)$$

$$+ (-1.3)(1) + (-1.2)(1) + (-0.5)(1) + (-0.8)(1) + (0.3)(1) + (1.2)(1)$$

$$= 2.6$$

where the values of $E(t)$ are approximated from the graph. Therefore, the net 12-hour APC energy use is

$$\int_6^{18} E(t) \, dt \approx 2.6 \text{ kilowatt-hours}$$

**Properties of the Definite Integral**

In the rest of this section, we discuss some basic properties of definite integrals, beginning with the linearity properties.

**Theorem 3 Linearity of the Definite Integral** If $f$ and $g$ are integrable over $[a, b]$, then $f + g$ and $Cf$ are integrable (for any constant $C$), and

- \[ \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]
- \[ \int_a^b Cf(x) \, dx = C \int_a^b f(x) \, dx \]

**Proof** These properties follow from the corresponding linearity properties of sums and limits. For example, Riemann sums are additive:

$$R(f + g, P, C) = \sum_{i=1}^{N} (f(c_i) + g(c_i))\Delta x_i = \sum_{i=1}^{N} f(c_i)\Delta x_i + \sum_{i=1}^{N} g(c_i)\Delta x_i$$

$$= R(f, P, C) + R(g, P, C)$$
By the additivity of limits,
\[
\int_a^b (f(x) + g(x)) \, dx = \lim_{||P|| \to 0} R(f + g, P, C) = \lim_{||P|| \to 0} R(f, P, C) + \lim_{||P|| \to 0} R(g, P, C) = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx
\]

The second property is proved similarly.

**EXAMPLE 5** Calculate \( \int_0^3 (2x^2 - 3x - 7) \, dx \).

Solution
\[
\int_0^3 (2x^2 - 3x - 7) \, dx = 2 \int_0^3 x^2 \, dx - 3 \int_0^3 x \, dx + \int_0^3 (-7) \, dx \quad \text{(linearity)}
\]
\[
= 2 \left( \frac{3^3}{3} \right) - 3 \left( \frac{3^2}{2} \right) - 7(3 - 0) = -\frac{33}{2}
\]

[Eq. (5)]

So far we have used the notation \( \int_a^b f(x) \, dx \) with the understanding that \( a < b \).

Can we make sense of and compute integrals like \( \int_{-1}^1 f(x) \, dx \)? The answer is yes. To do so rigorously involves expanding the idea of Riemann sum to allow \( \Delta x_i \) to be negative because we are essentially integrating in a negative direction. We do not pursue the details here. For our purposes, the following definition captures the idea properly.

**DEFINITION** Reversing the Limits of Integration For \( a < b \), we set
\[
\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx
\]

For example, by Eqs. (5) and (6),
\[
\int_0^5 x^2 \, dx = - \int_0^5 x^2 \, dx = -\frac{5^3}{3} = -\frac{125}{3}
\]

When \( a = b \), the interval \([a, b] = [a, a]\) has length zero and we define the definite integral to be zero:
\[
\int_a^a f(x) \, dx = 0
\]

**EXAMPLE 6** Prove that, for all \( b \) (negative, zero, and positive),
\[
\int_0^b x \, dx = \frac{1}{2} b^2 \quad \text{and} \quad \int_0^b x^2 \, dx = \frac{1}{3} b^3
\]

Solution These integral formulas hold for \( b > 0 \) by Eq. (5), and they hold for \( b = 0 \) by definition of a definite integral on an interval of length zero.
Now, we consider the situation where \( b < 0 \). In this case, the integrals can be determined from the signed area of the dark shaded regions in Figure 9. The signed area of the dark shaded triangle is \( \int_{b}^{0} x \, dx \). By symmetry and Eq. (5),

\[
\int_{b}^{0} x \, dx = - \int_{0}^{|b|} x \, dx = - \frac{1}{2} |b|^2
\]

Therefore,

\[
\int_{0}^{b} x \, dx = \frac{1}{2} |b|^2 = \frac{1}{2} b^2
\]

In a similar way, by symmetry and Eq. (5), we have

\[
\int_{b}^{0} x^2 \, dx = \int_{0}^{|b|} x^2 \, dx = \frac{1}{3} |b|^3
\]

Therefore,

\[
\int_{0}^{b} x^2 \, dx = \frac{1}{3} |b|^3 = \frac{1}{3} b^3
\]

Definite integrals satisfy an important additivity property: If \( f \) is an integrable function and \( a \leq b \leq c \) as in Figure 10, then the integral from \( a \) to \( c \) is equal to the integral from \( a \) to \( b \) plus the integral from \( b \) to \( c \). We state this in the next theorem (a formal proof can be given using Riemann sums).

**Theorem 4** Additivity for Adjacent Intervals Let \( a \leq b \leq c \), and assume that \( f \) is integrable. Then

\[
\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx
\]

This theorem remains true as stated even if the condition \( a \leq b \leq c \) is not satisfied (Exercise 87).

**Example 7** Calculate \( \int_{4}^{7} x^2 \, dx \).

**Solution** Using Theorem 4, we can write

\[
\int_{0}^{4} x^2 \, dx + \int_{4}^{7} x^2 \, dx = \int_{0}^{7} x^2 \, dx
\]

Solving this equation for the desired integral, and using Eq. (5), we have

\[
\int_{4}^{7} x^2 \, dx = \int_{0}^{7} x^2 \, dx - \int_{0}^{4} x^2 \, dx = \left( \frac{1}{3} \right) 7^3 - \left( \frac{1}{3} \right) 4^3 = 93
\]

Another basic property of the definite integral is that if \( f(x) \geq g(x) \), then the integral of \( f \) is greater than the integral of \( g \) (Figure 11).

**Theorem 5** Comparison Theorem If \( f \) and \( g \) are integrable and \( g(x) \leq f(x) \) for \( x \) in \([a, b]\), then

\[
\int_{a}^{b} g(x) \, dx \leq \int_{a}^{b} f(x) \, dx
\]
Proof If \( g(x) \leq f(x) \), then for any partition and choice of sample points, we have
\[
\sum_{i=1}^{N} g(c_i) \Delta x_i \leq \sum_{i=1}^{N} f(c_i) \Delta x_i
\]
for all \( i \). Therefore, the Riemann sums satisfy
\[
\int_{a}^{b} g(x) \, dx = \lim_{\|P\| \to 0} \sum_{i=1}^{N} g(c_i) \Delta x_i \leq \lim_{\|P\| \to 0} \sum_{i=1}^{N} f(c_i) \Delta x_i = \int_{a}^{b} f(x) \, dx
\]

**EXAMPLE 8** Prove the inequality \( \int_{1}^{4} \frac{1}{x^2} \, dx \leq \int_{1}^{4} \frac{1}{x} \, dx \).

**Solution** If \( x \geq 1 \), then \( x^2 \geq x \), and \( x^{-2} \leq x^{-1} \) (Figure 12). Therefore, the inequality follows from the Comparison Theorem, applied with \( g(x) = x^{-2} \) and \( f(x) = x^{-1} \).

Suppose there are numbers \( m \) and \( M \) such that \( m \leq f(x) \leq M \) for \( x \) in \([a, b]\). We call \( m \) and \( M \) lower and upper bounds for \( f(x) \) on \([a, b]\). By the Comparison Theorem,
\[
\int_{a}^{b} m \, dx \leq \int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} M \, dx
\]
By Theorem 2, it follows that
\[
m(b-a) \leq \int_{a}^{b} f(x) \, dx \leq M(b-a)
\]
This says simply that if \( f(x) \geq 0 \) over \([a, b]\), then (as in Figure 13) the integral of \( f \) lies between the areas of two rectangles, one of height \( M \) enclosing the region associated with the integral, and one of height \( m \) enclosed in the region.

**EXAMPLE 9** Prove the inequalities \( \frac{3}{4} \leq \int_{1/2}^{2} \frac{1}{x} \, dx \leq 3 \).

**Solution** Because \( f(x) = x^{-1} \) is decreasing (Figure 14), its minimum value on \([\frac{1}{2}, 2]\) is \( m = f(2) = \frac{1}{2} \) and its maximum value is \( M = f(\frac{1}{2}) = 2 \). By Eq. (8),
\[
\int_{1/2}^{2} \frac{1}{x} \, dx \leq \frac{3}{4} \leq 2 \left( \frac{1}{2} - \frac{1}{2} \right) = 3
\]

**CONCEPTUAL INSIGHT** Keep in mind that a definite integral \( \int_{a}^{b} f(x) \, dx \) is defined as a limit of Riemann sums over finer and finer partitions of \([a, b]\), and each Riemann sum \( R(f, P, C) \) is a sum of terms \( f(c_i) \Delta x_i \) determined by a partition \( P \). The situation that we saw with the derivative in Chapters 3 and 4 is being repeated in our development of the theory of the definite integral:

- **Definition:** We define the definite integral via a limit definition. Limits are needed in the definition in order to properly capture the concept.
- **Properties:** We establish properties of the definite integral by proving theorems based on the limit definition. Some properties provide insight into the workings of the definite integral, and some provide computational tools.
• **Computation:** Computing the definite integral directly from the limit definition is messy at best and generally very difficult. We establish rules that aid us significantly in carrying out definite integral computations. The most important is the first part of the Fundamental Theorem of Calculus that we introduce in Section 5.4.

• **Application:** While most of the definite integral computations that we do are simplified by computation rules, we cannot lose sight of its definition. Knowing that a definite integral is a limit of sums defined over finer and finer partitions of the domain will help us identify when to use this tool in applications. We will see plenty of instances in the sections and chapters ahead.

### 5.2 SUMMARY

• A Riemann sum \( R(f, P, C) \) for the interval \([a, b]\) is defined by choosing a partition

\[
P : a = x_0 < x_1 < x_2 < \cdots < x_N = b
\]

and sample points \( C = \{c_i\} \), where \( c_i \in [x_{i-1}, x_i] \). Let \( \Delta x_i = x_i - x_{i-1} \). Then

\[
R(f, P, C) = \sum_{i=1}^{N} f(c_i) \Delta x_i
\]

• The maximum of the widths \( \Delta x_i \) is called the norm \( ||P|| \) of the partition.

• The **definite integral** is the limit of the Riemann sums (if it exists):

\[
\int_a^b f(x) \, dx = \lim_{||P|| \to 0} R(f, P, C)
\]

We say that \( f \) is integrable over \([a, b]\) if the limit exists.

• Theorem: If \( f \) is continuous on \([a, b]\), then \( f \) is integrable over \([a, b]\).

• The **signed area** of the region between the graph of \( f \) and the \( x \)-axis over \([a, b]\) is

\[
\int_a^b f(x) \, dx
\]

• Using geometry: When the geometry of the corresponding region is simple, \( \int_a^b f(x) \, dx \) can be computed using geometric formulas to determine the signed areas involved.

\[
\int_0^b x \, dx = \frac{1}{2} b^2 \quad \text{and} \quad \int_0^b x^2 \, dx = \frac{1}{3} b^3
\]

• Properties of definite integrals:

\[
\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx
\]

\[
\int_a^b C f(x) \, dx = C \int_a^b f(x) \, dx \quad \text{for any constant } C
\]

\[
\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx
\]

\[
\int_a^a f(x) \, dx = 0
\]

\[
\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx \quad \text{for all } a, b, c
\]
- Comparison Theorem: If \( f(x) \leq g(x) \) on \([a, b]\), then

\[
\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx
\]

- If \( m \leq f(x) \leq M \) on \([a, b]\), then

\[
m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)
\]

### 5.2 EXERCISES

#### Preliminary Questions

1. What is \( \int_3^6 dx \) [the function is \( f(x) = 1 \)]?

2. Let \( I = \int_2^7 f(x) \, dx \), where \( f \) is continuous. State whether the following are true or false:
   (a) \( I \) is the area between the graph and the x-axis over \([2, 7]\).
   (b) If \( f(x) \geq 0 \), then \( I \) is the area between the graph and the x-axis over \([2, 7]\).

3. If \( f(x) \leq 0 \), then \( -I \) is the area between the graph of \( f \) and the x-axis over \([2, 7]\).

4. Explain graphically: \( \int_0^\pi \cos x \, dx = 0 \).

5. Which is negative, \( \int_{-1}^5 8 \, dx \) or \( \int_{-1}^1 8 \, dx \)?

#### Exercises

In Exercises 1–10, draw a graph of the signed area represented by the integral and compute it using geometry.

1. \( \int_{-3}^2 2x \, dx \)
2. \( \int_{-2}^1 (2x + 4) \, dx \)
3. \( \int_{-2}^1 (3x + 4) \, dx \)
4. \( \int_{-2}^1 4x \, dx \)
5. \( \int_0^5 (7 - x) \, dx \)
6. \( \int_{-\pi/2}^{\pi/2} \sin x \, dx \)
7. \( \int_{-3}^5 \sqrt{25 - x^2} \, dx \)
8. \( \int_{-1}^3 |x| \, dx \)
9. \( \int_{-2}^1 (2 - |x|) \, dx \)
10. \( \int_{-2}^3 (3 + x - 2|x|) \, dx \)

11. Calculate \( \int_0^4 (8 - x) \, dx \) in two ways:
   (a) As the limit \( \lim_{N \to \infty} R_N \)
   (b) By sketching the relevant signed area and using geometry

12. Calculate \( \int_{-1}^4 (4x - 8) \, dx \) in two ways:
   (a) As the limit \( \lim_{N \to \infty} R_N \)
   (b) By using geometry

In Exercises 13 and 14, refer to Figure 15.

13. Evaluate: (a) \( \int_0^2 f(x) \, dx \) (b) \( \int_0^6 f(x) \, dx \)

14. Evaluate: (a) \( \int_1^4 f(x) \, dx \) (b) \( \int_1^6 |f(x)| \, dx \)

![Figure 15](image)

The two parts of the graph are semicircles.

In Exercises 15 and 16, refer to Figure 16.

15. Evaluate \( \int_0^3 g(t) \, dt \) and \( \int_3^5 g(t) \, dt \).

16. Find \( a, b, \) and \( c \) such that \( \int_0^a g(t) \, dt \) and \( \int_b^c g(t) \, dt \) are as large as possible.

![Figure 16](image)
17. Describe the partition $P$ and the set of sample points $C$ for the Riemann sum shown in Figure 17. Compute the value of the Riemann sum.

In Exercises 18–22, calculate the Riemann sum $R(f, P, C)$ for the given function, partition, and choice of sample points. Also, sketch the graph of $f$ and the rectangles corresponding to $R(f, P, C)$.

18. $f(x) = x, \quad P = \{1, 1, 2, 1, 5, 2\}, \quad C = \{1, 1, 1, 4, 1, 9\}$

19. $f(x) = 2x + 3, \quad P = \{-4, -1, 1, 4, 8\}, \quad C = \{-3, 0, 2, 5\}$

20. $f(x) = x^2 + x, \quad P = \{2, 3, 4, 5, 5\}, \quad C = \{2, 3, 5, 5\}$

21. $f(x) = \sin x, \quad P = \{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\}, \quad C = \{0, 0, 0.7, 1.2\}$

22. $f(x) = x^2 + x, \quad P = \{0, 1, 2, 5, 3, 2, 5\}, \quad C = \{0.5, 3, 5, 4\}$

23. In Example 4, approximate the net APC energy use from midnight to noon.

24. In Example 4, approximate the net APC energy use from noon to midnight.

In Exercises 25–30, sketch the signed area represented by the integral. Indicate the regions of positive and negative areas.

25. $\int_0^5 (4x - x^2)\,dx$

26. $\int_{-\pi/4}^{\pi/4} \tan x\,dx$

27. $\int_0^{2\pi} \sin x\,dx$

28. $\int_0^{2\pi} \sin x\,dx$

29. $\int_0^{1} (1 - 2x)\,dx$

30. $\int_0^{1} (x^2 - 1)(x^2 - 4)\,dx$

In Exercises 31–34, determine the sign of the integral without calculating it. Draw a graph if necessary.

31. $\int_{-2}^{1} x^2\,dx$

32. $\int_{-2}^{1} x^3\,dx$

33. $\int_{0}^{2\pi} x\sin x\,dx$

34. $\int_{0}^{2\pi} x\sin x\,dx$

In Exercises 35–44, use properties of the integral and the formulas in the summary to calculate the integrals.

35. $\int_{0}^{4} (6x - 3)\,dx$

36. $\int_{0}^{2} (4x + 7)\,dx$

37. $\int_{0}^{2} x^2\,dx$

38. $\int_{1/2}^{1/2} x^2\,dx$

39. $\int_{0}^{1} (x^2 - 2x)\,dx$

40. $\int_{0}^{1} (12y^2 + 6y)\,dy$

41. $\int_{0}^{1} (7t^2 - t + 1)\,dt$

42. $\int_{0}^{1} (9x - 4x^2)\,dx$

43. $\int_{0}^{1} (x^2 + x)\,dx$

44. $\int_{0}^{2} x^2\,dx$

In Exercises 45–48, calculate the integral, assuming that $\int_{0}^{5} f(x)\,dx = 5$ and $\int_{0}^{5} g(x)\,dx = 12$.

45. $\int_{0}^{5} (f(x) + g(x))\,dx$

46. $\int_{0}^{5} \left(2f(x) - \frac{1}{3}g(x)\right)\,dx$

47. $\int_{0}^{5} g(x)\,dx$

48. $\int_{0}^{5} (f(x) - x)\,dx$

49. Assume $a < b$ and $H$ is the Heaviside function given by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Find an expression for $\int_{a}^{b} H(x)\,dx$ in terms of $a$ and $b$.

50. By computing the limit of right-endpoint approximations, prove that if $b > 0$, then $\int_{0}^{b} x^2\,dx = \frac{b^4}{4}$.

51. Using the result of Exercise 50, prove that for all $b$ (negative, zero, and positive),

$$\int_{0}^{b} x^3\,dx = \frac{b^4}{4}$$

In Exercises 52–56, evaluate the integral using the formulas in the summary and Eq. (9).

52. $\int_{1}^{3} x^3\,dx$

53. $\int_{0}^{2} (x - x^2)\,dx$

54. $\int_{0}^{1} (2x^3 - x + 4)\,dx$

55. $\int_{0}^{1} (12x^3 + 24x^2 - 8x)\,dx$

56. $\int_{-2}^{2} (x^3 - 3x^3)\,dx$

In Exercises 57–60, calculate the integral, assuming that $\int_{0}^{1} f(x)\,dx = 1$, $\int_{0}^{2} f(x)\,dx = 4$, $\int_{0}^{4} f(x)\,dx = 7$.

57. $\int_{0}^{4} f(x)\,dx$

58. $\int_{0}^{2} f(x)\,dx$

59. $\int_{0}^{4} f(x)\,dx$

60. $\int_{0}^{4} f(x)\,dx$

In Exercises 61–64, express each integral as a single integral.

61. $\int_{0}^{3} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

62. $\int_{0}^{3} f(x)\,dx - \int_{0}^{9} f(x)\,dx$

63. $\int_{0}^{3} f(x)\,dx - \int_{0}^{5} f(x)\,dx$

64. $\int_{0}^{3} f(x)\,dx + \int_{0}^{9} f(x)\,dx$
In Exercises 65–66, prove the relationship for arbitrary $a$ and $b$ using the formulas in the summary.

65. $\int_a^b x \, dx = \frac{b^2 - a^2}{2}$

66. $\int_a^b x^3 \, dx = \frac{b^4 - a^4}{4}$

67. Explain the difference in graphical interpretation between $\int_a^b f(x) \, dx$ and $\int_a^b |f(x)| \, dx$.

68. Use the graphical interpretation of the definite integral to explain the inequality

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx$$

where $f$ is continuous. Explain also why equality holds if and only if either $f(x) \geq 0$ for all $x$ or $f(x) \leq 0$ for all $x$.

69. Let $f(x) = x$. Find an interval $[a, b]$ such that

$$\left| \int_a^b f(x) \, dx \right| = \frac{1}{2} \quad \text{and} \quad \int_a^b |f(x)| \, dx = \frac{3}{2}$$

70. Evaluate $I = \int_0^\pi \sin^2 x \, dx$ and $J = \int_0^\pi \cos^2 x \, dx$ as follows. First, show with a graph that $I = J$. Then, prove that $I + J = 2\pi$.

In Exercises 71–74, calculate the integral.

71. $\int_0^3 |3 - x| \, dx$

72. $\int_1^3 |2x - 4| \, dx$

73. $\int_{-1}^1 |x^2| \, dx$

74. $\int_0^4 |x^2 - 1| \, dx$

75. Use the Comparison Theorem to show that

$$\int_0^1 x^3 \, dx \leq \int_0^2 x^3 \, dx, \quad \int_1^2 x^4 \, dx \leq \int_1^2 x^2 \, dx$$

76. Prove that $\frac{1}{3} \leq \int_a^b ax \, dx \leq \frac{1}{2}$.

77. Prove that $0.0198 \leq \int_0^{0.3} \sin x \, dx \leq 0.2096$. Hint: Show that $0.2098 \leq \sin x \leq 0.2598$ for $x$ in $[0.2, 0.3]$.

78. Prove that $0.277 \leq \int_{\pi/4}^{\pi/2} \sqrt{x} \, dx \leq 0.363$.

79. Prove that $0 \leq \int_{\pi/4}^{3\pi/2} \sqrt{x} \, dx \leq \sqrt{\pi} / 2$.

80. Find upper and lower bounds for $\int_0^1 1 / \sqrt{5x^2 + 4} \, dx$.

81. Suppose that $f(x) \leq g(x)$ on $[a, b]$. By the Comparison Theorem, $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$. Is it also true that $f'(x) \leq g'(x)$ for $x \in [a, b]$? If not, give a counterexample.

82. State whether the following statement is true or false. If false, sketch the graph of a counterexample.

(a) If $f(x) > 0$, then $\int_a^b f(x) \, dx > 0$.

(b) If $\int_a^b f(x) \, dx > 0$, then $f(x) > 0$.

**Further Insights and Challenges**

83. Explain graphically: If $f$ is an odd function, then $\int_{-a}^a f(x) \, dx = 0$.

84. Compute $\int_0^1 (\sin x)(\sin^2 x + 1) \, dx$.

85. Let $k$ and $b$ be positive. Show, by comparing the right-endpoint approximations, that

$$\int_0^b x^k \, dx = b^{k+1} \int_0^1 x^k \, dx$$

86. Suppose that $f$ and $g$ are continuous functions such that, for all $a$,

$$\int_a^b f(x) \, dx = \int_a^b g(x) \, dx$$

Give an intuitive argument showing that $f(0) = g(0)$. Explain your idea with a graph.

87. Theorem 4 remains true without the assumption $a \leq b \leq c$. Verify this for the case $a < b < c$. Show that the integral from $a$ to $b$ is the same as the difference of the integrals from $a$ to $c$ and from $b$ to $c$.

### 5.3 The Indefinite Integral

In earlier chapters, we have seen how useful it is to be able to find the derivative of a function. But what about the inverse problem? Given the derivative of an unknown function, can we find the function itself? For example, in physics we may know the velocity $v(t)$ (the derivative) and wish to compute the position $s(t)$ of an object. Since $s'(t) = v(t)$, this amounts to finding a function whose derivative is $v(t)$. A function $F$ whose derivative is $f$ is called an antiderivative of $f$. Antiderivatives will turn out to be the key to evaluating definite integrals.

**Definition** Antiderivatives. A function $F$ is an antiderivative of $f$ on an open interval $(a, b)$ if $F'(x) = f(x)$ for all $x$ in $(a, b)$.
Examples:

- \( F(x) = -\cos x \) is an antiderivative of \( f(x) = \sin x \) because for all values of \( x \),
  \[
  F'(x) = \frac{d}{dx} (-\cos x) = \sin x = f(x)
  \]

- \( F(x) = \frac{1}{3}x^3 \) is an antiderivative of \( f(x) = x^2 \) because for all values of \( x \),
  \[
  F'(x) = \frac{d}{dx} \left( \frac{1}{3}x^3 \right) = x^2 = f(x)
  \]

One critical observation is that antiderivatives are not unique. We are free to add a constant \( C \) because the derivative of a constant is zero, and so, if \( F'(x) = f(x) \), then \( (F(x) + Cy) = f(x) \). For example, each of the following is an antiderivative of \( x^2 \):

\[
\frac{1}{3}x^3, \quad \frac{2}{3}x^3 + 5, \quad \frac{1}{3}x^3 - 4
\]

Are there any antiderivatives of \( f \) other than those obtained by adding a constant to a given antiderivative \( F \)? Our next theorem says that the answer is no if \( f \) is defined on an open interval \((a, b)\).

**THEOREM 1** The General Antiderivative Let \( y = F(x) \) be an antiderivative of \( y = f(x) \) on \((a, b)\). Then every antiderivative on \((a, b)\) is of the form \( y = F(x) + C \) for some constant \( C \).

**Proof** Assume \( y = G(x) \) is an antiderivative of \( y = f(x) \), and set \( H(x) = G(x) - F(x) \). Then \( H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0 \). By the Corollary to the Mean Value Theorem in Section 4.3, \( H(x) \) must be a constant—say, \( H(x) = C \)—and therefore \( G(x) = F(x) + C \).

**GRAPHICAL INSIGHT** The graph of \( y = F(x) + C \) is obtained by shifting the graph of \( y = F(x) \) vertically by \( C \) units. Since vertical shifting moves the tangent lines without changing their slopes, it makes sense that the functions \( y = F(x) + C \), for all possible \( C \), have the same derivative (Figure 1). Theorem 1 asserts that these functions are all of the functions with the same derivative as \( F \). That is, they all are in the form \( y = F(x) + C \) and have a graph that is a vertical shift of the graph of \( y = F(x) \).

We often describe the general antiderivative of a function in terms of an arbitrary constant \( C \), as in the following example.

**EXAMPLE 1** Find the general antiderivative of \( f(x) = \cos x \).

**Solution** The function \( F(x) = \sin x \) is an antiderivative of \( f(x) = \cos x \). The general antiderivative is \( F(x) = \sin x + C \), where \( C \) is any constant.

The process of finding an antiderivative is called **antidifferentiation** or **integration**. In the next section we will see why the term "integration" is used when we discuss the connection between antiderivatives and areas under curves given by the Fundamental Theorem of Calculus. Anticipating this result, we begin using the following notation and additional terminology for the general antiderivative:

**NOTATION** Indefinite Integral The notation

\[
\int f(x)\,dx = F(x) + C \quad \text{means that} \quad F'(x) = f(x)
\]

We say that \( y = F(x) + C \) is the general antiderivative or **indefinite integral** of \( y = f(x) \).
The expression \( f(x) \) appearing in the integral sign is called the \textit{Integrand}. The symbol \( dx \) is a differential. It is part of the integral notation and serves to indicate the independent variable. The constant \( C \) is called the \textit{constant of integration}.

Some indefinite integrals can be evaluated by reversing the familiar derivative formulas. For example, we obtain the indefinite integral of \( y = x^n \) by reversing the Power Rule for derivatives.

**THEOREM 2** Power Rule for Integrals

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1
\]

**Proof** We just need to verify that \( F(x) = \frac{x^{n+1}}{n+1} \) is an antiderivative of \( f(x) = x^n \):

\[
F'(x) = \frac{d}{dx} \left( \frac{x^{n+1}}{n+1} + C \right) = \frac{1}{n+1} (n+1)x^n = x^n
\]

In words, the Power Rule for Integrals says that to integrate a power of \( x \), "increase the power by one and divide by the new power." Here are some examples:

\[
\int x^5 \, dx = \frac{1}{6}x^6 + C, \quad \int x^{-9} \, dx = -\frac{1}{8}x^{-8} + C, \quad \int x^{3/5} \, dx = \frac{5}{8}x^{8/5} + C
\]

The Power Rule is not valid for \( n = -1 \) because when \( n = -1 \), the expression \( \frac{x^{n+1}}{n+1} \) is undefined.

It turns out that the natural logarithm function, which we introduce in Section 7.3, is an antiderivative of \( f(x) = x^{-1} \).

The indefinite integral obeys the usual linearity rules that allow us to integrate term by term. These rules follow from the linearity rules for the derivative (see Exercise 79).  

**THEOREM 3** Linearity of the Indefinite Integral

- **Sum Rule:** \( \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \)
- **Multiples Rule:** \( \int cf(x) \, dx = c \int f(x) \, dx \)

**EXAMPLE 2** Evaluate \( \int (3x^4 - 5x^{2/3} + x^{-3}) \, dx \).

**Solution** We integrate term by term and use the Power Rule:

\[
\int (3x^4 - 5x^{2/3} + x^{-3}) \, dx = \int 3x^4 \, dx - \int 5x^{2/3} \, dx + \int x^{-3} \, dx \quad \text{(Sum Rule)}
\]

\[
= \frac{3}{5}x^5 - \frac{5}{5/3}x^{5/3} + \frac{1}{2}x^{-2} + C \quad \text{(Multiples Rule)}
\]

When we break up an indefinite integral into a sum of several integrals as in Example 2, it is not necessary to include a separate constant of integration for each integral.
To check the answer, we verify that the derivative is equal to the integrand:

\[
\frac{d}{dx} \left( \frac{3}{5} x^5 - 3x^{5/3} - \frac{1}{2} x^{-2} + C \right) = 3x^4 - 5x^{2/3} + x^{-3}
\]

While we do not always do so in the text, it is often beneficial to check your work when computing antiderivatives. It is simply a matter of verifying that the derivative of your result is the function that you were antidifferentiating.

Although the linearity rules for the derivative carry over to linearity rules for indefinite integrals, there are no rules for directly computing indefinite integrals of products, quotients, and compositions of functions. At this point the best approach, if possible, is to convert the integrand algebraically so that the result is an integral that can be computed with the rules we have.

**EXAMPLE 3** Evaluate \( \int \left( \frac{5}{x^2} - 3x^{-10} \right) \, dx \).

Solution

\[
\int \left( \frac{5}{x^2} - 3x^{-10} \right) \, dx = 5 \int \frac{dx}{x^2} - 3 \int x^{-10} \, dx
\]

\[
= 5 \left( -x^{-1} \right) - 3 \left( \frac{x^{-9}}{-9} \right) + C = -5x^{-1} + \frac{1}{3} x^{-9}
\]

The differentiation formulas for the trigonometric functions give us the following integration formulas. Each formula can be checked by differentiation.

**Basic Trigonometric Integrals**

<table>
<thead>
<tr>
<th>Integral</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int \sin x , dx )</td>
<td>( -\cos x + C )</td>
</tr>
<tr>
<td>( \int \cos x , dx )</td>
<td>( \sin x + C )</td>
</tr>
<tr>
<td>( \int \sec^2 x , dx )</td>
<td>( \tan x + C )</td>
</tr>
<tr>
<td>( \int \csc^2 x , dx )</td>
<td>( -\cot x + C )</td>
</tr>
<tr>
<td>( \int \sec x \tan x , dx )</td>
<td>( \sec x + C )</td>
</tr>
<tr>
<td>( \int \csc x \cot x , dx )</td>
<td>( -\csc x + C )</td>
</tr>
</tbody>
</table>

**EXAMPLE 4** Evaluate \( \int (\sin t + 20 \sec^2 t) \, dt \).

Solution

\[
\int (\sin t + 20 \sec^2 t) \, dt = \int \sin t \, dt + 20 \int \sec^2 t \, dt
\]

\[
= -\cos t + 20 \tan t + C
\]

**CONCEPTUAL INSIGHT** Definite Versus Indefinite Integrals While the definite integral and the indefinite integral have similar names and notation, it is important to realize that they are very different objects. They are about as similar as apples and bricks. The definite integral is a numerical value obtained as a limit of Riemann sums. The indefinite integral is a family of functions whose derivative is a given function. Make sure you understand these differences and use the names and notations properly.

Even though the definite integral and indefinite integral are very different, they are closely related to each other via the Fundamental Theorem of Calculus (FTC). We will learn more about the FTC relationships in the next two sections.
Differential Equations

We can think of an antiderivative as a solution to the differential equation

$$\frac{dy}{dx} = f(x)$$

In general, a differential equation is an equation relating an unknown function and its derivatives. The unknown in Eq. (1) is a function $y = F(x)$ whose derivative is $f(x)$.

There are infinitely many solutions to Eq. (1)—all functions in the form $y = F(x) + C$, where $F(x)$ is an antiderivative of $y = f(x)$. However, we can specify a particular solution by imposing an initial condition—that is, by requiring that the solution satisfies $y(x_0) = y_0$ for some fixed values $x_0$ and $y_0$. A differential equation with an initial condition is called an initial value problem.

EXAMPLE 5 Solve $\frac{dy}{dx} = 4x^7$ subject to the initial condition $y(0) = 4$.

Solution First, find the general antiderivative:

$$y(x) = \int 4x^7 \, dx = \frac{1}{2} x^8 + C$$

Next, choose $C$ so that the initial condition is satisfied: From $y(x) = \frac{1}{2} x^8 + C$ we have $y(0) = 0 + C$, and from the initial condition, we have $y(0) = 4$. This yields $C = 4$, and our solution is $y = \frac{1}{2} x^8 + 4$.

EXAMPLE 6 Solve the initial value problem $\frac{dy}{dt} = \sin t$, $y(0) = 2$.

Solution First, find the general antiderivative:

$$y(t) = \int \sin t \, dt = -\cos t + C$$

Then solve for $C$: From $y(t) = -\cos t + C$ we have

$$y(0) = -\cos(0) + C = -1 + C$$

and from the initial condition we have $y(0) = 2$. So, $-1 + C = 2$, implying that $C = 3$. Therefore, the solution of the initial value problem is $y(t) = -\cos t + 3$.

EXAMPLE 7 A car traveling with velocity 24 m/s begins to slow down at time $t = 0$ s with a constant acceleration of $a = -6$ m/s$^2$. Find (a) the velocity $v(t)$ at time $t$, and (b) the distance traveled before the car comes to a halt.

Solution (a) The derivative of velocity is acceleration, so velocity is the antiderivative of acceleration:

$$v(t) = \int a \, dt = \int (-6) \, dt = -6t + C$$

The initial condition $v(0) = 24$ yields $C = 24$ and therefore $v(t) = -6t + 24$ m/s.

(b) Position is the antiderivative of velocity, so the car's position in meters is

$$s(t) = \int v(t) \, dt = \int (-6t + 24) \, dt = -3t^2 + 24t + C_1$$

where $C_1$ is a constant. We are not told where the car is at $t = 0$, so let us set $s(0) = 0$ for convenience, obtaining $C_1 = 0$. With this choice, $s(t) = -3t^2 + 24t$. This is the distance traveled from time $t = 0$.  

Scanned with CamScanner
The car comes to a halt when its velocity is zero, so we solve
\[ 0 = v(t) = -6t + 24 \quad \Rightarrow \quad t = 4 \text{ s} \]
The distance traveled before coming to a halt is \[ s(4) = -3(4^2) + 24(4) = 48 \text{ m}. \]

Antidifferentiation is generally a more difficult process than differentiation. We will be developing indefinite integral formulas and techniques in the sections and chapters that follow. In this section, we derived a collection of integral formulas shown in the summary. At the end of this text there is a table of a few dozen integral formulas. Computer algebra systems such as Wolfram Alpha are excellent tools for computing integrals; they have made the process much easier than it used to be. Previously, extensive tables of integrals were relied on for computation of antiderivatives. Such tables are published by the CRC Press in their *Book of Standard Mathematical Tables*, containing over 50 pages of integral formulas.

### 5.3 SUMMARY

- \( F \) is called an antiderivative of \( f \) if \( F'(x) = f(x) \).
- Any two antiderivatives of \( f \) on an interval \((a, b)\) differ by a constant.
- The general antiderivative is denoted by the indefinite integral:
  \[ \int f(x) \, dx = F(x) + C \]
- Some integration formulas:
  \[
  \begin{align*}
  \int 0 \, dx &= C \\
  \int k \, dx &= kx + C \\
  \int x^n \, dx &= \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \\
  \int \sin x \, dx &= -\cos x + C \\
  \int \cos x \, dx &= \sin x + C \\
  \int \sec^2 x \, dx &= \tan x + C \\
  \int \csc^2 x \, dx &= -\cot x + C \\
  \int \sec x \tan x \, dx &= \sec x + C \\
  \int \csc x \cot x \, dx &= -\csc x + C \\
  \int cf(x) \, dx &= c \int f(x) \, dx \\
  \int (f(x) + g(x)) \, dx &= \int f(x) \, dx + \int g(x) \, dx
  \end{align*}
  \]

- To solve an initial value problem \[ \frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \], first, find the general antiderivative \( y = F(x) + C \). Then, determine \( C \) using the initial condition \( F(x_0) + C = y_0 \).

### 5.3 EXERCISES

**Preliminary Questions**

1. Find an antiderivative of the function \( f(x) = 0 \).

2. Is there a difference between finding the general antiderivative of a function \( f \) and evaluating \( \int f(x) \, dx \)?

3. Jacques was told that \( f \) and \( g \) have the same derivative, and he wonders whether \( f(x) = g(x) \). Does Jacques have sufficient information to answer his question?

4. Suppose that \( F'(x) = f(x) \) and \( G'(x) = g(x) \). Which of the following statements are true? Explain.
   (a) If \( f = g \), then \( F = G \).
   (b) If \( F \) and \( G \) differ by a constant, then \( f = g \).
   (c) If \( f \) and \( g \) differ by a constant, then \( F = G \).

5. Is \( y = x \) a solution of the following initial value problem?
   \[ \frac{dy}{dx} = 1, \quad y(0) = 1 \]
Exercises

In Exercises 1–8, find the general antiderivative of $f$ and check your answer by differentiating.

1. $f(x) = 18x^2$
2. $f(x) = x^{-3/5}$
3. $f(x) = 2x^5 - 24x^3$
4. $f(x) = 9x + 15x^{-2}$
5. $f(x) = 2\cos x - 9\sin x$
6. $f(x) = 4x^2 - 3\cos x$
7. $f(x) = \sin 2x + 12\cos 3x$
8. $f(x) = \sin(4 - 9x)$

9. Match functions (a)–(d) with their antiderivatives (i)–(iv).

(a) $f(x) = \sin x$
(b) $f(x) = x\sin(x^2)$
(c) $f(x) = \sec(x - \pi)$
(d) $f(x) = x\sin x$

(i) $F(x) = \cos(1 - x)$
(ii) $F(x) = -\cos x$
(iii) $F(x) = \frac{1}{2}\cos(x^2)$
(iv) $F(x) = \sin x - x\cos x$

In Exercises 10–37, evaluate the indefinite integral. Remember, there are no Product, Quotient, or Chain Rules for integration.

10. $\int (9x + 2)\,dx$
11. $\int (4 - 18x)\,dx$
12. $\int x^{-3}\,dx$
13. $\int \frac{\sin t}{t}\,dt$
14. $\int (5t^3 - t^{-3})\,dt$
15. $\int (18t^5 - 10t^4 - 28t)\,dt$
16. $\int 14x^{9/5}\,dx$
17. $\int (x^{-4/3} - x^{2/3} + x^{1/3})\,dx$
18. $\int \frac{3}{2}\,dx$
19. $\int \frac{1}{\sqrt{x}}\,dx$
20. $\int \frac{dx}{x^{7/3}}$
21. $\int \frac{36\,dt}{t^7}$
22. $\int (x^2 - 4)\,dx$
23. $\int (t^{1/2} + 1)(t + 1)\,dt$
24. $\int \frac{12 - z}{\sqrt{z}}\,dz$
25. $\int \frac{3\,dx}{x^2 - 4}$
26. $\int \left(\frac{1}{2}\sin x - \frac{1}{4}\cos x\right)\,dx$
27. $\int 12\sec x\tan x\,dx$
28. $\int (\tan^2\theta)\,d\theta$
29. $\int \csc x\sec x\,d\theta$
30. $\int (x - \sin x)\,dt$
31. $\int (x^2 - \sec^2 x)\,dx$
32. $\int \sec^2 x\,d\theta$
33. $\int \sec x\tan x\,d\theta$
34. $\int \sec 12t\tan 12t\,dt$
35. $\int 5\sec^2(3t - 4t)\,d\theta$
36. $\int \sec 4x(3\sec 4x - 5\tan 4x)\,dx$
37. $\int \sec(x + 5)\tan(x + 5)\,dx$
38. In Figure 2, is graph (A) or graph (B) the graph of an antiderivative of $y = f(x)$?

![Figure 2](image)

39. In Figure 3, which of graphs (A), (B), and (C) is not the graph of an antiderivative of $y = f(x)$? Explain.

![Figure 3](image)

40. Verify that the function $F(x) = \frac{1}{2}(x + 13)^3$ is an antiderivative of $f(x) = (x + 13)^2$.

In Exercises 41–44, verify by differentiation.

41. $\int (x + 13)^6\,dx = \frac{1}{7}(x + 13)^7 + C$
42. $\int (x + 13)^{-5}\,dx = -\frac{1}{4}(x + 13)^{-4} + C$
43. $\int (4x + 13)^2\,dx = \frac{1}{12}(4x + 13)^3 + C$
44. $\int (ax + b)^n\,dx = \frac{1}{a(n + 1)}(ax + b)^{n+1} + C$ (for $n \neq -1$)

In Exercises 45–46, we demonstrate that, in general, you cannot obtain an antiderivative of a product of functions by taking a product of antiderivatives of each.

45. Show that $G(x) = 4x^3(-\cos x)$ is not an antiderivative of $f(x) = 8\sin x$ but $H(x) = -8\cos x + 8\sin x$ is.

46. Show that $G(x) = 3x^2\sin x$ is not an antiderivative of $f(x) = 6x\cos x$ but $H(x) = 6x\sin x + 6\cos x$ is.

In Exercises 47–60, solve the initial value problem.

47. $\frac{dy}{dx} = x^4$, $y(0) = 4$
48. $\frac{dy}{dx} = 3 - 2t$, $y(0) = -5$
49. \( \frac{dy}{dt} = 2t + 9t^2 \), \( y(1) = 2 \)
50. \( \frac{dy}{dt} = 8x^3 + 3x^2 \), \( y(2) = 0 \)
51. \( \frac{dy}{dt} = \sqrt{t} \), \( y(1) = 1 \)
52. \( \frac{dz}{dt} = t^{-3/2} \), \( z(4) = -1 \)
53. \( \frac{dy}{dx} = (3x + 2)^3 \), \( y(0) = 1 \)
54. \( \frac{dy}{dt} = (4t + 3)^2 \), \( y(1) = 0 \)
55. \( \frac{dy}{dx} = \sin x \), \( y\left(\frac{\pi}{2}\right) = 1 \)
56. \( \frac{dy}{dx} = \sec^2 x \), \( y\left(\frac{\pi}{4}\right) = 2 \)
57. \( \frac{dy}{dt} = 6 \sec 3\theta \tan 3\theta \), \( y\left(\frac{\pi}{12}\right) = -4 \)
58. \( \frac{dy}{dt} = 4t - \sin 2t \), \( y(0) = 2 \)
59. \( \frac{dy}{dt} = \cos \left(3\pi - \frac{\pi}{2}\right) \), \( y(3\pi) = 8 \)
60. \( \frac{dy}{dx} = \frac{1}{x^2} - \csc^2 x \), \( y\left(\frac{\pi}{2}\right) = 0 \)

In Exercises 61–67, first find \( f' \) and then find \( f \).
61. \( f''(x) = 12x \), \( f'(0) = 1 \), \( f(0) = 2 \)
62. \( f''(x) = x^3 - 2x \), \( f'(1) = 0 \), \( f(1) = 2 \)
63. \( f''(x) = x^3 - 2x + 1 \), \( f'(0) = 1 \), \( f(0) = 0 \)
64. \( f''(x) = x^3 - 2x + 1 \), \( f'(1) = 0 \), \( f(1) = 4 \)
65. \( f''(t) = t^{-3/2} \), \( f'(4) = 1 \), \( f(4) = 4 \)
66. \( f''(\theta) = \cos \theta \), \( f'(\frac{\pi}{2}) = 1 \), \( f\left(\frac{\pi}{2}\right) = 6 \)
67. \( f''(\theta) = \sec \theta \), \( f'(1) = 0 \), \( f(1) = -2 \)
68. Show that \( F(x) = \tan^2 x \) and \( G(x) = \sec^2 x \) have the same derivative. What can you conclude about the relation between \( F \) and \( G \)? Verify this conclusion directly.

69. A particle located at the origin at \( t = 1 \) second moves along the \( x \)-axis with velocity \( v(t) = (6t^2 - t) \) m/s. State the differential equation with its initial condition satisfied by the position \( s(t) \) of the particle, and find \( s(t) \).
70. A particle moves along the \( x \)-axis with velocity \( u(t) = (6t^2 - t) \) m/s. Find the particle’s position \( s(t) \), assuming that \( s(2) = 4 \) m.
71. A water balloon is dropped from a high building. It falls for 5 seconds before hitting the ground. Determine the velocity it is traveling when it is about to hit the ground, assuming an acceleration due to gravity of \(-9.8 \text{ m/s}^2 \) and no wind resistance.
72. A hammer is dropped and it falls for 2 seconds before hitting the ground. Determine how far it falls, assuming an acceleration due to gravity of \(-9.8 \text{ m/s}^2 \) and no wind resistance.
73. A mass oscillates at the end of a spring. Let \( s(t) \) be the displacement of the mass from the equilibrium position at time \( t \). Assuming that the mass is located at the origin at \( t = 0 \) and has velocity \( u(t) = \sin t \) m/s, state the differential equation with initial condition satisfied by \( s(t) \), and find \( s(t) \).
74. Beginning at \( t = 0 \) with initial velocity \( v_0 \) m/s, a particle moves along a straight line with acceleration \( a(t) = 3t \) \text{ m/s}^2. Find the distance traveled after 25 s.
75. At time \( t = 0 \) a car traveling 25 m/s begins to accelerate at a constant rate of \(-4 \text{ m/s}^2 \). After how many seconds does the car come to a stop and how far will the car have traveled between \( t = 0 \) and the time it stopped?
76. At time \( t = 1 \) second, a particle is traveling at 72 m/s and begins to accelerate at the rate \( a(t) = -t \) until it stops. How far does the particle travel from \( t = 1 \) until the time it stopped?
77. A 900-kg rocket is released from a space station. As it burns fuel, the rocket’s mass decreases and its velocity increases. Let \( u(m) \) be the velocity (in meters per second) as a function of mass \( m \). Find the velocity when \( m = 729 \) kg if \( du/dm = -50m^{-1/2} \). Assume that \( u(900) = 0 \) m/s.
78. As water flows through a tube of radius \( R = 10 \text{ cm} \), the velocity \( v \) of an individual water particle depends only on its distance \( r \) from the center of the tube. The particles at the walls of the tube have zero velocity and \( dv/dr = -0.06r \). Determine \( v(r) \).
79. Verify the linearity properties of the indefinite integral stated in Theorem 3.

Further Insights and Challenges
80. Find constants \( c_1 \) and \( c_2 \) such that \( F(x) = c_1 \sin 3x + c_2 \cos 3x \) is an antiderivative of \( f(x) = 2x \sin 3x \).
81. Find constants \( c_1 \) and \( c_2 \) such that \( F(x) = c_1 x \cos x + c_2 \sin x \) is an antiderivative of \( f(x) = x \sin x \).
82. Suppose that \( F(x) = f(x) \) and \( G(x) = g(x) \). Is it true that \( y = F(x)G(x) \) is an antiderivative of \( y = f(x)g(x) \)? Confirm or provide a counterexample.
83. Suppose that \( F'(x) = f(x) \).
(a) Show that \( y = \frac{1}{2}F(2x) \) is an antiderivative of \( y = f(2x) \).
(b) Find the general antiderivative of \( y = f(kx) \) for \( k \neq 0 \).
84. Find an antiderivative for \( f(x) = \lvert x \rvert \).
85. Using Theorem 1, prove that if \( F(x) = f(x) \), where \( f \) is a polynomial of degree \( n - 1 \), then \( F \) is a polynomial of degree \( n \). Then prove that if \( g \) is an arbitrary function such that \( g^{(n)}(x) = 0 \), then \( g \) is a polynomial of degree at most \( n \).
86. The Power Rule for antiderivatives does not apply to \( f(x) = x^{-1} \). Which of the graphs in Figure 4 could plausibly represent an antiderivative of \( f(x) = x^{-1} \)?

![Figure 4](image-url)
5.4 The Fundamental Theorem of Calculus, Part I

Since we have so far introduced both derivatives and integrals, a very reasonable question is why they appear together in this topic called calculus. The answer is the Fundamental Theorem of Calculus (FTC), which is one of the most important theorems in all of mathematics. This foundational result reveals an unexpected connection between the two main operations of calculus: differentiation and integration. The theorem has two parts. Although they are closely related, we discuss them in separate sections to emphasize the different ways they are used. The first part of the Fundamental Theorem of Calculus will allow us to compute definite integrals without having to take limits of Riemann sums.

To explain FTC I, recall a result from Example 7 of Section 5.2:

\[
\int_4^7 x^2 \, dx = \left( \frac{1}{3} \right) 7^3 - \left( \frac{1}{3} \right) 4^3 = 93
\]

Now, observe that \( F(x) = \frac{1}{2} x^2 \) is an antiderivative of \( f(x) = x^2 \), so we can write

\[
\int_4^7 x^2 \, dx = F(7) - F(4)
\]

According to FTC I, this is no coincidence; this relation between the definite integral and the antiderivative holds in general.

**Theorem 1** The Fundamental Theorem of Calculus, Part I

Assume that \( a < b \) and that \( f \) is continuous on \([a, b]\). If \( F \) is an antiderivative of \( f \) on \([a, b]\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a) \tag{1}
\]

**Proof** The quantity \( F(b) - F(a) \) is the total change in \( F \) (also called the net change) over the interval \([a, b]\). Our task is to relate it to the integral of \( f' = f \). There are two main steps.

Write total change as a sum of small changes: Given any partition \( P \) of \([a, b]\):

\[
P : x_0 = a < x_1 < x_2 < \cdots < x_N = b
\]

we can break up \( F(b) - F(a) \) as a sum of changes over the intervals \([x_{i-1}, x_i]\):

\[
F(b) - F(a) = (F(b) - F(x_{N-1})) + (F(x_{N-1}) - F(x_{N-2})) + \cdots + (F(x_2) - F(x_1)) + (F(x_1) - F(a))
\]

On the right-hand side, \( -F(x_{N-1}) \) is canceled by \( F(x_{N-1}) \) in the second term, \( -F(x_{N-2}) \) is canceled by \( F(x_{N-2}) \) in the third term, and so on (Figure 1). In summation notation,

\[
F(b) - F(a) = \sum_{i=1}^{N} (F(x_i) - F(x_{i-1})) \tag{2}
\]

![Figure 1](image_url)

Note the cancellation when we write \( F(b) - F(a) \) as a sum of small changes \( F(x_i) - F(x_{i-1}) \).
Interpret Eq. (2) as a Riemann sum: The Mean Value Theorem tells us that there is a point $c^*_i$ in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c^*_i)(x_i - x_{i-1}) = f(c^*_i)(x_i - x_{i-1}) = f(c^*_i) \Delta x_i$$

Therefore, Eq. (2) can be written

$$F(b) - F(a) = \sum_{i=1}^{N} f(c^*_i) \Delta x_i$$

This sum is the Riemann sum $R(f, P, C^*)$ with sample points $C^* = \{c^*_i\}$.

Now, $f$ is integrable (Theorem 1, Section 5.2), so $R(f, P, C^*)$ approaches $\int_a^b f(x) \, dx$ as the norm $\|P\|$ tends to zero. On the other hand, $R(f, P, C^*)$ is equal to $F(b) - F(a)$ with our particular choice $C^*$ of sample points. This proves the desired result:

$$F(b) - F(a) = \lim_{\|P\| \to 0} R(f, P, C^*) = \int_a^b f(x) \, dx$$

**CONCEPTUAL INSIGHT** A Tale of Two Graphs In the proof of FTC I, we used the MVT to write a small change in $y$ in the graph of $y = F(x)$ in terms of the derivative $F'(x) = f(x)$:

$$F(x_i) - F(x_{i-1}) = f(c^*_i) \Delta x_i$$

But $f(c^*_i) \Delta x_i$ is the signed area of a thin rectangle that approximates a sliver of signed area under the graph of $f$ (Figure 2). This is the essence of the Fundamental Theorem:

- The total change, $F(b) - F(a)$, is
- The sum of small changes, $F(x_i) - F(x_{i-1})$, which is
- The sum of the signed areas of rectangles from the graph of $f$, and that is
- A Riemann sum for $f$.

The Fundamental Theorem itself is then obtained by taking the limit as the widths of the rectangles tend to zero.

![Figure 2](Image)

**FIGURE 2**

FTC I tells us that if we can find an antiderivative of $f$, then we can compute the definite integral easily, without calculating any limits. It is for this reason that we use the integral sign $\int$ for both the definite integral $\int_a^b f(x) \, dx$ and the indefinite integral (antiderivative) $\int f(x) \, dx$. 

---

Scanned with CamScanner
While Theorem 1 is stated with the assumption that \( a < b \), it holds for general \( a \) and \( b \) as well. See Exercise 49.

**NOTATION** \( F(b) - F(a) \) is denoted \( F(x)\bigg|_a^b \). In this notation, the FTC reads

\[
\int_a^b f(x) \, dx = F(x)\bigg|_a^b \quad \text{where} \quad \int f(x) \, dx = F(x) + C
\]

This form of FTC I suggests a two-step approach for evaluating the definite integral

\[
\int_a^b f(x) \, dx:
\]

- Compute \( \int_a^b f(x) \, dx = F(x) + C \)
- Evaluate \( \int_a^b f(x) \, dx = F(x)\bigg|_a^b = F(b) - F(a) \)

In examples that follow, we will use this two-step process. As we become more familiar with antiderivatives, we will usually carry out both steps at once.

**EXAMPLE 1** Calculate the area under the graph of \( f(x) = x^2 \) over \([2, 4]\).

**Solution** Since \( \int x^3 \, dx = \frac{1}{4}x^4 + C \), we have

\[
\int_2^4 x^3 \, dx = \frac{1}{4}x_2^4 = \frac{1}{4}4^4 - \frac{1}{4}2^4 = 60
\]

**EXAMPLE 2** Find the area under \( g(x) = x^{-3/4} + 3x^{5/3} \) over \([1, 3]\) (Figure 3).

**Solution** To begin,

\[
\int_1^3 (x^{-3/4} + 3x^{5/3}) \, dx = 4x^{1/4} + \frac{9}{8}x^{8/3} + C
\]

Therefore, the area is equal to

\[
\int_1^3 (x^{-3/4} + 3x^{5/3}) \, dx = \left( 4x^{1/4} + \frac{9}{8}x^{8/3} \right)_1^3
\]

\[
= \left( 4 \cdot 3^{1/4} + \frac{9}{8} \cdot 3^{8/3} \right) - \left( 4 \cdot 1^{1/4} + \frac{9}{8} \cdot 1^{8/3} \right)
\]

\[
= 26.325 - 5.125 = 21.2
\]

**CONCEPTUAL INSIGHT** Which Antiderivative? As we have seen, antiderivatives are not unique. Does it matter then, which antiderivative is used in the FTC? The answer is no. If \( F \) and \( G \) are both antiderivatives of a continuous function \( f \) on \([a, b]\), then \( F(x) = G(x) + C \) for some constant \( C \), and

\[
F(b) - F(a) = (G(b) + C) - (G(a) + C) = G(b) - G(a)
\]

The constant cancels.

The two antiderivatives yield the same value for the definite integral:

\[
\int_a^b f(x) \, dx = F(b) - F(a) = G(b) - G(a)
\]

In Section 5.2, we showed that \( \int_0^b x \, dx = \frac{1}{2}b^2 \) and \( \int_0^b x^2 \, dx = \frac{1}{3}b^3 \). We asked you to conjecture what the result is for general \( f(x) = x^a \). We are now in a position to consider the general case.
EXAMPLE 3 Calculate $\int_0^b x^n \, dx$, assuming that $n \neq -1$.

Solution First, $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$. Therefore,

$$\int_0^b x^n \, dx = \frac{1}{n+1} x^{n+1} \bigg|_0^b = \frac{1}{n+1} b^{n+1} - 0$$

We know that the definite integral is equal to the signed area between the graph and the $x$-axis. Needless to say, the FTC "knows" this also: When you evaluate an integral using the FTC, you obtain the signed area.

EXAMPLE 4 Evaluate (a) $\int_0^\pi \sin x \, dx$ (b) $\int_0^{2\pi} \sin x \, dx$ (c) $\int_{\pi/4}^{3\pi/4} \sin x \, dx$.

Solution

(a) $\int_0^\pi \sin x \, dx = \left[-\cos x\right]_0^\pi = -\cos \pi - (-\cos 0) = -(-1) - (-1) = 2$

(b) $\int_0^{2\pi} \sin x \, dx = \left[-\cos x\right]_0^{2\pi} = -\cos(2\pi) - (-\cos 0) = -1 - (-1) = 0$

(c) $\int_{\pi/4}^{3\pi/4} \sin x \, dx = -\cos x \bigg|_{\pi/4}^{3\pi/4} = -\cos(3\pi/4) - (-\cos(\pi/4))$

$$= -\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

It is an interesting fact that the area under one hump of the sine graph (Figure 4) is a nice whole-number value, 2. That is certainly worth remembering. Also, it is no surprise that the result of the second integral in this example is zero since it corresponds to the signed area from 0 to $2\pi$, and that value is zero since the contribution from the hump below the axis exactly cancels the contribution from the hump above.

Finally, in Example 5 in Section 5.1, we introduced a general right-hand sum $R_N$ for approximating the area associated with the integral in (c). We indicated that directly taking the limit of $R_N$ to get an exact value is a difficult task, but now with FTC I, the process of obtaining the area couldn't be easier: Antidifferentiate $\sin x$, evaluate the result at the endpoints, and take the difference.

EXAMPLE 5 Evaluate $\int_{-1}^4 (4 - 2t) \, dt$.

Solution The function $F(x) = 4t - t^2$ is an antiderivative of $f(x) = 4 - 2t$, so the definite integral (the signed area under the graph in Figure 5) is

$$\int_{-1}^4 (4 - 2t) \, dt = (4t - t^2) \bigg|_{-1}^4 = (4 \cdot 4 - 4^2) - (4 \cdot (-1) - (-1)^2) = 0 - (-5) = 5$$

FIGURE 4 The area of one hump is 2. The signed area over $[0, 2\pi]$ is zero.

FIGURE 5

Scanned with CamScanner
CONCEPTUAL INSIGHT FTC I reveals a valuable relationship for computing definite integrals, but it does not always help. For example, to use FTC I for \( \int_0^4 \sqrt{1 + \cos x} \, dx \), we need an antiderivative formula for \( f(x) = \sqrt{1 + \cos x} \). Unfortunately, even though \( f \) has an antiderivative, there is no formula for the antiderivative that helps determine the definite integral. (We elaborate on these points further in the next section.)

This does not mean that we cannot obtain a value for this definite integral. Remember, a definite integral is a limit of Riemann sums, so we can obtain an approximate value to the definite integral using Riemann sums directly. In this case, partitioning \([0, 4]\) into 1000 subintervals and computing (using technology!) the left-endpoint approximation, we obtain \( \int_0^4 \sqrt{1 + \cos x} \, dx \approx 3.190 \). In Section 8.8, we introduce other techniques for approximating definite integrals.

5.4 SUMMARY

- The Fundamental Theorem of Calculus, Part I, states that

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

where \( F \) is an antiderivative of \( f \).

- Two-step approach for using FTC I to evaluate the definite integral \( \int_a^b f(x) \, dx \):
  - Compute \( \int_a^b f(x) \, dx = F(x) + C \).
  - Evaluate \( \int_a^b f(x) \, dx = F(b) - F(a) \).

- Antiderivative formulas are helpful for evaluating definite integrals. See the summary in Section 5.3 and the table of integrals in this text's endleaf.

5.4 EXERCISES

Preliminary Questions

1. Suppose that \( F(x) = f(x) \) and \( F(0) = 3 \), \( F(2) = 7 \).
   (a) What is the area under \( y = f(x) \) over \([0, 2]\) if \( f(x) \geq 0 \)?
   (b) What is the graphical interpretation of \( F(2) - F(0) \) if \( f(x) \) takes on both positive and negative values?

2. Suppose that \( f \) is a negative function with antiderivative \( F \) such that \( F(1) = 7 \) and \( F(3) = 4 \). What is the area (a positive number) between the \( x \)-axis and the graph of \( f \) over \([1, 3]\)?

3. Are the following statements true or false? Explain.
   (a) FTC I is valid only for positive functions.
   (b) To use FTC I, you have to choose the right antiderivative.
   (c) If you cannot find an antiderivative of \( f \), then the definite integral does not exist.

4. Evaluate \( \int_a^b f'(x) \, dx \), where \( f \) is differentiable and \( f(2) = f(9) = 4 \).

Exercises

In Exercises 1–4, sketch the region under the graph of the function and find its area using FTC I.

1. \( f(x) = x^2 \), \([0, 1]\)
2. \( f(x) = 2x - x^2 \), \([0, 2]\)
3. \( f(x) = x^2 \), \([1, 2]\)
4. \( f(x) = \cos x \), \([0, \frac{\pi}{2}]\)

In Exercises 5–30, evaluate the integral using FTC I.

5. \( \int_0^6 x \, dx \)
6. \( \int_0^2 2x \, dx \)
7. \( \int_0^1 (4x - x^2) \, dx \)
8. \( \int_3^4 u^2 \, du \)
9. \( \int_0^3 (2x^2 + 3x^3 - 4x) \, dx \)
10. \( \int_2^3 (10x^9 + 3x^5) \, dx \)
11. \( \int_0^6 (2x^3 - 6x^2) \, dx \)
12. \( \int_1^4 (5u^4 + u^2 - u) \, du \)
13. \( \int_0^2 \sqrt{x} \, dx \)
14. \( \int_0^8 x^4 \, dx \)
15. \( \int_{0.16}^{1} \frac{1}{t^{1/4}} \, dt \)
16. \( \int_4^1 t^{5/2} \, dt \)
17. \[ \int_1^3 \frac{dt}{t^2} \]
18. \[ \int_1^4 x^{-4} \, dx \]
19. \[ \int_{1/2}^1 \frac{8}{x^3} \, dx \]
20. \[ \int_{-2}^{-1} \frac{1}{x^3} \, dx \]
21. \[ \int_{x^2}^{x^2 - 2x} \, dx \]
22. \[ \int_1^0 t^{-1/2} \, dt \]
23. \[ \int_1^n \frac{1 + x}{\sqrt[3]{x}} \, dx \]
24. \[ \int_0^1 \frac{10x^4 + 8x^{1/3}}{t} \, dt \]
25. \[ \int_{\pi/4}^{\pi} \sin \theta \, d\theta \]
26. \[ \int_0^\pi \sin x \, dx \]
27. \[ \int_0^{\pi/3} \cos t \, dt \]
28. \[ \int_0^{\pi/6} \sec \theta \tan \theta \, d\theta \]
29. \[ \int_{x^{1/4}}^{x^{3/4}} (2 - \cos^2 x) \, dx \]
30. \[ \int_0^{157010} \sec^2 x \, dx \]

In Exercises 31–36, write the integral as a sum of integrals without absolute values and evaluate.
31. \[ \int_0^1 |x| \, dx \]
32. \[ \int_0^5 (3 - |x|) \, dx \]
33. \[ \int_2^3 |x^3| \, dx \]
34. \[ \int_0^3 |x^2 - 1| \, dx \]
35. \[ \int_0^\pi |\cos x| \, dx \]
36. \[ \int_0^2 |x^2 - 4x + 3| \, dx \]

In Exercises 37–40, evaluate the integral in terms of the constants.
37. \[ \int_1^6 x^9 \, dx \]
38. \[ \int_0^8 x^9 \, dx \]
39. \[ \int_0^b x^3 \, dx \]
40. \[ \int_{-a}^1 (x^3 + r) \, dx \]

41. Calculate \[ \int_{-2}^2 f(x) \, dx \], where
   \[ f(x) = \begin{cases} 12 - x^2 & \text{for } x \leq 2 \\ x^3 & \text{for } x > 2 \end{cases} \]

42. Calculate \[ \int_0^{2\pi} f(x) \, dx \], where
   \[ f(x) = \begin{cases} \sin x & \text{for } x \leq \pi \\ -2\sin x & \text{for } x > \pi \end{cases} \]

43. Use FTC I to show that \[ \int_{-1}^1 x^4 \, dx = 0 \] if n is an odd whole number. Explain graphically.

44. CAS Plot the function \( f(x) = 3 \sin x - x \). Find the positive root of \( f \) to three decimal places and use it to find the area under the graph of \( f \) in the first quadrant.

45. Calculate \( F(4) \) given that \( F(1) = 3 \) and \( F'(x) = x^2 \). Hint: Express \( F(4) - F(1) \) as a definite integral.

46. Calculate \( G(16) \), where \( dG/dt = r^{-1/2} \) and \( G(9) = -5 \).

47. With \( n > 0 \), does \[ \int_0^n x^n \, dx \] get larger or smaller as \( n \) increases? Explain graphically.

48. With \( k > 1 \), does \[ \int_0^k x^k \, dx \] get larger or smaller as \( k \) increases? Explain graphically.

49. Theorem 1 is stated with the assumption that \( a < b \). Prove that the FTC I relationship
   \[ \int_a^b f(x) \, dx = F(b) - F(a) \]
   also holds for \( a = b \) and for \( b < a \) assuming that \( F \) is an antiderivative of \( f \) on \([b, a]\).

50. Show that the area of the shaded parabolic arch in Figure 6 is equal to four-thirds the area of the triangle shown.

**Further Insights and Challenges**

51. Prove a famous result of Archimedes (generalizing Exercise 50): For \( r < s \), the area of the shaded region in Figure 7 is equal to four-thirds the area of triangle \( \triangle ACE \), where \( C \) is the point on the parabola at which the tangent line is parallel to secant line \( AE \).
   (a) Show that \( C \) has \( x \)-coordinate \( (r + s)/2 \).
   (b) Show that \( ABDE \) has area \( (s - r)^2/4 \) by viewing it as a parallelogram of height \( s - r \) and base of length \( CF \).
   (c) Show that \( \triangle ACE \) has area \( (s - r)^2/8 \) by observing that it has the same base and height as the parallelogram.
   (d) Compute the shaded area as the area under the graph minus the area of a trapezoid, and prove Archimedes's result.

52. (a) Apply the Comparison Theorem (Theorem 5 in Section 5.2) to the inequality \( \sin x \leq x \) (valid for \( x \geq 0 \)) to prove that
   \[ 1 - \frac{x^2}{2} \leq \cos x \leq 1 \]
(b) Apply it again to prove that
\[ x - \frac{x^3}{3} \leq \sin x \leq x \quad (\text{for } x \geq 0) \]
(c) Verify these inequalities for \( x = 0.3 \).

53. Use the method of Exercise 52 to prove that
\[ 1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} \]
\[ x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120} \quad (\text{for } x \geq 0) \]
Verify these inequalities for \( x = 0.1 \). Why have we specified \( x \geq 0 \) for \( \sin x \) but not for \( \cos x \)?

54. Calculate the next pair of inequalities for \( \sin x \) and \( \cos x \) by integrating the results of Exercise 53. Can you guess the general pattern?

55. Use Part I of the Fundamental Theorem of Calculus to prove that if \( |f'(x)| \leq K \) for \( x \in [a, b] \), then \( |f(x) - f(a)| \leq K(x - a) \) for \( x \in [a, b] \).

56. (a) Use Exercise 55 to prove that \( |\sin a - \sin b| \leq |a - b| \) for all \( a, b \).
(b) Let \( f(x) = \sin(x + \alpha) - \sin x \). Use part (a) to show that the graph of \( f \) lies between the horizontal lines \( y = \pm \alpha \).
(c) \( \mathbf{GU} \) Plot \( y = f(x) \) and the lines \( y = \pm \alpha \) to verify (b) for \( \alpha = 0.5 \) and \( \alpha = 0.2 \).

### 5.5 The Fundamental Theorem of Calculus, Part II

Part I of the Fundamental Theorem says that we can compute definite integrals using indefinite integrals (antiderivatives). Part II does the opposite, providing a way to compute antiderivatives using definite integrals. Another interpretation of Part II of the FTC demonstrates how definite integration and differentiation are inverse processes. We will focus on that aspect of FTC II first, considering a motivating example before stating the theorem.

The idea behind this inverse relationship is as follows:

- Start with a function \( f \).
- Create a new function \( A \) via definite integration of \( f \). The function \( A \) is called an area function of \( f \).
- Differentiate \( A \) to return to the original function \( f \).

To begin, we introduce the area function of \( f \) with lower limit \( a \):

\[
A(x) = \int_a^x f(t) \, dt = \text{signed area from } a \text{ to } x
\]

For \( A(x) \) to be defined, \( f \) must be integrable over \([a, x]\) when \( x > a \) or over \([x, a]\) when \( x < a \). The definite integral defining \( A(x) \) yields a function because we regard the upper limit \( x \) as a variable.

Let us consider an example with \( f(x) = 4 - x^2 \) and \( a = 1 \). The resulting area function is

\[
A(x) = \int_1^x (4 - t^2) \, dt
\]

We can obtain an expression for \( A(x) \) by using FTC I to evaluate the definite integral:

\[
A(x) = \left[ 4t - \frac{1}{3}t^3 \right]_1^x = \left( 4x - \frac{1}{3}x^3 \right) - \left( 4(1) - \frac{1}{3}(1)^3 \right) = 4x - \frac{1}{3}x^3 - \frac{11}{3}
\]

Thus, \( A(x) = 4x - \frac{1}{3}x^3 - \frac{11}{3} \). Now, notice that if we differentiate this area function, the result is \( f \), the function we started with:

\[
A'(x) = 4 - x^2 = f(x)
\]

Thus, if we start with \( f \), create an area function \( A \), and then differentiate \( A \), we return to \( f \). Part II of the Fundamental Theorem of Calculus asserts that if \( f \) is continuous, then this relationship always holds.
**Theorem 1** Fundamental Theorem of Calculus, Part II Assume that $f$ is continuous on an open interval $I$ and let $a$ be in $I$. Then the area function

$$A(x) = \int_a^x f(t) \, dt$$

is an antiderivative of $f$ on $I$; that is, $A'(x) = f(x)$. Equivalently,

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$

**Proof** For simplicity we assume that $f$ is nonnegative and increasing. (For the general case, see Exercise 51.) First, we use the additivity property of the definite integral to write the change in $A$ over $[x, x + h]$ as an integral:

$$A(x + h) - A(x) = \int_x^{x+h} f(t) \, dt = \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt = \int_x^{x+h} f(t) \, dt$$

In other words, $A(x + h) - A(x)$ is equal to the area of the thin slice between the graph and the $x$-axis from $x$ to $x + h$ in Figure 1.

![Figure 1](image1.png)  
**Figure 1** The area of the thin slice equals $A(x + h) - A(x)$.

Since $f$ is nondecreasing, when $h > 0$, this thin slice lies between the two rectangles of heights $f(x)$ and $f(x + h)$ in Figure 2, and we have

$$h f(x) \leq A(x + h) - A(x) \leq h f(x + h)$$

Now divide by $h$ to squeeze the difference quotient between $f(x)$ and $f(x + h)$:

$$f(x) \leq \frac{A(x + h) - A(x)}{h} \leq f(x + h)$$

We have $\lim_{h \to 0^+} f(x + h) = f(x)$ because $f$ is continuous, and $\lim_{h \to 0^+} f(x) = f(x)$, so the Squeeze Theorem gives us

$$\lim_{h \to 0^+} \frac{A(x + h) - A(x)}{h} = f(x)$$

A similar argument shows that for $h < 0$,

$$f(x + h) \leq \frac{A(x + h) - A(x)}{h} \leq f(x)$$

Again, the Squeeze Theorem gives us

$$\lim_{h \to 0^-} \frac{A(x + h) - A(x)}{h} = f(x)$$

Equations (1) and (2) show that $A'(x)$ exists and that $A'(x) = f(x)$.
EXAMPLE 1 Let \( f(x) = 1 - 6x - \cos x \) and \( a = -\pi \). Compute the area function \( A(x) = \int_a^x f(t) \, dt \), and then verify the FTC II inverse relationship by showing that \( A'(x) = f(x) \).

Solution
\[
A(x) = \int_{-\pi}^x (1 - 6t - \cos t) \, dt = (t - 3t^2 - \sin t)|_x^{\pi} = x - 3x^2 - \sin x + \pi + 3\pi^2
\]
Taking the derivative,
\[
A'(x) = \frac{d}{dx}(x - 3x^2 - \sin x + \pi + 3\pi^2) = 1 - 6x - \cos x = f(x)
\]
Thus, we see that when we start with \( f \), compute an area function of it, and differentiate the area function, we obtain the function \( f \) back.

EXAMPLE 2 A Numerically Approximate FTC II Verification Consider the function \( f \) that is presented graphically in Figure 3. Here we will approximate an area function and its derivative to demonstrate (at least approximately) the FTC II inverse relationship.

Let \( A \) be the area function defined by \( A(x) = \int_0^x f(t) \, dt \).

(a) By estimating the corresponding signed areas, approximate \( A(x) \) for \( x = 0, 1, 2, \ldots, 10 \).

(b) Use the symmetric difference quotient approximation, \( A'(x) \approx \frac{A(x+\Delta x) - A(x-\Delta x)}{2\Delta x} \)
with \( \Delta x = 1 \) to approximate \( A'(x) \) for \( x = 1, 2, \ldots, 9 \). Plot these values of \( A'(x) \) on the graph of \( f \) to demonstrate \( A' \approx f \).

Solution

(a) First note that \( A(0) = \int_0^0 f(t) \, dt = 0 \). For other \( x \), we can estimate \( A(x) \) by counting the \( 1 \times 1 \) squares (and partial squares) between the graph and the \( x \)-axis from \( 0 \) to \( x \). For example, for \( A(1) \) we have 4.4 squares between the graph and the \( x \)-axis from \( x = 0 \) to \( x = 1 \). Therefore, \( A(1) \approx 4.4 \). Continuing, we obtain the approximate values of \( A(x) \) shown in the table below. Note that in determining the values for \( A(8) \) through \( A(10) \), each square below the \( x \)-axis contributes a value of \( -1 \) to the signed area.

(b) As a couple of examples, for \( x = 1 \) and \( x = 9 \), we obtain the following difference quotient approximations:
\[
A'(1) \approx \frac{A(2) - A(0)}{2} = \frac{8.9 - 0}{2} = 4.45, \quad A'(9) \approx \frac{A(10) - A(8)}{2} = \frac{14.7 - 23.3}{2} = -4.3
\]

The remaining values of \( A'(x) \) are obtained similarly and are plotted with \( f \) in Figure 4.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(x) )</td>
<td>0</td>
<td>4.4</td>
<td>8.9</td>
<td>13.3</td>
<td>17.4</td>
<td>21.0</td>
<td>23.7</td>
<td>25.2</td>
<td>23.3</td>
<td>19.3</td>
<td>14.7</td>
</tr>
<tr>
<td>( A'(x) )</td>
<td>4.45</td>
<td>4.45</td>
<td>4.25</td>
<td>3.85</td>
<td>3.15</td>
<td>2.1</td>
<td>-0.2</td>
<td>-2.95</td>
<td>-4.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We can see in the graph, at least approximately, the FTC II inverse relationship: If we start with a function \( f \), compute an area function of \( f \), and then take the derivative of the area function, we return to \( f \).
CONCEPTUAL INSIGHT The FTC shows that integration and differentiation are inverse operations. By FTC II, if you start with a continuous function \( f \) and form the integral \( \int_{a}^{x} f(t) \, dt \), then you get back the original function by differentiating:

\[
f(x) \quad \text{Integrate} \quad \int_{a}^{x} f(t) \, dt \quad \text{Differentiate} \quad \frac{df}{dx} \quad \int_{a}^{x} f(t) \, dt = f(x)
\]

On the other hand, by FTC I, if you differentiate first and then integrate, you also recover \( f(x) \) (but only up to a constant \( f(a) \)):

\[
f(x) \quad \text{Differentiate} \quad f'(x) \quad \text{Integrate} \quad \int_{a}^{x} f'(t) \, dt = f(x) - f(a)
\]

In addition to showing the inverse relationship between integration and differentiation, FTC II also reveals how area functions may be used to compute antiderivatives. For example,

- \( A(x) = \int_{0}^{x} t^2 \, dt \) is an antiderivative of \( f(x) = x^2 \),
- \( A(x) = \int_{0}^{x} \cos t \, dt \) is an antiderivative of \( f(x) = \cos x \),
- \( A(x) = \int_{0}^{x} \sqrt{1 + \cos^2 t} \, dt \) is an antiderivative of \( f(x) = \sqrt{1 + \cos x} \).

Since we already have simple expressions for antiderivatives of \( f(x) = x^2 \) and \( f(x) = \cos x \), the area-function version might not be helpful. On the other hand, for a function like \( f(x) = \sqrt{1 + \cos x} \), the situation is different. Try as you may, you cannot find an elementary function whose derivative is \( f(x) = \sqrt{1 + \cos x} \). That does not mean that there is no antiderivative. In fact, FTC II guarantees that there is an antiderivative, but expressing it in the area-function form might be the best we can do.

Unfortunately, the antiderivative \( A(x) = \int_{0}^{x} \sqrt{1 + \cos t} \, dt \) for \( f(x) = \sqrt{1 + \cos x} \) does not yield a formula that we can use to compute definite integrals involving \( f \) via FTC I. Instead, we need to consider alternative approaches such as numerical approximation.

While it might seem unfortunate that \( A(x) = \int_{0}^{x} \sqrt{1 + \cos t} \, dt \) is the best that we can do for an antiderivative of \( f(x) = \sqrt{1 + \cos x} \), as the next example shows, there is plenty that we can do to understand the behavior of \( A \).

**EXAMPLE 3** Graphing an Area Function Let \( A(x) = \int_{0}^{x} \sqrt{1 + \cos t} \, dt \).

(a) For \( x = 1, 2, \ldots, 20 \), with \( \Delta x = 0.01 \), use technology to calculate a right-endpoint Riemann sum that approximates the definite integral defining \( A(x) \). Plot the points \((x, A(x))\) and connect them with a smooth curve to obtain a graph of \( A \).

(b) Examine \( A' \) to determine the critical points of \( A \) and the increasing/decreasing behavior of the graph of \( A \).

**Solution**

(a) We use \( \Delta x = 0.01 \) and a right-endpoint Riemann sum to approximate each definite integral. For example, to approximate \( A(4) \), we compute \( R_{100} \) for the definite integral over \([0, 4]\). We obtain the following values for \( A(x) \):

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>A(x)</td>
<td>1.22</td>
<td>2.24</td>
<td>2.80</td>
<td>3.19</td>
<td>4.10</td>
<td>5.29</td>
<td>6.54</td>
<td>7.63</td>
<td>8.35</td>
<td>8.66</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
</table>
These points are plotted and joined with a smooth curve to obtain a graph of \( A \) in Figure 5.

(b) By FTC II, \( A'(x) = \sqrt{1 + \cos x} \). Thus, \( A \) has critical points when \( \cos x = -1 \); that is, at \( nx \) for all odd \( n \). Furthermore, except at the critical points, \( A'(x) \) is positive, and therefore, \( A \) is an increasing function.

**CONCEPTUAL INSIGHT** Every continuous function on an open interval \( I \) is guaranteed by FTC II to have an antiderivative. Furthermore, we can obtain an antiderivative via an area function.

A number of important functions studied and employed by mathematicians, scientists, and engineers are designed by area functions. When no simple formula for such functions is available, we can examine their behavior by approximating the function via definite-integral estimates and by using the first and second derivatives to determine increasing and decreasing behavior and concavity.

**EXAMPLE 4** Given

\[
A(x) = \int_2^x \sqrt{1 + t^2} \, dt
\]

calculate or approximate \( A(2), A(3), A'(2) \), and \( A'(3) \).

**Solution** First, \( A(2) = \int_2^2 \sqrt{1 + t^2} \, dt = 0 \). For \( A(3) = \int_2^3 \sqrt{1 + t^2} \, dt \), we need to approximate the definite integral because we do not have a simple antiderivative that enables us to compute the integral using FTC I. We compute an approximating Riemann sum, \( R_{100} \), and find \( A(3) \approx 4.11 \).

By FTC II, \( A'(x) = \sqrt{1 + x^2} \). In particular,

\[
A'(2) = \sqrt{1 + 2^2} = 3 \quad \text{and} \quad A'(3) = \sqrt{1 + 3^2} = \sqrt{28}
\]

When the upper limit of the integral is a function of \( x \) rather than \( x \) itself, we use FTC II together with the Chain Rule to differentiate a function defined via an integral.

**EXAMPLE 5** The FTC and the Chain Rule

Find the derivative of

\[
G(x) = \int_{-2}^{x^2} \sin t \, dt
\]

**Solution** FTC II does not apply directly because the upper limit is \( x^2 \) rather than \( x \). It is necessary to recognize that \( G \) is a composite function with outer function

\[
A(x) = \int_{-2}^x \sin t \, dt
\]

\[
G(x) = A(x^2) = \int_{-2}^{x^2} \sin t \, dt
\]

FTC II tells us that \( A'(x) = \sin x \), so by the Chain Rule,

\[
G'(x) = A'(x^2) \cdot (x^2)' = \sin(x^2) \cdot 2x = 2x \sin(x^2)
\]

Alternatively, we may set \( u = x^2 \) and use the Chain Rule as follows:

\[
\frac{dG}{dx} = \frac{d}{dx} \int_{-2}^{x^2} \sin t \, dt = \left( \frac{du}{dx} \int_{-2}^{u} \sin t \, dt \right) \frac{du}{dx} = (2x) \cdot 2x = 2x \sin(x^2)
\]

**GRAPHICAL INSIGHT** Another Tale of Two Graphs

FTC II tells us that \( A'(x) = f(x) \), or in other words, \( f(x) \) is the rate of change of \( A(x) \). If we did not know this result, we might come to suspect it by comparing the graphs of \( A \) and \( f \). Consider the following:

- Figure 6 shows that the increase in area \( \Delta A \) for a given \( \Delta x \) is larger at \( x_2 \) than at \( x_1 \) because \( f(x_2) > f(x_1) \). So the size of \( f(x) \) determines how quickly \( A(x) \) changes, as we would expect if \( A'(x) = f(x) \).
Figure 7 shows that the sign of \( f(x) \) determines whether \( A \) is increasing or decreasing. If \( f(x) > 0 \), then \( A \) is increasing because positive area is added as we move to the right. When \( f(x) \) turns negative, \( A \) begins to decrease because we start adding negative area.

- \( A \) has a local max at points where \( f(x) \) changes sign from + to − (the points where the area turns negative), and has a local min when \( f(x) \) changes from − to +. This agrees with the First Derivative Test.

These observations show that \( f \) behaves like \( A' \), as claimed by FTC II.

### 5.5 SUMMARY

- The area function with lower limit \( a \): \( A(x) = \int_a^x f(t) \, dt \). It satisfies \( A(a) = 0 \).
- FTC II: \( A'(x) = f(x) \), or equivalently, \( \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \).
- FTC II shows that every continuous function on an open interval \( I \) has an antiderivative on \( I \)—namely, its area function (with any lower limit in \( I \)).
- To differentiate the function \( G(x) = \int_a^x f(t) \, dt \), write \( G(x) = A(g(x)) \), where \( A(x) = \int_a^x f(t) \, dt \). Then use the Chain Rule:

\[
G'(x) = A'(g(x))g'(x) = f(g(x))g'(x)
\]

### 5.5 EXERCISES

**Preliminary Questions**

1. Let \( G(x) = \int_a^x \sqrt{r^3 + 1} \, dt \).
   - (a) Is the FTC II needed to calculate \( G(4) \)?
   - (b) Is the FTC II needed to calculate \( G'(4) \)?

2. Which of the following is an antiderivative \( F \) of \( f(x) = x^2 \) satisfying \( F(2) = 0 \)?
   - (a) \( \int_2^x 2x \, dt \)
   - (b) \( \int_0^x t^2 \, dt \)
   - (c) \( \int_0^x t^2 \, dt \)

3. Does every continuous function have an antiderivative? Explain.

4. Let \( G(x) = \int_4^x \sin t \, dt \). Which of the following statements are correct?
   - (a) \( G \) is the composite function \( \sin(x^2) \).
   - (b) \( G \) is the composite function \( A(x^2) \), where \( A(x) = \int_0^x \sin t \, dt \).
   - (c) \( G(x) \) is too complicated to differentiate.
   - (d) The Product Rule is used to differentiate \( G \).
   - (e) The Chain Rule is used to differentiate \( G \).
   - (f) \( G'(x) = 3x^2 \sin(x^2) \).

**Exercises**

In Exercises 1–6 compute an area function \( A(x) \) of \( f(x) \) with lower limit \( a \). Then, to verify the FTC II inverse relationship, compute \( A'(x) \) and show that it equals \( f(x) \).

1. \( f(x) = 4 - 2x \), \( a = 0 \)
2. \( f(x) = 4 - 2x \), \( a = 5 \)
3. \( f(x) = 4x + 6x^2 \), \( a = -1 \)
4. \( f(x) = x^2 - 8 \), \( a = 3 \)
5. \( f(x) = x^2 - \sin x \), \( a = 0 \)

6. \( f(x) = 1 - x + \cos x \), \( a = 0 \)

In Exercises 7–10, compute or approximate the corresponding function values and derivative values for the given area function. In some cases, approximations will need to be done via a Riemann sum.

7. \( F(x) = \int_0^x \sqrt{t^2 + t} \, dt \). Find \( F(0) \), \( F(3) \), \( F'(0) \), and \( F'(3) \).
8. \( G(x) = \int_0^x \sqrt{4 - t^2} \, dt \). Find \( G(0) \), \( G(2) \), \( G'(0) \), and \( G'(1) \).
9. \( F(x) = \int_0^1 \frac{du}{u^2 + 1} \). Find \( F(-2) \), \( F(2) \), \( F'(0) \), and \( F'(2) \).

10. \( H(x) = \int_0^x (x + 2) \, dt \).

Scanned with CamScanner
10. \( T(x) = \int_0^x \tan \theta \, d\theta \). Find \( T(0), T(\pi/3), T'(0) \), and \( T'(\pi/3) \).

In Exercises 11–20, find formulas for the functions represented by the integrals.

11. \( \int_2^x u^4 \, du \)
12. \( \int_2^x (12t^3 - 3t) \, dt \)
13. \( \int_0^x \sin u \, du \)
14. \( \int_{\pi/4}^x \sec^2 \theta \, d\theta \)
15. \( \int_2^x \frac{dt}{t^2} \)
16. \( \int_{\sin \theta}^x (5t + 9) \, dt \)
17. \( \int_1^x t \, dt \)
18. \( \int_{x/2}^x \sec^2 u \, du \)
19. \( \int_1^x \frac{dt}{\sqrt[3]{t}} \)
20. \( \int_{-2x}^{x^2} \frac{dt}{t} \)

21. Verify \( \int_0^x |t| \, dt = \frac{1}{2} x^2 \). Hint: Consider \( x \leq 0 \) and \( x \geq 0 \) separately.

22. Verify \( \int_0^x |t|^3 \, dt = \frac{1}{4} x^4 \). Hint: Consider \( x \leq 0 \) and \( x \geq 0 \) separately.

In Exercises 23–26, calculate the derivative.

23. \( \frac{d}{dx} \int_0^x (t^2 - 9t^3) \, dt \)
24. \( \frac{d}{dx} \int_0^x \cos u \, du \)
25. \( \frac{d}{dx} \int_{-1}^x \sec(5x - 9) \, dx \)
26. \( \frac{d}{dx} \int_1^x \tan \left( \frac{1}{1 + u^2} \right) \, du \)

27. Let \( A(x) = \int_0^x f(t) \, dt \) for \( f(x) \) in Figure 8.
   (a) Calculate \( A(2), A(3), A'(2) \), and \( A'(3) \).
   (b) Find formulas for \( A(x) \) on \([0, 2]\) and \([2, 4]\), and sketch the graph of \( A \).

28. Make a rough sketch of the graph of \( A(x) = \int_0^x g(t) \, dt \) for the function \( g(x) \) in Figure 9.

### SECTION 5.5 The Fundamental Theorem of Calculus, Part II

In Exercises 29–30, do the following:

- For \( x = 0, 1, 2, \ldots, 10 \), approximate \( A(x) = \int_0^x f(t) \, dt \).
- For \( x = 1, 2, 3, \ldots, 9 \), approximate \( A'(x) \) using \( \Delta x = 1 \) and the symmetric difference quotient approximation, \( A'(x) \approx \frac{A(x + \Delta x) - A(x - \Delta x)}{2\Delta x} \)

- Plot the values of \( A'(x) \) on a graph of \( f \) to demonstrate \( A' \approx f \).

29. Use \( f(x) \) from Figure 10(A).
30. Use \( f(x) \) from Figure 10(B).

![Figure 10](image)

**FIGURE 10**

In Exercises 31–36, calculate the derivative.

31. \( \frac{d}{dx} \int_0^x \frac{t^2}{1 + t} \, dt \)
32. \( \frac{d}{dx} \int_0^{1/3} \cos t \, dt \)
33. \( \frac{d}{dx} \int_{-6}^0 u^4 \, du \)
34. \( \frac{d}{dx} \int_{-1}^1 \sqrt{1 + u^4} \, du \)

**HINT for Exercise 34:** \( F(x) = A(x^2) - A(x^2) \).

35. \( \frac{d}{dx} \int_{-3}^x \tan t \, dt \)
36. \( \frac{d}{dx} \int_{-6}^0 \sqrt{1 + u^4} \, du \)

In Exercises 37–40, with \( f(x) \) as in Figure 11, let

\( A(x) = \int_0^x f(t) \, dt \) and \( B(x) = \int_2^x f(t) \, dt \).

37. Find the min and max of \( A \) on \([0, 6]\).
38. Find the min and max of \( B \) on \([0, 6]\).
39. Find formulas for \( A(x) \) and \( B(x) \) valid on \([2, 4]\).
40. Find formulas for \( A(x) \) and \( B(x) \) valid on \([4, 5]\).

![Figure 11](image)

**FIGURE 11**

41. Let \( A(x) = \int_0^x f(t) \, dt \), with \( f(x) \) as in Figure 12.
   (a) Does \( A \) have a local maximum at \( P \)?
   (b) Where does \( A \) have a local minimum?
   (c) Where does \( A \) have a local maximum?
   (d) True or false? \( A(x) < 0 \) for all \( x \) in the interval shown.
42. Let \( A(x) = \int_a^x f(t) \, dt \), with \( f(x) \) as in Figure 12.
(a) Where does \( A \) have its absolute maximum over the interval \([P, S]\)?
(b) Where does \( A \) have its absolute minimum over the interval \([P, S]\)?
(c) On what interval is \( A \) increasing?

In Exercises 43–44, let \( A(x) = \int_a^x f(t) \, dt \).

43. **Area Functions and Concavity** Explain why the following statements are true. Assume \( f \) is differentiable.
(a) If \( A \) has an inflection point at \( x = c \), then \( f'(c) = 0 \).
(b) \( A \) is concave up if \( f \) is increasing.
(c) \( A \) is concave down if \( f \) is decreasing.

44. Match the property of \( A \) with the corresponding property of the graph of \( f \). Assume \( f \) is differentiable.

**Area function \( A \)**
(a) \( A \) is decreasing.
(b) \( A \) has a local maximum.
(c) \( A \) is concave up.
(d) \( A \) goes from concave up to concave down.

**Graph of \( f \)**
(i) Lies below the \( x \)-axis.
(ii) Crosses the \( x \)-axis from positive to negative.
(iii) Has a local maximum.
(iv) \( f \) is increasing.

45. Let \( A(x) = \int_a^x f(t) \, dt \), with \( f(x) \) as in Figure 13. Determine:
(a) The intervals on which \( A \) is increasing and decreasing
(b) The values \( x \) where \( A \) has a local min or max
(c) The values of \( x \) where there are inflection points of \( A \)
(d) The intervals where \( A \) is concave up or concave down

46. Let \( f(x) = x^2 - 5x - 6 \) and \( F(x) = \int_0^x f(t) \, dt \).
(a) Find the critical points of \( F \) and determine whether they are local minima or local maxima.
(b) Find the points of inflection of \( F \) and determine whether the concavity changes from up to down or down to up.
(c) **GU** Plot \( y = f(x) \) and \( y = F(x) \) on the same set of axes and confirm your answers to (a) and (b).

47. Sketch the graph of an increasing function \( f \) such that both \( f'(x) \) and \( A(x) = \int_a^x f(t) \, dt \) are decreasing.

48. **GU** Figure 14 shows the graph of \( f(x) = x \sin x \). Let \( F(x) = \int_0^x \sin t \, dt \).
(a) Locate the local max and absolute max of \( F \) on \([0, 3\pi]\).
(b) Justify graphically: \( F \) has precisely one zero in \([\pi, 2\pi]\).
(c) How many zeros does \( F \) have in \([0, 3\pi]\)?
(d) Find the inflection points of \( F \) on \([0, 3\pi]\). For each one, state whether the concavity changes from up to down or from down to up.

**Figure 14** Graph of \( f(x) = x \sin x \).

49. **GU** Find the smallest positive critical point of
\[
F(x) = \int_0^x \cos(t^2) \, dt
\]
and determine whether it is a local min or max. Then find the smallest positive inflection point of \( F(x) \) and use a graph of \( y = \cos(x^{1/2}) \) to determine whether the concavity changes from up to down or from down to up.

50. Let \( A(x) = \int_a^x \sqrt{4 - t^2} \, dt \).
(a) **CAS** For \( x = -3, -2, \ldots, 4 \), calculate a Riemann sum that approximates the definite integral defining \( A(x) \). Plot the points \((x, A(x))\) for \( x = -4, -3, -2, \ldots, 4 \) and connect them with a smooth curve to obtain a graph of \( A \).
(b) Examine \( A' \) to determine the critical points of \( A \) and the increasing/decreasing behavior of the graph of \( A \).

**Further Insights and Challenges**

51. **Proof of FTC II** The proof in the text assumes that \( f \) is nonnegative and increasing. To prove it for all continuous functions, let \( m(h) \) and \( M(h) \) denote the minimum and maximum of \( f \) on \([x, x+h]\) (Figure 15).

The continuity of \( f \) implies that \( \lim_{h \to 0} m(h) = \lim_{h \to 0} M(h) = f(x) \). Show that for \( h > 0 \),
\[
hm(h) \leq A(x+h) - A(x) \leq hM(h)
\]
For $h < 0$, the inequalities are reversed. Prove that $A'(x) = f(x)$.

**Figure 15** Graphical interpretation of $A(x + h) - A(x)$.

52. Proof of FTC I FTC I asserts that $\int_a^b f(t)dt = F(b) - F(a)$ if $F'(x) = f(x)$. Use FTC II to give a new proof of FTC I as follows. Set $A(x) = \int_a^x f(t)dt$.

(a) Show that $F(x) = A(x) + C$ for some constant.
(b) Show that $F(b) - F(a) = A(b) - A(a) = \int_a^b f(t)dt$.

53. Can Every Antiderivative Be Expressed as an Integral? The area function $A(x) = \int_a^x f(t)dt$ is an antiderivative of $f$ for every value of $a$. However, not all antiderivatives are obtained in this way. The general antiderivative of $f(x) = x$ is $F(x) = \frac{1}{2}x^2 + C$. Show that $F$ is an area function if $C \leq 0$ but not if $C > 0$.

54. Prove the formula

$$\frac{d}{dx} \int_a^x f(t)dt = f(u(x))u'(x) - f(u(x))u'(x)$$

---

### 5.6 Net Change as the Integral of a Rate of Change

So far, we have seen how the definite integral can be used to compute area. That application barely scratches the surface of ways that this important tool from calculus can be applied. In this section, we use the integral to compute net change, a concept that arises in a broad range of applications.

Consider the following problem: Water flows into an empty bucket at a rate of $r(t)$ liters per second. How much water is in the bucket after 4 seconds? If the rate of water flow were constant—say, 1.5 L/s—we would have

<table>
<thead>
<tr>
<th>Quantity of water</th>
<th>Flow rate</th>
<th>Time elapsed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.5 L/s</td>
<td>4 s</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6 L</td>
</tr>
</tbody>
</table>

Suppose, however, that the flow rate $r(t)$ varies as in Figure 1. Then the **quantity of water is equal to the area under the graph of $y = r(t)$**. To prove this, let $s(t)$ be the amount of water in the bucket at time $t$. Then $s'(t) = r(t)$ because $s'(t)$ is the rate at which the quantity of water is changing, that is, the rate that water is entering the bucket, $r(t)$. Furthermore, $s(0) = 0$ because the bucket is initially empty. By FTC I,

\[
\int_0^4 s'(t)dt = s(4) - s(0) = \frac{s(4)}{4}
\]

More generally, $s(t_2) - s(t_1)$ is the **net change in $s(t)$ over the interval $[t_1, t_2]$**. FTC I yields the following result:

**Theorem 1** Net Change as the Integral of a Rate of Change The net change in $s(t)$ over an interval $[t_1, t_2]$ is given by the integral

\[
\int_{t_1}^{t_2} s'(t)dt = \frac{s(t_2) - s(t_1)}{t_2 - t_1}
\]

**Example 1** Water leaks from a tank at a rate of $2 + 5t$ L/hour, where $t$ is the number of hours after 7 AM. How much water is lost between 9 and 11 AM?
Solution  Let \( s(t) \) be the quantity of water in the tank at time \( t \). Since \( 2 + 5t \) represents the rate that the water is leaving the tank, the rate of change of the water in the tank is \( -(2 + 5t) \). So \( s'(t) = -(2 + 5t) \). Since 9 AM and 11 AM correspond to \( t = 2 \) and \( t = 4 \), respectively, the net change in \( s(t) \) between 9 and 11 AM is

\[
s(4) - s(2) = \int_{2}^{4} s'(t) \, dt = - \int_{2}^{4} (2 + 5t) \, dt
\]

\[
= - \left[ (2t + \frac{5}{2} t^2) \right]_{2}^{4} = (-48) - (-14) = -34 \text{ liters}
\]

The tank lost 34 L between 9 and 11 AM.

In the next example, we estimate an integral using numerical data. We shall compute the average of the left- and right-endpoint approximations, because this is usually more accurate than either endpoint approximation alone. This method of approximation is called the Trapezoidal Approximation; we investigate it further in Section 8.8.

**EXAMPLE 2  Traffic Flow** The number of cars per hour passing an observation point along a highway is called the traffic flow rate \( q(t) \) (in cars per hour).

(a) Which quantity is represented by the integral \( \int_{t_1}^{t_2} q(t) \, dt \)?

(b) The flow rate on the Jocoro Highway is recorded at 15-minute intervals on Monday morning between 7:00 and 9:00 AM. Estimate the number of cars using the highway during this 2-hour period.

<table>
<thead>
<tr>
<th>( t )</th>
<th>7:00</th>
<th>7:15</th>
<th>7:30</th>
<th>7:45</th>
<th>8:00</th>
<th>8:15</th>
<th>8:30</th>
<th>8:45</th>
<th>9:00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q(t) )</td>
<td>1044</td>
<td>1297</td>
<td>1478</td>
<td>1844</td>
<td>1451</td>
<td>1378</td>
<td>1155</td>
<td>802</td>
<td>542</td>
</tr>
</tbody>
</table>

Solution

(a) The integral \( \int_{t_1}^{t_2} q(t) \, dt \) represents the total number of cars that passed the observation point during the time interval \([t_1, t_2] \).

(b) The data values are spaced at intervals of \( \Delta t = 0.25 \) h. Thus,

\[
L_N = 0.25 (1044 + 1297 + 1478 + 1844 + 1451 + 1378 + 1155 + 802) \approx 2612
\]

\[
R_N = 0.25 (1297 + 1478 + 1844 + 1451 + 1378 + 1155 + 802 + 542) \approx 2487
\]

We estimate the number of cars that passed the observation point between 7 and 9 AM by taking the average of \( R_N \) and \( L_N \):

\[
\int_{7}^{9} q(t) \, dt \approx \frac{1}{2} (R_N + L_N) = \frac{1}{2} (2612 + 2487) \approx 2550
\]

Approximately 2550 cars used the Jocoro Highway during the time period.

**The Integral of Velocity**

Let \( s(t) \) be the position at time \( t \) of an object in linear motion. Then the object's velocity is \( v(t) = s'(t) \), and the integral of \( v \) is equal to the net change in position or displacement over a time interval \([t_1, t_2] \):

\[
\int_{t_1}^{t_2} v(t) \, dt = \int_{t_1}^{t_2} s'(t) \, dt = s(t_2) - s(t_1)
\]

Displacement or net change in position
We must distinguish between displacement and distance traveled. If you travel 10 km and then return to your starting point, your displacement is zero but your distance traveled is 20 km. To compute distance traveled rather than displacement, we integrate the speed \( v(t) \).

**Theorem 2** The Integral of Velocity

For an object in linear motion with velocity \( v(t) \), then

\[
\text{displacement during } [t_1, t_2] = \int_{t_1}^{t_2} v(t) \, dt
\]

\[
\text{distance traveled during } [t_1, t_2] = \int_{t_1}^{t_2} |v(t)| \, dt
\]

**Example 3** A particle has velocity \( v(t) = t^3 - 10t^2 + 24t \) m/s. Compute:

(a) Displacement over \([0, 6]\)

(b) Total distance traveled over \([0, 6]\)

Indicate the particle’s trajectory with a motion diagram.

**Solution** First, we compute the indefinite integral:

\[
\int v(t) \, dt = \int (t^3 - 10t^2 + 24t) \, dt = \frac{1}{4}t^4 - \frac{10}{3}t^3 + 12t^2 + C
\]

(a) The displacement over the time interval \([0, 6]\) is

\[
\int_0^6 v(t) \, dt = \left( \frac{1}{4}t^4 - \frac{10}{3}t^3 + 12t^2 \right)_0^6 = 36 \text{ m}
\]

(b) The factorization \( v(t) = t(t-4)(t-6) \) shows that \( v(t) \) changes sign at \( t = 4 \). It is positive on \([0, 4]\) and negative on \([4, 6]\), as we see in Figure 2. Therefore, the total distance traveled is

\[
\int_0^6 |v(t)| \, dt = \int_0^4 v(t) \, dt - \int_4^6 v(t) \, dt
\]

We evaluate these two integrals separately:

\[
[0, 4]: \int_0^4 v(t) \, dt = \left( \frac{1}{4}t^4 - \frac{10}{3}t^3 + 12t^2 \right)_0^4 = \frac{128}{3} \text{ m}
\]

\[
[4, 6]: \int_4^6 v(t) \, dt = \left( \frac{1}{4}t^4 - \frac{10}{3}t^3 + 12t^2 \right)_4^6 = -\frac{20}{3} \text{ m}
\]

The total distance traveled is \( \frac{128}{3} - \left(-\frac{20}{3}\right) = \frac{148}{3} = 49\frac{2}{3} \text{ m} \).

**Total Versus Marginal Cost**

Let \( C(x) \) represent a manufacturer’s cost to produce \( x \) units of a particular product or commodity. The derivative \( C'(x) \) is called the marginal cost. The cost of increasing production from \( a \) units to \( b \) units is the net change \( C(b) - C(a) \), which is equal to the integral of the marginal cost:

\[
\text{cost of increasing production from } a \text{ units to } b \text{ units} = \int_a^b C'(x) \, dx
\]
**EXAMPLE 4** The marginal cost of producing \( x \) computer chips (in units of 1000) is \( C'(x) = 300x^2 - 4000x + 40,000 \) (dollars per thousand chips).

(a) Find the cost of increasing production from 10,000 to 15,000 chips.

(b) Determine the total cost of producing 15,000 chips, assuming that it costs $30,000 to set up the manufacturing run [i.e., \( C(0) = 30,000 \)].

**Solution**

(a) The cost of increasing production from 10,000 \( (x = 10) \) to 15,000 \( (x = 15) \) is

\[
C(15) - C(10) = \int_{10}^{15} (300x^2 - 4000x + 40,000) \, dx
\]

\[
= \left. (100x^3 - 2000x^2 + 40,000x) \right|_{10}^{15}
\]

\[
= 487,500 - 300,000 = $187,500
\]

(b) The cost of increasing production from 0 to 15,000 chips is

\[
C(15) - C(0) = \int_{0}^{15} (300x^2 - 4000x + 40,000) \, dx
\]

\[
= \left. (100x^3 - 2000x^2 + 40,000x) \right|_{0}^{15} = $487,500
\]

The total cost of producing 15,000 chips includes the set-up costs of $30,000:

\[
C(15) = C(0) + 487,500 = 30,000 + 487,500 = $517,500
\]

---

**5.6 SUMMARY**

- Many applications are based on the following principle: *The net change in a quantity \( s(t) \) is equal to the integral of its rate of change:*

\[
s(t_2) - s(t_1) = \int_{t_1}^{t_2} s'(t) \, dt
\]

- For an object traveling in a straight line at velocity \( v(t) \),

\[
\text{displacement during } [t_1, t_2] = \int_{t_1}^{t_2} v(t) \, dt
\]

\[
\text{total distance traveled during } [t_1, t_2] = \int_{t_1}^{t_2} |v(t)| \, dt
\]

- If \( C(x) \) is the cost of producing \( x \) units of a commodity, then \( C'(x) \) is the marginal cost and the total cost of producing from \( a \) units to \( b \) units is

\[
C(b) - C(a) = \int_{a}^{b} C'(x) \, dx
\]

---

**5.6 EXERCISES**

**Preliminary Questions**

1. A hot metal object is submerged in cold water. The rate at which the object cools (in degrees per minute) is a function \( f(t) \) of time. Which quantity is represented by the integral \( \int_{0}^{T} f(t) \, dt \)?

2. A plane travels 560 km from Los Angeles to San Francisco in 1 hour. If the plane's velocity at time \( t \) is \( v(t) \) km/h, what is the value of \( \int_{0}^{1} v(t) \, dt \)?
3. Which of the following quantities would be naturally represented as derivatives and which as integrals?
(a) Velocity of a train
(b) Rainfall during a 6-month period
(c) Mileage per gallon of an automobile
(d) Increase in the U.S. population from 1990 to 2010

Exercises
1. Water flows into an empty reservoir at a rate of 3000 + 20t L per hour (t is in hours). What is the quantity of water in the reservoir after 5 h?
2. A population of insects increases at a rate of 200 + 10t + 0.25t² insects per day (t in days). Find the insect population after 3 days, assuming that there are 35 insects at t = 0.
3. A survey shows that a mayoral candidate is gaining votes at a rate of 2000 + 1000 votes per day, where t is the number of days since she announced her candidacy. How many supporters will the candidate have after 60 days, assuming that she had no supporters at t = 0?
4. A factory produces bicycles at a rate of 95 + 3t² - t bicycles per week (t in weeks). How many bicycles were produced from the beginning of week 2 to the end of week 3?
5. Find the displacement of a particle moving in a straight line with velocity v(t) = 4t - 3 m/s over the time interval [2, 5].
6. Find the displacement over the time interval [1, 6] of a helicopter whose (vertical) velocity at time t is v(t) = 0.02t² + t m/s.
7. A cat falls from a tree (with zero initial velocity) at time t = 0. How far does the cat fall between t = 0.5 second and t = 1 s? Use Galileo’s formula s(t) = -9.8t² for the distance traveled during the first 15 seconds.

In Exercises 8-12, a particle moves in a straight line with the given velocity (in meters per second). Find the displacement and distance traveled over the time interval, and draw a motion diagram like Figure 3 (with distance and time labels).
8. v(t) = 12 - 4t, [0, 5]
9. v(t) = 36 - 24t + 3t², [0, 10]
10. v(t) = 4t² - 1, 0 ≤ t ≤ 2
11. v(t) = cos t, 0 ≤ t ≤ 3π
12. v(t) = cos t, 0 ≤ t ≤ π
13. Find the net change in velocity over [1, 4] of an object with s(t) = 8t - t² m/s².
14. Show that if acceleration is constant, then the change in velocity is proportional to the length of the time interval.
15. The traffic flow rate past a certain point on a highway is q(t) = 3000 + 200t - 0.005t² (t in hours), where t = 0 is 8 AM. How many cars pass by in the time interval from 8 to 10 AM?
16. The marginal cost of producing x tablet computers is
   \[ C'(x) = 120 - 0.06x + 0.00001x^2 \]
   What is the additional cost of producing 3000 units if the set-up cost is $50,000? If production is set at 3000 units, what is the cost of producing 200 additional units?
17. A small boutique produces wool sweaters at a marginal cost of $40 - 5(x/5) for 0 ≤ x ≤ 20, where \( \lfloor x \rfloor \) is the greatest integer function. Find the cost of producing 20 sweaters. Then compute the average cost of the first 10 sweaters and the last 10 sweaters.
18. The rate (in liters per minute) at which water drains from a tank is recorded at half-minute intervals. Compute the average of the left- and right-endpoint approximations to estimate the total amount of water drained during the first 3 min.

<table>
<thead>
<tr>
<th>t (min)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>r (L/min)</td>
<td>50</td>
<td>48</td>
<td>46</td>
<td>44</td>
<td>42</td>
<td>40</td>
<td>38</td>
</tr>
</tbody>
</table>

19. The velocity of a car is recorded at half-second intervals (in feet per second). Use the average of the left- and right-endpoint approximations to estimate the total distance traveled during the first 4 s.

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>v(t)</td>
<td>0</td>
<td>20</td>
<td>29</td>
<td>38</td>
<td>44</td>
<td>32</td>
<td>35</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

20. To model the effects of a carbon tax on CO2 emissions, policymakers study the marginal cost of abatement \( B(x) \), defined as the cost of increasing CO2 reduction from \( x \) to \( x + 1 \) tons (in units of ten thousand tons)—Figure 4. Which quantity is represented by the area under the curve over \([0,3]\) in Figure 4?

![Figure 4](image)

**FIGURE 4** Marginal cost of abatement \( B(x) \).

21. The snowfall rate \( R \) (in inches per hour) was tracked during a major 24-hour lake effect snowstorm in Buffalo, New York. The graph in Figure 5 shows \( R \) as a function of \( t \) (hours) during the storm. What quantity does \( \int_{0}^{24} R(t) \, dt \) represent? Approximate the integral.

![Figure 5](image)

**FIGURE 5** The snowfall rate during a major 24-h storm.
22. Figure 6 shows the migration rate $M(t)$ of Ireland in the period 1988–1998. This is the rate at which people (in thousands per year) moved into or out of the country.

(a) Is the following integral positive or negative? What does this quantity represent?

$$\int_{1988}^{1998} M(t) \, dt$$

(b) Did migration in the period 1988–1998 result in a net influx of people into Ireland or a net outflow of people from Ireland?

(c) During which 2 years could the Irish prime minister announce, "We've hit an inflection point. We are still losing population, but the trend is now improving."

![Irish migration rate (thousands per year).](image)

**FIGURE 6** Irish migration rate (thousands per year).

23. Let $N(d)$ be the number of asteroids of diameter $\leq d$ kilometers. Data suggest that the diameters are distributed according to a piecewise power law:

$$N'(d) = \begin{cases} 1.9 \times 10^9 d^{-2.3} & \text{for } d < 70 \\ 2.6 \times 10^7 d^{-4} & \text{for } d \geq 70 \end{cases}$$

(a) Compute the number of asteroids with a diameter between 0.1 and 10 km.

(b) Using the approximation $N(d + 1) - N(d) \approx N'(d)$, estimate the number of asteroids of diameter 50 km.

24. Heat Capacity The heat capacity $C(T)$ of a substance is the amount of energy (in joules) required to raise the temperature of 1 g by 1°C at constant volume.

(a) Explain why the energy required to raise the temperature from $T_1$ to $T_2$ is the area under the graph of $C(T)$ over $[T_1, T_2]$.

(b) How much energy is required to raise the temperature from 50°C to 100°C if $C(T) = 6 + 0.2\sqrt{T}$?

25. Figure 7 shows the rate $R(t)$ of natural gas consumption (billions of cubic feet per day) in the mid-Atlantic states (New York, New Jersey, Pennsylvania). Express the total quantity of natural gas consumed in 2009 as an integral (with respect to time $t$ in days). Then estimate this quantity, given the following average monthly values of $R(t)$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(t)$</td>
<td>0.4</td>
<td>2.8</td>
<td>6.5</td>
<td>9.8</td>
<td>8.9</td>
<td>6.1</td>
<td>4.2</td>
<td>2.3</td>
<td>1.1</td>
<td>0</td>
</tr>
</tbody>
</table>

![Natural gas consumption (10^9 ft³/day) in 2009 in the mid-Atlantic states.](image)

**FIGURE 7** Natural gas consumption in 2009 in the mid-Atlantic states.

26. Cardiac output is the rate $R$ of volume of blood pumped by the heart per unit time (in liters per minute). Doctors measure $R$ by injecting a mg of dye into a vein leading into the heart at $t = 0$ and recording the concentration $c(t)$ of dye (in milligrams per liter) pumped out at short regular time intervals (Figure 8).

(a) Explain: The quantity of dye pumped out in a small time interval $[t, t + \Delta t]$ is approximately $R(t) \Delta t$.

(b) Show that $A = R \int_{0}^{T} c(t) \, dt$, where $T$ is large enough that all of the dye is pumped through the heart but not large enough that the dye returns by recirculation.

(c) Assume $A = 5$ mg. Estimate $R$ using the following values of $c(t)$ recorded at 1-second intervals

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(t)$</td>
<td>0</td>
<td>0.4</td>
<td>2.8</td>
<td>6.5</td>
<td>9.8</td>
<td>8.9</td>
<td>6.1</td>
<td>4.2</td>
<td>2.3</td>
<td>1.1</td>
<td>0</td>
</tr>
</tbody>
</table>

![Dye concentration here](image)

**FIGURE 8** Dye concentration here.

Exercises 27 and 28: A study suggests that the extinction rate $r(t)$ of marine animal families during the Phanerozoic Eon can be modeled by the function $r(t) = 3130/(t + 262)$ for $0 \leq t \leq 544$, where $t$ is time elapsed (millions of years) since the beginning of the eon 544 million years ago. Thus, $t = 544$ refers to the present time, $t = 540$ is 4 million years ago, and so on.

27. Compute the average of $R_5$ and $L_N$ with $N = 5$ to estimate the total number of families that became extinct in the periods $100 \leq t \leq 150$ and $350 \leq t \leq 400$.

28. Estimate the total number of extinct families from $t = 0$ to the present, using $M_N$ with $N = 544$.

29. Show that a particle, located at the origin at $t = 1$ and moving along the $x$-axis with velocity $v(t) = t^{-2}$, will never pass the point $x = 2$.

30. Show that a particle, located at the origin at $t = 1$ and moving along the $x$-axis with velocity $v(t) = t^{-1/2}$, moves arbitrarily far from the origin after sufficient time has elapsed.
31. In a free market economy, the demand curve is the graph of the function $D$ that represents the demand for a specific product by the consumers in the economy at price $q$. It is not surprising that the curve is decreasing, as the demand drops as the price goes up. The supply curve is the graph of the function $S$ that represents the supply of the product that the producers are willing to produce as a function of the price $q$. As the price goes up, the producers are willing to produce more of the item, and therefore, this curve is increasing. The point $(p^*, q^*)$ at which the two curves cross is called the equilibrium point, where the supply and demand balance. Tradition in economics is to make the horizontal axis the quantity $q$ of the item and the vertical axis the price $p$. We define $p = S(q)$ to correspond to the supply curve and $p = D(q)$ to correspond to the demand curve. In other words, we have inverts the formula for these two functions from giving quantities in terms of price to giving price in terms of quantity. The areas depicted in Figure 9 represent the excess supply and excess demand.

(a) The consumer surplus represents the savings on the part of consumers if they pay price $p^*$ rather than the price greater than $p^*$ that many were willing to pay. Write a formula for this consumer surplus. The formula will include a definite integral and it will depend on $p^*$ and $q^*$.

(b) The producer surplus represents the savings on the part of producers if they sell at price $p^*$ rather than the price less than $p^*$ that some producers were willing to accept. Write a formula for this producer surplus.

(e) A variety of coffee shops in a town sell mocha late supreme coffees. If the supply curve is given by $p = \frac{q}{100} + 1$ and the demand curve is given by $p = \frac{10}{q/100 + 1}$, determine the equilibrium point $(p^*, q^*)$ and determine or approximate the consumer surplus and producer surplus when the mocha late supreme coffees are sold at price $p^*$.

![Figure 9 The supply and demand curves.](image)

### 5.7 The Substitution Method

Integration (antidifferentiation) is generally more difficult than differentiation. There are no sure-fire methods, and many antiderivatives cannot be expressed in terms of elementary functions. However, there are a few important general techniques. One such technique is the Substitution Method, which is essentially an inverse of the Chain Rule for Differentiation.

Consider the integral $\int 2x \cos(x^2) dx$. We can evaluate it if we remember the Chain Rule calculation:

$$\frac{d}{dx} \sin(x^2) = 2x \cos(x^2)$$

This tells us that $\sin(x^2)$ is an antiderivative of $2x \cos(x^2)$, and therefore,

$$\int 2x \cos(x^2) dx = \sin(x^2) + C$$

A similar Chain Rule calculation shows that

$$\int (1 + 3x^2) \cos(x + x^3) dx = \sin(x + x^3) + C$$

In both cases, the integrand is the product of a composite function and the derivative of the inside function. The Chain Rule does not help if the derivative of the inside function is missing. For instance, we cannot use the Chain Rule to compute $\int \cos(x + x^3) dx$ because the factor $(1 + 3x^2)$ does not appear.

In general, if $F'(u) = f(u)$, then by the Chain Rule,

$$\frac{d}{dx} F(u(x)) = F'(u(x))u'(x) = f(u(x))u'(x)$$

**REMINDER** A composite function has the form $f(g(x))$. For convenience, we call $u = g(x)$ the inside function and $f(u)$ the outside function.
This translates into the following integration formula:

**THEOREM 1 The Substitution Method** If \( F'(x) = f(x) \), and \( u \) is a differentiable function whose range includes the domain of \( f \), then

\[
\int f(u(x))u'(x) \, dx = F(u(x)) + C
\]

**Substitution Using Differentials**

Before proceeding to the examples, we discuss the procedure for carrying out substitution using differentials. We can treat the symbols \( du \) and \( dx \) that occur in integration as differentials, and in that way can use the relationship (introduced in Section 4.1 on linear approximation)

\[
du = \frac{du}{dx} \, dx
\]

to symbolically substitute into integrals in order to simplify them.

Equivalently, \( du \) and \( dx \) are related by

\[
du = u'(x) \, dx
\]

For example,

If \( u = x^2 \), then \( du = 2x \, dx \)

If \( u = \cos(x^3) \), then \( du = -3x^2 \sin(x^3) \, dx \)

Now when the integrand has the form \( f(u(x))u'(x) \), we can use Eq. (1) to rewrite the entire integral (including the \( dx \) term) in terms of \( u \) and its differential \( du \):

\[
\int \frac{f(u(x))}{u'} \, dx = \int f(u) \, du
\]

This equation is called the **Change of Variables Formula**. It transforms an integral in the variable \( x \) into a (hopefully simpler) integral in the new variable \( u \).

**EXAMPLE 1** Evaluate \( \int 3x^2 \sin(x^3) \, dx \).

**Solution** The integrand contains the composite function \( \sin(x^3) \), so we set \( u = x^3 \). The differential \( du = 3x^2 \, dx \) also appears, so we can carry out the substitution:

\[
\int 3x^2 \sin(x^3) \, dx = \int \sin(u) \frac{3x^2}{du} \, dx = \int \sin u \, du
\]

Now evaluate the integral in the \( u \)-variable and replace \( u \) by \( x^3 \) in the answer:

\[
\int 3x^2 \sin(x^3) \, dx = \int \sin u \, du = -\cos u + C = -\cos(x^3) + C
\]

Let's check our answer by differentiating:

\[
\frac{d}{dx}(-\cos(x^3)) = \sin(x^3) \frac{d}{dx} x^3 = 3x^2 \sin(x^3)
\]
EXAMPLE 2 Multiplying \( du \) by a Constant  Evaluate \( \int x(x^2 + 9)^5 \, dx \).

Solution  We let \( u = x^2 + 9 \) because the composite \( u^2 = (x^2 + 9)^5 \) appears in the integrand. The differential \( du = 2x \, dx \) does not appear as is, but we can multiply by \( \frac{1}{2} \) to obtain \( \frac{1}{2} \, du = x \, dx \).

Now we can apply substitution:

\[
\int x(x^2 + 9)^5 \, dx = \frac{1}{2} \int u^5 \, du = \frac{1}{12} u^6 + C
\]

Finally, we express the answer in terms of \( x \) by substituting \( u = x^2 + 9 \):

\[
\int x(x^2 + 9)^5 \, dx = \frac{1}{12} u^6 + C = \frac{1}{12} (x^2 + 9)^6 + C
\]

EXAMPLE 3 Evaluate \( \int \frac{(x^2 + 2x) \, dx}{(x^3 + 3x^2 + 12)^6} \).

Solution  The appearance of \((x^3 + 3x^2 + 12)^{-6}\) in the integrand suggests that we try \( u = x^3 + 3x^2 + 12 \). With this choice,

\[
du = (3x^2 + 6x) \, dx = 3(x^2 + 2x) \, dx \quad \Rightarrow \quad \frac{1}{3} \, du = (x^2 + 2x) \, dx
\]

\[
\int \frac{(x^2 + 2x) \, dx}{(x^3 + 3x^2 + 12)^6} = \int \frac{(x^3 + 3x^2 + 12)^{-6} (x^2 + 2x) \, dx}{u^{-6}}
\]

\[
= \frac{1}{3} \int u^{-6} \, du = \left( \frac{1}{3} \right) \left( \frac{u^{-5}}{-5} \right) + C
\]

\[
= -\frac{1}{15} (x^3 + 3x^2 + 12)^{-5} + C
\]

CONCEPTUAL INSIGHT  An integration method that works for a given function may fail if we change the function even slightly. In the previous example, if we replace 2 by 2.1 and consider instead \( \int \frac{(x^2 + 2.1x) \, dx}{(x^3 + 3x^2 + 12)^6} \), the Substitution Method does not work. The problem is that \((x^2 + 2.1x) \, dx\) is not a multiple of \(du = (3x^2 + 6x) \, dx\).

EXAMPLE 4 Evaluate \( \int \sin(7\theta + 5) \, d\theta \).

Solution  Let \( u = 7\theta + 5 \). Then \( du = 7 \, d\theta \) and \( \frac{1}{7} \, du = d\theta \). We obtain

\[
\int \frac{\sin(7\theta + 5) \, d\theta}{\sin u} = \frac{1}{7} \int \sin u \, du = -\frac{1}{7} \cos u + C = -\frac{1}{7} \cos(7\theta + 5) + C
\]

EXAMPLE 5 Evaluate \( \int \frac{\sin(t^{1/3}) \, dt}{t^{1/3}} \).

Solution  It makes sense to try \( u = t^{1/3} \) because \( du = \frac{1}{3} t^{-2/3} \, dt \), and thus the multiple \( 3 \, du \) appears in the integrand. In other words,
where 

\[ u = t^{1/3}, \quad \frac{dt}{t^{2/3}} = 3\, du \]

\[ \int \frac{\sin(t^{1/3})dt}{t^{2/3}} = \int \sin(u) (3du) \]

\[ = -3\cos u + C = -3\cos(t^{1/3}) + C \]

**EXAMPLE 6**  Additional Step Necessary  Evaluate \( \int x\sqrt{5x + 1}\, dx \).

**Solution**  Since \( \sqrt{5x + 1} \) appears, we are tempted to set \( u = 5x + 1 \). Then

\[ du = 5\, dx \quad \Rightarrow \quad \sqrt{5x + 1}\, dx = \frac{1}{5}\sqrt{u}\, du \]

Unfortunately, the integrand is not \( \sqrt{5x + 1} \) but \( x\sqrt{5x + 1} \). To take care of the extra factor of \( x \), we solve \( u = 5x + 1 \) to obtain \( x = \frac{1}{5}(u - 1) \). Then

\[ x\sqrt{5x + 1}\, dx = \left( \frac{1}{5}(u - 1) \right) \frac{1}{5}\sqrt{u}\, du = \frac{1}{25}(u^{3/2} - u^{1/2})\, du \]

\[ \int x\sqrt{5x + 1}\, dx = \frac{1}{25} \int (u^{3/2} - u^{1/2})\, du \]

\[ = \frac{1}{25} \left( \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \]

\[ = \frac{2}{125} (5x + 1)^{5/2} - \frac{2}{75} (5x + 1)^{3/2} + C \]

**Change of Variables Formula for Definite Integrals**

The Change of Variables Formula can be applied to definite integrals provided that the limits of integration are changed, as indicated in the next theorem.

**THEOREM 2**  Change of Variables Theorem for Definite Integrals  If \( u' \) is continuous on \( [a, b] \), and \( f \) is continuous on the range of \( u \), then

\[ \int_{a}^{b} f(u(x))u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du \]

**Proof**  If \( F(x) \) is an antiderivative of \( f(x) \), then \( F(u(x)) \) is an antiderivative of \( f(u(x))u'(x) \). FTC I shows that the two integrals are equal:

\[ \int_{a}^{b} f(u(x))u'(x)\, dx = F(u(b)) - F(u(a)) \]

\[ \int_{u(a)}^{u(b)} f(u)\, du = F(u(b)) - F(u(a)) \]

**EXAMPLE 7**  Evaluate \( \int_{0}^{2} x^2\sqrt{x^3 + 1}\, dx \).

**Solution**  We use the substitution \( u = x^3 + 1 \). Thus, \( du = 3x^2\, dx \), and therefore \( x^2\, dx = \frac{1}{3} du \). By Eq. (2), the new limits of integration are
\[ u(0) = 0^3 + 1 = 1 \quad \text{and} \quad u(2) = 2^3 + 1 = 9 \]

Thus,

\[ \int_0^2 x^2\sqrt{x^3 + 1} \, dx = \frac{1}{3} \int_1^9 \sqrt{u} \, du = \frac{2}{9} u^{3/2} \bigg|_1^9 = \frac{52}{9} \]

This substitution shows that the area in (A) in Figure 1 is equal to one-third of the area in (B).

In the previous example, we can avoid changing the limits of integration by evaluating the indefinite integral in terms of \( x \):

\[ \int x^2\sqrt{x^3 + 1} \, dx = \frac{1}{3} \int \sqrt{u} \, du = \frac{2}{9} u^{3/2} + C = \frac{2}{9} (x^3 + 1)^{3/2} + C \]

This leads to the same result:

\[ \int_0^2 x^2\sqrt{x^3 + 1} \, dx = \frac{2}{9} (x^3 + 1)^{3/2} \bigg|_0^2 = \frac{52}{9} \]

**EXAMPLE 8** Evaluate \( \int_0^{\pi/4} \tan^3 \theta \sec^2 \theta \, d\theta \).

**Solution** The substitution \( u = \tan \theta \) makes sense because \( du = \sec^2 \theta \, d\theta \), and therefore, \( u^3 \, du = \tan^3 \theta \sec^2 \theta \, d\theta \). The new limits of integration are

\[ u(0) = \tan 0 = 0 \quad \text{and} \quad u \left( \frac{\pi}{4} \right) = \tan \left( \frac{\pi}{4} \right) = 1 \]

Thus,

\[ \int_0^{\pi/4} \tan^3 \theta \sec^2 \theta \, d\theta = \int_0^1 u^3 \, du = \frac{u^4}{4} \bigg|_0^1 = \frac{1}{4} \]

**EXAMPLE 9** Calculate the area under the graph of \( y = \frac{x}{(x^2 + 1)^2} \) over \([1, 3]\).

**Solution** The area (Figure 2) is equal to the integral \( \int_1^3 \frac{x}{(x^2 + 1)^2} \, dx \). We use the substitution

\[ u = x^2 + 1, \quad du = 2x \, dx, \quad \frac{1}{2} \frac{du}{u^2} = \frac{x \, dx}{(x^2 + 1)^2} \]

The new limits of integration are \( u(1) = 1^2 + 1 = 2 \) and \( u(3) = 3^2 + 1 = 10 \), so

\[ \int_1^3 \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2} \int_2^{10} \frac{du}{u^2} = \frac{1}{2} [ \frac{1}{u} ]_2^{10} = \frac{1}{20} + \frac{1}{4} = \frac{1}{5} \]

**5.7 SUMMARY**

- Try the Substitution Method when the integrand has the form \( f(u(x))u'(x) \). If \( F \) is an antiderivative of \( f \), then

\[ \int f(u(x))u'(x) \, dx = F(u(x)) + C \]

- The differential of \( u(x) \) is related to \( dx \) by \( du = u'(x) \, dx \).
- The Substitution Method is expressed by the Change of Variables Formula:

\[ \int f(u(x))u'(x) \, dx = \int f(u) \, du \]
• Change of Variables Formula for definite integrals:
\[ \int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du \]

• There are two approaches that can be taken to compute a definite integral in \( x \) by a change of variables to \( u \).
  
  - Change the limits of integration to the corresponding limits of integration in \( u \) and compute the definite integral in \( u \).
  
  - Compute the antiderivative first, expressing it in terms of \( x \). Then use that antiderivative to compute the definite integral in \( x \).

## 5.7 EXERCISES

### Preliminary Questions

1. Which of the following integrals is a candidate for the Substitution Method?

(a) \( \int 5x^4 \sin(x^5)\,dx \)

(b) \( \int \sin^3 x \cos x\,dx \)

(c) \( \int x^5 \sin x\,dx \)

2. Find an appropriate choice of \( u \) for evaluating the following integrals by substitution:

(a) \( \int (x^2 + 9)^4\,dx \)

(b) \( \int x^2 \sin(x^3)\,dx \)

(c) \( \int \sin x \cos x^2\,dx \)

3. Which of the following is equal to \( \int_0^2 x^2(x^3 + 1)\,dx \) for a suitable substitution?

(a) \( \frac{1}{3} \int_0^2 u\,du \)

(b) \( \int_0^9 u\,du \)

(c) \( \frac{1}{3} \int_1^9 u\,du \)

---

### Exercises

In Exercises 1–6, calculate \( du \).

1. \( u = x^3 - x^2 \)

2. \( u = 2x^4 + 8x^{-1} \)

3. \( u = \cos(x^2) \)

4. \( u = \tan x \)

5. \( u = \sin^4 x \)

6. \( u = \frac{t}{r + 1} \)

In Exercises 7–22, write the integral in terms of \( u \) and \( du \). Then evaluate.

7. \( \int (x + 8)^6\,dx, \ u = x + 8 \)

8. \( \int (x + 25)^{-2}\,dx, \ u = x + 25 \)

9. \( \int (3t - 4)^2\,dt, \ u = 3t - 4 \)

10. \( \int (8 - x)^{2/3}\,dx, \ u = 8 - x \)

11. \( \int \sqrt{t^2 + 1}\,dt, \ u = t^2 + 1 \)

12. \( \int (x^4 + 1)\cos(x^4 + 4x)\,dx, \ u = x^4 + 4x \)

13. \( \int \frac{t^3}{(t^2 + 4)^2}\,dt, \ u = t^2 + 4 \)

14. \( \int 4x - 1\,dx, \ u = 4x - 1 \)

15. \( \int x(x + 1)^9\,dx, \ u = x + 1 \)

16. \( \int x\sqrt{4x - 1}\,dx, \ u = 4x - 1 \)

17. \( \int x^2\sqrt{4 - x}\,dx, \ u = 4 - x \)

18. \( \int \sin(4\theta - 7)\,d\theta, \ u = 4\theta - 7 \)

19. \( \int \sin \theta \cos^2 \theta\,d\theta, \ u = \cos \theta \)

20. \( \int \sec^2 x\tan x\,dx, \ u = \tan x \)

21. \( \int x\sec^2(x^2)\,dx, \ u = x^2 \)

22. \( \int \sec^2(\cos x)\sin x\,dx, \ u = \cos x \)

In Exercises 23–26, evaluate the integral in the form \( a \sin(u(x)) + C \) for an appropriate choice of \( u(x) \) and constant \( a \).

23. \( \int x^3 \cos(x^4)\,dx \)

24. \( \int x^2 \cos(x^3 + 1)\,dx \)

25. \( \int x^{1/2}\cos(x^{3/2})\,dx \)

26. \( \int \cos x\cos(x^3)\,dx \)

In Exercises 27–64, evaluate the indefinite integral.

27. \( \int (4x + 5)^6\,dx \)

28. \( \int \frac{dx}{(x - 9)^2} \)

29. \( \int \frac{dt}{\sqrt{t + 1}} \)

30. \( \int (9t^2 + 1)^{1/3}\,dt \)

31. \( \int \frac{x + 1}{(x^2 + 2x)^{1/4}}\,dx \)

32. \( \int (x + 1)(x^2 + 2x)^{1/4}\,dx \)
33. \[ \int \frac{x}{\sqrt{x^2 + 9}} \, dx \]
34. \[ \int \frac{2x^2 + x}{(4x^4 + 3x^3)^3} \, dx \]
35. \[ \int (2x^3 - 7)^2 \, dx \]
36. \[ \int (4 - x^2)^3 \, dx \]
37. \[ \int x(x^2 + 1)^{11} \, dx \]
38. \[ \int 6x^2(4 - x)^4 \, dx \]
39. \[ \int (3 + 8)^{11} \, dx \]
40. \[ \int x(3x + 8)^{11} \, dx \]
41. \[ \int x^4 \sqrt{x^2 + 1} \, dx \]
42. \[ \int \sqrt{x^4 + 1} \, dx \]
43. \[ \int \frac{dx}{x + 5} \]
44. \[ \int \frac{x^3 \, dx}{(x + 5)^3} \]
45. \[ \int e^2(x^3 + 1)^2 \, dx \]
46. \[ \int (x^2 + 4e(x^3 + 1)^2) \, dx \]
47. \[ \int (x + 2)(x + 1)^4 \, dx \]
48. \[ \int (x^2 - 1)^3 \, dx \]
49. \[ \int \sin(8 - 3x) \, dx \]
50. \[ \int \sin(3x^2) \, dx \]
51. \[ \int \frac{dx}{\sqrt{x^4}} \]
52. \[ \int \frac{dx}{\sqrt{x^2 + 1}} \]
53. \[ \int \sin x \cos x \sqrt{\sin x + 1} \, dx \]
54. \[ \int \frac{dx}{\sqrt{x^2 + 9}} \]
55. \[ \int \sec^2 x \left(12 \tan^3 x - 6 \tan^2 x \right) \, dx \]
56. \[ \int x^{-1/3} \sec \left( x^{1/3} \right) \, dx \]
57. \[ \int \sec^2(4x + 9) \, dx \]
58. \[ \int \sec^2 x \tan^4 x \, dx \]
59. \[ \int \frac{dx}{\sqrt{x^4 - 2x^2 + 5}} \]
60. \[ \int \frac{dx}{\sqrt{x^4 + 1}} \]
61. \[ \int \sin 4x \sqrt{\cos 4x} + 1 \, dx \]
62. \[ \int \cos x(3 \sin x - 1) \, dx \]
63. \[ \int \sec \theta \tan \theta (\sec \theta - 1) \, d\theta \]
64. \[ \int \tan \theta + \sin \theta \cos \theta \, d\theta \]

Evaluate \[ \int \frac{dx}{1 + \sqrt{x}} \] using \( u = 1 + \sqrt{x} \). Hint: Show that \( dx = \frac{2\sqrt{x}}{u^2} \, du \).

66. Can They Both Be Right? Hannah uses the substitution \( u = \tan x \) and Akiva uses \( u = \sec x \) to evaluate \( \int \tan x \sec^2 x \, dx \). Show that they obtain different answers, and explain the apparent contradiction.

67. Evaluate \( \int \cos x \, dx \) using substitution in two different ways: first using \( u = \sin x \) and then using \( u = \cos x \). Resolve the two different answers.

68. Same Choices Are Better Than Others Evaluate \( \int \sin x \cos^2 x \, dx \) in two ways. First use \( u = \cos x \) to show that \( \int \sin x \cos^2 x \, dx = \int \frac{1}{u^2} \, du \) and evaluate the integral on the right by a further substitution. Then show that \( u = \cos x \) is a better choice than \( u = \sin x \) to begin with.

69. What are the new limits of integration if we apply the substitution \( u = 3x + 4 \) to the integral \( \int_0^\pi \sin(3x + 4) \, dx \)?

70. Which of the following is the result of applying the substitution \( u = 4x - 9 \) to the integral \( \int_2^1 (4x - 9)^{10} \, dx \)?

(a) \( \int_2^1 u^{10} \, du \)
(b) \( \frac{1}{4} \int_2^1 u^{10} \, du \)
(c) \( \frac{1}{4} \int_2^1 3u^{10} \, du \)
(d) \( \frac{1}{4} \int_2^1 3u^{10} \, du \)

In Exercises 71–72, compute the definite integral two ways:

- Multiply out the integrand and then integrate directly without a substitution.
- Use the Change of Variables Formula with the substitution provided.

71. \( \int_0^2 (2x - 3)^3 \, dx \)
72. \( \int_0^2 x(x^2 + 4)^2 \, dx \)

In Exercises 73–84, use the Change of Variables Formula to evaluate the definite integral.

73. \( \int_0^6 \frac{dx}{\sqrt{x + 3}} \)
74. \( \int_0^6 \frac{dx}{x + 3} \)
75. \( \int_0^1 \frac{dx}{(x + 1)^2} \)
76. \( \int_0^1 \frac{dx}{(x + 1)^3} \)
77. \( \int_0^2 \frac{dx}{(x^2 + 4x + 9)^2} \)
78. \( \int_0^2 \frac{dx}{(x^2 + 4x + 9)^3} \)
79. \( \int_0^3 \frac{dx}{(x + 1)(x^2 + 2x)^2} \)
80. \( \int_0^2 \frac{dx}{(x - 9)^2 \sin x + 1} \)
81. \( \int_0^\pi \cos^2 x \, dx \)
82. \( \int_0^\pi \cos^2 2x \, dx \)
83. \( \int_0^\pi \cos^2 x \, dx \)
84. \( \int_0^\pi \cos^2 x \, dx \)

Evaluate \( \int_0^\pi \sqrt{\sin x} \, dx \) and evaluate. Hint: Use the identity \( \sin^2 \theta = \tan^2 \theta + 1 \).

85. Evaluate \( \int_0^\pi \sin^2 x \, dx \) for \( n \geq 0 \).
86. Evaluate \( \int_0^\pi \tan^2 x \, dx \) for \( n \geq 0 \).

In Exercises 89–92, use substitution to evaluate the integral in terms of \( f(x) \).

89. \( \int f(x)^3 f'(x) \, dx \)
90. \( \int f'(x) f(x)^2 \, dx \)
91. \( \int f(x) \, dx \)
92. \( \int f(-x + 7) \, dx \)
93. Show that \( \int_0^{\pi/2} \sin x \, dx = \int_0^{\pi/2} \frac{1}{\sqrt{1 - u^2}} \, du \).
(a) Prove that $R_N = \frac{64(N + 1)^2}{N^2}$.
(b) Prove that the area of the region within the right-endpoint rectangles above the graph is equal to $\frac{64(2N + 1)}{N^2}$.

**Figure 2** Approximation $R_N$ for $f(x) = x^3$ on $[0, 4]$.

11. Which approximation to the area is represented by the shaded rectangles in Figure 3? Compute $R_5$ and $L_5$.

**Figure 3**

12. Calculate any two Riemann sums for $f(x) = x^2$ on the interval $[2, 5]$, but choose partitions with at least five subintervals of unequal widths and intermediate points that are neither endpoints nor midpoints.

In Exercises 13–16, express the limit as an integral (or multiple of an integral) and evaluate.

13. $\lim_{N \to \infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin \left( \frac{\pi j}{6N} \right)$
14. $\lim_{N \to \infty} \frac{3}{N} \sum_{k=0}^{N-1} \left( 10 + \frac{3k}{N} \right)$
15. $\lim_{N \to \infty} \frac{5}{N} \sum_{j=1}^{N} \sqrt{4 + \frac{5j}{N}}$
16. $\lim_{N \to \infty} \frac{1^k + 2^k + \cdots + N^k}{N^{k+1}}$ $(k > 0)$

In Exercises 17–30, calculate the indefinite integral.

17. $\int (4x^3 - 2x^2) \, dx$
18. $\int x^{2/3} \, dx$
19. $\int \sin(\theta - 8) \, d\theta$
20. $\int \cos(5 - 7\theta) \, d\theta$
21. $\int (4r^{-3} - 12r^{-4}) \, dr$
22. $\int (9r^{-2/3} + 4r^{1/3}) \, dr$
23. $\int \sec^2 x \, dx$
24. $\int \tan 3\theta \sec 3\theta \, d\theta$
25. $\int (y + 2)^6 \, dy$
26. $\int \frac{3x^3 - 9}{x^2} \, dx$
27. $\int (\cos \theta - \theta) \, d\theta$
28. $\int \sec^2(12 - 25\theta) \, d\theta$
29. $\int \frac{8 \, dx}{x^3}$
30. $\int \sin(4x - 9) \, dx$

In Exercises 31–36, solve the differential equation with the given initial condition.

31. $\frac{dy}{dx} = 4x^3$, $y(1) = 4$
32. $\frac{dy}{dt} = 3t^2 + \cos t$, $y(0) = 12$
33. $\frac{dy}{dx} = x^{-1/2}$, $y(1) = 1$
34. $\frac{dy}{dx} = \sec^2 x$, $y(\pi) = 2$
35. $\frac{dy}{dx} = 1 + \pi \sin 3t$, $y(\pi) = \pi$
36. $\frac{dy}{dt} = \cos 3\pi t + \sin 4\pi t$, $y(\pi) = 0$
37. Find $f(t)$ if $f''(t) = 1 - 2t$, $f(0) = 2$, and $f'(0) = -1$.
38. At time $t = 0$, a driver begins accelerating at a constant rate of $-10$ m/s² and comes to a halt after traveling 500 m. Find the velocity at $t = 0$.

In Exercises 39–42, use the given substitution to evaluate the integral.

39. $\int_0^1 \frac{dt}{\sqrt{4t + 12}}$, $u = 4t + 12$
40. $\int_0^\pi (x^2 + 1) \, dx$ $(x^2 + 3x)^4$, $u = x^3 + 3x$
41. $\int_0^{\pi/6} \sin x \cos x \, dx$, $u = \cos x$
42. $\int \sec^2(2\theta) \tan(2\theta) \, d\theta$, $u = \tan(2\theta)$

In Exercises 43–70, evaluate the integral.

43. $\int \left(20x^4 - 9x^2 - 2x\right) \, dx$
44. $\int_0^2 \left(12x^3 - 3x^2\right) \, dx$
45. $\int (2x^2 - 3x^2) \, dx$
46. $\int_0^1 (x^{7/3} - 2x^{1/4}) \, dx$
47. $\int \frac{\sqrt{5} + 3x^4}{x^2} \, dx$
48. $\int_1^3 r^{-4} \, dr$
49. $\int_3^{10} |x^2 - 4| \, dx$
50. $\int_2^{1} \left[ (x - 1)(x - 3)\right] \, dx$
51. $\int_4^4 |2t| \, dt$
52. $\int_0^2 t - (1) \, dt$
53. \[ \int (10t - 7)^4 \, dt \]
54. \[ \int_2^3 \sqrt{7y} - 5 \, dy \]
55. \[ \int (2x^3 + 3x) \, dx \]
56. \[ \int_{-3}^{-1} \frac{x \, dx}{(x^2 + 5)^2} \]
57. \[ \int_0^5 15x \sqrt{x^3 + 4} \, dx \]
58. \[ \int_2^1 t^2 \sqrt{t + 8} \, dt \]
59. \[ \int_0^1 \cos \left( \frac{\pi}{3} (t + 2) \right) \, dt \]
60. \[ \int_\pi/4^n \sin \left( \frac{5\theta - \pi}{6} \right) \, d\theta \]
61. \[ \int t^2 \sec^2(9\theta^3 + 1) \, dt \]
62. \[ \int \sin^3(3\theta) \cos(3\theta) \, d\theta \]
63. \[ \int \csc^2(9 - 2\theta) \, d\theta \]
64. \[ \int \sin \theta \sqrt{4 - \cos \theta} \, d\theta \]
65. \[ \int_0^{\pi/3} \frac{\sin \theta}{\cos^2(\frac{\pi}{6} \theta)} \, d\theta \]
66. \[ \int \frac{\sec^2 t \, dt}{(\tan t - 1)^2} \]
67. \[ \int_0^{\pi/2} y \sqrt{2y + 3} \, dy \]
68. \[ \int_0^8 t^2 \sqrt{t + 8} \, dt \]
69. \[ \int_0^{\pi/2} \sec^2(\cos \theta) \sin \theta \, d\theta \]
70. \[ \int_0^1 \frac{12x \, dx}{(x^2 + 2)^3} \]

71. Combine to write as a single integral:
\[ \int_0^8 f(x) \, dx + \int_{-2}^0 f(x) \, dx + \int_8^6 f(x) \, dx \]

72. Let \( A(x) = \int_0^x f(x) \, dx \), where \( f \) is the function shown in Figure 4. Identify the location of the local minima, the local maxima, and points of inflection of \( A \) on the interval \([0, E]\), as well as the intervals where \( A \) is increasing, decreasing, concave up, or concave down. Where does the absolute maximum of \( A \) occur?

![Figure 4](image_url)

73. Find the local minima, the local maxima, and the inflection points of \( A(x) = \int_0^x f(t) \, dt \).

74. A particle starts at the origin at time \( t = 0 \) and moves with velocity \( v(t) \) as shown in Figure 5.
(a) How many times does the particle return to the origin in the first 12 seconds?
(b) What is the particle's maximum distance from the origin?
(c) What is the particle's maximum distance to the left of the origin?

![Figure 5](image_url)

75. For the function \( f \) illustrated in Figure 6 do the following:
(a) For \( x = 0, 1, 2, \ldots, 10 \), approximate \( A(x) = \int_0^x f(t) \, dt \).
(b) For \( x = 1, 2, 3, \ldots, 9 \), approximate \( A'(x) \) using \( \Delta x = 1 \) and the symmetric difference quotient approximation,
\[ A'(x) \approx \frac{A(x + \Delta x) - A(x - \Delta x)}{2\Delta x} \]
(c) Plot the values of \( A'(x) \) on a graph of \( f \) to demonstrate \( A' \approx f \).

![Figure 6](image_url)

76. The sine integral function \( Si \) is an area function defined by
\[ Si(x) = \int_0^x \frac{\sin t}{t} \, dt \]

(a) Explain why \( Si \) has critical points at \( n\pi \) for all nonzero integers \( n \).
(b) (CAS) Use Riemann sums to approximate \( Si(x) \) for \( x = \pi, 2\pi, \ldots, 8\pi \) and sketch a graph of \( Si(x) \) for \( 0 \leq x \leq 8\pi \).

77. On a typical day, a city consumes water at the rate of \( r(t) = 100 + 72t - 3t^2 \) (in thousands of gallons per hour), where \( t \) is the number of hours past midnight. What is the daily water consumption? How much water is consumed between 6PM and midnight?

78. The learning curve in a certain bicycle factory is \( L(x) = 12x^{-1/5} \) (in hours per bicycle), which means that it takes a bike mechanic \( L(n) \) hours to assemble the \( n \)th bicycle. If a mechanic has produced 24 bicycles, how long does it take her or him to produce a subsequent batch of 12?

79. Cost engineers at NASA have the task of projecting the cost \( P \) of major space projects. It has been found that the cost \( C \) of developing a projection increases with \( P \) at the rate \( \frac{dC}{dP} = 21P^{-0.66} \), where \( C \) is in thousands of dollars and \( P \) in millions of dollars. What is the cost of developing a projection for a project whose cost turns out to be \( P = 35 \) million?
80. An astronomer estimates that in a certain constellation, the number of stars of magnitude \( m \), per degree-squared of sky, is equal to \( A(m) = 2.4 \times 10^{-6}m^{7.4} \) (fainter stars have higher magnitudes). Estimate the total number of stars of magnitude between 6 and 15 in a 1-degree-squared region of sky.

81. Evaluate \( \int_{-\infty}^{\infty} \frac{x^{15}}{3 + \cos^2 x} \, dx \), using the properties of odd functions.

82. Evaluate \( \int_{0}^{1} f(x) \, dx \), assuming that \( f \) is an even continuous function such that

\[
\int_{-1}^{1} f(x) \, dx = 5, \quad \int_{-2}^{2} f(x) \, dx = 8
\]

83. (GU) Plot the graph of \( f(x) = \sin mx \sin nx \) on \([0, \pi]\) for the pairs \((m,n) = (2, 4), (3, 5)\) and in each case guess the value of \( I = \int_{0}^{\pi} f(x) \, dx \). Experiment with a few more values (including two cases with \( m = n \)) and formulate a conjecture for when \( I \) is zero.

84. Show that

\[
\int x \, f(x) \, dx = xF(x) - G(x)
\]

where \( F'(x) = f(x) \) and \( G'(x) = F(x) \). Use this to evaluate \( \int x \cos x \, dx \).

85. Prove

\[
2 \leq \int_{1}^{2} 2x \, dx \leq 4 \quad \text{and} \quad \frac{1}{9} \leq \int_{1}^{2} 3 - x \, dx \leq \frac{1}{3}
\]

86. (GU) Plot the graph of \( f(x) = x^{-2} \sin x \), and show that

\[
0.2 \leq \int_{1}^{2} f(x) \, dx \leq 0.9.
\]

87. Find upper and lower bounds for \( \int_{a}^{b} f(x) \, dx \), for \( y = f(x) \) in Figure 7.

88. In Exercises 88–93, find the derivative.

88. \( A'(x) \), where \( A(x) = \int_{3}^{x} \sin(t^3) \, dt \)

89. \( A'(a) \), where \( A(x) = \int_{2}^{x} \frac{\cos t}{1 + t} \, dt \)

90. \( \frac{d}{dy} \int_{-2}^{y} 3x \, dx \)

91. \( G'(x) \), where \( G(x) = \int_{-2}^{x} t^3 \, dt \)

92. \( G'(2) \), where \( G(x) = \int_{0}^{x} \sqrt{t + 1} \, dt \)

93. \( H'(1) \), where \( H(x) = \int_{0}^{9} \frac{1}{\sqrt{t}} \, dt \)

94. Explain with a graph: If \( f \) is increasing and concave up on \([a, b]\), then \( L_{N} \) is more accurate than \( R_{N} \). Which is more accurate if \( f \) is increasing and concave down?

95. Explain with a graph: If \( f \) is linear on \([a, b]\), then the \( \int_{a}^{b} f(x) \, dx = \frac{1}{2}(R_{N} + L_{N}) \) for all \( N \).
6 APPLICATIONS OF THE INTEGRAL

In the previous chapter, we used the integral to compute areas under curves and net change. In this chapter, we discuss some of the other quantities that are represented by integrals, including volume, average value, work, total mass, population, and fluid flow.

6.1 Area Between Two Curves

Sometimes we are interested in the area between two curves. Figure 1 shows projected electric power generation in the United States through renewable resources (wind, solar, biofuels, etc.) under two scenarios: with and without government stimulus spending. The area of the shaded region between the two graphs represents the additional energy projected to result from stimulus spending. How can we compute such an area?

![U.S. Renewable Generating Capacity Forecast Through 2030](image)

**Figure 1** The area of the shaded region (which has units of power \( \times \) time, or energy) represents the additional energy from renewable generating capacity projected to result from government stimulus spending in 2009–2010. *Source: Energy Information Agency.*

Now suppose that we are given two functions \( y = f(x) \) and \( y = g(x) \) such that \( f(x) \geq g(x) \) for all \( x \) in an interval \( [a, b] \) (Figure 2). We call such a region vertically simple since any vertical line that intersects the region does so in a single point or a single vertical line segment with its lower endpoint on the graph of \( y = g(x) \) and upper endpoint on the graph of \( y = f(x) \). Then the graph of \( y = f(x) \) lies above the graph of \( y = g(x) \), and the area between the graphs is equal to the area under the top function minus the area under the bottom function (Figure 3):

\[
\text{area between the graphs} = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b (f(x) - g(x)) \, dx
\]

**Figure 3** illustrates this formula in the case that both graphs lie above the \( x \)-axis. We see that the region between the graphs is obtained by removing the region under \( y = g(x) \) from the region under \( y = f(x) \).

![Area Between Two Curves](image)

**Figure 2** A vertically simple region.

**Figure 3** The area between the graphs is a difference of two areas.
EXAMPLE 1 Find the area of the region between the graphs of the functions as shown in Figure 4:

\[ f(x) = x^2 - 4x + 10, \quad g(x) = 4x - x^2, \quad 1 \leq x \leq 3 \]

Solution Figure 4 shows that \( f(x) \geq g(x) \) over \([1, 3]\), and therefore the region is vertically simple. By Eq. (1), the area between the graphs is

\[
\int_1^3 (f(x) - g(x)) \, dx = \int_1^3 ((x^2 - 4x + 10) - (4x - x^2)) \, dx
\]

\[
= \int_1^3 (2x^2 - 8x + 10) \, dx = \left[ \frac{2}{3}x^3 - 4x^2 + 10x \right]_1^3
\]

\[ = 12 - \frac{20}{3} = \frac{16}{3} \]

Before continuing with more examples, we note that Eq. (1) remains valid whenever \( f(x) \geq g(x) \), even if \( f(x) \) and \( g(x) \) are not assumed to be positive. As in Figure 5, we can simply shift the two functions up by adding to each a constant \( C \) big enough so that both functions are positive over the interval \([a, b]\). This does not change the area between them. Then by our previous result, we have

area between the graphs \( = \int_a^b ((f(x) + C) - (g(x) + C)) \, dx \)

\[ = \int_a^b (f(x) - g(x)) \, dx \]

Writing \( y_{\text{top}} = f(x) \) for the top curve and \( y_{\text{bot}} = g(x) \) for the bottom curve, we obtain

\[ \text{area between the graphs} = \int_a^b (y_{\text{top}} - y_{\text{bot}}) \, dx = \int_a^b (f(x) - g(x)) \, dx \]

EXAMPLE 2 Find the area between the graphs of \( f(x) = x^2 - 5x - 7 \) and \( g(x) = x - 12 \) over \([-2, 5]\).

Solution First, we must determine which graph lies on top.

Step 1. Sketch the region (especially, find any points of intersection).

We know that \( y = f(x) \) is a parabola with \( y \)-intercept \(-7\) and that \( y = g(x) \) is a line with \( y \)-intercept \(-12\) (Figure 6). To determine where the graphs intersect, we look for values of \( x \) where \( f(x) = g(x) \), or equivalently, where \( f(x) - g(x) = 0 \):

\[ f(x) - g(x) = (x^2 - 5x - 7) - (x - 12) = x^2 - 6x + 5 = (x - 1)(x - 5) \]

The graphs intersect where \( (x - 1)(x - 5) = 0 \), that is, at \( x = 1 \) and \( x = 5 \).

Step 2. Set up the integrals and evaluate.

We see that \( f(x) - g(x) \geq 0 \) for \(-2 \leq x \leq 1 \), and \( f(x) - g(x) \leq 0 \) for \( 1 \leq x \leq 5 \). Thus,

\[ f(x) \geq g(x) \text{ on } [-2, 1] \quad \text{and} \quad g(x) \geq f(x) \text{ on } [1, 5] \]

This tells us to subdivide our region into separate vertically simple regions, one over \([-2, 1]\) and the other over \([1, 5]\). Therefore, we write the area as a sum of integrals over the two intervals:
\[ \int_{-2}^{5} (y_{\text{top}} - y_{\text{bot}}) \, dx = \int_{-2}^{1} (f(x) - g(x)) \, dx + \int_{1}^{5} (g(x) - f(x)) \, dx \]
\[ = \int_{-2}^{1} ((x^2 - 5x - 7) - (x - 12)) \, dx \]
\[ + \int_{1}^{5} ((x - 12) - (x^2 - 5x - 7)) \, dx \]
\[ = \int_{-2}^{1} (x^2 - 6x + 5) \, dx + \int_{1}^{5} (-x^2 + 6x - 5) \, dx \]
\[ = \left( \frac{1}{3}x^3 - 3x^2 + 5x \right) \bigg|_{-2}^{1} + \left( -\frac{1}{3}x^3 + 3x^2 - 5x \right) \bigg|_{1}^{5} \]
\[ = \left( \frac{7}{3} - \frac{-74}{3} \right) + \left( \frac{25}{3} - \frac{-7}{3} \right) = \frac{113}{3} \]

**EXAMPLE 3** Calculating Area by Dividing the Region

Find the area of the region bounded by the graphs of \( y = \frac{8}{x^2} \), \( y = 8x \), and \( y = x \).

Solution

**Step 1. Sketch the region (especially, find any points of intersection).**

The curve \( y = \frac{8}{x^2} \) cuts off a region in the sector between the two lines \( y = 8x \) and \( y = x \) (Figure 7). We find the intersection of \( y = \frac{8}{x^2} \) and \( y = 8x \) by solving

\[ \frac{8}{x^2} = 8x \quad \Rightarrow \quad x^3 = 1 \quad \Rightarrow \quad x = 1 \]

and the intersection of \( y = \frac{8}{x^2} \) and \( y = x \) by solving

\[ \frac{8}{x^2} = x \quad \Rightarrow \quad x^3 = 8 \quad \Rightarrow \quad x = 2 \]

**Step 2. Set up the integrals and evaluate.**

Figure 7 shows that \( y_{\text{bot}} = x \), but \( y_{\text{top}} \) changes at \( x = 1 \) from \( y_{\text{top}} = 8x \) to \( y_{\text{top}} = \frac{8}{x^2} \). So, the region is not vertically simple. Therefore, we break up the regions into two parts, \( A \) and \( B \), each vertically simple, and compute their areas separately.

The area of \( A \) is

\[ \text{area of } A = \int_{0}^{1} (y_{\text{top}} - y_{\text{bot}}) \, dx = \int_{0}^{1} (8x - x) \, dx = \int_{0}^{1} 7x \, dx = \frac{7}{2} x^2 \bigg|_{0}^{1} = \frac{7}{2} \]

The area of \( B \) is

\[ \text{area of } B = \int_{1}^{2} (y_{\text{top}} - y_{\text{bot}}) \, dx = \int_{1}^{2} \left( \frac{8}{x^2} - x \right) \, dx = \left( \frac{8}{x} \right) x^2 \bigg|_{1}^{2} = \frac{5}{2} \]

The total area bounded by the curves is the sum \( \frac{7}{2} + \frac{5}{2} = 6 \).
Integration Along the y-Axis

Suppose we are given \( x \) as a function of \( y \), say, \( x = g(y) \). What is the meaning of the integral \( \int_c^d g(y) \, dy \)? This integral can be interpreted as signed area, where regions to the right of the \( y \)-axis have positive area and regions to the left have negative area:

\[
\int_c^d g(y) \, dy = \text{signed area between graph and } y\text{-axis for } c \leq y \leq d
\]

In Figure 8(A), the part of the shaded region to the left of the \( y \)-axis has a negative signed area. The signed area of the entire region is

\[
\int_{-6}^6 (y^2 - 9) \, dy = \left[ \frac{1}{3}y^3 - 9y \right]_{-6}^6 = 36
\]

Area to the right of \( y \)-axis minus area to the left of \( y \)-axis

More generally, if \( g(y) \geq h(y) \) as in Figure 8(B), then the graph of \( x = g(y) \) lies to the right of the graph of \( x = h(y) \). In this case, we write \( x_{\text{right}} = g(y) \) and \( x_{\text{left}} = h(y) \). We call the region horizontally simple, since every horizontal line that intersects the region in more than a single point does so in a single line segment such that the left endpoint is on the curve \( x = h(y) \) and the right endpoint is on the curve \( x = g(y) \), as in Figure 9. The formula for the area corresponding to Eq. (2) is

\[
\text{area between the graphs} = \int_c^d (x_{\text{right}} - x_{\text{left}}) \, dy = \int_c^d (g(y) - h(y)) \, dy
\]

Example 4 Calculate the area enclosed by the graphs of \( h(y) = y^2 - 1 \) and \( g(y) = y^2 - \frac{1}{8}y^4 + 1 \).

Solution Figure 10 shows that the enclosed region stretches between the two points of intersection of the graphs and that \( g(y) \geq h(y) \) over the region. Therefore, the region is horizontally simple with \( x_{\text{right}} = g(y) \) and \( x_{\text{left}} = h(y) \). To set up the integral for the area, we need to determine the points of intersection.

We do that by solving \( g(y) = h(y) \) for \( y \):

\[
y^2 - \frac{1}{8}y^4 + 1 = y^2 - 1 \quad \Rightarrow \quad \frac{1}{8}y^4 - 2 = 0 \quad \Rightarrow \quad y = \pm 2
\]
It would be more difficult to calculate the area of the region in Figure 10 as an integral with respect to \( x \) because the curves are not graphs of functions of \( x \).

Now, we have

\[
x_{\text{right}} - x_{\text{left}} = \left( y^2 - \frac{1}{8} y^4 + 1 \right) - (y^2 - 1) = 2 - \frac{1}{8} y^4
\]

The enclosed area is

\[
\int_{-2}^{2} (x_{\text{right}} - x_{\text{left}}) \, dy = \int_{-2}^{2} \left( 2 - \frac{1}{8} y^4 \right) \, dy = \left. \left( 2y - \frac{1}{40} y^5 \right) \right|_{-2}^{2} = \frac{16}{5} - \left( -\frac{16}{5} \right) = \frac{32}{5}
\]

For many regions, we have a choice of whether to find the area by integrating with respect to \( x \) or with respect to \( y \). The decision is usually based on two factors:

- How easy it is to obtain the curves as functions of one variable in terms of the other
- How easy it is to subdivide the region into simple regions and to integrate the functions involved

In the next example, we carry out the area computation, both integrating with respect to \( x \) and integrating with respect to \( y \), demonstrating how these factors are involved.

**EXAMPLE 5** Find the area of the region that is bounded by the three curves \( y = x^2 \), \( y = (x - 2)^2 \), and \( y = 0 \).

**Solution** The area appears in Figure 11. Notice immediately that it is not vertically simple, since the top function changes over the interval \([0, 2]\). It is horizontally simple, but to calculate the area using the fact it is a horizontally simple region takes some work to set up the expressions for \( x_{\text{right}} \) and \( x_{\text{left}} \).

We first compute the area of the region by splitting it into two vertically simple regions. In this case, the area is given by

\[
\int_{0}^{1} x^2 \, dx + \int_{1}^{2} (x - 2)^2 \, dx = \left. \frac{x^3}{3} \right|_{0}^{1} + \left. \frac{(x - 2)^3}{3} \right|_{1}^{2} = \frac{1}{3} + 0 - \left( -\frac{1}{3} \right) = \frac{2}{3}
\]

Now, let's redo the problem using the fact the region is horizontally simple. To determine \( x_{\text{right}} \) and \( x_{\text{left}} \), we must invert the formulas for each of the parabolas. The left boundary of the region is the right side of the parabola given by \( y = x^2 \). Solving for \( x \), we have \( x = \pm \sqrt{y} \). The right side of the parabola corresponds to the positive choice; therefore, \( x_{\text{right}} = \sqrt{y} \). The right boundary of the region is the left side of the parabola \( y = (x - 2)^2 \). We solve for \( x \):

\[
x - 2 = \pm \sqrt{y}
\]

\[
x = 2 \pm \sqrt{y}
\]

The left side of the parabola is obtained by choosing the minus sign, and therefore \( x_{\text{right}} = 2 - \sqrt{y} \).

Then the area is given by

\[
\int_{0}^{1} (2 - \sqrt{y} - \sqrt{y}) \, dy = \int_{0}^{1} (2 - 2y^{1/2}) \, dy = \left. \left( 2y - \frac{4}{3} y^{3/2} \right) \right|_{0}^{1} = \frac{2}{3}
\]

**6.1 SUMMARY**

- If \( f(x) \geq g(x) \) on \([a, b]\), then the region between the graphs is vertically simple and we have

\[
\text{area between the graphs} = \int_{a}^{b} (y_{\text{top}} - y_{\text{bot}}) \, dx = \int_{a}^{b} (f(x) - g(x)) \, dx
\]
To calculate the area between $y = f(x)$ and $y = g(x)$, sketch the region to find $y_{\text{top}}$. If necessary, find points of intersection by solving $f(x) = g(x)$.

- Integral along the $y$-axis: $\int_{c}^{d} g(y) \, dy$ is equal to the signed area between the graph and the $y$-axis for $c \leq y \leq d$. The signed area to the right of the $y$-axis is positive and the signed area to the left is negative.
- If $g(y) \geq h(y)$ on $[c, d]$, then $x = g(y)$ lies to the right of $x = h(y)$ and the region is horizontally simple:

$$\text{area between the graphs} = \int_{c}^{d} (x_{\text{right}} - x_{\text{left}}) \, dy = \int_{c}^{d} (g(y) - h(y)) \, dy$$

### 6.1 Exercises

#### Preliminary Questions

1. What is the area interpretation of $\int_{a}^{b} (f(x) - g(x)) \, dx$ if $f(x) \geq g(x)$?

2. Is $\int_{a}^{b} (f(x) - g(x)) \, dx$ equal to the area between the graphs of $f$ and $g$ if $f(x) \geq 0$ but $g(x) \leq 0$?

3. Suppose that $f(x) \geq g(x)$ on $[0, 3]$ and $g(x) \geq f(x)$ on $[3, 5]$. Express the area between the graphs over $[0, 5]$ as a sum of integrals.

4. Suppose that the graph of $x = f(y)$ lies to the left of the $y$-axis. Is $\int_{a}^{b} f(y) \, dy$ positive or negative?

5. Explain what $\int_{a}^{b} |f(x) - g(x)| \, dx$ represents.

6. Draw a region that is both vertically simple and horizontally simple.

#### Exercises

1. Find the area of the region between $y = 3x^2 + 12$ and $y = 4x + 4$ over $[-3, 3]$ (Figure 12).

2. Find the area of the region between the graphs of $f(x) = 3x + 8$ and $g(x) = x^2 + 2x + 2$ over $[0, 2]$ (Figure 13).

3. Find the area of the region enclosed by the graphs of $f(x) = x^2 + 2$ and $g(x) = 2x + 5$ (Figure 14).

4. Find the area of the region enclosed by the graphs of $f(x) = x^3 - 10x$ and $g(x) = 6x$ (Figure 15).
In Exercises 5 and 6, sketch the region between \( y = \sin x \) and \( y = \cos x \) over the interval and find its area.

5. \([\frac{\pi}{4}, \frac{\pi}{2}]\) 

6. \([0, \pi]\)

In Exercises 7 and 8, let \( f(x) = 20 + x - x^2 \) and \( g(x) = x^2 - 5x \).

7. Sketch the region enclosed by the graphs of \( f \) and \( g \), and compute its area.

8. Sketch the region between the graphs of \( f \) and \( g \) over \([4, 8]\), and compute its area as a sum of two integrals.

9. (GU) Find the points of intersection of the graph of \( y = x(x^2 - 1) \) and the graph of \( y = 1 - x^2 \). Sketch the region enclosed by these curves over \([-1, 1]\) and compute its area.

10. (GU) Find the points of intersection of the graph of \( y = x(4 - x) \) and the graph of \( y = x^2(4 - x) \). Sketch the region enclosed by these curves over \([0, 4]\) and compute its area.

11. Sketch the region bounded by the line \( y = 2 \) and the graph of \( y = \sec^2 x \) for \(-\frac{\pi}{2} < x < \frac{\pi}{2}\) and find its area.

12. Sketch the region bounded by

\[
y = \sqrt{4 - x^2} \quad \text{and} \quad y = -\sqrt{4 - x^2}
\]

for \(-2 \leq x \leq 2\). Give a definite integral for the area of the region, but do not compute the integral. Instead, find the area using geometry.

In Exercises 13–16, determine whether or not the region bounded by the curves is vertically simple and/or horizontally simple.

13. \( x = y^2, x = 2 - y^2 \)

14. \( y = x^2, x = y^2 \)

15. \( y = x, y = 2x, y = \frac{1}{x} \)

16. In the first quadrant, \( y = x, y = \sin \left( \frac{x}{2} \right) \)

In Exercises 17–20, find the area of the shaded region in Figures 16–19.

17. \[
y = x^3 - 2x^2 + 10
\]

18. \[
y = \frac{1}{2}x
\]

19. \[
y = \cos x
\]

20. \[
y = \sin x
\]

21. \( 0 \leq y \leq \frac{\pi}{2} \)

22. \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \)

23. Find the area of the region lying to the right of \( x = y^2 + 4y - 22 \) and to the left of \( x = 3y + 8 \).

24. Find the area of the region lying to the right of \( x = y^2 - 5 \) and to the left of \( x = 3 - y^2 \).

25. Figure 21 shows the region enclosed by \( x = y^3 - 26y + 10 \) and \( x = 40 - 6y^2 - y^3 \). Match the equations with the curves and compute the area of the region.
26. Figure 22 shows the region enclosed by \( y = x^3 - 6x \) and \( y = 8 - 3x^2 \). Match the equations with the curves and compute the area of the region.

**Figure 22** Region between \( y = x^3 - 6x \) and \( y = 8 - 3x^2 \).

In Exercises 27 and 28, find the area enclosed by the graphs in two ways: by integrating along the \( x \)-axis and by integrating along the \( y \)-axis.

27. \( x = 9 - y^2 \), \( x = 5 \)

28. The semicircular parabola \( y^2 = x^3 \) and the line \( x = 1 \)

In Exercises 29 and 30, find the area of the region using the method (integration along either the \( x \)- or the \( y \)-axis) that requires you to evaluate just one integral.

29. Region between \( y^2 = x + 5 \) and \( y^2 = 3 - x \)

30. Region between \( y = x \) and \( x + y = 8 \) over \([2, 3]\)

In Exercises 31–48, sketch the region enclosed by the curves and compute its area as an integral along the \( x \)- or \( y \)-axis.

31. \( y = 25 - x^2 \), \( y = x^2 - 25 \)

32. \( y = x^2 - 6 \), \( y = 6 - x^2 \), \( x = 0 \)

33. \( x + y = 4 \), \( x - y = 0 \), \( y + 3x = 4 \)

34. \( y = 8 - 3x \), \( y = 6 - x \), \( y = 2 \)

35. \( y = 15 - \sqrt{x} \), \( y = 2\sqrt{x} \), \( x = 0 \)

36. \( y = |x^2 - 4| \), \( y = 5 \)

37. \( x = |y| \), \( x = 1 - |y| \)

38. \( y = |x| \), \( y = \frac{x}{2} + 3 \)

39. \( x = y^3 - 18y \), \( y + 2x = 0 \)

40. \( y = x\sqrt{x} - 2 \), \( y = -x\sqrt{x} - 2 \), \( x = 4 \)

41. \( x = 2y \), \( x + 1 = (y - 1)^2 \)

42. \( x + y = 1 \), \( x^{1/2} + y^{1/2} = 1 \)

43. \( y = \cos x \), \( y = \cos 2x \), \( x = 0 \), \( x = \frac{2\pi}{3} \)

44. \( y = \sin(2x) \), \( y = \sin(4x) \), \( x = 0 \), \( x = \frac{\pi}{6} \)

45. \( y = \sin x \), \( y = \cos^2 x \), \( x = \frac{\pi}{4} \)

46. \( x = \sin y \), \( y = \frac{2}{\pi} y \)

47. \( y = \sin x \), \( y = x \sin(x^2) \), \( 0 \leq x \leq 1 \)

48. \( y = \frac{\sin(\sqrt{x})}{\sqrt{x}} \), \( y = 0 \), \( \pi^2 \leq x \leq 9\pi^2 \)

49. **CAS** Plot

\[ y = \frac{x}{\sqrt{x^2 + 1}} \quad \text{and} \quad y = (x - 1)^2 \]

on the same set of axes. Use a computer algebra system to find the points of intersection numerically and compute the area between the curves.

50. Sketch a region whose area is represented by

\[ \int_{-\sqrt[3]{2}}^{\sqrt[3]{2}} \left( \sqrt{1 - x^2} - |x| \right) \, dx \]

and determine the area using geometry.

51. **CAS** Beginning at the same time and location, Athletes 1 and 2 run for 30 seconds along a straight track with velocities \( v_1(t) \) and \( v_2(t) \) (in meters per second) as shown in Figure 23.

(a) What is represented by the area of the shaded region over \([0, 10]\)?

(b) Which of the following is represented by the area of the shaded region over \([10, 20]\)?

i. How far Athlete 2 is ahead of Athlete 1 at \( t = 30 \).

ii. How much further Athlete 2 ran than Athlete 1 did over the last 20 seconds.

(c) Who is ahead at the end of each 5-second interval, \( t = 5, 10, \ldots, 30 \)?

![Figure 23](image)

52. Express the area of the shaded region in Figure 24 as a sum of three integrals involving \( f(x) \) and \( g(x) \).

![Figure 24](image)

53. Find the area enclosed by the curves \( y = c - x^2 \) and \( y = x^2 - c \) as a function of \( c \). Find the value of \( c \) for which this area is equal to 1.

54. Set up (but do not evaluate) an integral that expresses the area between the circles \( x^2 + y^2 = 2 \) and \( x^2 + (y - 1)^2 = 1 \).

55. Set up (but do not evaluate) an integral that expresses the area between the graphs of \( y = (1 + x^2)^{-1} \) and \( y = x^2 \).

56. **CAS** Find a numerical approximation to the area above \( y = 1 - (x/\pi) \) and below \( y = \sin x \) (find the points of intersection numerically).
57. **CAS** Find a numerical approximation to the area above \( y = |x| \) and below \( y = \cos x \).

58. **CAS** Use a computer algebra system to find a numerical approximation to the number \( c \) (besides zero) in \( \left[ 0, \frac{\pi}{2} \right] \), where the curves \( y = \sin x \) and \( y = \tan^2 x \) intersect. Then find the area enclosed by the graphs over \( [0, c] \).

59. Lauren and Harvey own a field that is bordered by Route 271, Rogadzo Road, and the Riemann River (Figure 25). To estimate the area of the field, at 50-ft intervals along Route 271 they measured the distance from Route 271 to the river, parallel to Rogadzo Road. Their measurements (in feet) are shown in the figure and in the following table where \( x \) represents the measurement location along Route 271 and \( y \) represents the distance from Route 271 to the Riemann River (both in feet).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
<th>350</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>260</td>
<td>265</td>
<td>215</td>
<td>205</td>
<td>250</td>
<td>305</td>
<td>295</td>
<td>240</td>
<td>150</td>
</tr>
</tbody>
</table>

Compute right-endpoint and left-endpoint Riemann sums to obtain approximations of the area of the field.

![Figure 25](image)

60. Referring to Figure 1 at the beginning of this section, estimate the projected number of additional joules produced in the years 2009-2030 as a result of government stimulus spending in 2009-2010. **Note:** One watt (W) is equal to 1 joule/second (J/s), and 1 gigawatt (GW) is \( 10^9 \) watts.

Exercises 61 and 62 use the notation and results of Exercises 47-49 of Section 3.4. For a given country, \( F(r) \) is the fraction of total income that goes to the bottom \( r \)th fraction of households. The graph of \( y = F(r) \) is called the Lorenz curve.

![Figure 26](image)

61. **CAS** Let \( A \) be the area between \( y = r \) and \( y = F(r) \) over the interval \( [0, 1] \) (Figure 26). The **Gini Index** is the ratio \( G = \frac{A}{B} \), where \( B \) is the area under \( y = r \) over \( [0, 1] \).

(a) Show that

\[
G = 2 \int_0^1 (r - F(r)) \, dr
\]

(b) Calculate \( G \) if

\[
F(r) = \begin{cases} 
3r & \text{for } 0 \leq r \leq \frac{1}{2} \\
6r - 3 & \text{for } \frac{1}{2} \leq r \leq 1 
\end{cases}
\]

(c) The Gini index is a measure of income distribution, with a lower value indicating a more equal distribution. Calculate \( G \) if \( F(r) = r \) (in this case, all households have the same income by Exercise 49(b) of Section 3.4).

(d) What is \( G \) if all of the income goes to one household? **Hint:** In this extreme case, \( F(r) = 0 \) for \( 0 \leq r < 1 \).

62. Calculate the Gini index of the United States in the year 2010 from the Lorenz curve in Figure 26, which consists of segments joining the data points in the following table:

<table>
<thead>
<tr>
<th>( r )</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(r) )</td>
<td>0.033</td>
<td>0.118</td>
<td>0.264</td>
<td>0.480</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

![Figure 27](image)

63. Find the line \( y = mx \) that divides the area under the curve \( y = x(1 - x) \) over \([0, 1]\) into two regions of equal area.

64. **CAS** Let \( c \) be the number such that the area under \( y = \sin x \) over \([0, \pi]\) is divided in half by the line \( y = cx \) (Figure 27). Find an equation for \( c \) and solve this equation **numerically** using a computer algebra system.

**Further Insights and Challenges**

63. Find the line \( y = mx \) that divides the area under the curve \( y = x(1 - x) \) over \([0, 1]\) into two regions of equal area.

64. **CAS** Let \( c \) be the number such that the area under \( y = \sin x \) over \([0, \pi]\) is divided in half by the line \( y = cx \) (Figure 27). Find an equation for \( c \) and solve this equation **numerically** using a computer algebra system.

**6.2 Setting Up Integrals: Volume, Density, Average Value**

Which quantities are represented by integrals? Roughly speaking, integrals represent quantities that are the "total amount" of something such as area, volume, or mass. We can approximate each quantity as a sum obtained by dividing an object into small pieces.
over which the quantity is easy to compute. We obtain an exact value by taking a limit; that is, by using an integral. There is a two-step procedure for computing such quantities: (1) approximate the quantity by a sum of $N$ terms, and (2) pass to the limit as $N \to \infty$ to obtain an integral. We'll use this procedure often in this and other sections.

**Volume**

Our first example is the **volume** of a solid body. Before proceeding, let's recall that the volume of a right cylinder (Figure 1) is $Ah$, where $A$ is the area of the base and $h$ is the height, measured perpendicular to the base. Here, we use the "right cylinder" in the general sense; the base does not have to be circular, but the sides are perpendicular to the base.

Suppose that a solid body extends from height $y = a$ to $y = b$ along the $y$-axis as in Figure 2. Furthermore, assume that the area of the **horizontal cross sections** (the intersections of the solid with horizontal planes) vary from level to level within the solid. Let $A(y)$ be the cross-sectional area at height $y$.

![Figure 1: The volume of a right cylinder is $Ah$.](image)

![Figure 2: Divide the solid into thin horizontal slices. Each slice is nearly a right cylinder whose volume can be approximated as area times height.](image)

To compute the volume $V$ of the body, divide the body into $N$ horizontal slices of thickness $\Delta y = (b - a)/N$. The $i$th slice extends from $y_{i-1}$ to $y_i$, where $y_i = a + i \Delta y$. Let $V_i$ be the volume of the slice.

If $N$ is very large, then $\Delta y$ is very small and the slices are very thin. In this case, the $i$th slice is nearly a right cylinder of base $A(y_{i-1})$ and height $\Delta y$, and therefore $V_i \approx A(y_{i-1})\Delta y$. The whole volume is obtained by summing the volumes of the slices, and therefore,

$$V = \sum_{i=1}^{N} V_i \approx \sum_{i=1}^{N} A(y_{i-1})\Delta y$$

The sum on the right is a left-endpoint approximation to the integral $\int_a^b A(y)\,dy$. If we assume that $A$ is a continuous function, then the approximation improves in accuracy and converges to the integral as $N \to \infty$. We conclude that the **volume of the solid is equal to the integral of its cross-sectional area**.

**Volume as the Integral of Cross-Sectional Area** Let $A(y)$ be the area of the horizontal cross section at height $y$ of a solid body extending from $y = a$ to $y = b$. Then

$$\text{Volume of the solid body} = \int_a^b A(y)\,dy$$
**SECTION 6.2 Setting Up Integrals: Volume, Density, Average Value**

**EXAMPLE 1 Volume of a Pyramid**  
Calculate the volume $V$ of a pyramid of height 12 m whose base is a square of side 4 m.

**Solution** To use Eq. (1), we need a formula for the horizontal cross section $A(y)$.

**Step 1. Find a formula for $A(y)$.**

Figure 3 shows that the horizontal cross section at height $y$ is a square. To find the side $s$ of this square, apply the law of similar triangles to $\Delta ABC$ and to the triangle of height $12 - y$ whose base of length $\frac{1}{2}s$ lies on the cross section:

$$\frac{\text{base}}{\text{height}} = \frac{2}{12} = \frac{s}{12-y} \quad \Rightarrow \quad 2(12-y) = 6s$$

We find that $s = \frac{1}{3}(12-y)$, and therefore, $A(y) = s^2 = \frac{1}{9}(12-y)^2$.

**Step 2. Compute $V$ as the integral of $A(y)$.**

$$V = \int_0^{12} A(y) \, dy = \int_0^{12} \frac{1}{9}(12-y)^2 \, dy = -\frac{1}{27}(12-y)^3 \bigg|_0^{12} = 64 \text{ m}^3$$

We found the antiderivative $-\frac{1}{27}(12-y)^3$ using a substitution $u = 12-y$. The resulting relation $du = -dy$ introduces the negative sign appearing in the result.

This volume of 64 m$^3$ agrees with the result obtained using the formula $V = \frac{1}{3}Ah$ for the volume of a pyramid of base $A$ and height $h$, since $\frac{1}{3}Ah = \frac{1}{3}(4^2)(12) = 64$.

**EXAMPLE 2** Compute the volume $V$ of the solid in Figure 4, whose base (shown at the bottom in the figure) is the region between the parabola $y = 4-x^2$ and the $x$-axis, and whose vertical cross sections perpendicular to the $y$-axis are semicircles.

**Solution** To find a formula for the area $A(y)$ of the cross section, observe that $y = 4-x^2$ can be written $x = \pm\sqrt{4-y}$. We see in Figure 4 that the cross section at $y$ is a semicircle of radius $r = \sqrt{4-y}$. This semicircle has area $A(y) = \frac{1}{2}\pi r^2 = \frac{1}{2}(4-y)$.

Therefore,

$$V = \int_0^4 A(y) \, dy = \frac{\pi}{2} \int_0^4 (4-y) \, dy = \frac{\pi}{2} \left[ 4y - \frac{1}{2}y^2 \right]^4_0 = 4\pi$$

In the next example, we compute volume using vertical rather than horizontal cross sections. This leads to an integral with respect to $x$ rather than $y$.

**EXAMPLE 3 Volume of a Sphere: Vertical Cross Sections**  
Compute the volume of a sphere of radius $R$.

**Solution** As we see in Figure 5, the vertical cross section of the sphere at $x$ is a circle whose radius $r$ satisfies $x^2 + r^2 = R^2$ or $r = \sqrt{R^2-x^2}$. Note that $R$ is the fixed radius of the sphere, while $r$ varies as the cross sections vary. The area of the cross section is $A(x) = \pi r^2 = \pi (R^2-x^2)$. Therefore, the sphere has volume

$$\int_{-R}^{R} \pi(R^2-x^2) \, dx = \pi \left( R^2x - \frac{x^3}{3} \right) \bigg|_{-R}^{R} = 2 \left( \pi R^3 - \frac{\pi R^3}{3} \right) = \frac{4}{3} \pi R^3$$

![Figure 3](image_url)  
A horizontal cross section of the pyramid is a square.

![Figure 4](image_url)  
Cross section is a semicircle of radius $\sqrt{4-y}$.

![Figure 5](image_url)  
Length $\sqrt{4-y}$.
CONCEPTUAL INSIGHT  Cavallieri’s principle states: Solids with equal cross-sectional areas have equal volume. It is often illustrated convincingly with two stacks of coins (Figure 6). Our formula $V = \int_a^b A(y) \, dy$ includes Cavallieri’s principle, because the volumes $V$ are certainly equal if the cross-sectional areas $A(y)$ are equal.

Density

Next, we show that the total mass of an object can be expressed as the integral of its mass density. Consider a rod of length $\ell$. The rod’s linear mass density $\rho$ is defined as the mass per unit length. If $\rho$ is constant, then by definition,

$$\text{total mass} = \text{linear mass density} \times \text{length} = \rho \cdot \ell$$

For example, if $\ell = 10 \text{ cm}$ and $\rho = 9 \text{ g/cm}$, then the total mass is $9 \cdot 10 = 90 \text{ g}$.

Now, consider a rod extending along the $x$-axis from $x = a$ to $x = b$ whose density $y = \rho(x)$ is a continuous function of $x$, as in Figure 7. To compute the total mass $M$, we break up the rod into $N$ small segments of length $\Delta x = (b - a)/N$. Then $M = \sum_{i=1}^{N} M_i$, where $M_i$ is the mass of the $i$th segment.

We cannot use Eq. (2) directly to find the mass of the rod because $\rho(x)$ is not constant. However, we can argue that if $\Delta x$ is small, then $\rho(x)$ is nearly constant along the $i$th segment, and we can use Eq. (2) to obtain the mass of the $i$th segment. If the $i$th segment extends from $x_{i-1}$ to $x_i$ and $c_i$ is any sample point in $[x_{i-1}, x_i]$, then $M_i \approx \rho(c_i) \Delta x$ and

$$\text{total mass} M = \sum_{i=1}^{N} M_i \approx \sum_{i=1}^{N} \rho(c_i) \Delta x$$

As $N \to \infty$, the accuracy of the approximation improves. However, the sum on the right is a Riemann sum whose value approaches $\int_a^b \rho(x) \, dx$, and thus, it makes sense to define the total mass of a rod as the integral of its linear mass density:

$$\text{Total mass} M = \int_a^b \rho(x) \, dx$$

Note the similarity in the way we use thin slices to compute volume and small pieces to compute total mass.

**EXAMPLE 4** Total Mass Find the total mass $M$ of a 2-m rod of density $\rho(x) = 1 + 2x - x^2 \text{ kg/m}$, where $x$ is the distance from one end of the rod.

Solution

$$M = \int_0^2 \rho(x) \, dx = \int_0^2 (1 + 2x - x^2) \, dx = \left( x + x^2 - \frac{1}{3} x^3 \right)_0^2 = \frac{10}{3} \text{ kg} $$
In general, population density is a function $p(x, y)$ that depends not just on the distance to the origin (city center) but also on the coordinates $(x, y)$ (the specific location). Total population is then computed using double integration, a topic in multivariable calculus.

In some situations, density is a function of distance to the origin. For example, in the study of urban populations, it might be assumed that the population density $\rho(r)$ (in people per square kilometer) depends only on the distance $r$ from the center of a city. Such a density function is called a radial density function.

We now derive a formula for the total population $P$ within a radius $R$ of the city center, assuming a radial density $\rho(r)$. First, divide the circle of radius $R$ into $N$ thin rings of equal width $\Delta r = R/N$ as in Figure 8.

Let $P_i$ be the population within the $i$th ring, so that the total population is given by

$$P = \sum_{i=1}^{N} P_i.$$  

If the outer radius of the $i$th ring is $r_i$, then the circumference is $2\pi r_i$, and if $\Delta r$ is small, the area of this ring is approximately $2\pi r_i \Delta r$ (outer circumference times width). Furthermore, the population density within the thin ring is nearly constant with value $\rho(r_i)$. With these approximations,

$$P_i \approx 2\pi r_i \Delta r \times \rho(r_i) = 2\pi r_i \rho(r_i) \Delta r$$

Adding up the $P_i$, we obtain

$$P = \sum_{i=1}^{N} P_i \approx 2\pi \sum_{i=1}^{N} r_i \rho(r_i) \Delta r$$

This last sum is a right-endpoint approximation to the integral $2\pi \int_0^R r \rho(r) \, dr$. As $N$ tends to $\infty$, the approximation improves in accuracy and the sum converges to the integral. Thus, for a population with a radial density function $\rho$,

$$\text{Population in a radius } R = 2\pi \int_0^R r \rho(r) \, dr$$

**EXAMPLE 5 Computing Total Population**  The population in the city of Isaactonia and its surrounding suburbs has radial density function $\rho(r) = 15(1 + r^2)^{-1/2}$, where $r$ is the distance from the city center in kilometers and $\rho$ has units of thousands of people per square kilometer. How many people live in the ring between 10 and 30 km from the city center?

**Solution**  The population $P$ (in thousands) within the ring is

$$P = 2\pi \int_{10}^{30} r(15(1 + r^2)^{-1/2}) \, dr = 2\pi(15) \int_{10}^{30} \frac{r}{(1 + r^2)^{1/2}} \, dr$$

Now use the substitution $u = 1 + r^2$, $du = 2r \, dr$. The limits of integration become $u(10) = 101$ and $u(30) = 901$;

$$P = 30\pi \int_{101}^{901} u^{-1/2} \left(\frac{1}{2}\right) du = 30\pi u^{1/2} \bigg|_{101}^{901} \approx 1882 \text{ thousand}$$

In other words, the population in the ring is approximately 1.9 million people.

**Flow Rate**  

When fluid flows through a tube, the **flow rate** $Q$ is the **volume per unit time** of fluid passing through the tube (Figure 9). The flow rate depends on the velocity of the fluid particles. If all particles of the fluid travel with the same velocity $v$ (say, in units of cubic meters per minute), and the tube has radius $R$, then
flow rate \( Q \) = cross-sectional area \( \times \) velocity = \( \pi R^2 v \) cm\(^2\)/min

Volume per unit time.

Why is this formula true? Let's fix an observation point \( P \) in the tube and ask: Which fluid particles flow past \( P \) in a 1-min interval? A particle travels \( v \) centimeters each minute, so it flows past \( P \) during this minute if it is located not more than \( v \) centimeters to the left of \( P \) (assuming the fluid flows from left to right). Therefore, the column of fluid flowing past \( P \) in a 1-min interval is a cylinder of radius \( R \), length \( v \), and volume \( \pi R^2 v \) (Figure 9).

In reality, the fluid particles do not all travel at the same velocity because of friction. However, for a slowly moving fluid, the flow is laminar, by which we mean that the velocity \( v(r) \) depends only on the distance \( r \) from the center of the tube. The particles at the center of the tube travel most quickly, and the velocity tapers off to zero near the walls of the tube (Figure 10).

If the flow is laminar, we can express the flow rate \( Q \) as an integral. We divide the circular cross section of the tube into \( N \) thin concentric rings of width \( \Delta r = R/N \) (Figure 11). The area of the \( i \)th ring is approximately \( 2\pi r_i \Delta r \) and the fluid particles flowing past this ring have a velocity that is nearly constant with value \( v(r_i) \). Therefore, we can approximate the flow rate \( Q_i \) through the \( i \)th ring by

\[
Q_i \approx \text{cross-sectional area} \times \text{velocity} \approx (2\pi r_i \Delta r) v(r_i)
\]

We obtain

\[
Q = \sum_{i=1}^{N} Q_i \approx 2\pi \sum_{i=1}^{N} r_i v(r_i) \Delta r
\]

The sum on the right is a right-endpoint approximation to the integral \( 2\pi \int_{0}^{R} r v(r) \, dr \). Once again, we let \( N \) tend to \( \infty \) to obtain the formula for laminar flow with velocity \( v(r) \),

\[
\text{Flow rate } Q = 2\pi \int_{0}^{R} r v(r) \, dr
\]

Note the similarity of this formula and its derivation to that of population with a radial density function.

**EXAMPLE 6 Laminar Flow** According to Poiseuille’s Law, the velocity of blood flowing in a blood vessel of radius \( R \) centimeters is \( v(r) = k(R^2 - r^2) \), where \( r \) is the distance from the center of the vessel (in centimeters) and \( k \) is a constant. Calculate the flow \( Q \) as function of \( R \), assuming that \( k = 0.5 \) cm-s\(^{-1}\).

**Solution** By Eq. (5),

\[
Q = 2\pi \int_{0}^{R} (0.5) r (R^2 - r^2) \, dr = \pi \left( R^4 \frac{r^2}{2} - \frac{r^4}{4} \right) \bigg|_{0}^{R} = \frac{\pi}{4} R^4 \text{ cm}^3/\text{s}
\]

Note that \( Q \) is proportional to \( R^4 \) (this is true for any value of \( k \)).
CONCEPTUAL INSIGHT In this section, we saw a number of examples of Riemann sums and the definite integral at work. In each case there was a quantity that we were interested in computing over a whole domain. We cut the domain into small pieces over which the quantity was straightforward to compute. Adding those results yielded a Riemann sum approximation of the desired whole-domain quantity, and then passing to the limit resulted in a definite integral formula for determining the exact value.

In the remainder of this text we will see many more examples where this approach is carried out and Riemann sums and the definite integral are employed to compute a desired quantity.

**Average Value**

Next, we discuss the average value of a function. Recall that the average of $N$ numbers $a_1, a_2, \ldots, a_N$ is the sum divided by $N$:

$$\frac{a_1 + a_2 + \cdots + a_N}{N} = \frac{1}{N} \sum_{j=1}^{N} a_j$$

We cannot define the average value of a function $f$ on an interval $[a, b]$ as a sum because there are infinitely many values of $x$ to consider. But recall the formula for the right-endpoint approximation $R_N$ (Figure 12):

$$R_N = \frac{b-a}{N} \left( f(x_1) + f(x_2) + \cdots + f(x_N) \right)$$

where $x_1, \ldots, x_N$ are the right endpoints of the subintervals. We see that $R_N$ divided by $(b-a)$ is equal to the average of the equally spaced function values $f(x_i)$:

$$\frac{1}{b-a} R_N = \frac{f(x_1) + f(x_2) + \cdots + f(x_N)}{N} = \text{Average of the function values}$$

If $N$ is large, it is reasonable to think of this quantity as an approximation to the average of $f(x)$ on $[a, b]$. Therefore, we define the average value itself as the limit:

$$\text{average value} = \lim_{N \to \infty} \frac{1}{b-a} R_N(f) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

**DEFINITION Average Value** The average value of an integrable function $f$ on $[a, b]$ is the quantity

$$\text{Average value} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

The average value of a function is also called the mean value.

**GRAPHICAL INSIGHT** The average value $M$ of a nonnegative function is the average height of its graph (Figure 13). The region under the graph has the same area as the rectangle of height $M$, because $\int_a^b f(x)\,dx = M(b-a)$.

**EXAMPLE 7** Find the average value of $f(x) = \sin x$ on $[0, \pi]$.

**Solution** The average value of $f(x) = \sin x$ on $[0, \pi]$ is

$$\frac{1}{\pi} \int_0^\pi \sin x\,dx = -\frac{1}{\pi} \cos x\bigg|_0^\pi = -\frac{1}{\pi} (\cos \pi - \cos 0) = \frac{2}{\pi} \approx 0.637$$
This answer is reasonable because \( \sin x \) varies from 0 to 1 on the interval \([0, \pi]\) and the average 0.637 lies somewhere between the two extremes (Figure 13).

**EXAMPLE 8** Vertical Jump of a Bushbaby  
The bushbaby (*Galago senegalensis*) is a small primate with remarkable jumping ability. Find the average speed during a jump if the initial vertical velocity is \( v_0 = 600 \text{ cm/s} \). Use Galileo's formula for the height 

\[
h(t) = v_0 t - \frac{1}{2} gt^2 \quad \text{(in centimeters, where } g = 980 \text{ cm/s}^2)\]

Solution  

The bushbaby's height is 

\[
h(t) = v_0 t - \frac{1}{2} gt^2 = (v_0 - \frac{1}{2} gt) \]  
The height is zero at \( t = 0 \) and at \( t = 2v_0/g = \frac{1200}{980} = \frac{60}{49} \text{ seconds, when the jump ends.} \)

The bushbaby's velocity is 

\[
h'(t) = v_0 - gt = 600 - 980t. \]

The velocity is negative for \( t > v_0/g = \frac{60}{49} \), so as we see in Figure 14, the integral of speed \( |h'(t)| \) is equal to the sum of the areas of two triangles of base \( \frac{60}{49} \) and height 600:

\[
\int_0^{6/49} |600 - 980t| \, dt = \frac{1}{2} \left( \frac{6}{49} \right) (600) + \frac{1}{2} \left( \frac{6}{49} \right) (600) = \frac{3600}{9.8}
\]

The average speed \( \bar{v} \) is

\[
\bar{v} = \frac{1}{\frac{60}{49}} \int_0^{6/49} |600 - 980t| \, dt = \frac{1}{\frac{6}{49}} \left( \frac{3600}{9.8} \right) = 300 \text{ cm/s}
\]

There is an important difference between the average of a list of numbers and the average value of a continuous function. If the average score on an exam is 84, then 84 lies between the highest and lowest scores, but it is possible that no student received a score of 84. By contrast, the Mean Value Theorem (MVT) for Integrals asserts that a continuous function always takes on its average value somewhere in its interval (Figure 15).

**THEOREM 1** Mean Value Theorem for Integrals  
If \( f \) is continuous on \([a, b]\), then there exists a value \( c \in [a, b] \) such that

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

For example, the average of \( f(x) = \sin x \) on \([0, \pi]\) is 2/\( \pi \) by Example 7. We have \( f(c) = 2/\pi \) for \( c = \sin^{-1}(2/\pi) \approx 0.69 \). Since 0.69 lies in \([0, \pi]\), \( f(x) = \sin x \) indeed takes on its average value at a point in the interval.

Proof  
Let \( M = \frac{1}{b-a} \int_a^b f(x) \, dx \) be the average value. Because \( f \) is continuous, we can apply Theorem 1 of Section 4.2 to conclude that \( f \) takes on a minimum value \( m_{\text{min}} \) and a maximum value \( M_{\text{max}} \) on the closed interval \([a, b]\). Furthermore, by Eq. (8) of Section 5.2,

\[
m_{\text{min}}(b-a) \leq \int_a^b f(x) \, dx \leq M_{\text{max}}(b-a)
\]

Dividing by \( b-a \), we find

\[
m_{\text{min}} \leq M \leq M_{\text{max}}
\]

In other words, the average value \( M \) lies between \( m_{\text{min}} \) and \( M_{\text{max}} \). The Intermediate Value Theorem guarantees that \( f(x) \) takes on every value between its min and max, so \( f(c) = M \) for some \( c \in [a, b] \).
6.2 SUMMARY

- Formulas:
  Volume \( V = \int_{a}^{b} A(y) \, dy \), \( A(y) = \) cross-sectional area
  Total Mass \( M = \int_{a}^{b} \rho(x) \, dx \), \( \rho(x) = \) linear mass density
  Total Population \( P = 2\pi \int_{0}^{R} r p(r) \, dr \), \( p(r) = \) radial density
  Laminar Flow Rate \( Q = 2\pi \int_{0}^{R} r v(r) \, dr \), \( v(r) = \) velocity at radius \( r \)
  Average value \( M = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \), \( f = \) any continuous function

- The MVT for Integrals: If \( f \) is continuous on \([a, b]\) with average (or mean) value \( M \), then \( f(c) = M \) for some \( c \in [a, b] \).

6.2 EXERCISES

Preliminary Questions

1. What is the average value of \( f \) on \([0, 4]\) if the area between the graph of \( f \) and the \( x \)-axis is equal to 12?
2. Find the volume of a solid extending from \( y = 2 \) to \( y = 5 \) if every cross section has area \( A(y) = 5 \).
3. What is the definition of flow rate?
4. Which assumption about fluid velocity did we use to compute the flow rate as an integral?
5. The average value of \( f \) on \([1, 4]\) is 5. Find \( \int_{1}^{4} f(x) \, dx \).

Exercises

1. Let \( V \) be the volume of a pyramid of height 20 whose base is a square of side 8.
   (a) Use similar triangles as in Example 1 to find the area of the horizontal cross section at a height \( y \).
   (b) Calculate \( V \) by integrating the cross-sectional area.

2. Let \( V \) be the volume of a right circular cone of height 10 whose base is a circle of radius 4 [Figure 16(A)].
   (a) Use similar triangles to find the area of a horizontal cross section at a height \( y \).
   (b) Calculate \( V \) by integrating the cross-sectional area.

3. Use the method of Exercise 2 to find the formula for the volume of a right circular cone of height \( h \) whose base is a circle of radius \( R \) [Figure 16(B)].

4. Calculate the volume of the ramp in Figure 17 in three ways by integrating the area of the cross sections:
   (a) Perpendicular to the \( x \)-axis (rectangles)
   (b) Perpendicular to the \( y \)-axis (triangles)
   (c) Perpendicular to the \( z \)-axis (rectangles)

FIGURE 16 Right circular cones.

FIGURE 17 Ramp of length 6, width 4, and height 2.
5. Find the volume of liquid needed to fill a sphere of radius \( R \) to height \( h \) (Figure 18).

![Figure 18 Sphere filled with liquid to height \( h \).](image)

6. Find the volume of the wedge in Figure 19(A) by integrating the area of vertical cross sections.

![Figure 19](image)

7. Derive a formula for the volume of the wedge in Figure 19(B) in terms of the constants \( a \), \( b \), and \( c \).

8. Let \( B \) be the solid whose base is the unit circle \( x^2 + y^2 = 1 \) and whose vertical cross sections perpendicular to the \( x \)-axis are equilateral triangles. Show that the vertical cross sections have area \( A(x) = \sqrt{3}(1 - x^2) \) and compute the volume of \( B \).

In Exercises 9–14, find the volume of the solid with the given base and cross sections.

9. The base is the unit circle \( x^2 + y^2 = 1 \), and the cross sections perpendicular to the \( x \)-axis are triangles whose height and base are equal.

10. The base is the triangle enclosed by \( x + y = 1 \), the \( x \)-axis, and the \( y \)-axis. The cross sections perpendicular to the \( y \)-axis are semicircles.

11. The base is the semicircle \( y = \sqrt{9 - x^2} \), where \( -3 \leq x \leq 3 \). The cross sections perpendicular to the \( x \)-axis are squares.

12. The base is a square, one of whose sides is the interval \([0, 1]\) along the \( x \)-axis. The cross sections perpendicular to the \( x \)-axis are rectangles of height \( f(x) = x^2 \).

13. The base is the region enclosed by \( y = x^2 \) and \( y = 3 \). The cross sections perpendicular to the \( y \)-axis are squares.

14. The base is the region enclosed by \( y = x^2 \) and \( y = 3 \). The cross sections perpendicular to the \( x \)-axis are rectangles of height \( y^2 \).

15. Find the volume of the solid whose base is the region \( |x| + |y| \leq 1 \) and whose vertical cross sections perpendicular to the \( y \)-axis are semicircles (with diameter along the base).

16. Show that a pyramid of height \( h \) whose base is an equilateral triangle of side \( s \) has volume \( \frac{\sqrt{2}}{12}hs^2 \).

17. The area of an ellipse is \( \pi ab \), where \( a \) and \( b \) are the lengths of the semimajor and semiminor axes (Figure 20). Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis \( a = 6 \) and semiminor axis \( b = 4 \).

18. Find the volume \( V \) of a regular tetrahedron (Figure 21) whose face is an equilateral triangle of side \( s \). The tetrahedron has height \( h = \sqrt{2/3}s \).

![Figure 21](image)

19. A frustum of a pyramid is a pyramid with its top cut off (Figure 22(A)). Let \( V \) be the volume of a frustum of height \( h \) whose base is a square of side \( a \) and whose top is a square of side \( b \) with \( a > b > 0 \).

(a) Show that if the frustum were continued to a full pyramid, it would have height \( ha/(a - b) \) (Figure 22(B)).

(b) Show that the cross section at height \( x \) is a square whose side length is given by \( s(x) = (1/h)(a(h - x) + bx) \).

(c) Show that \( V = \frac{1}{3}h(a^2 + ab + b^2) \). A papyrus dating to the year 1850 BCE indicates that Egyptian mathematicians had discovered this formula almost 4000 years ago.

20. A plane inclined at an angle of 45° passes through a diameter of the base of a cylinder of radius \( r \). Find the volume of the region within the cylinder and below the plane (Figure 23).

![Figure 23](image)
21. The solid \( S \) in Figure 24 is the intersection of two cylinders of radius \( r \) whose axes are perpendicular.
(a) The horizontal cross section of each cylinder at distance \( y \) from the central axis is a rectangular strip. Find the strip’s width.
(b) Find the area of the horizontal cross section of \( S \) at distance \( y \).
(c) Find the volume of \( S \) as a function of \( r \).

![Figure 24](image)

**FIGURE 24** Two cylinders intersecting at right angles.

22. Let \( S \) be the intersection of two cylinders of radius \( r \) whose axes intersect at an angle \( \theta \). Find the volume of \( S \) as a function of \( r \) and \( \theta \).

23. Calculate the volume of a cylinder inclined at an angle \( \theta = 30^\circ \) with height 10 and base of radius 4 (Figure 25).

![Figure 25](image)

**FIGURE 25** Cylinder inclined at an angle \( \theta = 30^\circ \).

24. The areas of cross sections of Lake Ngorobow at 5-m intervals are given in the table below. Figure 26 shows a contour map of the lake. Estimate the volume \( V \) of the lake by taking the average of the right- and left-endpoint approximations to the integral of cross-sectional area.

<table>
<thead>
<tr>
<th>Depth (m)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area (million m(^2))</td>
<td>2.1</td>
<td>1.5</td>
<td>1.1</td>
<td>0.835</td>
<td>0.217</td>
</tr>
</tbody>
</table>

![Figure 26](image)

**FIGURE 26** Depth contour map of Lake Ngorobow.

25. Find the total mass of a 2-m rod whose linear density function is \( p(x) = 10(x + 1)^{-2} \) kg/m for \( 0 \leq x \leq 1 \).

26. Find the total mass of a 3-m rod whose linear density function is \( p(x) = 3 \cos(x) \) kg/m for \( 0 \leq x \leq 3 \).

27. A mineral deposit along a strip of length 6 cm has density \( x(x) = 0.01x(6 - x) \) g/cm for \( 0 \leq x \leq 6 \). Calculate the total mass of the deposit.

28. Charge is distributed along a glass tube of length 10 cm with linear charge density \( p(x) = x(x^2 + 1)^{-2} \times 10^{-14} \) coulombs per centimeter (C/cm) for \( 0 \leq x \leq 10 \). Calculate the total charge.

29. Calculate the population within a 10-mile radius of the city center if the radial population density is \( p(r) = 4(1 + r)^{-3} \) (in thousands per square mile).

30. Odzala National Park in the Republic of the Congo has a high density of gorillas. Suppose that the population density is given by the radial density function \( p(r) = 52(1 + r)^{-2} \) gorillas/km\(^2\), where \( r \) is the distance from a grassy clearing with a source of water. Calculate the number of gorillas within a 5-km radius of the clearing.

31. Table 1 lists the population density (in people per square kilometer) as a function of distance \( r \) (in kilometers) from the center of a rural town. Estimate the total population within a 1.2-km radius of the center by taking the average of the left- and right-endpoint approximations.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( p(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>125.0</td>
</tr>
<tr>
<td>0.2</td>
<td>102.3</td>
</tr>
<tr>
<td>0.4</td>
<td>83.8</td>
</tr>
<tr>
<td>0.6</td>
<td>68.6</td>
</tr>
</tbody>
</table>

32. Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function \( p(r) = 0.03 + 0.01 \cos(\pi r^2) \) g/cm\(^2\).

33. Assume that the density of deer in a forest is given by the radial function \( p(r) = 150(r^2 + 2)^{-2} \) deer per square kilometer, where \( r \) is the distance (in kilometers) to a small meadow. Calculate the number of deer in the region \( 2 \leq r \leq 5 \) km.

34. Show that a circular plate of radius 2 cm with radial mass density \( p(r) = \frac{1}{2} \) g/cm\(^2\) has finite total mass, even though the density becomes infinite at the origin.

35. Find the flow rate through a tube of radius 4 cm, assuming that the velocity of fluid particles at a distance \( r \) centimeters from the center is \( v(r) = (16 - r^2) \) cm/s.

36. The velocity of fluid particles flowing through a tube of radius 5 cm is \( v(r) = (10 - 0.3r - 0.3r^2) \) cm/s, where \( r \) centimeters is the distance from the center. What quantity per second of fluid flows through the portion of the tube where \( 0 \leq r \leq 2 \)?

37. A solid rod of radius 1 cm is placed in a pipe of radius 3 cm so that their axes are aligned. Water flows through the pipe and around the rod. Find the flow rate if the velocity of the water is given by the radial function \( v(r) = 0.5(r - 1)(3 - r) \) cm/s.

38. Let \( v(r) \) be the velocity of blood in an arterial capillary of radius \( R = 4 \times 10^{-4} \) m. Use Poiseuille's Law (Example 6) with \( k = 10^5 \) (m s\(^{-1}\)) to determine the velocity at the center of the capillary and the flow rate (use correct units).

In Exercises 39–48, calculate the average over the given interval.
39. \( f(x) = x^3 \), \([-0.4, 0.4]\)
40. \( f(x) = x^3 \), \([-1, 1]\)
41. \( f(x) = \cos x \), \([0, \frac{\pi}{6}]\)
42. \( f(x) = \sec^2 x \), \([\frac{\pi}{6}, \frac{\pi}{3}]\)
43. \( f(x) = x^2 \), \([2, 5]\)
44. \( f(x) = \frac{\sin(x)}{x^2} \), \([1, 2]\)
45. \( f(x) = 2x^3 - 6x^2 \), \([-1, 3]\)
46. \( f(x) = \frac{x}{(x^2 + 16)^2} \), \([0, 3]\)
47. \( f(x) = \sin x \), \([0, \frac{\pi}{2}]\)
48. \( f(x) = \sin(\pi x) \), \([0, \pi]\)

49. The temperature (in degrees Celsius) at time \( t \) (in hours) in an art museum varies according to \( T(t) = 20 + 5\cos(\frac{\pi}{6})t \). Find the average temperature over the time periods \([0, 24]\) and \([2, 6]\).

50. A steel bar of length 3 m experiences extreme heat at its center, so that the temperature at coordinate \( x \) on the bar is given by \( T(x) = 40\sin\left(\frac{\pi x}{3}\right) + 30^\circ\text{C} \) where the bar sits along the interval \([0, 3]\) on the \( x \)-axis. Determine the average temperature of the bar.

51. The temperature in the town of Walla Walla during the month of July follows a pattern given by \( T(t) = 10\sin\left(\frac{\pi t}{3}\right) + 14\sin\left(\frac{\pi t}{4}\right) + 73^\circ\text{F} \). Here, \( t \) is measured in days, and there are 31 days in July. Explain why you might see a pattern like this and compute the average temperature during the month of July.

52. The door to the garage is left open, and over the next 4 hours the temperature in a house is in degrees Celsius is given by \( T(t) = 20 + 0.25t^2 \). Determine the average temperature over those 4 hours.

53. A 10 cm copper wire with one end in an ice bath is heated at the other end, so that the temperature at each point \( x \) along the wire in degrees Celsius is given by \( T(x) = 50\cos\left(\frac{\pi x}{2}\right) \). Find the average temperature over the wire.

54. A ball thrown in the air vertically from ground level with initial velocity 18 m/s has height \( h(t) = 18t - 9.8t^2 \) at time \( t \) (in seconds). Find the average height and the average speed over the time interval, extending from the ball’s release to its return to ground level.

55. Find the average speed over the time interval \([1, 5]\) (in seconds) of a particle whose position at time \( t \) is \( x(t) = t^3 - 6t^2 \).

56. An object with zero initial velocity accelerates at a constant rate of 10 m/s². Find its average velocity during the first 15 seconds.

57. The acceleration of a particle is \( \alpha(t) = 60t - 4t^3 \) m/s². Compute the average acceleration and the average speed over the time interval \([2, 6]\), assuming that the particle’s initial velocity is zero.

58. What is the average area of circles whose radii vary from 0 to \( R \)?

59. Let \( M \) be the average value of \( f(x) = x^4 \) on \([0, 3]\). Find a value of \( c \) in \([0, 3]\) such that \( f(c) = M \).

60. Let \( f(x) = \sqrt{x} \). Find a value of \( c \) in \([4, 9]\) such that \( f(c) \) is equal to the average of \( f \) on \([4, 9]\).

61. Let \( M \) be the average value of \( f(x) = x^3 \) on \([0, A]\), where \( A > 0 \). Which theorem guarantees that \( f(c) = M \) has a solution \( c \) in \([0, A]\)? Find \( c \).

62. (CAS) Let \( f(x) = 2\sin x - x \). Use a computer algebra system to plot \( f \) and estimate:
   (a) The positive root \( a \) of \( f \)
   (b) The average value \( M \) of \( f \) on \([0, a]\)
   (c) A value \( c \in [0, a] \) such that \( f(c) = M \)

63. Which of \( f(x) = x\sin^2 x \) and \( g(x) = x^2 \sin^2 x \) has a larger average value over \([0, 1]\)? Over \([1, 2]\)?

64. Find the average of \( f(x) = ax + b \) over the interval \([-M, M]\), where \( a, b, \) and \( M \) are arbitrary constants.

65. Sketch the graph of a function \( f \) such that \( f(x) \geq 0 \) on \([0, 1]\) and \( f(x) \leq 0 \) on \([1, 2]\), whose average on \([0, 2]\) is negative.

66. Give an example of a function (necessarily discontinuous) that does not satisfy the conclusion of the MVT for Integrals.

Further Insights and Challenges

67. An object is tossed into the air vertically from ground level with initial velocity 18 m/s at time \( t = 0 \). Find the average speed of the object over the time interval \([0, T]\), where \( T \) is the time the object returns to Earth.

68. (CAS) Review the MVT stated in Section 4.3 (Theorem 1) and show how it can be used, together with the Fundamental Theorem of Calculus, to prove the MVT for Integrals.

6.3 Volumes of Revolution: Disks and Washers

A solid of revolution is a solid obtained by rotating a region in the plane about an axis. The sphere and right circular cone are familiar examples of such solids. Each of these is "swept out" as a plane region revolves around an axis (Figure 1).

![Figure 1](image-url) The right circular cone and the sphere are solids of revolution.
This method for computing the volume is referred to as the disk method because the vertical slices of the solid are circular disks.

Suppose that \( f(x) \geq 0 \) for \( a \leq x \leq b \). The solid obtained by rotating the region under the graph about the \( x \)-axis has a special feature: All vertical cross sections are circles (Figure 2). In fact, the vertical cross section at location \( x \) is a circle of radius \( R = f(x) \) and thus,

\[
\text{area of the vertical cross section} = \pi R^2 = \pi f(x)^2
\]

We know from Section 6.2 that the total volume \( V \) is equal to the integral of cross-sectional area. Therefore,

\[
V = \pi \int_a^b R^2 \, dx = \pi \int_a^b f(x)^2 \, dx
\]

**Figure 2**

**Volume of Revolution: Disk Method** If \( f \) is continuous and \( f(x) \geq 0 \) on \([a, b]\), then the solid obtained by rotating the region under the graph about the \( x \)-axis has volume

\[
V = \pi \int_a^b R^2 \, dx = \pi \int_a^b f(x)^2 \, dx
\]

**Example 1** Calculate the volume \( V \) of the solid obtained by rotating the region under \( y = x^2 \) about the \( x \)-axis for \( 0 \leq x \leq 2 \).

**Solution** The solid is shown in Figure 3. By Eq. (1) with \( f(x) = x^2 \), its volume is

\[
V = \pi \int_0^2 R^2 \, dx = \pi \int_0^2 (x^2)^2 \, dx = \pi \int_0^2 x^4 \, dx = \pi \left[ \frac{x^5}{5} \right]_0^2 = \pi \frac{32}{5} = \frac{32}{5} \pi
\]

There are some useful variations on the formula for a volume of revolution. First, consider the region between two curves \( y = f(x) \) and \( y = g(x) \), where \( f(x) \geq g(x) \geq 0 \) as in Figure 5(A). When this region is rotated about the \( x \)-axis, segment \( AB \) sweeps out the washer shown in Figure 5(B). The inner and outer radii of this washer (also called an annulus; see Figure 4) are

\[
R_{\text{outer}} = f(x), \quad R_{\text{inner}} = g(x)
\]

The washer has area \( \pi R_{\text{outer}}^2 - \pi R_{\text{inner}}^2 \) or \( \pi (f(x))^2 - (g(x))^2 \), and the volume of the solid of revolution [Figure 5(C)] is the integral of this cross-sectional area:

\[
V = \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2) \, dx = \pi \int_a^b (f(x))^2 - (g(x))^2 \, dx
\]
EXAMPLE 2 Region Between Two Curves Find the volume \( V \) obtained by revolving the region between \( y = x^2 + 4 \) and \( y = 2 \) about the \( x \)-axis for \( 1 \leq x \leq 3 \).

Solution The graph of \( y = x^2 + 4 \) lies above the graph of \( y = 2 \) (Figure 6). Therefore, \( R_{\text{outer}} = x^2 + 4 \) and \( R_{\text{inner}} = 2 \). By Eq. (2),

\[
V = \pi \int_{1}^{3} (R_{\text{outer}}^2 - R_{\text{inner}}^2) \, dx = \pi \int_{1}^{3} (x^2 + 4)^2 - 2^2 \, dx
\]

\[
= \pi \int_{1}^{3} (x^4 + 8x^2 + 12) \, dx = \pi \left( \frac{1}{5}x^5 + \frac{8}{3}x^3 + 12x \right) \bigg|_{1}^{3} = \frac{2126}{15} \pi
\]

EXAMPLE 3 Revolving About a Horizontal Axis Find the volume \( V \) of the "wedding band" (Figure 7(C)) obtained by rotating the region between the graphs of \( f(x) = x^2 + 2 \) and \( g(x) = 4 - x^2 \) about the horizontal line \( y = -3 \).

Solution First, let's find the points of intersection of the two graphs by solving

\[
f(x) = g(x) \quad \Rightarrow \quad x^2 + 2 = 4 - x^2 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1
\]

Figure 7(A) shows that \( g(x) \geq f(x) \) for \(-1 \leq x \leq 1\).
If we wanted to revolve about the x-axis, we would use Eq. (2). Since we want to revolve around \( y = -3 \), we must determine how the radii are affected. Figure 7(B) shows that when we rotate about \( y = -3 \), \( \overline{AB} \) generates a washer whose outer and inner radii are both 3 units longer, and therefore we have

- \( R_{\text{outer}} = g(x) - (-3) = (4 - x^2) + 3 = 7 - x^2 \)
- \( R_{\text{inner}} = f(x) - (-3) = (x^2 + 2) + 3 = x^2 + 5 \)

The volume of revolution (about \( y = -3 \)) is equal to the integral of the area of this washer:

\[
V = \pi \int_{-1}^{1} \left( R_{\text{outer}}^2 - R_{\text{inner}}^2 \right) dx
\]

\[
= \pi \int_{-1}^{1} \left( (7 - x^2)^2 - (x^2 + 5)^2 \right) dx
\]

\[
= \pi \int_{-1}^{1} \left( 49 - 14x^2 + x^4 - (x^4 + 10x^2 + 25) \right) dx
\]

\[
= \pi \int_{-1}^{1} (24 - 24x^2) dx = \pi (24x - 8x^3) \bigg|_{-1}^{1} = 32\pi
\]

EXAMPLE 4 Find the volume obtained by rotating the graphs of \( f(x) = 9 - x^2 \) and \( y = 12 \) for \( 0 \leq x \leq 3 \) about

(a) The line \( y = 12 \)  
(b) The line \( y = 15 \)

The resulting solids are illustrated in Figure 8.

Solution To set up the integrals, let's visualize the cross section. Is it a disk or a washer?

(a) Figure 8(B) shows that \( \overline{AB} \) rotated about \( y = 12 \) generates a disk of radius

\[
R = \text{length of } \overline{AB} = 12 - f(x) = 12 - (9 - x^2) = 3 + x^2
\]

The volume when we rotate about \( y = 12 \) is

\[
V = \pi \int_{0}^{3} R^2 dx = \pi \int_{0}^{3} (3 + x^2)^2 dx = \pi \int_{0}^{3} (9 + 6x^2 + x^4) dx
\]

\[
= \pi \left( 9x + 2x^3 + \frac{1}{5} x^5 \right) \bigg|_{0}^{3} = \frac{648}{5} \pi
\]
Figure 8 Segment \( \overline{AB} \) generates a disk when rotated about \( y = 12 \), but it generates a washer when rotated about \( y = 15 \).

(b) Figure 8(C) shows that \( \overline{AB} \) rotated about \( y = 15 \) generates a washer. The outer radius of this washer is the distance from \( B \) to the line \( y = 15 \):

\[
R_{\text{outer}} = 15 - f(x) = 15 - (9 - x^2) = 6 + x^2
\]

The inner radius is \( R_{\text{inner}} = 3 \), so the volume of revolution about \( y = 15 \) is

\[
V = \pi \int_0^3 (R_{\text{outer}}^2 - R_{\text{inner}}^2) \, dx = \pi \int_0^3 ((6 + x^2)^2 - 3^2) \, dx
\]

\[
= \pi \int_0^3 (27 + 12x^2 + x^4) \, dx
\]

\[
= \pi \left( 27x + 4x^3 + \frac{1}{5}x^5 \right)_0^3 = \frac{1188}{5} \pi
\]

We can use the disk and washer methods for solids of revolution about vertical axes, but it is necessary to describe the graph as a function of \( y \)—that is, \( x = g(y) \).

**Example 5** Revolving About a Vertical Axis  
Find the volume of the solid obtained by rotating the region under the graph of \( f(x) = 9 - x^2 \) for \( 0 \leq x \leq 3 \) about the vertical axis \( x = -2 \).

**Solution** Figure 9 shows that \( \overline{AB} \) sweeps out a horizontal washer when rotated about the vertical line \( x = -2 \). We are going to integrate with respect to \( y \), so we need the inner and outer radii of this washer as functions of \( y \). Solving for \( x \) in \( y = 9 - x^2 \), we obtain \( x^2 = 9 - y \), or \( x = \pm\sqrt{9 - y} \). Since we are rotating the right half of the parabola, we choose the positive square root. Therefore,

\[
R_{\text{outer}} = \sqrt{9 - y} + 2, \quad R_{\text{inner}} = 2
\]

\[
R_{\text{outer}}^2 - R_{\text{inner}}^2 = (\sqrt{9 - y} + 2)^2 - 2^2 = (9 - y) + 4\sqrt{9 - y} + 4 - 4
\]

\[
= 9 - y + 4\sqrt{9 - y}
\]

The region extends from \( y = 0 \) to \( y = 9 \) along the \( y \)-axis, so

\[
V = \pi \int_0^9 (R_{\text{outer}}^2 - R_{\text{inner}}^2) \, dy = \pi \int_0^9 (9 - y + 4\sqrt{9 - y}) \, dy
\]

\[
= \pi \left( 9y - \frac{1}{2}y^2 - \frac{8}{3}(9 - y)^{3/2} \right)_0^9 = \frac{225}{2} \pi
\]
SECTION 6.3 Volumes of Revolution: Disks and Washers

**CONCEPTUAL INSIGHT** A few different volume formulas were introduced in this section. Note that they all arise from the formula in the previous section for the volume as an integral of cross-sectional area $A(y)$:

$$ V = \int_a^b A(y) \, dy $$

To compute the volumes in this section, we developed the cross-sectional area formulas for the different situations: disks or washers, and horizontal or vertical rotation axes. While they might appear to be distinct cases, we do not need to consider disks and washers separately. A disk is simply a washer with $R_{\text{inner}} = 0$.

### 6.3 SUMMARY

- **Disk method:** When you rotate the region between two graphs about an axis, the segments perpendicular to the axis generate disks or washers. The volume $V$ of the solid of revolution is the integral of the areas of these disks or washers.
- Sketch the graphs to visualize the disks or washers.
- **Figure 10(A):** Region between $y = f(x)$ and the $x$-axis, rotated about the $x$-axis.
  - Vertical cross section: a circle of radius $R = f(x)$ and area $\pi R^2 = \pi f(x)^2$:
    $$ V = \pi \int_a^b f(x)^2 \, dx $$
  - **Figure 10(B):** Region between $y = f(x)$ and $y = g(x)$, with $f(x) \geq g(x)$, rotated about the $x$-axis.
    - Vertical cross section: a washer of outer radius $R_{\text{outer}} = f(x)$ and inner radius $R_{\text{inner}} = g(x)$:
      $$ V = \pi \int_a^b (f(x)^2 - g(x)^2) \, dx $$

**FIGURE 10**
6.3 EXERCISES

Preliminary Questions

1. Which of the following is a solid of revolution? (a) sphere (b) pyramid (c) cylinder (d) cube
2. True or false? When the region under a single graph is rotated about the x-axis, the cross sections of the solid perpendicular to the x-axis are circular disks.
3. True or false? When the region between two graphs is rotated about the x-axis, the cross sections of the solid perpendicular to the x-axis are circular disks.

Exercises

In Exercises 1-4, (a) sketch the solid obtained by revolving the region under the graph of f about the x-axis over the given interval, (b) describe the cross section perpendicular to the x-axis located at x, and (c) calculate the volume of the solid.

1. \( f(x) = x + 1 \), \([0, 3]\)  
2. \( f(x) = x^2 \), \([1, 3]\)  
3. \( f(x) = \sqrt{x + 1} \), \([1, 4]\)  
4. \( f(x) = x^{-1} \), \([1, 4]\)

In Exercises 5-12, find the volume of revolution about the x-axis for the given function and interval.

5. \( f(x) = 3x - x^2 \), \([0, 3]\)  
6. \( f(x) = \frac{1}{x^2} \), \([1, 4]\)  
7. \( f(x) = x^{5/3} \), \([1, 8]\)  
8. \( f(x) = 4 - x^2 \), \([0, 2]\)  
9. \( f(x) = \frac{2}{x + 1} \), \([1, 3]\)  
10. \( f(x) = \sqrt{x^2 + 1} \), \([1, 3]\)  
11. \( f(x) = \sqrt{3} \cos x \), \([0, \frac{\pi}{2}]\)  
12. \( f(x) = \sqrt{\cos x \sin x} \), \([0, \frac{\pi}{2}]\)

In Exercises 13 and 14, R is the shaded region in Figure 11.

13. Which of the integrals (i)-(iv) is used to compute the volume obtained by rotating region R about y = -2?
   (i) \( \int (f(x)^2 + 2^2) \) 
   (ii) \( \int (f(x) + 2)^2 \) 
   (iii) \( \int (f(x)^2 - 2^2) \) 
   (iv) \( \int (f(x) - 2)^2 \)

14. Which of the integrals (i)-(iv) is used to compute the volume obtained by rotating R about y = 9 in Figure 11?
   (i) \( \int (9 + f(x))^2 \) 
   (ii) \( \int (9 + g(x))^2 \) 
   (iii) \( \int (9 - f(x))^2 \) 
   (iv) \( \int (9 - g(x))^2 \)

In Exercises 15-20, (a) sketch the region enclosed by the curves, (b) describe the cross section perpendicular to the x-axis located at x, and (c) find the volume of the solid obtained by rotating the region about the x-axis.

15. \( y = x^2 + 2 \), \( y = 10 - x^2 \)  
16. \( y = x^2 \), \( y = 2x + 3 \)  
17. \( y = 16 - x \), \( y = 3x + 12 \), \( x = -1 \)  
18. \( y = \frac{1}{x} \), \( y = \frac{5}{2} - x \)
19. \( y = \sec x, \quad y = 0, \quad x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4} \)
20. \( y = \sec x, \quad y = 0, \quad x = 0, \quad x = \frac{\pi}{4} \)

In Exercises 21–24, find the volume of the solid obtained by rotating the region enclosed by the graphs about the y-axis over the given interval.

21. \( x = \sqrt{y}, \quad x = 0; \quad 1 \leq y \leq 4 \)
22. \( x = \sqrt{\sin y}, \quad x = 0; \quad 0 \leq y \leq \pi \)
23. \( x = y^2, \quad x = \sqrt{y} \)
24. \( x = 4 - y, \quad x = 16 - y^2 \)

25. Rotation of the region in Figure 12 about the y-axis produces a solid with two types of different cross sections. Compute the volume as a sum of two integrals, one for \(-12 \leq y \leq 4\) and one for \(4 \leq y \leq 12\).

\[ y = 12 - 4x \]
\[ y = 8x - 12 \]
\[ y = 12 \]
\[ y = 4 \]

**Figure 12**

26. Let \( R \) be the region enclosed by \( y = x^2 + 2, \ y = (x - 2)^2 \) and the axes \( x = 0 \) and \( y = 0 \). Compute the volume \( V \) obtained by rotating \( R \) about the x-axis. **Hint:** Express \( V \) as a sum of two integrals.

In Exercises 27–32, find the volume of the solid obtained by rotating region \( A \) in Figure 13 about the given axis.

27. x-axis
28. \( y = -2 \)
29. \( y = 2 \)
30. y-axis
31. \( x = -3 \)
32. \( x = 2 \)

**Figure 13**

In Exercises 33–38, find the volume of the solid obtained by rotating region \( B \) in Figure 13 about the given axis.

33. x-axis
34. \( y = -2 \)
35. \( y = 6 \)
36. y-axis

**Hint for Exercise 36:** Express the volume as a sum of two integrals along the y-axis or use Exercise 30.

37. \( x = 2 \)
38. \( x = -3 \)

In Exercises 39–52, find the volume of the solid obtained by rotating the region enclosed by the graphs about the given axis.

39. \( y = x^2, \ y = 12 - x, \ x = 0; \quad \text{about } y = -2 \ x \geq 0 \)
40. \( y = x^2, \ y = 12 - x, \ x = 0; \quad \text{about } y = 15 \)
41. \( y = 16 - 2x, \ y = 6, \ x = 0; \quad \text{about } x = 0 \)
42. \( y = 32 - 2x, \ y = 2 + 4x, \ x = 0; \quad \text{about } y = 0 \)
43. \( y = \sec x, \ y = 1 + \frac{3}{\pi}x, \ x = 0; \quad \text{about } y = x \)
44. \( x = 2, \ x = 3, \ y = 16 - x^2, \ y = 0; \quad \text{about } y = 0 \)
45. \( y = 2\sqrt{x}, \ y = x, \ x = 0; \quad \text{about } y = -2 \)
46. \( y = 2\sqrt{x}, \ y = x, \ x = 0; \quad \text{about } y = 4 \)
47. \( y = x^3, \ y = x^{1/2}, \ x \geq 0; \quad \text{about } y = 0 \)
48. \( y = x^2, \ y = x^{1/2}, \ x = 0; \quad \text{about } y = -2 \)
49. \( y = \frac{9}{x^2}, \ y = 10 - x^2, \ x \geq 0; \quad \text{about } y = 12 \)
50. \( y = \frac{9}{x^2}, \ y = 10 - x^2, \ x \geq 0; \quad \text{about } x = -1 \)
51. \( y = \frac{1}{x}, \ y = 0; \quad \text{about } y = -2 \)
52. \( y^2 = 4x, \ y = x, \ x = 0; \quad \text{about } y = 8 \)

53. The bowl in Figure 14(A) is 21 cm high, obtained by rotating the curve in Figure 14(B) as indicated. Estimate the volume capacity of the bowl shown by taking the average of right- and left-endpoint approximations to the integral with \( n = 7 \). The inner radii (in centimeters) starting from the top are 0, 4, 7, 8, 10, 13, 14, 20.

**Figure 14**

54. The region between the graphs of \( f \) and \( g \) over \([0, 1]\) is revolved about the line \( y = -3 \). Use the midpoint approximation with values from the following table to estimate the volume \( V \) of the resulting solid:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0.1 )</th>
<th>( 0.3 )</th>
<th>( 0.5 )</th>
<th>( 0.7 )</th>
<th>( 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>( g(x) )</td>
<td>2</td>
<td>3.5</td>
<td>4</td>
<td>3.5</td>
<td>2</td>
</tr>
</tbody>
</table>

55. With the following barrel circumference measurements, estimate the volume of the barrel in gallons:

<table>
<thead>
<tr>
<th>Dist from Bottom (in.)</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>21</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumference (in.)</td>
<td>30</td>
<td>36</td>
<td>38</td>
<td>40</td>
<td>41</td>
<td>39</td>
<td>38</td>
<td>35</td>
<td>28</td>
</tr>
</tbody>
</table>
56. With the following barrel circumference measurements, estimate the volume of the barrel in gallons.

<table>
<thead>
<tr>
<th>Dist from Bottom (in.)</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
<th>28</th>
<th>32</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumference (in.)</td>
<td>62</td>
<td>70</td>
<td>75</td>
<td>79</td>
<td>83</td>
<td>83</td>
<td>77</td>
<td>74</td>
<td>68</td>
<td>62</td>
</tr>
</tbody>
</table>

57. Find the volume of the cone obtained by rotating the region under the segment joining (0, h) and (r, 0) about the y-axis.

58. The torus (doughnut-shaped solid) in Figure 15 is obtained by rotating the circle \((x - a)^2 + y^2 = b^2\) around the y-axis (assume that \(a > b\)). Show that it has volume \(2\pi^2ab^2\). Hint: After simplifying it, evaluate the integral by interpreting it as the area of a circle.

![Figure 15 Torus obtained by rotating a circle about the y-axis.](image)

59. Sketch the hypocycloid \(x^{2/3} + y^{2/3} = 1\) and find the volume of the solid obtained by revolving it about the x-axis.

60. The solid generated by rotating the region between the branches of the hyperbola \(y^2 - x^2 = 1\) about the x-axis is called a hyperboloid (Figure 16). Find the volume of the hyperboloid for \(-a \leq x \leq a\).

![Figure 16 The hyperbola with equation \(y^2 - x^2 = 1\).](image)

61. A "bead" is formed by removing a cylinder of radius \(r\) from the center of a sphere of radius \(R\) (Figure 17). Find the volume of the bead with \(r = 1\) and \(R = 2\).

![Figure 17 A bead is a sphere with a cylinder removed.](image)

### Further Insights and Challenges

62. Find the volume \(V\) of the bead (Figure 17) in terms of \(r\) and \(R\). Then show that \(V = \frac{2}{3}R^3\), where \(h\) is the height of the bead. This formula has a surprising consequence: Since \(V\) can be expressed in terms of \(h\) alone, it follows that two beads of height 1 cm, one formed from a sphere the size of an orange and the other from a sphere the size of the earth, would have the same volume! Can you explain intuitively how this is possible?

63. The solid generated by rotating the region inside the ellipse with equation \((\frac{x}{a})^2 + (\frac{y}{b})^2 = 1\) around the x-axis is called an ellipsoid. Show that the ellipsoid has volume \(\frac{4}{3}\pi ab^2\). What is the volume if the ellipse is rotated around the y-axis?

64. The curve \(y = f(x)\) in Figure 18, called a tractrix, has the following property: The tangent line at each point \((x, y)\) on the curve has slope

\[
\frac{dy}{dx} = -\frac{y}{\sqrt{1-y^2}}
\]

Let \(R\) be the shaded region under the graph of \(y = f(x)\) for \(0 \leq x \leq a\) in Figure 18. Compute the volume \(V\) of the solid obtained by revolving \(R\) around the x-axis in terms of the constant \(e = f(a)\). Hint: Use the substitution \(u = f(x)\) to show that

\[
V = \pi \int_0^1 u \sqrt{1 - u^2} \, du
\]

65. Verify the formula

\[
\int_{x_1}^{x_2} (x - x_1)(x - x_2) \, dx = \frac{1}{6} (x_1 - x_2)^3
\]

Then prove that the solid obtained by rotating the shaded region in Figure 19 about the x-axis has volume \(V = \frac{2}{3}BH^2\), with \(B\) and \(H\) as in the figure. Hint: Let \(x_1\) and \(x_2\) be the roots of \(f(x) = ax + b - (mx + c)^2\), where \(x_1 < x_2\). Show that

\[
V = \pi \int_{x_1}^{x_2} f(x) \, dx
\]
and use Eq. (3).

![Figure 19](image)

**Figure 19** The line \( y = mx + c \) intersects the parabola \( y^2 = ax + b \) at two points above the \( x \)-axis.

66. Let \( R \) be the region in the unit circle lying above the cut with the line \( y = mx + b \) (Figure 20). Assume that the points where the line intersects the circle lie above the \( x \)-axis. Use the method of Exercise 65 to show that the solid obtained by rotating \( R \) about the \( x \)-axis has volume \( V = \frac{\pi}{6} hd^2 \), with \( h \) and \( d \) as in the figure.

![Figure 20](image)

**6.4 Volumes of Revolution: Cylindrical Shells**

In the previous two sections, we computed the volume of solids by slicing them into parallel planar cross sections and integrating the cross-sectional area. The **Shell Method**, based on dividing a solid into concentric cylindrical shells, is more convenient in some cases, depending on the geometry of the solid under consideration.

Consider a cylindrical shell (Figure 1) of height \( h \), with outer radius \( R \) and inner radius \( r \). Because the shell is obtained by removing a cylinder of radius \( r \) from the wider cylinder of radius \( R \), it has volume

\[
\pi R^2 h - \pi r^2 h = \pi h (R^2 - r^2) = \pi h (R + r)(R - r) = \pi h (R + r) \Delta r
\]

where \( \Delta r = R - r \) is the width of the shell. If the shell is very thin, then \( R \) and \( r \) are nearly equal and we may approximate \((R + r)\) with \(2R\) to obtain

\[
\text{volume of shell} \approx 2\pi Rh \Delta r = 2\pi (\text{radius}) \times (\text{height of shell}) \times (\text{thickness})
\]

This is the product of surface area of the outer cylinder with the thickness \( \Delta r \).

Now, let us rotate the region under \( y = f(x) \) from \( x = a \) to \( x = b \) about the \( y \)-axis as in Figure 2. The resulting solid can be divided into thin concentric shells. More precisely, we divide \([a, b]\) into \(N\) subintervals of length \( \Delta x = (b - a) / N \) with endpoints \( x_0, x_1, \ldots, x_N \). When we rotate the thin strip of area above \([x_{i-1}, x_i]\) about the \( y \)-axis, we obtain a thin shell whose volume we denote by \( V_i \). The volume of the solid is equal to the sum

\[
V = \sum_{i=1}^{N} V_i.
\]

![Figure 2](image)

**Figure 2** The shaded strip, when rotated about the \( y \)-axis, generates a "thin shell."

The top rim of the \( i \)th thin shell in Figure 2 is curved. However, when \( \Delta x \) is small, we can approximate this thin shell by the cylindrical shell (with flat rim) of height \( f(x_i) \) and radius \( x_i \). Then, using Eq. (1), we obtain

[Refer to the original document for the continuation of the explanation or exercises involving cylindrical shells.]
\[ V_i \approx 2\pi (\text{radius})(\text{height of shell})(\text{thickness}) = 2\pi x_i f(x_i) \Delta x \]

\[ V = \sum_{i=1}^{N} V_i \approx 2\pi \sum_{i=1}^{N} x_i f(x_i) \Delta x \]

The sum on the right is the volume of a cylindrical shell approximation that converges to \( V \) as \( N \to \infty \) (Figure 3). This sum is also a right-endpoint approximation that converges to \( 2\pi \int_{a}^{b} x f(x) \, dx \). Thus, we obtain Eq. (2) for the volume of the solid.

**Figure 3** Cylindrical shell approximations as \( N \to \infty \).

**Volume of Revolution: The Shell Method**  The solid obtained by rotating the region under \( y = f(x) \) over the interval \([a, b]\) about the \( y \)-axis has volume

\[ V = 2\pi \int_{a}^{b} (\text{radius})(\text{height of shell}) \, dx = 2\pi \int_{a}^{b} x f(x) \, dx \]

**EXAMPLE 1**  Find the volume \( V \) of the solid obtained by rotating the region under the graph of \( f(x) = 1 - 2x + 3x^2 - 2x^3 \) over \([0, 1]\) about the \( y \)-axis.

**Solution**  The solid is shown in Figure 4. By Eq. (2),

\[ V = 2\pi \int_{0}^{1} x f(x) \, dx = 2\pi \int_{0}^{1} x(1 - 2x + 3x^2 - 2x^3) \, dx \]

\[ = 2\pi \int_{0}^{1} (x - 2x^2 + 3x^3 - 2x^4) \, dx \]

\[ = 2\pi \left[ \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{2}{5}x^5 \right]_{0}^{1} = \frac{11}{30}\pi \]

**Figure 4** The graph of \( f(x) = 1 - 2x + 3x^2 - 2x^3 \) rotated about the \( y \)-axis.
CONCEPTUAL INSIGHT  Shells Versus Disks and Washers

- **Shell Method**: To calculate a volume, you must find the shell height, which is always parallel to the axis of rotation (Figure 5).
- **Disk and Washer Method**: To calculate a volume, you must find the disk radius or washer radii, which are always perpendicular to the axis of rotation.

Some volumes can be computed equally well using either the Shell Method or the Disk and Washer Method. In Example 1, however, the Shell Method is much easier because the shell height is \( f(x) \). Using the Disk Method would have been more challenging because we would need to find an expression for the radius of the disk perpendicular to the \( y \)-axis (Figure 5). This would require finding the inverse \( g(y) = f^{-1}(y) \), and that could be difficult or impossible.

In general: Use the Shell Method if finding the shell height is easier than finding the disk radius or washer radii. Use the Disk and Washer Method when finding the disk radius or washer radii is easier.

When we rotate the region between the graphs of two functions \( f \) and \( g \) satisfying \( f(x) \geq g(x) \), the vertical segment at location \( x \) generates a cylindrical shell of radius \( x \) and height \( f(x) - g(x) \) (Figure 6). Therefore, the volume is

\[
V = 2\pi \int_a^b (\text{radius})(\text{height of shell}) \, dx = 2\pi \int_a^b x(f(x) - g(x)) \, dx
\]

**EXAMPLE 2 Region Between Two Curves**  Find the volume \( V \) obtained by rotating the region enclosed by the graphs of \( f(x) = x(5 - x) \) and \( g(x) = 8 - x(5 - x) \) about the \( y \)-axis.

**Solution** First, find the points of intersection by solving \( x(5 - x) = 8 - x(5 - x) \). We obtain \( 0 = x^2 - 5x + 4 = (x - 1)(x - 4) \), so the curves intersect at \( x = 1, 4 \). Sketching the graphs (Figure 7), we see that \( f(x) \geq g(x) \) on the interval \([1, 4]\) and the height of shell \( f(x) - g(x) = x(5 - x) - (8 - x(5 - x)) = 10x - 2x^2 - 8 \)

\[
V = 2\pi \int_1^4 (\text{radius})(\text{height of shell}) \, dx = 2\pi \int_1^4 x(10x - 2x^2 - 8) \, dx = 2\pi \left( \frac{54}{3} - \frac{4}{2} \right) = 45\pi
\]

**EXAMPLE 3 Rotating About a Vertical Axis** Use the Shell Method to calculate the volume \( V \) obtained by rotating the region under the graph of \( f(x) = x^{-1/2} \) over \([1, 4]\) about the axis \( x = -3 \).

**Solution** If we were rotating this region about the \( y \)-axis (i.e., \( x = 0 \)), we would use Eq. (3). To rotate it around the line \( x = -3 \), we must take into account that the radius of revolution is now 3 units longer.
Figure 8 shows that the radius of the shell at \( x \) is now \( x - (-3) = x + 3 \). The height of the shell is still \( f(x) = x^{-1/2} \), so

\[
V = 2\pi \int_{1}^{4} (\text{radius})(\text{height of shell}) \, dx = 2\pi \int_{1}^{4} (x + 3)x^{-1/2} \, dx = 2\pi \left( \left. \frac{2}{3}x^{3/2} + 6x^{1/2} \right|_{1}^{4} \right) = \frac{64\pi}{3}
\]

![Figure 8](image)

The method of cylindrical shells can be applied to rotations about horizontal axes, but in this case, the graph must be described in the form \( x = g(y) \).

**EXAMPLE 4 Rotating About the x-Axis** Use the Shell Method to compute the volume \( V \) obtained by rotating the region in the first quadrant between \( y = 9 - x^2 \) and the x-axis about the x-axis (Figure 9).

Solution When we rotate about the x-axis, the cylindrical shells are generated by horizontal segments (\( AB \) in Figure 9) and the Shell Method gives us an integral with respect to \( y \). The radius of the shell is \( y \), the distance from the rotation axis to the segment. The length of \( AB \) is the height of the shell (we use the term "height" even though the shell is horizontal). Therefore, the length of \( AB \) is given by the positive value of \( x \) on the parabola associated with \( y \), that is, by \( x = \sqrt{9 - y} \). The volume is then obtained as follows, where we use a substitution \( u = 9 - y \), \( du = -dy \), in the integral computation:

\[
V = 2\pi \int_{0}^{9} (\text{radius})(\text{height of shell}) \, dy = 2\pi \int_{0}^{9} y\sqrt{9 - y} \, dy
\]

\[
= -2\pi \int_{0}^{9} (9 - u)^{1/2} \, du = 2\pi \int_{0}^{9} (9u^{1/2} - u^{3/2}) \, du
\]

\[
= 2\pi \left( \left. \frac{6u^{3/2}}{3} - \frac{2}{5}u^{5/2} \right|_{0}^{9} \right) = \frac{648\pi}{5}
\]

![Figure 9](image)
6.4 SUMMARY

- Shell Method: When you rotate the region between two graphs about an axis, the segments parallel to the axis generate cylindrical shells [Figure 10(A)]. The volume \( V \) of the solid of revolution is the integral of the surface areas of these shells:

\[
\text{surface area of shell} = 2\pi (\text{radius})(\text{height of shell})
\]

- Sketch the graphs to visualize the shells.

- Figure 10(B): Region between \( y = f(x) \) (with \( f(x) \geq 0 \)) and the \( y \)-axis, rotated about the \( y \)-axis:

\[
V = 2\pi \int_a^b (\text{radius})(\text{height of shell}) \, dx = 2\pi \int_a^b xf(x) \, dx
\]

- Figure 10(C): Region between \( y = f(x) \) and \( y = g(x) \) (with \( f(x) \geq g(x) \geq 0 \)), rotated about the \( y \)-axis:

\[
V = 2\pi \int_a^b (\text{radius})(\text{height of shell}) \, dx = 2\pi \int_a^b x(f(x) - g(x)) \, dx
\]

- Rotation about a vertical axis \( x = c \).
  - Figure 10(D): \( c \leq a \), radius of shell is \( (x - c) \):

\[
V = 2\pi \int_a^b (x - c)f(x) \, dx
\]

  - Figure 10(E): \( c \geq a \), radius of shell is \( (c - x) \):

\[
V = 2\pi \int_a^b (c - x)f(x) \, dx
\]

- Rotation about the \( x \)-axis using the Shell Method: Write the graph as \( x = g(y) \):

\[
V = 2\pi \int_c^d (\text{radius})(\text{height of shell}) \, dy = 2\pi \int_c^d yg(y) \, dy
\]
6.4 EXERCISES

Preliminary Questions
1. Consider the region under the graph of the constant function \( f(x) = h \) over the interval \([0, r]\). Give the height and the radius of the cylinder generated when the region is rotated about
(a) The x-axis
(b) The y-axis
2. Let \( V \) be the volume of a solid of revolution about the y-axis.
(a) Does the Shell Method for computing \( V \) lead to an integral with respect to \( x \) or \( y \)?
(b) Does the Disk or Washer Method for computing \( V \) lead to an integral with respect to \( x \) or \( y \)?
3. If we rotate the region under the curve \( y = 8 \) between \( x = 2 \) and \( x = 3 \) about the x-axis, what answer should the Shell Method give us?

Exercises

In Exercises 1–6, sketch the solid obtained by rotating the region underneath the graph of the function over the given interval about the y-axis, and find its volume.

1. \( f(x) = x^2, \quad [0, 1] \)
2. \( f(x) = \sqrt{x}, \quad [0, 4] \)
3. \( f(x) = x^{-1}, \quad [1, 3] \)
4. \( f(x) = 4 - x^2, \quad [0, 2] \)
5. \( f(x) = \sqrt{x^2 + 9}, \quad [0, 3] \)
6. \( f(x) = \frac{x}{\sqrt{1 + x^2}}, \quad [1, 4] \)

In Exercises 7–14, use the Shell Method to compute the volume obtained by rotating the region enclosed by the graphs as indicated, about the y-axis.

7. \( y = 3x - 2, \quad y = 6 - x, \quad x = 0 \)
8. \( y = \sqrt{x}, \quad y = x^2 \)
9. \( y = x^2, \quad y = 8 - x^2, \quad x = 0, \quad \text{for } x \geq 0 \)
10. \( y = 8 - x^2, \quad y = 8 - 4x, \quad \text{for } x \geq 0 \)
11. \( y = (x^2 + 1)^{-2}, \quad y = 2 - (x^2 + 1)^{-2}, \quad x = 2 \)
12. \( y = 1 - |x - 1|, \quad y = 0 \)
13. \( y = 2 - x^2, \quad y = x^2, \quad x = 0, \quad \text{for } x \geq 0 \)
14. \( y = \sqrt{x^2 + 9}, \quad y = 0, \quad x = 0, \quad x = 4 \)

In Exercises 15 and 16, use a graphing utility to find the points of intersection of the curves numerically, and then compute the volume of rotation of the enclosed region about the y-axis.

15. \( \textbf{GU} \quad y = \frac{1}{2}x^2, \quad y = \sin(x^2), \quad x \geq 0 \)
16. \( \textbf{GU} \quad y = \cos(x^2), \quad y = x, \quad x = 0 \)

In Exercises 17–22, sketch the solid obtained by rotating the region underneath the graph of \( f \) over the interval about the given axis, and calculate its volume using the Shell Method.

17. \( f(x) = x^2, \quad [0, 1], \quad \text{about } x = 2 \)
18. \( f(x) = x^3, \quad [0, 1], \quad \text{about } x = -2 \)
19. \( f(x) = x^{-1}, \quad [-3, -1], \quad \text{about } x = 4 \)
20. \( f(x) = \frac{1}{\sqrt{x^2 + 1}}, \quad [0, 2], \quad \text{about } x = 0 \)

21. \( f(x) = a - x \) with \( a > 0 \), \( [0, a] \), about \( x = -1 \)
22. \( f(x) = 1 - x^2, \quad [-1, 1], \quad x = c \) with \( c > 1 \)

In Exercises 23–28, sketch the enclosed region and use the Shell Method to calculate the volume of rotation about the x-axis.

23. \( x = y, \quad y = 0, \quad x = 1 \)
24. \( x = \frac{1}{4}y + 1, \quad x = 3 - \frac{1}{4}y, \quad y = 0 \)
25. \( x = y(4 - y), \quad x = 0 \)
26. \( x = y(4 - y), \quad x = (y - 2)^2 \)
27. \( y = 4 - x^2, \quad x = 0, \quad y = 0 \)
28. \( y = x^{1/3} - 2, \quad y = 0, \quad x = 27 \)

29. Determine which of the following is the appropriate integrand needed to determine the volume of the solid obtained by rotating around the vertical axis given by \( x = -1 \) the area that is between the curves \( y = f(x) \) and \( y = g(x) \) over the interval \([a, b]\), where \( a \geq 0 \) and \( f(x) \geq g(x) \) over that interval.
   (a) \( x(f(x) - g(x)) \)
   (b) \( (x + 1)(f(x) - g(x)) \)
   (c) \( (f(x) - 1) - (g(x) - 1) \)
   (d) \( (x - 1)(f(x) - g(x)) \)
   (e) \( f(x) - g(x) \)

30. Let \( y = f(x) \) be a decreasing function on \([0, b]\), such that \( f(b) = 0 \). Explain why \( 2\pi \int_0^b x f(x) \, dx = \pi \int_0^\infty (h(x))^2 \, dx \), where \( h \) denotes the inverse of \( f \).

31. Use both the Shell and Disk Methods to calculate the volume obtained by rotating the region under the graph of \( f(x) = 8 - x^3 \) for \( 0 \leq x \leq 2 \) about
   (a) The x-axis
   (b) The y-axis

32. Sketch the solid of rotation about the y-axis for the region under the graph of the constant function \( f(x) = c \) (where \( c > 0 \)) for \( 0 \leq x \leq r \).
   (a) Find the volume without using integration.
   (b) Use the Shell Method to compute the volume.
33. The graph in Figure 11(A) can be described by both \( y = f(x) \) and \( x = h(y) \), where \( h \) is the inverse of \( f \). Let \( V \) be the volume obtained by rotating the region under the graph about the \( y \)-axis.
(a) Describe the figures generated by rotating segments \( AB \) and \( CB \) about the \( y \)-axis.
(b) Set up integrals that compute \( V \) by the Shell and Disk Methods.

\[
\begin{align*}
\text{(A)} & & \quad \text{(B)} \\
\begin{array}{l}
1.3 \\
A \\
B \\
C \\
\end{array} & & \\
\begin{array}{l}
1.3 \\
A' \\
B' \\
C' \\
\end{array} \\
\begin{array}{c}
y = f(x) \\
x = h(y) \\
y = g(x) \\
x \\
\end{array}
\end{align*}
\]

**Figure 11**

34. Let \( W \) be the volume of the solid obtained by rotating the region under the graph in Figure 11(B) about the \( y \)-axis.
(a) Describe the figures generated by rotating segments \( A'B' \) and \( A'C' \) about the \( y \)-axis.
(b) Set up an integral that computes \( W \) by the Shell Method.
(c) Explain the difficulty in computing \( W \) by the Washer Method.

35. Let \( R \) be the region under the graph of \( y = 9 - x^2 \) for \( 0 \leq x \leq 2 \). Use the Shell Method to compute the volume of rotation of \( R \) about the \( x \)-axis as a sum of two integrals along the \( y \)-axis. **Hint:** The shells generated depend on whether \( x \in [0, 5] \) or \( x \in [5, 9] \).

36. Let \( R \) be the region under the graph of \( y = 4x^{-1} \) for \( 1 \leq y \leq 4 \). Use the Shell Method to compute the volume of rotation of \( R \) about the \( y \)-axis as a sum of two integrals along the \( x \)-axis.

In Exercises 37–42, use the Shell Method to find the volume obtained by rotating region \( A \) in Figure 12 about the given axis.

37. \( y \)-axis
38. \( x = -3 \)
39. \( x = 2 \)
40. \( x \)-axis
41. \( y = -2 \)
42. \( y = 6 \)

\[
\begin{align*}
\text{FIGURE 12} \\
\begin{array}{c}
y = x^2 + 2 \\
A \\
B \\
1 \\
2 \\
x \\
\end{array}
\end{align*}
\]

In Exercises 43–48, use the most convenient method (Disk or Shell Method) to find the volume obtained by rotating region \( B \) in Figure 12 about the given axis.

43. \( y \)-axis
44. \( x = -3 \)
45. \( x = 2 \)
46. \( x \)-axis
47. \( y = -2 \)
48. \( y = 8 \)

In Exercises 49–56, use the most convenient method (Disk or Shell Method) to find the given volume of rotation.

49. Region between \( x = y(5 - y) \) and \( x = 0 \), rotated about the \( y \)-axis
50. Region between \( x = y(5 - y) \) and \( x = 0 \), rotated about the \( x \)-axis
51. Region bounded by \( y = x^2 \) and \( x = y^2 \), rotated about the \( y \)-axis
52. Region bounded by \( y = x^2 \) and \( x = y^2 \), rotated about \( x = 3 \)
53. Region in Figure 13, rotated about the \( x \)-axis
54. Region in Figure 13, rotated about the \( y \)-axis

\[
\begin{align*}
\text{FIGURE 13} & & \quad \text{FIGURE 14} \\
\begin{array}{c}
y = x^2 + 2 \\
1 \\
2 \\
x \\
\end{array} & & \\
\begin{array}{c}
y = 4 - x^2 \\
1 \\
2 \\
x \\
\end{array}
\end{align*}
\]

55. Region in Figure 14, rotated about \( x = 4 \)
56. Region in Figure 14, rotated about \( y = -2 \)

In Exercises 57–60, use the Shell Method to find the given volume of rotation.

57. A sphere of radius \( r \)
58. The "bead" formed by removing a cylinder of radius \( r \) from the center of a sphere of radius \( R \) (compare with Exercise 61 in Section 6.3)
59. The torus obtained by rotating the circle \( (x - a)^2 + y^2 = b^2 \) about the \( y \)-axis, where \( a > b \) (compare with Exercise 58 in Section 6.3). **Hint:** Evaluate the integral by interpreting part of it as the area of a circle.
60. The "paraboloid" obtained by rotating the region between \( y = x^2 \) and \( y = c \) \( (c > 0) \) about the \( y \)-axis
61. Given \( a \) and \( b \), \( 0 \leq a \leq b \), find a function \( f \) such that the volume obtained by rotating about the \( x \)-axis the region \( R \) under the graph of \( y = f(x) \) over the interval \([a, b] \) equals the volume obtained by rotating that same region \( R \) about the \( y \)-axis.

(b) Approximate \( V \) by decomposing the sphere of radius \( R \) into \( N \) thin spherical shells of thickness \( \Delta r = R/N \).
(c) Show that the approximation is a Riemann sum that converges to an integral. Evaluate the integral.

**Further Insights and Challenges**

62. The surface area of a sphere of radius \( r \) is \( 4\pi r^2 \). Use this to derive the formula for the volume \( V \) of a sphere of radius \( R \) in a new way.
(a) Show that the volume of a thin spherical shell of inner radius \( r \) and thickness \( \Delta r \) is approximately \( 4\pi r^2 \Delta r \).
(b) Approximate \( V \) by decomposing the sphere of radius \( R \) into \( N \) thin spherical shells of thickness \( \Delta r = R/N \).
(c) Show that the approximation is a Riemann sum that converges to an integral. Evaluate the integral.
63. Show that the solid (an ellipsoid) obtained by rotating the region \( R \) in Figure 15 about the \( y \)-axis has volume \( \frac{4}{3} \pi a^2 b \).

![Figure 15](image)

For those who want some proof that physicists are human, the proof is in the idiocy of all the different units which they use for measuring energy.

—Richard Feynman, The Character of Physical Law

64. The bell-shaped curve \( y = f(x) \) in Figure 16 satisfies \( dy/dx = -xy \). Use the Shell Method and the substitution \( u = f(x) \) to show that the solid obtained by rotating the region \( R \) about the \( y \)-axis has volume \( V = 2 \pi (1 - c) \), where \( c = f(a) \). Observe that as \( c \to 0 \), the region \( R \) becomes infinite but the volume \( V \) approaches \( 2 \pi \).

![Figure 16](image)

6.5 Work and Energy

All physical tasks, from running up a hill to turning on a computer, require an expenditure of energy. When a force is applied to an object to move it, the energy expended is called work. When a constant force \( F \) is applied to move the object a distance \( d \) in the direction of the force, the work \( W \) is defined as "force times distance" (Figure 1):

\[
W = F \cdot d
\]

The International System (SI) unit of force is the newton (abbreviated N), defined as 1 kg·m/s\(^2\). Energy and work are both measured in units of the joule (J), equal to 1 N·m. In the British system, the unit of force is the pound, and both energy and work are measured in foot-pounds. Another unit of energy is the calorie. One ft-lb is approximately 1.356 J or 0.324 calories.

To become familiar with the units, let’s calculate the work \( W \) required to lift a 2-kg stone 3 m above the ground. Gravity acts on the stone of mass \( m \) with a force equal to \(-mg\), where \( g = 9.8\) m/s\(^2\). Therefore, lifting the stone requires an upward vertical force \( F = mg \), and the work expended is

\[
W = (mg)h = (2\text{ kg})(9.8\text{ m/s}^2)(3\text{ m}) = 58.8\text{ J}
\]

The kilogram is a unit of mass in the SI system. In the British system, we typically work with weight, which is a force rather than a mass. Consequently, the factor \( g \) does not appear when work against gravity is computed in the British system because, essentially, this factor is incorporated in the weight. For example, the work required to lift a 2-lb stone 3 ft is

\[
W = (2\text{ lb})(3\text{ ft}) = 6\text{ ft-lb}
\]

We are interested in the case where the force \( F(x) \) varies as the object moves from \( a \) to \( b \) along the \( x \)-axis. Eq. (1) does not apply directly, but we can break up the task into a large number of smaller tasks for which Eq. (1) gives a good approximation. Divide \([a, b]\) into \( N \) subintervals of length \( \Delta x = (b - a)/N \) as in Figure 2 and let \( W_i \) be the work required to move the object from \( x_{i-1} \) to \( x_i \). If \( \Delta x \) is small, then the force \( F(x) \) is nearly constant on the interval \([x_{i-1}, x_i]\) with value \( F(x_i) \), so \( W_i \approx F(x_i) \Delta x \). Summing the contributions, we obtain

\[
W = \sum_{i=1}^{N} W_i \approx \sum_{i=1}^{N} F(x_i) \Delta x
\]

Right-endpoint approximation.
The sum on the right is a right-endpoint approximation that converges to \( \int_{a}^{b} F(x) \, dx \). This leads to the following definition.

**Definition** Work The work performed in moving an object along the x-axis from \( a \) to \( b \) by applying a force \( F(x) \) is

\[
W = \int_{a}^{b} F(x) \, dx
\]

One typical calculation involves finding the work required to stretch or compress a spring. Assume that the free end of the spring has position \( x = 0 \) at equilibrium, when no force is acting (Figure 3). According to Hooke's Law, when the spring is stretched or compressed to position \( x \), it exerts a restoring spring force \( F(x) = -kx \), where \( k > 0 \) is the spring constant.

If we want to stretch the spring from \( x = a \) to \( x = b \), with \( 0 < a < b \), we must apply a force \( F(x) = kx \) to counteract the force exerted by the spring. The work required to stretch the spring is \( \int_{a}^{b} kx \, dx \). Similarly, if we wish to compress the spring from \( x = a \) to \( x = b \) with \( b < a < 0 \), the work required is also \( \int_{a}^{b} kx \, dx \). In this latter case the integration is in the negative direction, but the applied force is also in the negative direction, so the result is a positive value of work.

**Example 1** Hooke's Law Assuming a spring constant of \( k = 400 \, \text{N/m} \), find the work required to

(a) Stretch the spring 10 cm beyond equilibrium

(b) Compress the spring 2 cm more when it is already compressed 3 cm

**Solution** A force \( F(x) = 400x \) N is required to stretch the spring (with \( x \) in meters). Note that centimeters must be converted to meters.

(a) The work required to stretch the spring 10 cm (0.1 m) beyond equilibrium is

\[
W = \int_{0}^{0.1} 400x \, dx = 200x^2 \bigg|_{0}^{0.1} = 2 \, \text{J}
\]

(b) If the spring is at position \( x = -3 \) cm, then the work \( W \) required to compress it further to \( x = -5 \) cm is

\[
W = \int_{-0.03}^{-0.05} 400x \, dx = 200x^2 \bigg|_{-0.03}^{-0.05} = 0.5 - 0.18 = 0.32 \, \text{J}
\]

In the next two examples, we are not moving a single object through a fixed distance, so we cannot apply Eq. (2). Rather, each thin layer of the object is moved through a different distance. The work performed is computed by "summing" (i.e., integrating) the work performed on the thin layers.

**Example 2** Building a Concrete Column Compute the work (against gravity) required to build a concrete column of height 5 m and square base of side 2 m. Assume that concrete has density 1500 kg/m³.

**Solution** Think of the column as a stack of \( n \) thin layers of width \( \Delta y = 5/n \). The work consists of lifting up these layers and placing them on the stack (Figure 4), but the work performed on a given layer depends on how high we lift it. First, let us compute the gravitational force on a thin layer of width \( \Delta y \):
On the earth's surface, work against gravity is equal to the force \( mg \) times the vertical distance through which the object is lifted. No work against gravity is done when an object is moved sideways.

**Figure 4** Total work is the sum of the work performed on each layer of the column.

\[
\text{volume of layer} = \text{area} \times \text{width} = 4 \Delta y \text{ m}^3
\]

\[
\text{mass of layer} = \text{density} \times \text{volume} = 1500 \cdot 4 \Delta y \text{ kg}
\]

\[
\text{force on layer} = g \times \text{mass} = 9.8 \cdot 1500 \cdot 4 \Delta y = 58,800 \Delta y \text{ N}
\]

The work performed in lifting this layer to height \( y \) is equal to the force times the distance \( y \), which is \((58,800 \Delta y) y\). Setting \( L(y) = 58,800 y \), we have

\[
\text{Work lifting layer to height } y \approx (58,800 \Delta y) y = L(y) \Delta y
\]

This is only an approximation (although a very good one if \( \Delta y \) is small) because the layer has nonzero width and we are not taking into account that, for example, the cement particles at the top are lifted \( \Delta y \) prior to lifting the whole layer to height \( y \). The \( i \)th layer is lifted to height \( y_i \), so the total work performed is

\[
W \approx \sum_{i=1}^{n} L(y_i) \Delta y
\]

This sum is a right-endpoint approximation to \( \int_{0}^{5} L(y) \, dy \). Letting \( n \rightarrow \infty \), we obtain

\[
W = \int_{0}^{5} L(y) \, dy = \int_{0}^{5} 58,800 y \, dy = \frac{58,800 \cdot 5^2 \cdot 1}{2} = 735,000 \text{ J}
\]

**EXAMPLE 3** Pumping Water out of a Tank

A spherical tank of radius 5 m is filled with water. Calculate the work \( W \) performed (against gravity) in pumping out the water through a spout of height 1 m at the top. The density of water is 1000 kg/m\(^3\).

**Solution** The first step, as in the previous example, is to compute the work against gravity performed on a thin layer of water of width \( \Delta y \). We place the origin of our coordinate system at the center of the sphere because this leads to a simple formula for the radius \( r \) of the cross section at height \( y \) (Figure 5).

**Step 1. Compute work performed on a layer.**

Figure 5 shows that the cross section at height \( y \) is a circle of radius \( r = \sqrt{25 - y^2} \) and area \( A(y) = \pi r^2 = \pi (25 - y^2) \). A thin layer has volume \( A(y) \Delta y \) and mass obtained by multiplying this volume by the density 1000 kg/m\(^3\). To lift this layer, we must exert a force against gravity equal to

\[
\text{force on layer} = g \times \text{density} \times A(y) \Delta y \approx (9.8)1000\pi(25 - y^2) \Delta y
\]

\[
\text{Mass}
\]

Water exits from spout at the top.

Radius at height \( y \) is \( r = \sqrt{25 - y^2} \). This layer is pumped up a vertical distance \( 6 - y \).
The layer has to be lifted a vertical distance \(6 - y\), so

\[
\text{Work on layer} \approx 9800\pi \left(25 - y^2\right) \Delta y \times (6 - y) = L(y) \Delta y
\]

where \(L(y) = 9800\pi (25 - y^2)(6 - y) = 9800\pi (150 - 25y - 6y^2 + y^3)\).

**Step 2. Compute total work.**

Now, divide the sphere into \(N\) layers and let \(y_i\) be the height of the \(i\)th layer. The work performed on \(i\)th layer is approximately \(L(y_i) \Delta y\), and therefore,

\[
W \approx \sum_{i=1}^{N} L(y_i) \Delta y
\]

This sum approaches the integral of \(L(y)\) as \(N \to \infty\) (i.e., \(\Delta y \to 0\)), so

\[
W = \int_{-5}^{5} L(y) \, dy = 9800\pi \int_{-5}^{5} (150 - 25y - 6y^2 + y^3) \, dy
\]

\[
= 9800\pi \left(150y - \frac{25}{2}y^2 - 2y^3 + \frac{1}{4}y^4\right) \Bigg|_{-5}^{5} = 9,800,000\pi \approx 3.1 \times 10^7 \text{ J}
\]

Note that the integral extends from \(-5\) to \(5\) because the \(y\)-coordinate along the sphere varies from \(-5\) to \(5\).

How much energy is \(3.1 \times 10^7\) joules? A liter of gasoline has an energy content of approximately \(3.4 \times 10^7\) joules. Hence, the work required to pump the water out of the spout is equal to the energy content of roughly 0.9 L of gasoline.

### 6.5 SUMMARY

- Work performed to move an object:
  
  Constant force: \( W = F \cdot d \), \quad Variable force: \( W = \int_{a}^{b} F(x) \, dx \)

- Hooke’s Law: A spring stretched or compressed to position \(x\) from equilibrium exerts a restoring force \(-kx\). An applied force \(F(x) = kx\) is required to stretch or compress the spring further.

  To stretch a spring from \(a\) to \(b\) with \(0 < a < b\) or to compress a spring from \(a\) to \(b\) with \(b < a < 0\), the work performed is \(W = \int_{a}^{b} kx \, dx\).

- To compute work against gravity by decomposing an object into \(N\) thin layers of thickness \(\Delta y\), express the work performed on a thin layer as \(L(y) \Delta y\), where

  \[ L(y) = g \times \text{density} \times A(y) \times (\text{vertical distance lifted}) \]

  The total work performed is \(W = \int_{a}^{b} L(y) \, dy\).

### 6.5 EXERCISES

**Preliminary Questions**

1. Why is integration needed to compute the work performed in stretching a spring?

2. Why is integration needed to compute the work performed in pumping water out of a tank but not to compute the work performed in lifting up the tank?

3. Which of the following represents the work required to stretch a spring (with spring constant \(k\)) a distance \(x\) beyond its equilibrium position: \(kx\), \(-kx\), \(\frac{1}{2}mk^2\), \(\frac{1}{2}kx^2\), or \(\frac{1}{2}mx^2\)?

4. What does it mean when the integral used to calculate work gives a negative answer?
Exercises

1. How much work is done raising a 4-kg mass to a height of 16 m above ground?

2. How much work is done raising a 4-lb mass to a height of 16 ft above ground?

In Exercises 3-6, compute the work (in joules) required to stretch or compress a spring as indicated, assuming a spring constant of $k = 800$ N/m.

3. Stretching from equilibrium to 12 cm past equilibrium

4. Compressing from equilibrium to 4 cm past equilibrium

5. Stretching from 5 to 15 cm past equilibrium

6. Compressing 4 cm more when it is already compressed 5 cm

In Exercises 7-10 we investigate nonlinear springs. A spring is linear if it obeys Hooke's Law, which indicates that the applied force to stretch the spring is $F(x) = kx$. For a linear spring, $F'$ is constant. If, instead, $F'$ is not constant, then the spring is called nonlinear. Furthermore, if $F'(x)$ increases as $x$ increases, then the spring is said to be progressive, and if $F'(x)$ decreases as $x$ increases, then the spring is said to be degreessive.

7. Of the two statements (a) and (b), which describes a progressive spring, and which describes a degreessive spring?

(a) To stretch the spring a fixed additional distance, a greater change in force is needed farther from equilibrium than closer to it.

(b) To stretch the spring a fixed additional distance, a greater change in force is needed closer to equilibrium than farther from it.

8. (a) Of the two applied force graphs in Figure 6, which describes a progressive spring, and which describes a degreessive spring?

(b) For each, approximate the work required to stretch the spring from 4 to 9 cm.

9. Let $F(x) = 20\sqrt{x}$ be the applied force function for a spring (with $F(x)$ in N and $x$ in cm). Indicate whether the spring is progressive or degreessive. Compute the work required to stretch the spring from 6 to 12 cm.

10. Let $F(x) = 0.8x + 2.4x^{2/3}$ be the applied force function for a spring (with $F(x)$ in N and $x$ in cm). Indicate whether the spring is progressive or degreessive. Compute the work required to stretch the spring from 6 to 12 cm.

In Exercises 11-14, use the method of Examples 2 and 3 to calculate the work against gravity required to build the structure out of a lightweight material of density 600 kg/m³.

11. Solid box of height 3 m and square base of side 2 m

12. Cylindrical column of height 4 m and radius 0.8 m

13. Right circular cone of height 4 m and base of radius 1.2 m

14. Hemisphere of radius 0.8 m

15. Built around 2560 BCE, the Great Pyramid of Giza in Egypt (Figure 7) is 146 m high and has a square base of side 230 m. Find the work (against gravity) required to build the pyramid if the density of the stone is estimated at 2000 kg/m³.

**FIGURE 7** The Great Pyramid in Giza, Egypt.

16. Calculate the work (against gravity) required to build a box of height 3 m and square base of side 2 m out of material of variable density, assuming that the density at height $y$ is in $f(y) = 1000 - 100y$ kg/m³.

In Exercises 17-22, calculate the work (in joules) required to pump all of the water out of a full tank. Distances are in meters, and the density of water is 1000 kg/m³.

17. Rectangular tank in Figure 8; water exits from a small hole at the top

18. Rectangular tank in Figure 8; water exits through the spout

19. Hemisphere in Figure 9; water exits through the spout

20. Conical tank in Figure 10; water exits through the spout

**FIGURE 8**

**FIGURE 9**

**FIGURE 10**
21. Horizontal cylinder in Figure 11; water exits from a small hole at the top. *Hint:* Evaluate the integral by interpreting part of it as the area of a circle.

![Figure 11](image)

22. Trough in Figure 12; water exits by pouring over the sides.

![Figure 12](image)

23. Find the work \( W \) required to empty the tank in Figure 8 through the hole at the top if the tank is half full of water.

24. Assume the tank in Figure 8 is full of water and let \( W \) be the work required to pump out half of the water through the hole at the top. Do you expect \( W \) to equal the work computed in Exercise 23? Explain and then compute \( W \).

25. Assume the tank in Figure 10 is full. Find the work required to pump out half of the water. *Hint:* First, determine the level \( H \) at which the water remaining in the tank is equal to one-half the total capacity of the tank.

26. Assume that the tank in Figure 10 is full.
   (a) Calculate the work \( F(y) \) required to pump out water until the water level has reached level \( y \).
   (b) CAS Plot \( F \).
   (c) What is the significance of \( F'(y) \) as a rate of change?
   (d) CAS If your goal is to pump out all of the water, at which water level \( y \) will half of the work be done?

27. Calculate the work required to lift a 10-m chain over the side of a building (Figure 13). Assume that the chain has a density of 8 kg/m. *Hint:* Break up the chain into \( N \) segments, estimate the work performed on a segment, and compute the limit as \( N \to \infty \) as an integral.

![Figure 13](image)

28. How much work is done lifting a 3-m chain over the side of a building if the chain has mass density 4 kg/m?

29. A 6-m chain has mass 18 kg. Find the work required to lift the chain over the side of a building.

30. A 10-m chain with mass density 4 kg/m is initially coiled on the ground. How much work is performed in lifting the chain so that it is fully extended (and one end touches the ground)?

31. How much work is done lifting a 12-m chain that has mass density 3 kg/m (initially coiled on the ground) so that its top end is 10 m above the ground?

32. A 500-kg wrecking ball hangs from a 12-m cable of density 15 kg/m attached to a crane. Calculate the work done if the crane lifts the ball from ground level to 12 m in the air by drawing in the cable.

33. Calculate the work required to lift a 3-m chain over the side of a building if the chain has a variable density of \( \rho(x) = x^2 - 3x + 10 \) kg/m for \( 0 \leq x \leq 3 \).

34. A 3-m chain with linear mass density \( \rho(x) = 2x(4 - x) \) kg/m lies on the ground. Calculate the work required to lift the chain from its front end so that its bottom is 2 m above ground.

Exercises 35–37: The gravitational force between two objects of mass \( m \) and \( M \), separated by a distance \( r \), has magnitude \( GmM/r^2 \), where \( G = 6.67 \times 10^{-11} \) m\(^3\)kg\(^{-1}\)s\(^{-2}\).

35. Show that if two objects of mass \( M \) and \( m \) are separated by a distance \( r_1 \), then the work required to increase the separation to a distance \( r_2 \) is equal to \( W = GmM(r_1^{-2} - r_2^{-2}) \).

36. Use the result of Exercise 35 to calculate the work required to place a 2000-kg satellite in an orbit 1200 km above the surface of the earth. Assume that the earth is a sphere of radius \( R_e = 6.37 \times 10^6 \) m and mass \( M_e = 5.98 \times 10^{24} \) kg. Treat the satellite as a point mass.

37. Use the result of Exercise 35 to compute the work required to move a 1500-kg satellite from an orbit 1000 to an orbit 1500 km above the surface of the earth.

38. The pressure \( P \) and volume \( V \) of the gas in a cylinder of length 0.8 m and radius 0.2 m, with a movable piston, are related by \( PV^{1.4} = k \), where \( k \) is a constant (Figure 14). When the piston is fully extended, the gas pressure is 2000 kilopascals (kPa; 1 kilopascal is 10\(^3\) newtons per square meter).
   (a) Calculate \( k \).
   (b) The force on the piston is \( PA \), where \( A \) is the piston's area. Calculate the force as a function of the length \( x \) of the column of gas.
   (c) Calculate the work required to compress the gas column from 0.8 to 0.5 m.

![Figure 14](image)
Further Insights and Challenges

39. Work-Energy Theorem An object of mass \( m \) moves from \( x_1 \) to \( x_2 \) during the time interval \([t_1, t_2]\) due to a force \( F(x) \) acting in the direction of motion. Let \( x(t), v(t), \) and \( a(t) \) be the position, velocity, and acceleration at time \( t \). The object's kinetic energy is \( KE = \frac{1}{2}mv^2 \).

(a) Use the Change of Variables Formula to show that the work performed is equal to
\[
W = \int_{x_1}^{x_2} F(x) \, dx = \int_{t_1}^{t_2} F(x(t))v(t) \, dt
\]

(b) Use Newton's Second Law, \( F(x(t)) = ma(t) \), to show that
\[
\frac{d}{dt} \left( \frac{1}{2}mv^2(t) \right) = F(x(t))v(t)
\]

(c) Use the FTC to prove the Work-Energy Theorem: The change in kinetic energy during the time interval \([t_1, t_2]\) is equal to the work performed.

40. A model train of mass 0.5 kg is placed at one end of a straight 3-m electric track. Assume that a force \( F(x) = (3x - x^2) \) N acts on the train at distance \( x \) along the track. Use the Work-Energy Theorem (Exercise 39) to determine the velocity of the train when it reaches the end of the track.

41. With what initial velocity \( v_0 \) must we fire a rocket so it attains a maximum height \( r \) above the earth? Hints: Use the results of Exercises 25 and 39. As the rocket reaches its maximum height, its KE decreases from \( \frac{1}{2}mu_0^2 \) to zero.

42. With what initial velocity must we fire a rocket so it attains a maximum height of \( r = 20 \) km above the surface of the earth?

43. Calculate escape velocity, the minimum initial velocity of an object to ensure that it will continue traveling into space and never fall back to earth (assuming that no force is applied after takeoff). Hints: Take the limit as \( r \to \infty \) in Exercise 41.

CHAPTER REVIEW EXERCISES

1. Compute the area of the region in Figure 1(A) enclosed by \( y = 2 - x^2 \) and \( y = -2 \).

2. Compute the area of the region in Figure 1(B) enclosed by \( y = 2 - x^2 \) and \( y = x \).

3. \( y = x^3 - 2x^2 + x \), \( y = x^2 - x \)

4. \( y = 2x + 2x \), \( y = x^2 - 1 \), \( h(x) = x^2 + x - 2 \)

5. \( x = 4y, y = 24 - 8y \), \( y = 0 \)

6. \( y = x^3 - 9 \), \( x = 15 - 2y \)

7. \( y = 4 - x^2 \), \( y = 3x \), \( x = 4 \)

8. \( y = \frac{1}{2}x^2 \), \( y = y\sqrt{1-y^2} \), \( 0 \leq y \leq 1 \)

9. \( y = \sin x \), \( y = \cos x \), \( 0 \leq x \leq \frac{\pi}{4} \)

10. \( f(x) = \sin x, g(x) = \sin 2x, \frac{\pi}{3} \leq x \leq \pi \)

11. \( y = \sec^2 \left( \frac{\pi x}{4} \right) \), \( y = \sec^2 \left( \frac{\pi x}{8} \right) \), \( 0 \leq x \leq 1 \)

12. \( y = \frac{x}{\sqrt{x^2 + 4}} \), \( y = \frac{x}{\sqrt{x^2 + 4}} \), \( -1 \leq x \leq 1 \)

13. (GU) Use a graphing utility to locate the points of intersection of \( y = x^2 \) and \( y = \cos x \), and find the area between the two curves (approximately).

14. Figure 2 shows a solid whose horizontal cross section at height \( y \) is a circle of radius \( (1 + y)^{-2} \) for \( 0 \leq y \leq H \). Find the volume of the solid.

15. The base of a solid is the unit circle \( x^2 + y^2 = 1 \), and its cross sections perpendicular to the \( x \)-axis are rectangles of height 4. Find its volume.

16. The base of a solid is the triangle bounded by the axes and the line \( 2x + 3y = 12 \), and its cross sections perpendicular to the \( y \)-axis have area \( A(y) = (y + 2) \). Find its volume.

17. Find the total mass of a rod of length 1.2 m with linear density \( \rho(x) = (1 + 2x + \frac{1}{3}x^3) \) kg/m.

18. Find the flow rate (in the correct units) through a pipe of diameter 6 cm if the velocity of fluid particles at a distance \( r \) from the center of the pipe is \( v(r) = (3 - r) \) cm/s.

In Exercises 19–24, find the average value of the function over the interval.

19. \( f(x) = x^3 - 2x + 2 \), \([-1, 2]\)

20. \( f(x) = |x| \), \([-4, 4]\)

21. \( f(x) = (x + 1)(x^2 + 2x + 1)^{1/5} \), \([0, 4]\)

22. \( f(x) = |x^2 - 1| \), \([0, 4]\)

23. \( f(x) = \sqrt{9 - x^2} \), \([0, 3]\) Hints: Use geometry to evaluate the integral.
24. $f(x) = x \lfloor x \rfloor$, $[0, 3]$, where $\lfloor x \rfloor$ is the greatest integer function

25. Find $\int_2^5 g(t) \, dt$ if the average value of $g$ on $[2, 5]$ is 9.

26. The average value of $R$ over $[0, x]$ is equal to $x$ for all $x$. Use the FTC to determine $R(x)$.

27. Use the Washer Method to find the volume obtained by rotating the region in Figure 3 about the $x$-axis.

![Figure 3](image)

28. Use the Shell Method to find the volume obtained by rotating the region in Figure 3 about the $x$-axis.

In Exercises 29–40, use any method to find the volume of the solid obtained by rotating the region enclosed by the curves about the given axis.

29. $y = x^2 + 2$, $y = x + 4$, $x$-axis

30. $y = x^2 + 6$, $y = 8x - 1$, $y$-axis

31. $x = y^2 - 3$, $x = 2y$, axis $y = 4$

32. $y = 2x$, $y = 0$, $x = 8$, axis $x = -3$

33. $y = x^2 - 1$, $y = 2x - 1$, axis $x = -2$

34. $y = x^2 - 1$, $y = -x - 1$, axis $y = 4$

35. $y = -x^2 + 4x - 3$, $y = 0$, axis $y = -1$

36. $y = -x^2 + 4x - 3$, $y = 0$, axis $x = 4$

37. $x = 4y - y^3$, $y = 0$, $y \geq 0$, $x$-axis

38. $y^2 = x^{-1}$, $x = 1$, $x = 3$, axis $y = -3$

39. $y = \cos(x^2)$, $y = 0$, $0 \leq x \leq \sqrt{\pi/2}$, $y$-axis

40. $y = \sec x$, $y = \csc x$, $y = 0$, $x = 0$, $x = \frac{\pi}{2}$, $x$-axis

In Exercises 41–44, find the volume obtained by rotating the region about the given axis. The regions refer to the graph of the hyperbola $y^2 - x^2 = 1$ in Figure 4.

41. The shaded region between the upper branch of the hyperbola and the $x$-axis for $-c \leq x \leq c$, about the $x$-axis

42. The region between the upper branch of the hyperbola and the $x$-axis for $0 \leq x \leq c$, about the $y$-axis

43. The region between the upper branch of the hyperbola and the line $y = x$ for $0 \leq x \leq c$, about the $x$-axis

44. The region between the upper branch of the hyperbola and $y = 2$, about the $y$-axis

![Figure 4](image)

45. Let $R$ be the intersection of the circles of radius 1 centered at $(1, 0)$ and $(0, 1)$. Express as an integral (but do not evaluate): (a) the area of $R$ and (b) the volume of revolution of $R$ about the $x$-axis.

46. Let $R$ be the intersection of the circles of radius 1 centered at $(0, 0)$ and $(0, 1)$. Express an integral that gives the volume of revolution of $R$ about the $x$-axis. (Do not evaluate the integral.)

47. Let $a > 0$. Show that the volume obtained when the region between $y = a\sqrt{x - ax^2}$ and the $x$-axis is rotated about the $x$-axis is independent of the constant $a$.

48. If 12 J of work are needed to stretch a spring 20 cm beyond equilibrium, how much work is required to compress it 6 cm beyond equilibrium?

49. A spring whose equilibrium length is 15 cm exerts a force of 50 N when it is stretched to 20 cm. Find the work required to stretch the spring from 22 to 24 cm.

50. If 18 ft-lb of work are needed to stretch a spring 1.5 ft beyond equilibrium, how far will the spring stretch if a 12-lb weight is attached to its end?

51. Let $W$ be the work (against the Sun’s gravitational force) required to transport an 80-kg person from Earth to Mars when the two planets are aligned with the Sun at their minimal distance of $3.57 \times 10^8$ km. Use Newton’s Universal Law of Gravity (see Exercises 35–37 in Section 6.5) to express $W$ as an integral and evaluate it. The Sun has mass $M_S = 1.99 \times 10^{30}$ kg, and the distance from the Sun to Earth is $149.6 \times 10^8$ km.

In Exercises 52 and 53, water is pumped into a spherical tank of radius 2 m from a source located 1 m below a hole at the bottom (Figure 5). The density of water is $1000 \text{ kg/m}^3$.

52. Calculate the work required to fill the tank.

53. Calculate the work $F(h)$ required to fill the tank to level $h$ meters in the sphere.

54. A tank of mass 20 kg containing 100 kg of water (density $1000 \text{ kg/m}^3$) is raised vertically at a constant speed of 100 m/minute for 1 min, during which time it leaks water at a rate of 40 kg/min. Calculate the total work performed in raising the container.

![Figure 5](image)
7 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

This chapter is focused on exponential and logarithmic functions and their applications. These functions are used to model a remarkably wide range of phenomena, such as radioactive decay, population growth, interest rates, atmospheric pressure, and the diffusion of molecules across a cell membrane. Calculus gives us insight into why these functions play an important role in so many different situations. A key, it turns out, is the simple relationship between the exponential function and its derivative.

7.1 The Derivative of $f(x) = b^x$ and the Number $e$

An exponential function is a function of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$. The number $b$ is called the base. Some examples are $f(x) = 2^x$, $g(x) = (1.4)^x$, and $h(x) = 10^x$. The case $b = 1$ is excluded because $f(x) = 1^x$ is a constant function. Calculators give good decimal approximations to values of exponential functions:

$$2^4 = 16, \quad 2^{-3} = 0.125, \quad (1.4)^{0.8} \approx 1.309, \quad 10^{1.6} \approx 39,810.717$$

Three properties of exponential functions should be singled out from the start (see Figure 1 for the case $b = 2$):

- Exponential functions are positive: $b^x > 0$ for all $x$.
- The range of $f(x) = b^x$ is the set of all positive real numbers.
- $f(x) = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$.

If $b > 1$, the exponential function $f(x) = b^x$ is not merely increasing but is, in a certain sense, rapidly increasing. Although the term "rapid increase" is perhaps subjective, the following precise statement is true: For all $n$, if $x$ is positive and large enough, then $f(x) = b^x$ increases more rapidly than the power function $g(x) = x^n$ (we will prove this in Section 7.5). For example, Figure 2 shows that $f(x) = 3^x$ eventually overtakes and

![Figure 1](image)

![Figure 2](image)
increases faster than the power functions \( g(x) = x^2 \), \( g(x) = x^4 \), and \( g(x) = x^5 \). Table 1 compares \( f(x) = 3^x \) and \( g(x) = x^3 \).

**CONCEPTUAL INSIGHT** The expressions "exponential growth" and "increases exponentially" (and related expressions) describe the growth in or modeled by an exponential function. A function whose graph increases more and more steeply does not necessarily increase exponentially. What distinguishes true exponential growth from other forms of growth is the precise way the function values increase. In exponential growth, an increase in \( x \) by 1 causes an increase in the function value by the same fixed percent, regardless of the value of \( x \) involved. In contrast, power functions do not exhibit such increase.

For example, Table 2 demonstrates how \( f(x) = 3(2^x) \) increases by 100% with each increase by 1 in \( x \), but \( g(x) = 4x^2 \) does not exhibit growth by a fixed percent.

**TABLE 1**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x^3 )</th>
<th>( 3^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>3125</td>
<td>243</td>
</tr>
<tr>
<td>10</td>
<td>100,000</td>
<td>59,049</td>
</tr>
<tr>
<td>15</td>
<td>759,375</td>
<td>14,348,907</td>
</tr>
<tr>
<td>25</td>
<td>9,768,625</td>
<td>847,588,609,443</td>
</tr>
</tbody>
</table>

**TABLE 2**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 3(2^x) )</th>
<th>% increase</th>
<th>( g(x) = 4x^2 )</th>
<th>% increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3(2^1) = 6</td>
<td></td>
<td>4(1^2) = 4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3(2^2) = 12</td>
<td>100</td>
<td>4(2^2) = 16</td>
<td>300</td>
</tr>
<tr>
<td>3</td>
<td>3(2^3) = 24</td>
<td>100</td>
<td>4(3^2) = 36</td>
<td>125</td>
</tr>
<tr>
<td>4</td>
<td>3(2^4) = 48</td>
<td>100</td>
<td>4(4^2) = 64</td>
<td>78</td>
</tr>
<tr>
<td>5</td>
<td>3(2^5) = 96</td>
<td>100</td>
<td>4(5^2) = 100</td>
<td>56</td>
</tr>
</tbody>
</table>

**The Derivative of \( f(x) = b^x \)**

Next, we investigate the derivative of \( f(x) = b^x \). The rules of differentiation that we have developed so far are of no help because \( f(x) = b^x \) is neither a product, quotient, nor composition of functions with known derivatives. We must go back to the limit definition of the derivative. The difference quotient (for \( h \neq 0 \)) is

\[
\frac{f(x + h) - f(x)}{h} = \frac{b^{x+h} - b^x}{h} = \frac{b^x b^h - b^x}{h} = \frac{b^x (b^h - 1)}{h}
\]

Now, take the limit as \( h \to 0 \). The factor \( b^x \) does not depend on \( h \), so it may be taken outside the limit:

\[
\frac{d}{dx} b^x = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x (b^h - 1)}{h} = b^x \lim_{h \to 0} \left( \frac{b^h - 1}{h} \right)
\]

This last limit does not depend on \( x \). We denote its value by \( m(b) \). What we have shown, then, is that the derivative of \( f(x) = b^x \) is proportional to \( b^x \):

\[
\frac{d}{dx} b^x = m(b) b^x, \text{ where } m(b) = \lim_{h \to 0} \left( \frac{b^h - 1}{h} \right)
\]

At this point, this formula is not very helpful because we do not know much about \( m(b) \). In Section 7.3, we will learn that \( m(b) \) is equal to \( \ln b \), the natural logarithm of \( b \). To proceed further, we investigate \( m(b) \) numerically for a few values of \( b \).

**EXAMPLE 1** Estimate \( m(b) \) numerically for \( b = 2, 2.5, 3, \) and 10.

**Solution** We create a table of values of difference quotients to estimate \( m(b) \):
At least for the values of $b$ considered, the results in Example 1 suggest that $m(b)$ is defined (i.e., the limit exists) and that $m(b)$ increases as $b$ increases. In fact, the following properties of $m(b)$ can be shown, but the proofs are somewhat technical and we do not present them:

- $m(b)$ is defined for all $b > 0$.
- $m$ is a continuous and increasing function.

Given that $m(2.5) \approx 0.92$ and $m(3) \approx 1.10$, accurate to two decimal places, the Intermediate Value Theorem implies that there exists a value $b$ between 2.5 and 3 such that $m(b) = 1$. Furthermore, since $m$ is increasing, there is only one such value. This value is the number denoted by $e$. It can be shown that $e \approx 2.718$.

**The Number $e$**

There is a unique positive real number $e$ with the property

$$\lim_{h \to 0} \left( \frac{e^h - 1}{h} \right) = 1$$

The value of $e$ is approximately 2.718.

Now, we define the (natural) exponential function to be the function $f(x) = e^x$. By Eq. (1) and the definition of $e$, it follows that

$$\frac{d}{dx} e^x = e^x$$

Thus, the exponential function $f(x) = e^x$ is equal to its own derivative!

**GRAPHICAL INSIGHT**

The graph of $f(x) = b^x$ passes through $(0, 1)$ for all $b > 0$ because $b^0 = 1$ (Figure 3). The number $m(b)$ is simply the slope of the tangent line at $x = 0$:

$$\frac{d}{dx} b^x \bigg|_{x=0} = m(b) \cdot b^0 = m(b)$$

These tangent lines become steeper as $b$ increases and $b = e$ is the unique value for which the tangent line has slope 1.

**EXAMPLE 2** Find an equation of the tangent line to the graph of $f(x) = 3e^x - 5x^2$ at $x = 2$. 

![Figure 3](image-url) The tangent lines to $y = b^x$ at $x = 0$ grow steeper as $b$ increases.
Solution We compute both $f'(2)$ and $f(2)$:

$$f(x) = \frac{d}{dx}(3e^x - 5x^2) = 3\frac{d}{dx}e^x - 5\frac{d}{dx}x^2 = 3e^x - 10x$$

$$f'(2) = 3e^2 - 10(2) \approx 2.17$$

$$f(2) = 3e^2 - 5(2^2) \approx 2.17$$

An equation of the tangent line is $y = f(2) + f'(2)(x - 2)$. Using these approximate values, we write the equation as (Figure 4)

$$y = 2.17 + 2.17(x - 2) \quad \text{or} \quad y = 2.17x - 2.17$$

EXAMPLE 3 Calculate $f'(0)$, where $f(x) = e^x \cos x$.

Solution Use the Product Rule:

$$f'(x) = e^x \cdot \cos x)' + \cos x \cdot (e^x)' = -e^x \sin x + \cos x \cdot e^x = e^x (\cos x - \sin x)$$

Then $f'(0) = e^0(1 - 0) = 1$.

To compute the derivative of a function of the form $h(x) = e^{g(x)}$, we recognize $h(x)$ as a composite function $h(x) = e^{g(x)} = f(g(x))$, where $f(u) = e^u$, and we apply the Chain Rule:

$$h'(x) = \frac{d}{dx}(e^{g(x)}) = [f(g(x))]' = f'(g(x))g'(x) = e^{g(x)}g'(x)$$

A special case is $(e^{kx+b})' = ke^{kx+b}$, where $k$ and $b$ are constants.

$$\frac{d}{dx}e^{kx+b} = g'(x)e^{kx+b}, \quad \frac{d}{dx}e^{kx+b} = ke^{kx+b} \quad (k, b \text{ constants})$$

EXAMPLE 4 Differentiate:

(a) $f(x) = e^{9x^2-5}$ and (b) $f(x) = e^{\cos x}$

Solution Apply Eq. (3):

(a) $\frac{d}{dx}e^{9x^2-5} = 18xe^{9x^2-5}$ and (b) $\frac{d}{dx}e^{\cos x} = -\sin x e^{\cos x}$

EXAMPLE 5 Graph Sketching Involving $e^x$ Sketch the graph of $f(x) = xe^x$ on the interval $[-4, 2]$.

Solution As usual, the first step is to determine the critical points. The derivative of $f$ is

$$f'(x) = \frac{d}{dx}xe^x = xe^x + e^x = (x + 1)e^x$$

We need to solve $(x + 1)e^x = 0$. Since $e^x > 0$ for all $x$, there is a single critical point at $x = -1$ and

$$f'(x) = \begin{cases} < 0 & \text{for } x < -1 \\ > 0 & \text{for } x > -1 \end{cases}$$

Thus, $f'(x)$ changes sign from $-$ to $+$ at $x = -1$ and $f(-1)$ is a local minimum. For the second derivative, we have

$$f''(x) = (x + 1) \cdot (e^x)' + e^x \cdot (x + 1)' = (x + 1)e^x + e^x = (x + 2)e^x$$

$$f''(x) = \begin{cases} < 0 & \text{for } x < -2 \\ > 0 & \text{for } x > -2 \end{cases}$$
Thus, \( x = -2 \) is a point of inflection, where the graph changes from concave down to concave up at \( x = -2 \). Figure 5 shows the graph with its local minimum and point of inflection.

### Integrals Involving \( e^x \)

The formula \((e^x)' = e^x\) says that the function \( f(x) = e^x \) is its own derivative. But this means \( f(x) = e^x \) is also its own antiderivative. In other words,

\[
\int e^x\,dx = e^x + C
\]

More generally, for any constants \( b \) and \( k \) with \( k \neq 0 \),

\[
\int e^{kx+b}\,dx = \frac{1}{k} e^{kx+b} + C
\]

We prove this formula by noting that \( \frac{d}{dx}\left(\frac{1}{k} e^{kx+b}\right) = e^{kx+b} \).

**Example 6** Evaluate:

(a) \( \int e^{7x-5}\,dx \)

(b) \( \int xe^{2x^2}\,dx \)

(c) \( \int \frac{e^t}{1 + 2e^t + e^{2t}}\,dt \)

**Solution**

(a) \( \int e^{7x-5}\,dx = \frac{1}{7} e^{7x-5} + C \).

(b) Use the substitution \( u = 2x^2, du = 4x\,dx \):

\[
\int xe^{2x^2}\,dx = \frac{1}{4} \int e^u\,du = \frac{1}{4} e^u + C = \frac{1}{4} e^{2x^2} + C
\]

(c) We have \( 1 + 2e^t + e^{2t} = (1 + e^t)^2 \). The substitution \( u = 1 + e^t, du = e^t\,dt \) gives

\[
\int \frac{e^t}{1 + 2e^t + e^{2t}}\,dt = \int \frac{du}{u^2} = -u^{-1} + C = -(1 + e^t)^{-1} + C
\]

### Other Approaches to Defining \( e \)

Mathematicians first became aware of the special role played by the number \( e \) in the seventeenth century. The notation \( e \) was introduced around 1730 by Leonhard Euler, who discovered many fundamental properties of this important number.

In this section, we defined \( e \) as the unique value such that

\[
\lim_{h \to 0} \left( \frac{e^h - 1}{h} \right) = 1
\]

There are a number of other ways to define \( e \), all of which either directly or indirectly involve a limit. Here are three other approaches to defining \( e \). We will investigate each later in the text.

* \( e \) is given by each of the following limits, where the second is obtained from the first by substituting \( 1/t \) for \( x \):

\[
e = \lim_{x \to 0} (1 + x)^{1/x} \quad \text{and} \quad e = \lim_{t \to \infty} \left( 1 + \frac{1}{t} \right)^t
\]
7.1 SUMMARY

- $f(x) = b^x$ is the exponential function with base $b$ (where $b > 0$ and $b \neq 1$).
- $f(x) = b^x$ is increasing if $b > 1$ and decreasing if $b < 1$.
- The derivative of $f(x) = b^x$ is proportional to $b^x$:
  \[
  \frac{d}{dx} b^x = m(b) b^x
  \]
  where $m(b) = \lim_{h \to 0} \frac{b^h - 1}{h}$.
- There is a unique number $e \approx 2.718$ with the property $m(e) = 1$, so that
  \[
  \frac{d}{dx} e^x = e^x
  \]
- By the Chain Rule:
  \[
  \frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)} \quad \text{and} \quad \frac{d}{dx} e^{bx+b} = ke^{bx+b} \quad (k, b \text{ constants})
  \]
  \[
  \int e^{bx+b} \, dx = \frac{1}{k} e^{bx+b} + C \quad (k, b \text{ constants with } k \neq 0).
  \]

7.1 EXERCISES

Preliminary Questions

1. To which of the following does the Power Rule apply?
   (a) $f(x) = x^2$  (b) $f(x) = 2^x$  (c) $f(x) = x^e$
   (d) $f(x) = e^x$  (e) $f(x) = x^4$  (f) $f(x) = x^{-4/5}$
2. For which values of $b$ does $f(x) = b^x$ have a negative derivative?
3. For which values of $b$ is the graph of $y = b^x$ concave up?
4. Which point lies on the graph of $y = b^x$ for all $b$?
5. Which of the following statements is not true?
   (a) $(e^x)' = e^x$
   (b) $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$
   (c) The tangent line to $y = e^x$ at $x = 0$ has slope $e$.
   (d) The tangent line to $y = e^x$ at $x = 0$ has slope 1.

Exercises

In Exercises 1–4, determine the limit.

1. $\lim_{x \to \infty} 4^x$
2. $\lim_{x \to \infty} \frac{1}{4^x}$
3. $\lim_{x \to \infty} \left(\frac{1}{4}\right)^x$
4. $\lim_{x \to \infty} e^{-x^2}$

In Exercises 5–8, find the equation of the tangent line at the point indicated.

5. $y = 4e^x$, $x_0 = 0$
6. $y = e^{4x}$, $x_0 = 0$
7. $y = e^{x^2}$, $x_0 = -1$
8. $y = e^{x^2}$, $x_0 = 1$

In Exercises 9–30, find the derivative.

9. $f(x) = 7e^{2x} + 3e^{4x}$
10. $f(x) = e^{-3x}$
11. $f(x) = e^{x^2}$
12. $f(x) = e^x$
13. $f(x) = e^{-4x+9}$
14. $f(x) = 4e^{-x} + 7e^{-2x}$
15. \( f(x) = \frac{e^{x^2}}{x} \)

16. \( f(x) = x^2e^{2x} \)

17. \( f(x) = (1 + e^x)^4 \)

18. \( f(x) = (2e^{3x} + 2e^{-2x})^4 \)

19. \( f(x) = e^{x^2+2x-3} \)

20. \( f(x) = e^{1/x} \)

21. \( f(x) = e^{lnx} \)

22. \( f(x) = e^{(x^2+2x+3)^2} \)

23. \( f(t) = \sin(e^{2t}) \)

24. \( f(t) = e^{t^2} \)

25. \( f(t) = \frac{1}{1 - e^{-3t}} \)

26. \( f(t) = \cos(te^{-2t}) \)

27. \( f(x) = \frac{e^x}{3x+1} \)

28. \( f(x) = \tan(e^{x^2-6x}) \)

29. \( f(x) = e^{x+1} + x \)

30. \( f(x) = e^{x^2} \)

In Exercises 31–36, calculate the derivative indicated.

31. \( f''(x); \quad f(x) = e^{x^2-3} \)

32. \( f''(x); \quad f(x) = e^{12-3x} \)

33. \( \frac{d^2y}{dt^2}; \quad y = e^t \sin t \)

34. \( \frac{d^2y}{dt^2}; \quad y = e^{-2t} \sin 3t \)

35. \( \frac{d^3}{dt^3} e^{-t^2} \)

36. \( \frac{d^3}{dt^3} \cos(e^t) \)

In Exercises 37–42, find the critical points and determine whether they are local minima, maxima, or neither.

37. \( f(x) = e^x - x \)

38. \( f(x) = x + e^{-x} \)

39. \( f(x) = \frac{e^x}{x} \) for \( x > 0 \)

40. \( f(x) = x^2e^x \)

41. \( g(t) = \frac{e^{t^3}}{t^2+1} \)

42. \( g(t) = (t^3 - 2t)e^t \)

In Exercises 43–48, find the critical points and points of inflection. Then sketch the graph.

43. \( y = xe^{-x} \)

44. \( y = e^{-x} + e^x \)

45. \( y = e^{-x} \cos x \) on \([ -\frac{\pi}{2}, \frac{\pi}{2}] \)

46. \( y = e^{-x^2} \)

47. \( y = e^{-x} - x \)

48. \( y = x^2e^{-x} \)

49. Find \( a > 0 \) such that the tangent line to the graph of \( f(x) = x^2e^{-x} \) at \( x = a \) passes through the origin (Figure 7).

50. Use Newton's Method to find the two solutions of \( e^t = 5x \) to three decimal places (Figure 8).

51. Compute the linearization of \( f(x) = e^{-2x} \sin x \) at \( a = 0 \).

52. Compute the linearization of \( f(x) = xe^{6-3x} \) at \( a = 2 \).

53. Find the linearization of \( f(x) = e^x \) at \( a = 0 \) and use it to estimate \( e^{0.1} \).

54. Use the linear approximation to estimate \( f(1.03) - f(1) \), where \( y = x^{1/3}e^{x^2} \).

55. A 2005 study by the Fisheries Research Services in Aberdeen, Scotland, showed that the average length of the species Clupea harengus (Atlantic herring) as a function of age \( t \) (in years) can be modeled by \( L(t) = 3(1 - e^{-0.37t}) \) cm for \( 0 \leq t \leq 13 \).

(a) How fast is the average length changing at age \( t = 6 \) years?

(b) At what age is the average length changing at a rate of 5 cm/year?

(c) Calculate \( L' \) at \( t = 2 \).

56. According to a 1999 study by Starkey and Scarneccia, the average weight (in kg) of age \( t \) (in years) of channel catfish in the Lower Yellowstone River can be modeled by

\[ W(t) = (3.46293 - 3.32173e^{-0.03456})^{3.4026} \]

Find the rate at which weight is changing at age \( t = 10 \).

57. The functions in Exercises 55 and 56 are examples of the von Bertalanffy growth function

\[ M(t) = (a + (b - a)e^{kt})^{1/m} \]

introduced in the 1930s by Austrian-born biologist Karl Ludwig von Bertalanffy. Calculate \( M'(0) \) in terms of the constants \( a, b, k, \) and \( m \).

58. Find an approximation to \( m(4) \) using the limit definition and estimate the slope of the tangent line to \( y = e^x \) at \( x = 0 \) and \( x = 2 \).

59. Find an approximation to \( m(1/2) \) using the limit definition and estimate the slope of the tangent line to \( y = (1/2)^x \) at \( x = 0 \) and \( x = 1 \).

60. Find approximations to \( m(2.71) \) and \( m(2.72) \) using the limit definition.

In Exercises 61–79, evaluate the integral.

61. \( \int (e^x + 2) \, dx \)

62. \( \int e^{4x} \, dx \)

63. \( \int_0^1 e^{-3s} \, ds \)

64. \( \int_0^6 e^{-4s} \, ds \)

65. \( \int_0^3 e^{1-4t} \, dt \)

66. \( \int_0^3 e^{1-t} \, dt \)

67. \( \int (e^{4x} + 1) \, dx \)

68. \( \int (e^x + e^{-x}) \, dx \)

69. \( \int_0^1 xe^{-x/2} \, dx \)

70. \( \int_0^1 xe^{1/2} \, dx \)

FIGURE 8: Graphs of \( y = e^x \) and \( y = 5x \).
71. \[ \int e^{\sqrt{x^2 + 1}} \, dt \]  
72. \[ \int (e^{-x} - 4x) \, dx \]

73. \[ \int \frac{e^{2x} - e^{3x}}{e^x} \, dx \]  
74. \[ \int e^x \cos(e^x) \, dx \]

75. \[ \int \frac{e^x}{e^{2x} + 1} \, dx \]  
76. \[ \int e^x(e^{2x} + 1)^3 \, dx \]

77. \[ \int \frac{x^{1/2}}{e^{2x}} \, dx \]  
78. \[ \int x^{-1/2} e^{x^{1/3}} \, dx \]

79. Find the area between \( y = e^x \) and \( y = e^{2x} \) over \([0, 1]\).

80. Find the area between \( y = e^x \) and \( y = e^{-x} \) over \([0, 2]\).

81. Find the area bounded by \( y = e^x, y = e^{-x} \), and \( x = 0 \).

82. Find the volume obtained by revolving \( y = e^x \) about the \( x \)-axis for \( 0 \leq x \leq 1 \).

83. Wind engineers have found that wind speed \( v \) (in m/s) at a given location follows a Rayleigh distribution of the type

\[ W(v) = \frac{1}{32} v^2 e^{-v^2/64} \]

This means that the probability that \( v \) lies between \( a \) and \( b \) is equal to the shaded area in Figure 9.

(a) Show that the probability that \( v \in [0, b] \) is \( 1 - e^{-b^2/64} \).

(b) Calculate the probability that \( v \in [2, 5] \).

**FIGURE 9** The shaded area is the probability that \( v \) lies between \( a \) and \( b \).

84. The function \( f(x) = e^x \) satisfies \( f'(x) = f(x) \). Show that if \( g \) is another function satisfying \( g'(x) = g(x) \), then \( g(x) = Ce^x \) for some constant \( C \). Hint: Compute the derivative of \( g(x)e^{-x} \).

---

**Further Insights and Challenges**

85. Prove that \( f(x) = e^x \) is not a polynomial function. Hint: Differentiation lowers the degree of a polynomial by 1.

86. Recall the following property of integrals: If \( f(t) \geq g(t) \) for all \( t \geq 0 \), then for all \( x \geq 0 \),

\[ \int_{0}^{x} f(t) \, dt \geq \int_{0}^{x} g(t) \, dt \]

The inequality \( e^t \geq 1 \) holds for \( t \geq 0 \) because \( e > 1 \). Use (4) to prove that

\[ e^x \geq 1 + x \quad \text{for} \quad x \geq 0 \]

Then prove, by successive integration, the following inequalities (for \( x \geq 0 \)):

\[ e^x \geq 1 + x + \frac{1}{2} x^2 \]

\[ e^x \geq 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 \]

87. Generalize Exercise 86; that is, use induction (if you are familiar with this method of proof) to prove that for all \( x \geq 0 \),

\[ e^x \geq 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots + \frac{1}{n!} x^n \quad (x \geq 0) \]

88. Use Exercise 86 to show that \( \lim_{x \to \infty} e^x / x^n = 0 \) and conclude that

\[ \lim_{x \to \infty} e^x / x^n = 0 \quad \text{for all} \quad n \geq 1 \]

89. Calculate the first three derivatives of \( f(x) = xe^x \). Then guess the formula for \( f^{(n)}(x) \) (use induction to prove it if you are familiar with this method of proof).

90. Consider the equation \( e^x = \lambda x \), where \( \lambda \) is a constant.

(a) For which \( \lambda \) does it have a unique solution? For intuition, draw a graph of \( y = e^x \) and the line \( y = \lambda x \).

(b) For which \( \lambda \) does it have at least one solution?

91. Prove in two ways that the numbers \( m(a) \) satisfy

\[ m(ab) = m(a) + m(b) \]

(a) First method: Use the limit definition of \( m(b) \) and

\[ \frac{(ab)^h - 1}{h} = b^h \left( \frac{a^h - 1}{h} \right) + \frac{b^h - 1}{h} \]

(b) Second method: Apply the Product Rule to \( a^h b^h = (ab)^h \).

---

**7.2 Inverse Functions**

In this section, we discuss the concept of inverse functions and we develop an important theorem for computing their derivatives. In the next section, we will define logarithmic functions as inverses of exponential functions.

The inverse of \( f \), denoted \( f^{-1} \), is the function that reverses the effect of \( f \) (Figure 1). For example, the inverse of \( f(x) = x^3 \) is the cube root function \( f^{-1}(x) = x^{1/3} \).

Given a table of function values for \( f \), we obtain a table for \( f^{-1} \) by interchanging the \( x \) and \( y \) columns, assuming the resulting \( f^{-1} \) is a function:
In general, \( f^{-1}(x) \neq \frac{1}{f(x)} \). The expression \( f^{-1}(x) \) is simply a notation for the inverse function, and the \(-1\) does not represent an exponent.

<table>
<thead>
<tr>
<th>Function</th>
<th>Inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( f(x) = x^3 )</td>
</tr>
<tr>
<td>(-2)</td>
<td>(-8)</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
</tbody>
</table>

(Interchange columns)

If we apply both \( f \) and \( f^{-1} \) to a number \( x \) in either order, we get back \( x \). For instance,

Apply \( f \) and then \( f^{-1} \): \( 2 \) \((\text{apply } x^3)\) \( \rightarrow \) \( 8 \) \((\text{apply } x^{1/3})\) \( \rightarrow \) \( 2 \)

Apply \( f^{-1} \) and then \( f \): \( 8 \) \((\text{apply } x^{1/3})\) \( \rightarrow \) \( 2 \) \((\text{apply } x^3)\) \( \rightarrow \) \( 8 \)

This property is used in the formal definition of the inverse function:

**DEFINITION Inverse** Let \( f \) have domain \( D \) and range \( R \). If there is a function \( g \) with domain \( R \) such that

\[
g(f(x)) = x \quad \text{for } x \in D \quad \text{and} \quad f(g(x)) = x \quad \text{for } x \in R
\]

then \( f \) is said to be invertible. The function \( g \) is called the inverse function and is denoted \( f^{-1} \).

**EXAMPLE 1** Show that \( f(x) = 2x - 18 \) is invertible. What are the domain and range of \( f^{-1} \)?

**Solution** We show that \( f \) is invertible by computing the inverse function in two steps.

**Step 1.** Solve the equation \( y = f(x) \) for \( x \) in terms of \( y \).

\[
y = 2x - 18 \quad \Rightarrow \quad y + 18 = 2x \quad \Rightarrow \quad x = \frac{1}{2}y + 9
\]

This gives us the inverse as a function of the variable \( y \): \( f^{-1}(y) = \frac{1}{2}y + 9 \).

**Step 2. Interchange variables.**

We usually prefer to write the inverse as a function of \( x \), so we interchange the roles of \( x \) and \( y \):

\[
f^{-1}(x) = \frac{1}{2}x + 9
\]

Graphs of \( f \) and \( f^{-1} \) are shown in Figure 2.

To check our calculation, let's verify that \( f^{-1}(f(x)) = x \) and \( f(f^{-1}(x)) = x \):

\[
f^{-1}(f(x)) = f^{-1}(2x - 18) = \frac{1}{2}(2x - 18) + 9 = (x - 9) + 9 = x
\]

\[
f(f^{-1}(x)) = f\left(\frac{1}{2}x + 9\right) = 2\left(\frac{1}{2}x + 9\right) - 18 = (x + 18) - 18 = x
\]

Because \( f^{-1} \) is a linear function, its domain and range are \( \mathbb{R} \).
The inverse function, if it exists, is unique. However, some functions do not have an inverse. Consider $f(x) = x^2$. When we interchange the columns in a table of values (which should give us a table of values for $f^{-1}$), the resulting table does not define a function:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^2$</th>
<th>$f^{-1}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$-1$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

(Interchange columns)

$f^{-1}(1)$ has two values: 1 and $-1$.

The problem is that every positive number occurs twice as an output of $f(x) = x^2$. For example, 1 occurs twice as an output in the first table and therefore occurs twice as an input in the second table. So the second table gives us two possible values for $f^{-1}(1)$, namely $f^{-1}(1) = 1$ and $f^{-1}(1) = -1$. Neither value satisfies the inverse property. For instance, if we set $f^{-1}(1) = 1$, then $f^{-1}(f(-1)) = f^{-1}(1) = 1$, but an inverse would have to satisfy $f^{-1}(f(-1)) = -1$.

So when does a function $f$ have an inverse? The answer is: if $f$ is one-to-one, which means that $f$ takes on each value at most once (Figure 3). Here is the formal definition:

**DEFINITION One-to-One Function**  A function $f$ is one-to-one on a domain $D$ if, for every value $c$, the equation $f(x) = c$ has at most one solution for $x \in D$. Or, equivalently, if for all $a, b \in D$, if $a \neq b$, then $f(a) \neq f(b)$.

**FIGURE 3** A one-to-one function takes on each value at most once.

Think of a function as a device for "labeling" members of the range by members of the domain. When $f$ is one-to-one, this labeling is unique and $f^{-1}$ maps each number in the range back to its label.

When $f$ is one-to-one on its domain $D$, the inverse function $f^{-1}$ exists and its domain is equal to the range $R$ of $f$ (Figure 4). Indeed, for every $c \in R$, there is precisely one element $a \in D$ such that $f(a) = c$ and we may define $f^{-1}(c) = a$. With this definition, $f(f^{-1}(c)) = f(a) = c$ and $f^{-1}(f(a)) = f^{-1}(c) = a$. This proves the following theorem:

**THEOREM 1 Existence of Inverses**  The inverse function $f^{-1}$ exists if and only if $f$ is one-to-one on its domain $D$. Furthermore,

- Domain of $f = \text{range of } f^{-1}$
- Range of $f = \text{domain of } f^{-1}$
**Example 2** Show that \( f(x) = \frac{3x + 2}{5x - 1} \) is invertible. Determine the domain and range of \( f \) and \( f^{-1} \).

**Solution** The domain of \( f \) is \( D = \left\{ x : x \neq \frac{1}{5} \right\} \) (Figure 5). Assume that \( x \in D \), and let's solve \( y = f(x) \) for \( x \) in terms of \( y \):

\[
\begin{align*}
y &= \frac{3x + 2}{5x - 1} \\
y(5x - 1) &= 3x + 2 \\
5xy - y &= 3x + 2 \\
5xy - 3x &= y + 2 \\
x(5y - 3) &= y + 2 \\
x &= \frac{y + 2}{5y - 3}
\end{align*}
\]

(gather terms involving \( x \))

(factor out \( x \) in order to solve for \( x \))

(divide by \( 5y - 3 \))

The last step is valid if \( 5y - 3 \neq 0 \)—that is, if \( y \neq \frac{3}{5} \). But note that \( y = \frac{3}{5} \) is not in the range of \( f \). For if it were, Eq. (1) would yield the false equation \( 0 = \frac{3}{5} + 2 \). Now, Eq. (2) shows that for all \( y \neq \frac{3}{5} \), there is a unique value \( x \) such that \( f(x) = y \). Therefore, \( f \) is one-to-one on its domain. By Theorem 1, \( f \) is invertible. The range of \( f \) is \( R = \left\{ x : x \neq \frac{3}{5} \right\} \) and

\[ f^{-1}(x) = \frac{x + 2}{5x - 3} \]

The inverse function has domain \( R \) and range \( D \).

We can tell whether \( f \) is one-to-one from its graph. The horizontal line \( y = c \) intersects the graph of \( f \) at points \((a, f(a))\), where \( f(a) = c \) (Figure 6). There is at most one such point if \( f(x) = c \) has at most one solution. This gives us the following:

**Horizontal Line Test** A function of \( x \) is one-to-one if and only if every horizontal line intersects the graph of the function in at most one point.

In Figure 7, we see that \( f(x) = x^3 \) passes the Horizontal Line Test and therefore is one-to-one, whereas \( f(x) = x^2 \) fails the test and is not one-to-one.

**Example 3** Increasing Functions Are One-to-One Show that increasing functions are one-to-one. Then show that \( f(x) = x^3 + 4x + 3 \) is one-to-one.

**Solution** An increasing function satisfies \( f(a) < f(b) \) if \( a < b \). Therefore, \( f \) cannot take on any value more than once, and thus \( f \) is one-to-one.
CHAPTER 7 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Now, observe that
- If $n$ is odd and $c > 0$, then $f(x) = cx^n$ is increasing.
- A sum of increasing functions is increasing.

Thus, $g(x) = x^5$ and $h(x) = 4x$ are increasing, and therefore so is the sum $k(x) = x^5 + 4x$. It follows that the function $f(x) = x^5 + 4x$ is increasing and thus is one-to-one (Figure 8). However, determining an explicit formula for its inverse would be difficult.

Note that using an argument like the one in the previous example, we can prove that decreasing functions are also one-to-one and therefore have inverses.

We can make a function one-to-one by restricting its domain suitably.

EXAMPLE 4 Restricting the Domain Find a domain on which $f(x) = x^2$ is one-to-one and determine its inverse on this domain.

Solution The function $f(x) = x^2$ is one-to-one on the domain $D = \{x : x \geq 0\}$, for if $a^2 = b^2$, where $a$ and $b$ are both nonnegative, then $a = b$ (Figure 9). The inverse of $f$ on $D$ is the positive square root $f^{-1}(x) = \sqrt{x}$. Alternatively, we may restrict $f$ to the domain $\{x : x \leq 0\}$, on which the inverse function is $f^{-1}(x) = -\sqrt{x}$.

Next, we describe the graph of the inverse function. The reflection of a point $(a, b)$ through the line $y = x$ is defined to be the point $(b, a)$ (Figure 10). Note that if the $x$- and $y$-axes are drawn to the same scale, then $(a, b)$ and $(b, a)$ are equidistant from the line $y = x$ and the segment joining them is perpendicular to $y = x$.

The graph of $f^{-1}$ is the reflection of the graph of $f$ through $y = x$ (Figure 11). To check this, note that $(a, b)$ lies on the graph of $f$ if $f(a) = b$. But $f(a) = b$ if and only if $f^{-1}(b) = a$, and in this case, $(b, a)$ lies on the graph of $f^{-1}$.

EXAMPLE 5 Sketching the Graph of the Inverse Sketch the graph of the inverse of $f(x) = \sqrt[4]{-x}$.

Solution Let $g(x) = f^{-1}(x)$. Observe that the domain of $f$ is $\{x : x \leq 4\}$, and the range of $f$ is $\{x : x \geq 0\}$. We do not need a formula for $g(x)$ to draw its graph. We simply reflect the graph of $f$ through the line $y = x$, as in Figure 12. If desired, however, we can easily solve $y = \sqrt[4]{-x}$ to obtain $x = 4 - y^4$ and thus $g(x) = 4 - x^4$ with domain $\{x : x \geq 0\}$.

Derivatives of Inverse Functions

Next, we derive a formula for the derivative of the inverse $f^{-1}$, expressing how it is related to the derivative of $f$. 

Scanned with CamScanner
**THEOREM 2** Derivative of the Inverse  
Assume that \( f \) is differentiable and one-to-one with inverse \( g(x) = f^{-1}(x) \). If \( b \) belongs to the domain of \( g \) and \( f'(g(b)) \neq 0 \), then \( g'(b) \) exists and

\[
g'(b) = \frac{1}{f'(g(b))}
\]

**Proof**  
The proof that \( g \) is differentiable if \( f'(g(x)) \neq 0 \) is technical and therefore we omit it. To prove Eq. (3), note that \( f(g(x)) = x \) by definition of the inverse. Differentiate both sides of this equation, and apply the Chain Rule:

\[
\frac{d}{dx} f(g(x)) = \frac{d}{dx} x \Rightarrow f'(g(x))g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}
\]

Set \( x = b \) to obtain Eq. (3).

**GRAPHICAL INSIGHT**  
The formula for the derivative of the inverse function has a clear graphical interpretation. Consider a line \( L \) of slope \( m \) and let \( L' \) be its reflection through \( y = x \) as in Figure 13(A). Then the slope of \( L' \) is \( 1/m \). Indeed, if \((a, b)\) and \((c, d)\) are any two points on \( L \), then \((b, a)\) and \((d, c)\) lie on \( L' \) and

\[
\text{slope of } L = \frac{d - b}{c - a} \quad \text{slope of } L' = \frac{c - a}{d - b}
\]

The reciprocal slopes.

Now recall that the graph of the inverse \( g \) is obtained by reflecting the graph of \( f \) through the line \( y = x \). As we see in Figure 13(B), the tangent line to \( y = g(x) \) at \( x = b \) is the reflection of the tangent line to \( y = f(x) \) at \( x = a \) (where \( b = f(a) \) and \( a = g(b) \)). These tangent lines have reciprocal slopes, and thus \( g'(b) = 1/f'(a) = 1/f(g(b)) \), as claimed in Theorem 2.

**EXAMPLE 6**  
Using Equation (3)  
Calculate \( g'(x) \), where \( g \) is the inverse of the function \( f(x) = x^4 + 10 \) on the domain \( \{ x : x \geq 0 \} \).

**Solution**  
Solve \( y = x^4 + 10 \) for \( x \) to obtain \( x = (y - 10)^{1/4} \). Thus, \( g(x) = (x - 10)^{1/4} \). Since \( f'(x) = 4x^3 \), we have \( f'(g(x)) = 4g(x)^3 \), and by Eq. (3),

\[
g'(x) = \frac{1}{f'(g(x))} = \frac{1}{4g(x)^3} = \frac{1}{4(x - 10)^{3/4}} = \frac{1}{4}(x - 10)^{-3/4}
\]

We obtain this same result by differentiating \( g(x) = (x - 10)^{1/4} \) directly.
**Example 7** Calculating \( g'(x) \) Without Solving for \( g(x) \)

Calculate \( g'(1) \), where \( g \) is the inverse of \( f(x) = x + e^x \).

**Solution** In this case, we cannot solve for \( g(x) \) explicitly, but a formula for \( g(x) \) is not needed (Figure 14). All we need is the particular value \( g(1) \), which we can find by solving \( f(x) = 1 \). By inspection, \( x + e^x = 1 \) has solution \( x = 0 \). Therefore, \( f(0) = 1 \) and, by definition of the inverse, \( g(1) = 0 \). Since \( f'(x) = 1 + e^x \),

\[
g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}
\]

### 7.2 Summary

- A function \( f \) is one-to-one on a domain \( D \) if for every value \( c \), the equation \( f(x) = c \) has at most one solution for \( x \in D \), or, equivalently, if for all \( a, b \in D \), \( f(a) \neq f(b) \) unless \( a = b \).
- Let \( f \) have domain \( D \) and range \( R \). The inverse \( f^{-1} \) (if it exists) is the unique function with domain \( R \) and range \( D \) satisfying \( f(f^{-1}(x)) = x \) and \( f^{-1}(f(x)) = x \).
- The inverse of \( f \) exists if and only if \( f \) is one-to-one on its domain.
- To find the inverse function, solve \( y = f(x) \) for \( x \) in terms of \( y \) to obtain \( x = g(y) \). The inverse is the function \( g \).
- **Horizontal Line Test:** If \( f \) is one-to-one and only if every horizontal line intersects the graph of \( f \) in at most one point.
- The graph of \( f^{-1} \) is obtained by reflecting the graph of \( f \) through the line \( y = x \).
- **Derivative of the inverse:** If \( f \) is differentiable and one-to-one with inverse \( g \), then for \( x \) such that \( f'(g(x)) \neq 0 \),

\[
g'(x) = \frac{1}{f'(g(x))}
\]

### 7.2 Exercises

#### Preliminary Questions

1. Which of the following satisfy \( f^{-1}(x) = f(x) \)?
   - (a) \( f(x) = x \)
   - (b) \( f(x) = 1 - x \)
   - (c) \( f(x) = 1 \)
   - (d) \( f(x) = \sqrt{x} \)
   - (e) \( f(x) = |x| \)
   - (f) \( f(x) = x^{-1} \)

2. The graph of a function looks like the track of a roller coaster. Is the function one-to-one?

3. The function \( f \) maps teenagers in the United States to their last names. Explain why the inverse function \( f^{-1} \) does not exist.

4. The following fragment of a train schedule for the New Jersey Transit System defines a function \( f \) from towns to times. Is \( f \) one-to-one? What is \( f^{-1}(6:27) \)?

#### Exercises

1. Show that \( f(x) = 7x - 4 \) is invertible and find its inverse.

2. Is \( f(x) = x^2 + 2 \) one-to-one? If not, describe a domain on which it is one-to-one.

3. What is the largest interval containing zero on which \( f(x) = \sin x \) is one-to-one?

4. Show that \( f(x) = \frac{x - 2}{x + 3} \) is invertible and find its inverse.

5. A homework problem asks for a sketch of the graph of the inverse of \( f(x) = x + \cos x \). Frank, after trying but failing to find a formula for \( f^{-1}(x) \), says it's impossible to graph the inverse. Bianca hands in an accurate sketch without solving for \( f^{-1} \). How did Bianca complete the problem?

6. What is the slope of the line obtained by reflecting the line \( y = \frac{1}{3} \) through the line \( y = x \)?

7. Suppose that \( P = (2, 4) \) lies on the graph of \( f \) and that the slope of the tangent line through \( P \) is \( m = 3 \). Assuming that \( f^{-1} \) exists, what is the slope of the tangent line to the graph of \( f^{-1} \) at the point \( Q \) = (4, 2)?
(a) What is the domain of \( f \)? The range of \( f^{-1} \)?

(b) What is the domain of \( f^{-1} \)? The range of \( f \)?

5. Verify that \( f(x) = x^3 + 3 \) and \( g(x) = (x - 3)^{1/3} \) are inverses by showing that \( f(g(x)) = x \) and \( g(f(x)) = x \).

6. Repeat Exercise 5 for \( f(i) = \frac{t + 1}{t - 1} \) and \( g(t) = \frac{t + 1}{t - 1} \).

7. The escape velocity from a planet of radius \( R \) is \( v(R) = \sqrt{\frac{2GM}{R}} \), where \( G \) is the universal gravitational constant and \( M \) is the mass. Find the inverse of \( v \) expressing \( R \) in terms of \( v \).

8. Show that the power law relationship \( P(Q) = kQ^n \), for \( Q \geq 0 \) and \( k \neq 0 \), has an inverse that is also a power law, \( Q(P) = mP^n \), where \( m = k^{-1/n} \) and \( s = 1/n \).

9. The volume \( V \) of a cone that has height equal to its radius \( r \) is given by \( V(r) = \frac{1}{3} \pi r^3 \). Find the inverse of \( V(r) \), expressing \( r \) as a function of \( V \).

10. The surface area \( S \) of a sphere of radius \( r \) is given by \( S(r) = 4\pi r^2 \). Explain why, in the given context, \( S(r) \) has an inverse function. Find the inverse of \( S(r) \), expressing \( r \) as a function of \( S \).

In Exercises 11–17, find a domain on which \( f \) is one-to-one and a formula for the inverse of \( f \) restricted to this domain. Sketch the graphs of \( f \) and \( f^{-1} \).

11. \( f(x) = 4 - x \)

12. \( f(x) = \frac{1}{x + 1} \)

13. \( f(x) = \frac{1}{7x - 3} \)

14. \( f(s) = \frac{1}{s^2} \)

15. \( f(x) = \frac{1}{\sqrt{x^2 + 1}} \)

16. \( f(z) = z^3 \)

17. \( f(x) = \sqrt{x^2 + 9} \)

18. For each function shown in Figure 15, sketch the graph of the inverse (restrict the function’s domain if necessary).

19. Which of the graphs in Figure 16 is the graph of a function satisfying \( f^{-1} = f \)?

20. Let \( n \) be a nonzero integer. Find a domain on which \( f(x) = (1 - x^n)^{1/n} \) coincides with its inverse. Hint: The answer depends on whether \( n \) is even or odd.

21. Let \( f(x) = x^2 + x + 1 \).

(a) Show that \( f^{-1} \) exists (but do not attempt to find it). Hint: Show that \( f \) is increasing.

(b) What is the domain of \( f^{-1} \)?

(c) Find \( f^{-1}(3) \).

22. Show that \( f(x) = (x^3 + 1)^{1/3} \) is one-to-one on \((-\infty, 0] \), and find a formula for \( f^{-1} \) for this domain of \( f \).

23. Let \( f(x) = x^2 - 2x \). Determine a domain on which \( f^{-1} \) exists, and find a formula for \( f^{-1} \) for this domain of \( f \).

24. Show that the inverse of \( f(x) = e^{-x} \) exists (without finding it explicitly). What is the domain of \( f^{-1} \)?

25. Find the inverse \( g(x) = \sqrt{x^2 + 9} \) with domain \( x \geq 0 \) and calculate \( g'(x) \) in two ways: using Theorem 2 and by direct calculation.

26. Let \( g \) be the inverse of \( f(x) = x^3 + 1 \). Find a formula for \( g(x) \) and calculate \( g'(x) \) in two ways: using Theorem 2 and then by direct calculation.

In Exercises 27–32, use Theorem 2 to calculate \( g'(x) \), where \( g \) is the inverse of \( f \).

27. \( f(x) = 7x + 6 \)

28. \( f(x) = \sqrt{3 - x} \)

29. \( f(x) = x^{-1} \)

30. \( f(x) = 4x^3 - 1 \)

31. \( f(x) = \frac{-x}{x + 1} \)

32. \( f(x) = 2 + x^{-1} \)

33. Let \( g \) be the inverse of \( f(x) = x^2 + 2x + 4 \). Calculate \( g(7) \) (without finding a formula for \( g(x) \)), and then calculate \( g'(7) \).

34. Find \( g\left(-\frac{1}{4}\right) \), where \( g \) is the inverse of \( f(x) = \frac{x^2}{x^2 + 1} \).

In Exercises 35–40, calculate \( g(b) \) and \( g'(b) \), where \( g \) is the inverse of \( f \) (in the given domain, if indicated).

35. \( f(x) = x + \cos x \), \( b = 1 \)

36. \( f(x) = 4x^3 - 2x \), \( b = -2 \)

37. \( f(x) = \sqrt{x^2 + 6x} \) for \( x \geq 0 \), \( b = 4 \)

38. \( f(x) = \sqrt{x^2 + 6x} \) for \( x \leq -6 \), \( b = 4 \)

39. \( f(x) = \frac{1}{x + 1} \), \( b = \frac{1}{4} \)

40. \( f(x) = e^x \), \( b = e \)
41. Let \( f(x) = x^a \) and \( g(x) = x^{1/a} \). Compute \( g'(x) \) using Theorem 2 and check your answer using the Power Rule.

42. Show that \( f(x) = \frac{1}{1-x} \) and \( g(x) = \frac{1-x}{x} \) are inverses. Then compute \( g'(x) \) directly and verify that \( g'(x) = 1/f'(g(x)) \).

43. Use graphical reasoning to determine if the following statements are true or false. If false, modify the statement to make it correct.
   (a) If \( f \) is increasing, then \( f^{-1} \) is increasing.
   (b) If \( f \) is decreasing, then \( f^{-1} \) is decreasing.
   (c) If \( f \) is concave up, then \( f^{-1} \) is concave up.
   (d) If \( f \) is concave down, then \( f^{-1} \) is concave down.
   (e) Linear functions \( f(x) = ax + b (a \neq 0) \) are always one-to-one.
   (f) Quadratic polynomials \( f(x) = ax^2 + bx + c (a \neq 0) \) are always one-to-one.
   (g) \( f(x) = \sin x \) is not one-to-one.

Further Insights and Challenges

44. Show that if \( f \) is odd and \( f^{-1} \) exists, then \( f^{-1} \) is odd. Show, on the other hand, that an even function does not have an inverse.

45. Let \( g \) be the inverse of a function \( f \) satisfying \( f'(x) = f(x) \). Show that \( g'(x) = x^{-1} \). [Note that this shows that the inverse of the exponential function \( f(x) = e^x \) is an antiderivative of \( x^{-1} \). That inverse is the natural logarithm function that we define in the next section.]

7.3 Logarithmic Functions and Their Derivatives

Logarithmic functions are inverses of exponential functions. More precisely, if \( b > 0 \) and \( b \neq 1 \), then the logarithm to the base \( b \), denoted \( \log_b x \), is the inverse of \( f(x) = b^x \). By definition, \( y = \log_b x \) if \( b^y = x \), so we have

\[ g^{\log_b x} = x \quad \text{and} \quad \log_b (b^x) = x \]

In other words, \( \log_b x \) is the number to which \( b \) must be raised in order to get \( x \). For example,

\[ \log_2(8) = 3 \quad \text{because} \quad 2^3 = 8 \]
\[ \log_{10}(1) = 0 \quad \text{because} \quad 10^0 = 1 \]
\[ \log_3 \left( \frac{1}{9} \right) = -2 \quad \text{because} \quad 3^{-2} = \frac{1}{3^2} = \frac{1}{9} \]

The logarithm to the base \( e \), denoted \( \ln x \), plays a special role and is called the natural logarithm.

\[ \ln x = \log_e x \]

We use a calculator to evaluate logarithms numerically. For example,

\[ \ln 17 \approx 2.83321 \quad \text{because} \quad e^{2.83321} \approx 17 \]

As in Figure 1, \( f(x) = \ln x \) and \( g(x) = e^x \) are inverse functions, so we have

\[ e^{\ln x} = x \quad \text{and} \quad \ln(e^x) = x \]

Recall that the domain of \( f(x) = b^x \) is \( \mathbb{R} \) and its range is the set of positive real numbers \( \{x : x > 0\} \). Since the domain and range are reversed in the inverse function,

- The domain of \( f(x) = \log_b x \) is \( \{x : x > 0\} \).
- The range of \( f(x) = \log_b x \) is the set of all real numbers \( \mathbb{R} \).

If \( b > 1 \), then \( \log_b x \) is positive for \( x > 1 \) and negative for \( 0 < x < 1 \). Figure 1 illustrates these facts for the base \( b = e \). Keep in mind that the logarithm of a negative number does not exist. For example, \( \log_{10}(-2) \) does not exist because \( 10^y = -2 \) has no solution.
SECTION 7.3 Logarithmic Functions and Their Derivatives

For each law of exponents, there is a corresponding law for logarithms. The rule $b^{x+y} = b^x b^y$ corresponds to the rule

$$\log_b(xy) = \log_b x + \log_b y$$

In words: *The log of a product is the sum of the logs.* To verify this rule, observe that

$$b^{\log_b (xy)} = xy = b^{\log_b x} b^{\log_b y} = b^{\log_b x + \log_b y}$$

The exponents $\log_b (xy)$ and $\log_b x + \log_b y$ are equal as claimed because $f(x) = b^x$ is one-to-one. The logarithm laws are collected in the following table.

<table>
<thead>
<tr>
<th>Laws of Logarithms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Law</td>
</tr>
<tr>
<td>Log of 1</td>
</tr>
<tr>
<td>Log of $b$</td>
</tr>
</tbody>
</table>
| Products           | $\log_b(xy) = \log_b x + \log_b y$  
|                   | $\log_5(2 \cdot 3) = \log_5 2 + \log_5 3$ |
| Quotients          | $\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y$  
|                   | $\log_2 \left( \frac{3}{7} \right) = \log_2 3 - \log_2 7$ |
| Reciprocals        | $\log_b \left( \frac{1}{x} \right) = -\log_b x$  
|                   | $\log_2 \left( \frac{1}{7} \right) = -\log_2 7$ |
| Powers (any $n$)   | $\log_b(x^n) = n \log_b x$  
|                   | $\log_{10}(8^2) = 2 \cdot \log_{10} 8$ |

The following change-of-base formulas enable us to change bases in logarithms. For example, to convert from logarithm base $a$ to logarithm base $b$, you divide by $\log_a b$.

$$\log_b x = \frac{\log_a x}{\log_a b}, \quad \log_b x = \frac{\ln x}{\ln b}$$

We can prove these formulas using the Laws of Logarithms (see Exercise 114).

**EXAMPLE 1 Using the Logarithm Laws** Evaluate:

(a) $\log_6 9 + \log_6 4$  
(b) $\ln \left( \frac{1}{\sqrt{e}} \right)$  
(c) $10 \log_6 (b^3) - 4 \log_6 (\sqrt{b})$

**Solution**

(a) $\log_6 9 + \log_6 4 = \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = 2$

(b) $\ln \left( \frac{1}{\sqrt{e}} \right) = \ln(e^{-1/2}) = -\frac{1}{2} \ln(e) = -\frac{1}{2}$

(c) $10 \log_6 (b^3) - 4 \log_6 (\sqrt{b}) = 10(3) - 4 \log_6 (b^{1/2}) = 30 - 4 \left( \frac{1}{2} \right) = 28$

**EXAMPLE 2 Solving an Exponential Equation** The bacteria population in a bottle at time $t$ (in hours) has size $P(t) = 1000e^{0.35t}$. After how many hours will there be 5000 bacteria?
CHAPTER 7  EXPONENTIAL AND LOGARITHMIC FUNCTIONS

![Graph of bacteria population as a function of time.](image)

**Solution** We must solve \( P(t) = 1000e^{0.35t} = 5000 \) for \( t \) (Figure 2):

\[
e^{0.35t} = \frac{5000}{1000} = 5
\]

\[
\ln(e^{0.35t}) = \ln 5  \quad \text{(take logarithm of both sides)}
\]

\[
0.35t = \ln 5 \approx 1.609 \quad \text{[because } \ln(e^a) = a]\]

\[
t \approx \frac{1.609}{0.35} \approx 4.6 \text{ h}
\]

**The Derivative of Logarithmic Functions**

We would like to know the derivative of \( f(x) = \ln x \). Implicit differentiation makes this straightforward to determine. Letting \( y = \ln x \) and assuming \( x > 0 \) so it is in the domain of \( f(x) = \ln x \), our goal is to find \( dy/dx \). But then we have

\[
e^y = e^{\ln x}
\]

\[
e^y = x
\]

Implicitly differentiating, we obtain

\[
e^y \frac{dy}{dx} = 1
\]

\[
\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}
\]

**THEOREM 1** Derivative of the Natural Logarithm

\[
\frac{d}{dx} \ln x = \frac{1}{x} \quad \text{for } x > 0
\]

**EXAMPLE 3** Describe the graph of \( f(x) = \ln x \). Is \( f \) increasing or decreasing? Is the graph of \( f \) concave up or concave down?

**Solution** The derivative \( f'(x) = x^{-1} \) is positive on the domain \( \{x : x > 0\} \), so \( f(x) = \ln x \) is increasing. Furthermore, \( f'(x) = x^{-1} \) is decreasing, so the graph of \( f \) is concave down and grows flatter as \( x \to \infty \) (Figure 3).

**EXAMPLE 4** Differentiate: (a) \( y = x \ln x \) and (b) \( y = (\ln x)^2 \).

**Solution**

(a) Use the Product Rule:

\[
\frac{d}{dx}(x \ln x) = x \cdot (\ln x)' + (x)' \cdot \ln x
\]

\[
= x \cdot \frac{1}{x} + \ln x = 1 + \ln x
\]

(b) Use the General Power Rule:

\[
\frac{d}{dx}(\ln x)^2 = 2 \ln x \cdot \frac{d}{dx} \ln x = \frac{2 \ln x}{x}
\]
In Section 3.2, we proved the Power Rule for whole-number exponents. We can now prove it for all exponents \( n \) by writing \( x^n \) as an exponential and using the Chain Rule. For \( x > 0 \),
\[
 x^n = (e^{n \ln x})^n = e^{n \ln x} \\
 \frac{d}{dx} x^n = \frac{d}{dx} e^{n \ln x} = \left( \frac{d}{dx} n \ln x \right) e^{n \ln x} = n x^{n-1} 
\]

We obtain a useful formula for the derivative of \( \ln(f(x)) \) by applying the Chain Rule with \( u = f(x) \):
\[
 \frac{d}{dx} \ln(f(x)) = \frac{d}{du} \ln(u) \cdot \frac{du}{dx} = \frac{1}{u} \cdot f'(x) \\
 = \frac{f'(x)}{f(x)} 
\]

**EXAMPLE 5** Differentiate: (a) \( y = \ln(x^3 + 1) \) and (b) \( y = \ln(\sqrt{\sin x}) \).

**Solution** Use Eq. (3): (a) \( \frac{d}{dx} \ln(x^3 + 1) = \frac{(x^3 + 1)'}{x^3 + 1} = \frac{3x^2}{x^3 + 1} \)

(b) The algebra is simpler if we write \( \ln(\sqrt{\sin x}) = \ln((\sin x)^{1/2}) = \frac{1}{2} \ln(\sin x) \):
\[
 \frac{d}{dx} \ln(\sqrt{\sin x}) = \frac{d}{2 dx} \ln(\sin x) \\
 = \frac{1}{2} \cdot \frac{\cos x}{\sin x} = \frac{1}{2} \cdot \frac{\cos x}{\sin x} = \frac{1}{2} \cdot \cot x
\]

Next, we develop a derivative formula for general logarithmic functions \( f(x) = \log_b x \). The change-of-base formula for logarithms says \( \log_b x = \frac{\ln x}{\ln b} \). Differentiating this equation, we obtain
\[
 \frac{d}{dx} \log_b x = \frac{d}{dx} \frac{\ln x}{\ln b} = \left( \frac{1}{\ln b} \right) \frac{1}{x} = \frac{1}{x \ln b}
\]
Therefore, we have the following theorem:

**THEOREM 2** Derivative of \( f(x) = \log_b x \)

\[
 \frac{d}{dx} \log_b x = \frac{1}{x \ln b}
\]

For example, \( (\log_{10} x)' = \frac{1}{x \ln 10} \).

**The Derivative of \( f(x) = b^x \)**

In Section 7.1, we proved that for any base \( b > 0 \),
\[
 \frac{d}{dx} b^x = m(b) b^x, \quad \text{where} \quad m(b) = \lim_{h \to 0} \frac{b^h - 1}{h}
\]
but we were not able to identify the factor \( m(b) \) [other than to say that \( e \) is the unique number for which \( m(e) = 1 \)]. Now, we can use implicit differentiation to prove that \( m(b) = \ln b \).

Writing \( y = b^x \), our goal is to find \( \frac{dy}{dx} \). Taking the natural logarithm of both sides yields
\[
 \ln y = \ln b^x = x \ln b
\]
Then we implicitly differentiate each side of the equation with respect to \( x \), treating \( x \) as itself and treating \( y \) as a function of \( x \). This yields
\[
 \frac{1}{y} \frac{dy}{dx} = \ln b
\]

\[
 \frac{dy}{dx} = y \ln b
\]

\[
 \frac{d}{dx} b^x = b^x \ln b
\]
Thus, we have \( \frac{dy}{dx} = y \ln b = b^x \ln b \), where the last equality holds since \( y = b^x \).

We obtain the following theorem:

**THEOREM 3** Derivative of \( f(x) = b^x \)

\[
\frac{d}{dx} b^x = (\ln b)b^x \quad \text{for } b > 0
\]

For example, \( (10^x)' = (\ln 10)10^x \).

**EXAMPLE 6** Differentiate: (a) \( f(x) = 4^{3x} \) and (b) \( f(x) = 5^{x^2} \).

**Solution**

(a) The function \( f(x) = 4^{3x} \) is a composition of \( 4^u \) and \( u = 3x \):

\[
\frac{d}{dx} 4^{3x} = \left( \frac{d}{du} 4^u \right) \frac{du}{dx} = (\ln 4)4^u(3x)' = (\ln 4)4^{3x}(3) = (3 \ln 4)4^{3x}
\]

(b) The function \( f(x) = 5^{x^2} \) is a composition of \( 5^u \) and \( u = x^2 \):

\[
\frac{d}{dx} 5^{x^2} = \left( \frac{d}{du} 5^u \right) \frac{du}{dx} = (\ln 5)5^u(x^2)' = (\ln 5)5^{x^2}(2x) = (2 \ln 5)x \cdot 5^{x^2}
\]

**CONCEPTUAL INSIGHT** The change-of-base formulas for logarithmic functions indicate that we can freely change between bases. The same holds for exponential functions via the change-of-base relationship \( b^x = a^x \log_a b \). From an algebraic perspective, no particular base is generally preferred over any other. Now we see that from a calculus perspective, there is a preferred base, \( e \), for exponential and logarithmic functions. For \( f(x) = e^x \) and \( f(x) = \ln x \), we have the simple derivative formulas \( \frac{d}{dx} e^x = e^x \) and \( \frac{d}{dx} \ln x = \frac{1}{x} \), but for \( f(x) = b^x \) and \( f(x) = \log_b x \), we have to introduce an \( \ln b \) term in their more awkward derivative formulas \( \frac{d}{dx} b^x = (\ln b)b^x \) and \( \frac{d}{dx} \log_b x = \frac{1}{x \ln b} \).

**Logarithmic Differentiation**

The next example illustrates logarithmic differentiation. This technique saves work when the function is a product or quotient with several factors.

**EXAMPLE 7** Find the derivative of

\[ f(x) = \frac{(x + 1)^2(2x^2 - 3)}{\sqrt{x^2 + 1}} \]

**Solution** In logarithmic differentiation, we differentiate \( \ln(f(x)) \) rather than \( f(x) \) itself. First, we take the natural log of both sides of the equation:

\[
\ln f(x) = \ln \left( \frac{(x + 1)^2(2x^2 - 3)}{\sqrt{x^2 + 1}} \right)
\]

Then we expand the right-hand side using the logarithm rules:

\[
\ln(f(x)) = \ln((x + 1)^2) + \ln(2x^2 - 3) - \ln(\sqrt{x^2 + 1})
\]

\[= 2 \ln(x + 1) + 2 \ln(2x^2 - 3) - \frac{1}{2} \ln(x^2 + 1) \]
Next, use Eq. (3):

\[
\frac{f'(x)}{f(x)} = \frac{d}{dx} \ln(f(x)) = 2 \frac{d}{dx} \ln(x + 1) + \frac{d}{dx} \ln(2x^2 - 3) - \frac{1}{2} \frac{d}{dx} \ln(x^2 + 1)
\]

\[
\frac{f'(x)}{f(x)} = \frac{2}{x + 1} + \frac{4x}{2x^2 - 3} - \frac{1}{2} \frac{2x}{x^2 + 1}
\]

Finally, multiply through by \(f(x)\):

\[
f'(x) = \left( \frac{2}{x + 1} + \frac{4x}{2x^2 - 3} - \frac{x}{x^2 + 1} \right) \left( (x + 1)^2(2x^2 - 3) \right) \frac{1}{\sqrt{x^2 + 1}}
\]

Logarithmic differentiation also allows us to take the derivative of functions of the form \(y = f(x)^{g(x)}\), where both the base and the exponent depend on \(x\).

EXAMPLE 8 Differentiate (for \(x > 0\)): \(\textbf{a)} \ f(x) = x^x\) and \(\textbf{b)} \ g(x) = x^{\sin x}\).

\textbf{Solution} The graphs of \(f\) and \(g\) are shown in Figure 4. We illustrate two different methods; either method could be used in either case.

\textbf{a)} Method 1: Use the identity \(x = e^{\ln x}\) to rewrite \(f(x)\) as an exponential base \(e\):

\[
f(x) = x^x = (e^{\ln x})^x = e^{x \ln x}
\]

\[
f'(x) = (x \ln x)e^{x \ln x} = (1 + \ln x)x^{x \ln x} = (1 + \ln x)x^x
\]

\textbf{b)} Method 2: Apply Eq. (3) to \(\ln(g(x))\). Since \(\ln(g(x)) = \ln(x^{\sin x}) = (\sin x) \ln x\),

\[
g'(x) = \frac{d}{dx} \ln(g(x)) = \frac{d}{dx} (\sin x \ln x) = \frac{\sin x}{x} + (\cos x) \ln x
\]

\[
g'(x) = \left( \frac{\sin x}{x} + (\cos x) \ln x \right) g(x) = \left( \frac{\sin x}{x} + (\cos x) \ln x \right) x^{\sin x}
\]

Note that since \(f(x) = x^x\) has the variable \(x\) in both the base and the exponent, it is a rapidly growing function. It is evident from its graph that it decreases initially, reaches a minimum, and then increases from that point on. Where does that minimum occur and what is the minimum value? To answer this, we need to find the \(x\) where the slope of the graph of \(f\) is 0; that is, where \(f'(x) = 0\). We solve:

\[
(1 + \ln x)x^x = 0
\]

\[
1 + \ln x = 0 \quad \text{(since } x^x \text{ is never } 0)\]

\[
\ln x = -1
\]

\[
x = e^{-1} = 1/e
\]

So the minimum occurs at \(x = 1/e\), and the minimum value is \((1/e)^{1/e} \approx 0.6922\).

\textbf{Integrals Involving the Natural Logarithm}

In Chapter 5, we noted that the Power Rule for Integrals is valid for all exponents \(n \neq -1\):

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)
\]

This formula is not valid (or meaningful) for \(n = -1\), so the question remained: \textit{What is the antiderivative of } \(y = x^{-1}\)? We can now give the answer: the natural logarithm.
Indeed, the formula \((\ln x)' = \frac{1}{x}\) tells us that \(\ln x\) is an antiderivative of \(y = x^{-1}\) for \(x > 0\):

\[
\int \frac{dx}{x} = \ln x + C
\]

We would like to have an antiderivative of \(y = 1/x\) on its full domain, namely on the domain \(\{x : x \neq 0\}\). To achieve this end, we extend \(F\) to an even function by setting \(F(x) = \ln |x|\) (Figure 5). Then \(F(x) = F(-x)\), and by the Chain Rule, \(F'(x) = -F'(-x)\). For \(x < 0\), we obtain

\[
\frac{d}{dx} \ln |x| = F'(x) = -F'(-x) = -\frac{1}{-x} = \frac{1}{x}
\]

This proves that \(\frac{d}{dx} \ln |x| = \frac{1}{x}\) for all \(x \neq 0\).

**THEOREM 4** Antiderivative of \(y = \frac{1}{x}\). The function \(F(x) = \ln |x|\) is an antiderivative of \(y = \frac{1}{x}\) in the domain \(\{x : x \neq 0\}\), that is,

\[
\int \frac{dx}{x} = \ln |x| + C
\]

By the Fundamental Theorem of Calculus, the following formula is valid if both \(a\) and \(b\) are either both positive or both negative (Figure 6(A)):

\[
\int_{a}^{b} \frac{dx}{x} = \ln |b| - \ln |a| = \ln \frac{b}{a}
\]

**EXAMPLE 9** Evaluate: (a) \(\int_{2}^{8} \frac{dx}{x}\) (b) \(\int_{-4}^{-2} \frac{dx}{x}\) (c) \(\int_{1}^{e} \frac{dx}{x}\).

**Solution** By Eq. (7),

(a) \(\int_{2}^{8} \frac{dx}{x} = \ln \frac{8}{2} = \ln 4 \approx 1.39\)
(b) \( \int_{-4}^{-2} \frac{dx}{x} = \ln \left( \frac{-2}{-4} \right) = \ln \frac{1}{2} \approx -0.69 \)

(c) \( \int_{1}^{e} \frac{dx}{x} = \ln \left( \frac{e}{1} \right) = \ln e = 1 \)

The signed areas represented by (a)–(c) are shown in (B) and (C) in Figure 6. In Section 1 of this chapter, we indicated that \( e \) is the number for which the area under \( y = 1/x \) from 1 to \( e \) is equal to 1. Here, we have verified this fact via a definite integral.

**EXAMPLE 10** Evaluate:

(a) \( \int_{1}^{3} \frac{x}{x^2 + 1} \, dx \)

(b) \( \int \tan x \, dx \)

**Solution**

(a) Use the substitution \( u = x^2 + 1, \ \frac{1}{2} \, du = x \, dx \). In the \( u \)-variable, the limits of the integral become \( u(1) = 2 \) and \( u(3) = 10 \). The integral is equal to the area shown in Figure 7:

\[
\int_{1}^{3} \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \left[ \int_{2}^{10} \frac{du}{u} \right] = \frac{1}{2} \ln |u| \bigg|_{2}^{10} = \frac{1}{2} \ln 10 - \frac{1}{2} \ln 2 \approx 0.805
\]

(b) Use the substitution \( u = \cos x, \ \frac{du}{u} = -\sin x \):

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = -\ln |u| + C
\]

\[
= -\ln |\cos x| + C = \ln \left| \frac{1}{\cos x} \right| + C = \ln |\sec x| + C
\]

Antiderivatives of \( \sin x \) and \( \cos x \) came naturally to us from the corresponding derivative formulas. Here, using the Substitution Method, and an integral involving the natural logarithm, we obtained an antiderivative formula for \( \tan x \). We can similarly show that \( \int \cot x \, dx = \ln |\sin x| + C \).

Setting \( a = 1 \) and \( b = x \) in Eq. (7), we obtain a formula for the natural logarithm as an integral:

\[
\ln x = \int_{1}^{x} \frac{dt}{t}
\]

In some developments of the material in this chapter, Eq. (8) is used as the definition of the natural logarithm function. The resulting function is increasing and therefore has an inverse. That inverse is then defined as the exponential function. Exercises 116–118 consider these ideas in more detail.

**7.3 SUMMARY**

- For \( b > 0 \) with \( b \neq 1 \), the logarithm function \( f(x) = \log_{b} x \) is the inverse of \( g(x) = b^{x} \):
  \[
x = b^{y} \iff y = \log_{b} x
\]

- If \( b > 1 \), then \( \log_{b} x \) is positive for \( x > 1 \) and negative for \( 0 < x < 1 \), and
  \[
  \lim_{x \to +0} \ln x = -\infty, \quad \lim_{x \to +\infty} \ln x = \infty
  \]

- The **natural logarithm** is the logarithm to the base \( e \) and is denoted \( \ln x \).
• Important logarithm laws:
  (i) \( \log_b(xy) = \log_b x + \log_b y \)
  (ii) \( \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \)
  (iii) \( \log_b(x^n) = n \log_b x \)
  (iv) \( \log_b 1 = 0 \) and \( \log_b b = 1 \)

• Derivative formulas:
  \( (e^x)' = e^x \), \( \frac{d}{dx} \ln x = \frac{1}{x} \), \( (b^x)' = (\ln b)b^x \), \( \frac{d}{dx} \log_b x = \frac{1}{(\ln b)x} \)

• Integral formulas:
  \( \ln x = \int_1^x \frac{dt}{t} \) \( x > 0 \), \( \int \frac{dx}{x} = \ln |x| + C \)

### 7.3 Exercises

#### Preliminary Questions

1. Compute \( \log_6(b^4) \).
2. When is \( \ln x \) negative?
3. What is \( \ln(-3) \)? Explain.
4. Explain the statement "The logarithm converts multiplication into addition."
5. What are the domain and range of \( f(x) = \ln x \)?
6. Does \( f(x) = x^{-1} \) have an antiderivative for \( x < 0 \)? If so, describe one.
7. What is the slope of the tangent line to \( y = 4^x \) at \( x = 0 \)?
8. What is the rate of change of \( y = \ln x \) at \( x = 10? \)

#### Exercises

In Exercises 1–16, calculate without using a calculator.

1. \( \log_3 27 \)  
2. \( \log_5 \frac{15}{2} \)  
3. \( \ln 1 \)  
4. \( \log_3(5^5) \)  
5. \( \log_3(25/3) \)  
6. \( \log_3(85/3) \)  
7. \( \log_{49} 4 \)  
8. \( \log_7(49^2) \)  
9. \( \log_2 2 + \log_4 2 \)  
10. \( \log_2 3 + \log_2 \frac{5}{2} \)  
11. \( \log_4 48 - \log_4 12 \)  
12. \( \ln(\sqrt{e} \cdot 2^{7/5}) \)  
13. \( \ln(e^6) + \ln(e^6) \)  
14. \( \log_2 \frac{4}{9} + \log_2 24 \)  
15. \( \log_{3^{30}}(30) \)  
16. \( 83 \log_{49}(2) \)

17. Write as the natural log of a single expression:
   (a) \( 2 \ln 5 + 3 \ln 4 \)
   (b) \( 5 \ln(x^{1/2}) + \ln(9x) \)

18. Solve for \( x \): \( \ln(x^2 + 1) - 3 \ln x = \ln(2) \).

In Exercises 19–24, solve for the unknown.

19. \( e^{x^2} = 100 \)  
20. \( 6e^{-x} = 2 \)  
21. \( 2x^2 - 2x = 8 \)  
22. \( e^{2-x} = 9e^{-2x} \)  
23. \( \ln(x^4) - \ln(x^2) = 2 \)  
24. \( \log_3 y + 3 \log_3(y^2) = 14 \)

25. Show, by producing a counterexample, that \( \ln(ab) \) is not equal to \( \ln(a) + \ln(b) \).

26. What is \( b \) if \( \log_{ab} x^y = \frac{1}{3x} \)?

In Exercises 27–46, find the derivative.

27. \( y = x \ln x \)  
28. \( y = t \ln t - t \)  
29. \( y = (\ln x)^2 \)  
30. \( y = \ln(x^2) \)  
31. \( y = \ln(9x^2 - 8) \)  
32. \( y = \ln(5^r) \)  
33. \( y = \ln(\sin t + 1) \)  
34. \( y = x^2 \ln x \)  
35. \( y = \ln \left( \frac{x}{x} \right) \)  
36. \( y = e^{(\ln x)^2} \)  
37. \( y = \ln(\ln x) \)  
38. \( y = \ln(\cot x) \)  
39. \( y = \ln(\ln x)^3 \)  
40. \( y = \ln((\ln x)^3) \)  
41. \( y = \ln((x + 1)(2x + 9)) \)  
42. \( y = \ln \left( \frac{x + 1}{x^2 + 1} \right) \)  
43. \( y = 11^x \)  
44. \( y = 7^{x^2} \)  
45. \( y = \frac{2x^3 - 3x^2}{x} \)  
46. \( y = 16^{\ln x} \)

In Exercises 47–50, compute the derivative.

47. \( f'(x) \), \( f(x) = \log_2 x \)  
48. \( f'(3) \), \( f(x) = \log_4 x \)  
49. \( \frac{d}{dt} \log_5(\sin t) \)  
50. \( \frac{d}{dt} \log_{10}(t + 2) \)

In Exercises 51–62, find an equation of the tangent line at the point indicated.

51. \( f(t) = 6^t \), \( x = 2 \)  
52. \( f(y) = (\sqrt{2})^x \), \( x = 8 \)  
53. \( x(t) = 3^t \), \( t = 2 \)  
54. \( x(t) = \pi^{t^2 - 2}, t = 1 \)  
55. \( f(x) = 5^{x^2 - 2x}, x = 1 \)  
56. \( l(t) = \ln t \), \( t = 5 \)  
57. \( t(x) = \ln(8 - 4x), t = 1 \)  
58. \( f(x) = \ln(x^2), x = 4 \)  
59. \( R(x) = \log_3(2x^2 + 7), x = 3 \)  
60. \( y = \ln(\sin x), x = \frac{\pi}{4} \)
61. \( f(w) = \log_3 w, \quad w = \frac{1}{3} \)

62. \( y = \log_3(1 + 4x^{-1}), \quad x = 4 \)

In Exercises 63–70, find the derivative using logarithmic differentiation as in Example 7.

63. \( y = (x + 5)(x + 9) \)

64. \( y = (3x + 5)(4x + 9) \)

65. \( y = (x - 1)(x - 12)(x + 7) \)

66. \( y = \frac{x(x + 1)}{3x - 1} \)

67. \( y = \frac{x^2 + 1}{\sqrt{x + 1}} \)

68. \( y = (2x + 1)(4x^2) \sqrt{x - 9} \)

69. \( y = \frac{x(x + 2)}{(2x + 1)(3x + 2)} \)

70. \( y = (x^2 + 1)(x^4 + 2)(x^5 + 3)^2 \)

In Exercises 71–76, find the derivative using either method of Example 8.

71. \( f(x) = x^{3x} \)

72. \( f(x) = x^x \)

73. \( f(x) = x^x \)

74. \( f(x) = x^x \)

75. \( f(x) = x^3 \)

76. \( f(x) = e^x \)

In Exercises 77–80, find the local extreme values in the domain \((x : x > 0)\) and use the Second Derivative Test to determine whether these values are local minima or maxima.

77. \( g(x) = \ln x \)

78. \( g(x) = x \ln x \)

79. \( g(x) = \frac{\ln x}{x} \)

80. \( g(x) = x - \ln x \)

In Exercises 81 and 82, find the local extreme values and points of inflection, and sketch the graph of \( y = f(x) \) over the interval \([1, 4] \).

81. \( f(x) = \frac{10 \ln x}{x^2} \)

82. \( f(x) = x^3 - 8 \ln x \)

In Exercises 83–103, evaluate the indefinite integral, using substitution if necessary.

83. \( \int \frac{7 \, dx}{x} \)

84. \( \int \frac{dx}{x + 7} \)

85. \( \int \frac{dx}{x^2 + 4} \)

86. \( \int \frac{dx}{9x - 3} \)

87. \( \int \frac{dx}{x^2 + 4} \)

88. \( \int \frac{x^2 \, dx}{x^3 + 2} \)

89. \( \int \frac{(3x - 1) \, dx}{9 - 2x + 3x^2} \)

90. \( \int \frac{\tan(4x + 1) \, dx}{x} \)

91. \( \int \cot x \, dx \)

92. \( \int \frac{\cos x}{2 \sin x + 3} \, dx \)

**Further Insights and Challenges**

**113.** (a) Show that if \( f \) and \( g \) are differentiable, then

\[ \frac{d}{dx} \ln(f(x)g(x)) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \]

(b) Give a new proof of the Product Rule by observing that the left-hand side of Eq. (9) is equal to \( \frac{(f(x)g(x))'}{f(x)g(x)} \).

**114.** Prove the formula

\[ \log_b x = \frac{\log_a x}{\log_a b} \]

for all positive numbers \( a, b \) with \( a \neq 1 \) and \( b \neq 1 \).

**115.** Prove the formula \( \log_a b \log_b a = 1 \) for all positive numbers \( a, b \) with \( a \neq 1 \) and \( b \neq 1 \).
Exercises 116–118 develop an elegant approach to the exponential and logarithmic functions. Define a function \( G \) for \( x > 0 \):

\[
G(x) = \int_1^x \frac{1}{t} \, dt
\]

116. Defining \( x \) as an Integral

This exercise proceeds as if we didn’t know that \( G(x) = \ln x \) and shows directly that \( G \) has all the basic properties of the logarithm. Prove the following statements.

(a) \[ \int_a^b \frac{1}{t} \, dt = \int_{a/t}^{b/t} \frac{1}{t} \, dt \] for all \( a, b > 0 \). \( \text{Hint: Use the substitution } u = t/a. \)

(b) \[ G(ab) = G(a) + G(b). \] \( \text{Hint: Break up the integral from } 1 \text{ to } ab \text{ into two integrals and use (a).} \)

(c) \[ G(1) = 0 \text{ and } G(a^{-1}) = -G(a) \text{ for } a > 0. \]

(d) \[ G(a^r) = rG(a) \text{ for all } a > 0 \text{ and integers } n. \]

(e) \[ G(a^{1/n}) = \frac{1}{n} G(a) \text{ for all } a > 0 \text{ and integers } n \neq 0. \]

(f) \[ G(ab) = rG(a) \text{ for all } a > 0 \text{ and rational numbers } r. \]

(g) \( G \) is increasing. \( \text{Hint: Use FTC II.} \)

(h) There exists a number \( a \) such that \( G(a) > 1. \) \( \text{Hint: Show that } G(2) > 0 \text{ and take } a = 2^m \text{ for } m > 1/G(2). \)

(i) \[ \lim_{x \to \infty} G(x) = \infty \text{ and } \lim_{x \to 0^+} G(x) = -\infty. \]

(j) There exists a unique number \( E \) such that \( G(E) = 1. \)

(k) \( G(E^r) = r \) for every rational number \( r. \)

117. Defining \( e^x \)

Use Exercise 116 to prove the following statements.

(a) \( G \) has an inverse with domain \( R \) and range \( \{x : x > 0\}. \) Denote the inverse by \( F. \)

(b) \[ F(x + y) = F(x)F(y) \text{ for all } x, y. \] \( \text{Hint: It suffices to show that } G(F(x)F(y)) = G(F(x + y)). \)

(c) \[ F(r) = E^r \] for all numbers. In particular, \( F(0) = 1. \)

(d) \[ F'(x) = F(x). \] \( \text{Hint: Use the formula for the derivative of an inverse function.} \)

This shows that \( E = e \) and that \( F(x) \) is the function \( e^x \) as defined in the text.

118. Defining \( b^x \)

Let \( b > 0 \) and let \( f(x) = F(xG(b)) \) with \( F \) as in Exercise 117. Use Exercise 116 (f) to prove that \( f(r) = b^r \) for every rational number \( r. \) This gives us a way of defining \( b^x \) for irrational \( x, \) namely \( b^x = f(x). \) With this definition, \( f(x) = b^x \) is a differentiable function of \( x \) (because \( F \) is differentiable).

### 7.4 Applications of Exponential and Logarithmic Functions

In this section, we explore some applications involving exponential and logarithmic functions. To begin, consider a quantity \( P(t) \) that depends exponentially on time:

\[
P(t) = P_0 e^{kt}
\]

If \( k > 0, \) then \( P(t) \) grows exponentially and \( k \) is called the growth constant. Note that \( P_0 \) is the initial size (the size at \( t = 0)): \[
P(0) = P_0 e^{k \cdot 0} = P_0
\]

We can change the base and also write \( P(t) = P_0 b^t \) with \( b = e^k, \) because \( b^t = (e^k)^t = e^{kt}. \)

A quantity that decreases exponentially is said to have exponential decay. In this case, we write \( P(t) = P_0 e^{-kt} \) with \( k > 0, k \) is then called the decay constant.

Population is a typical example of a quantity that grows exponentially under suitable conditions, particularly in initial stages of growth. To understand why, consider a cell colony with initial population \( P_0 = 100 \) and assume that each cell divides into two cells after 1 hour. Then population \( P(t) \) doubles with each passing hour:

\[
P(0) = 100 \quad \text{(initial population)}
\]
\[
P(1) = 2(100) = 200 \quad \text{(population doubles)}
\]
\[
P(2) = 2(200) = 400 \quad \text{(population doubles again)}
\]

After \( t \) hours, \( P(t) = (100)2^t. \)

**Example 1**

During the Ebola virus outbreak in West Africa, from 2014 to 2016, there were approximately 750 cases reported up to July 1, 2014, and the number \( N \) of reported cases was increasing exponentially with a growth constant of approximately \( k = 0.019 \) (with time measured in days). Use \( N(t) = 750 e^{0.019t} \) as a model for the number of reported cases \( t \) days after July 1, 2014, and answer the following:

(a) What did the model indicate the number of reported cases would be 2 weeks later on July 15?
(b) According to the model, how many days would it take for the number of reported cases to double from 750 to 1500?

Solution

(a) July 15 corresponds to \( t = 14 \). At that time, \( N(14) = 750e^{0.019\cdot14} = 750e^{0.266} \approx 979 \) reported cases.

(b) The question asks for the time \( t \) such that \( N(t) = 1500 \), so we solve

\[
750e^{0.019t} = 1500 \quad \Rightarrow \quad e^{0.019t} = \frac{1500}{750} = 2
\]

Taking the natural logarithm of both sides, we obtain \( \ln(e^{0.019t}) = \ln 2 \), or

\[
0.019t = \ln 2 \quad \Rightarrow \quad t = \frac{\ln 2}{0.019} \approx 36.5
\]

Therefore, the model indicates that \( N(t) \) would have reached 1500 reported cases approximately 36.5 days after July 1 (Figure 1).

In exponential growth, the time it takes a quantity to double is called the **doubling time**. It depends only on the growth constant \( k \), and does not depend on the initial quantity. In general,

\[
\text{doubling time} = \frac{\ln 2}{k}
\]

Similarly in exponential decay, the **half-life** is the time it takes a quantity to decrease by half. If \( k \) is the decay constant, then

\[
\text{half-life} = \frac{\ln 0.5}{k}
\]

We discuss doubling time and half-life further in Section 10.1.

Exponential growth cannot continue over long periods of time. The number of Ebola cases in West Africa continued to grow exponentially until about the end of 2014, but through 2015 the growth slowed, and the number of reported cases leveled off at approximately 28,500.

The **logistic function** \( P(t) = \frac{M}{1 + Ae^{-kt}} \), with \( M, A, k \) all positive, is a function that is used to model phenomena that have an initial rapid increase but then level off toward some finite value (Figure 2).

**EXAMPLE 2** Logistic Functions and the Carrying Capacity of the Earth

From 1950 to 1980 to 2010 the human population of the earth grew from 2.65 to 4.45 to 6.90 billion people. Let \( t \) represent time in years since 1950 and \( P \) represent the population in billions. Using the \((t, P)\) values for 1950, 1980, and 2010, and employing a computer algebra system to solve for \( M, A, k \) in \( P(t) \), we obtain the logistic world-population model:

\[
P(t) = \frac{17.4}{1 + 5.56e^{-0.022t}}
\]

Find \( \lim_{t \to \infty} P(t) \).

**Solution** Since \( \lim_{t \to \infty} e^{-0.022t} = 0 \), then

\[
\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{17.4}{1 + 5.56e^{-0.022t}} = \frac{17.4}{1 + 0} = 17.4
\]
We can interpret this limiting value of 17.4 billion as a theoretical carrying capacity of the earth. Opinions vary on the earth’s actual human carrying capacity, and many approaches have been taken to develop estimates. This simply derived estimate is consistent with the carrying capacity models that are presented in a United Nations Environment Program 2012 review.

In Chapter 10, we further examine applications involving exponential growth and decay, as well as others involving the logistic function. In that setting, we demonstrate how the relationships arise via differential equation models. Exponential functions are used extensively in financial calculations. Two basic applications are compound interest and present value. An investment that earns continuously compounded interest at a fixed rate grows exponentially according to the following:

\[
P(t) = P_0 e^{rt}
\]

**Example 3** A principal of \( P_0 = $100,000 \) is deposited into an account paying 6\% interest compounded continuously. Find the balance after 3 years.

**Solution** After 3 years, the balance is \( 100,000e^{0.06\times3} \approx $119,722 \).

The concept of present value (PV) is used in business and finance to compare payments made at different times. Assume that there is an interest rate \( r \) (continuously compounded) at which an investor can lend or borrow money. By definition, the PV of \( P \) dollars to be received \( t \) years in the future is \( Pe^{-rt} \):

\[
\text{The PV of } P \text{ dollars received at time } t \text{ is } Pe^{-rt}.
\]

What is the reasoning behind this definition? When you invest at the rate \( r \) for \( t \) years, your principal increases by the factor \( e^t \), so if you invest \( Pe^{-rt} \) dollars, your principal grows to \( (Pe^{-rt})e^r = P \) dollars at time \( t \). The present value \( Pe^{-rt} \) is the amount you would have to invest today in order to have \( P \) dollars at time \( t \).

**Example 4** With an interest rate of \( r = 0.03 \), is it better to receive \( €2000 \) (euros) today or \( €2200 \) in 2 years? Does the answer change if the interest rate is \( r = 0.07 \) instead?

**Solution** We compare \( €2000 \) today with the PV of \( €2200 \) received in 2 years.

- If \( r = 0.03 \), the PV is \( 2200e^{-0.03\times2} \approx €2071.88 \). This is more than \( €2000 \), so a payment of \( €2200 \) in 2 years is preferable to a \( €2000 \) payment today.
- If \( r = 0.07 \), the PV is \( 2200e^{-0.07\times2} \approx €1912.59 \). This PV is less than \( €2000 \), so it is better to receive \( €2000 \) today if \( r = 0.07 \).

An income stream is a sequence of periodic payments that continue over an interval of \( T \) years. Consider an investment that produces income at a rate of \$800\$/year for 5 years. A total of \$4000 is paid out over 5 years, but the PV of the income stream is less. For instance, if \( r = 0.06 \) and payments are made at the end of the year, then the PV is

\[
800e^{-0.06} + 800e^{-0.06}\times2 + 800e^{-0.06}\times3 + 800e^{-0.06}\times4 + 800e^{-0.06}\times5 \approx $3353.12
\]

It is more convenient mathematically to assume that payments are made continuously at a rate of \( R(t) \) dollars per year. We can then calculate PV as an integral. Divide the time
interval \([0, T]\) into \(N\) subintervals of length \(\Delta t = T/N\). If \(\Delta t\) is small, the amount paid out between time \(t\) and \(t + \Delta t\) is approximately

\[
\frac{R(t)}{\text{Rate}} \times \frac{\Delta t}{\text{Time interval}} = R(t)\Delta t
\]

The PV of this payment is approximately \(e^{-rt}R(t)\Delta t\). Setting \(t_i = i\Delta t\), we obtain the approximation

\[
\text{PV of income stream} \approx \sum_{i=1}^{N} e^{-r t_i} R(t_i) \Delta t
\]

This is a Riemann sum whose value approaches \(\int_0^T R(t)e^{-rt} dt\) as \(\Delta t \to 0\).

**PV of an Income Stream** If the interest rate is \(r\), the present value of an income stream paying out \(R(t)\) dollars per year continuously for \(T\) years is

\[
\text{PV} = \int_0^T R(t)e^{-rt} dt
\]

**EXAMPLE 5** An investment pays out 800,000 Mexican pesos per year, continuously for 5 years. Find the PV of the investment for \(r = 0.04\) and \(r = 0.06\).

**Solution** In this case, \(R(t) = 800,000\). If \(r = 0.04\), the PV of the income stream is equal (in pesos) to

\[
\left. \int_0^5 800,000e^{-0.04t} dt \right|_0^5 = -800,000 \left[ e^{-0.04t} \right]_0^5 \\
\approx -16,374,615 - (-20,000,000) \\
= 3,625,385
\]

If \(r = 0.06\), the PV is equal (in pesos) to

\[
\left. \int_0^5 800,000e^{-0.06t} dt \right|_0^5 = -800,000 \left[ e^{-0.06t} \right]_0^5 \\
\approx -9,877,576 - (-13,333,333) \\
= 3,455,757
\]

**Applications of Logarithmic Functions**

Since logarithmic functions are inverses of exponential functions (which grow rapidly), logarithmic functions grow slowly, a property that is shown in Figure 3. Scientists exploit this slow growth of logarithmic functions to define logarithmic scales that measure phenomena that can have a very large range of values, such as the energy in earthquakes.

One scale for earthquakes is called the Moment Magnitude Scale, a logarithmic scale defined by

\[
M_w = \frac{2}{3} \log_{10} E - 10.7
\]

where \(M_w\) is the unitless moment magnitude of an earthquake and \(E\) is the energy (in ergs) released by the earthquake, which is directly related to the size of the earthquake fault and the distance the fault moved.

**EXAMPLE 6 Earthquake Measurement** The 2011 Tohoku earthquake in Japan, one of the strongest ever recorded, released \(3.9 \times 10^{29}\) ergs of energy, about 10 billion times
as much energy as a mild earthquake that rattles the windows in a house. What are the magnitudes of the mild (10^21 ergs of energy) and Tohoku earthquakes?

**Solution** If \( E = 10^{21} \), then

\[
M_w = \frac{2}{3} \log_{10} 10^{21} - 10.7 = \frac{2}{3}(21) - 10.7 = 3.3
\]

For the Tohoku earthquake, \( E = 3.9 \times 10^{29}, \) so

\[
M_w = \frac{2}{3} \log_{10}(3.9 \times 10^{29}) - 10.7 = \frac{2}{3}(\log_{10} 3.9 + 29) - 10.7 \approx 9.0
\]

The **log wind profile** is a formula for determining wind speeds at different heights near the surface of the earth. Over open agricultural land with few buildings and obstructions, it is expressed as

\[
v = v_0 \frac{\ln(h/h_0)}{\ln(h_0/0.03)}
\]

where \( v \) (in m/s) is the wind speed at height \( h \) (in m) and \( v_0 \) is the known wind speed at reference height \( h_0 \). Meteorologists believe this relationship is accurate for heights up to 200 m (Figure 4).

The value 0.03 is called a surface roughness factor. On different surfaces, this factor takes on different values. For example, 0.0002 is used over open water, and 0.4 is used over villages, forests, and rough uneven terrain.

**Example 7** Suppose \( v_0 = 10 \text{ m/s} \) at \( h_0 = 10 \text{ m} \) above a large corn field (thus with surface roughness factor 0.03). Determine \( v \) and \( dv/dh \) at \( h = 60 \text{ (a typical height of a wind-turbine tower).} \)

**Solution** With \( v_0 = 10 \) at \( h_0 = 10 \), it follows that

\[
10 \frac{\ln(h/0.03)}{\ln(10/0.03)} \approx 1.72 \ln \left( \frac{h}{0.03} \right) = 1.72(\ln h - \ln 0.03) \approx 1.72 \ln h + 6.03
\]

Therefore,

\[
v(h) = 1.72 \ln h + 6.03 \quad \text{and} \quad \frac{dv}{dh} = \frac{1.72}{h}
\]

Thus, at \( h = 60 \text{ m} \), we have \( v \approx 13.07 \text{ m/s} \) and \( dv/dh \approx 0.03 \text{ m/s per m} \). This rate of change indicates that at that height, the wind speed is increasing at three-hundredths of a meter per second for each meter increase in height.

In the next example, the natural logarithm appears in a relationship for the maximum height attained by a projectile when air resistance is taken into consideration. The relationship itself is derived via a differential equation model in Section 10.2.

**Example 8** Maximum Height Under Air Resistance A 1-kg bocce ball is launched straight upward at 30 m/s (Figure 5) and is acted on by gravity and an air resistance force. We assume that the latter force is in the form \(-kv(t)\), where \(v(t)\) is the ball's upward velocity and \(k\) is a positive constant reflecting the strength of the air resistance (the stronger the air resistance, the greater the value of \(k\)). The maximum height (in meters) that the ball reaches depends on the strength of the air resistance and is given by

\[
H(k) = \frac{30k - 9.8 \ln \left( \frac{150k}{49} + 1 \right)}{k^2}
\]

Values of \(k\) between 0 and 1 are physically reasonable. Intuitively, it makes sense that the stronger the air resistance, the lower the maximum height attained by the ball.
What is \( \lim_{k \to 0} H(k) \)? In other words, what happens to the maximum height with less and less air resistance, and ultimately with none at all?

**Solution** This limit has the indeterminate form \(0/0\). We cannot evaluate it by algebraic simplification, so we will estimate it here by examining it numerically. In the next section, we introduce L'Hôpital's Rule, which enables us to obtain an exact value for the limit. In the meantime, consider the following table of values:

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
</tr>
<tr>
<td>( H(k) )</td>
</tr>
</tbody>
</table>

The values in Table 1 suggest the limit is approximately 45.92. We can say that as the air resistance vanishes, the maximum height that the ball attains approaches approximately 45.92 m.

Note that the result in the previous example is consistent with the maximum projectile height obtained in Example 7 in Section 3.4, where the same initial velocity was used and air resistance was not taken into consideration.

**7.4 SUMMARY**

- **Exponential growth** with growth constant \( k > 0 \): \( P(t) = P_0e^{kt} \), doubling time = \( \frac{\ln 2}{k} \).
- **Exponential decay** with decay constant \( k > 0 \): \( P(t) = P_0e^{-kt} \), half-life = \( \frac{\ln(2)}{k} \).
- **Logistic function** with \( M, A, k \) positive: \( P(t) = \frac{M}{1 + Ae^{-kt}} \), which models initial rapid growth that eventually levels off to a limit of \( M \) as \( t \to \infty \).
- **Account value** with an interest rate \( r \), compounded continuously: \( P(t) = P_0e^{rt} \).
- **Present value** (PV) of \( P \) dollars (or other currency), to be paid \( t \) years in the future: \( PV = Pe^{-rt} \).
- **Present value** of an income stream paying \( R(t) \) dollars per year continuously for \( T \) years:

\[
PV = \int_0^T R(t)e^{-rt} \, dt
\]

- **Magnitude Scale** for earthquake measurement: \( M_w = \frac{3}{2} \log_{10} E - 10.7 \).
- **Log wind profile**: \( v = \frac{\ln(h/r)}{\ln(h_0/r)} \), where \( r \) is a surface roughness factor.

**7.4 EXERCISES**

**Preliminary Questions**

1. Two quantities increase exponentially with growth constants \( k = 1.2 \) and \( k = 3.4 \), respectively. Which quantity doubles more rapidly?

2. A cell population grows exponentially beginning with one cell. Which takes longer: increasing from one to two cells or increasing from 15 million to 20 million cells?

3. For the logistic function \( P(t) = \frac{M}{1 + Ae^{-kt}} \) with \( k > 0 \), what is \( \lim_{t \to \infty} P(t) \)?

4. The PV of \( N \) dollars received at time \( T \) is (choose the correct answer):
   (a) The value at time \( T \) of \( N \) dollars invested today (b) The amount you would have to invest today in order to receive \( N \) dollars at time \( T \)

5. In one year, you will be paid $1. Will the PV increase or decrease if the interest rate goes up at the start of the year?

6. For \( y = \log_{10} x \), if \( y \) increases by 2, then (choose the correct answer):
   (a) \( x \) increases by 20
   (b) \( x \) decreases by 20
   (c) \( x \) increases by a factor of 2
   (d) \( x \) increases by a factor of 100
Exercises

1. A certain population $P$ of bacteria obeys the exponential growth law $P(t) = P_0e^{kt}$ (in hours).
   (a) How many bacteria are present initially?
   (b) At what time will there be $10,000$ bacteria?

2. A quantity $P$ obeys the exponential growth law $P(t) = e^{kt}$ (in years).
   (a) At what time $t$ is $P = 10$?
   (b) At what time $t$ is $P = 20$?
   (c) At what time $t$ is $P = 40$?
   (d) How long does it take for $P$ to double?

3. Write $f(t) = 5(7)^t$ in the form $f(t) = P_0e^{kt}$ for some $P_0$ and $k$.

4. Write $f(t) = 9e^{4t}$ in the form $f(t) = P_0b^t$ for some $P_0$ and $b$.

5. A quantity $P$ obeys the exponential growth law $P(t) = Ce^{kt}$ (in years). Find the formula for $P(t)$, assuming that $P(0) = 100$ and that it takes $5$ years for $P$ to double.

6. A $10$-kg quantity of a radioactive isotope decays to $3$ kg after $17$ years. Find the decay constant of the isotope.

7. Assuming that population growth is approximately exponential, which of the following two sets of data is most likely to represent the population (in millions) of a city over a $5$-year period?

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set I</td>
<td>3.14</td>
<td>3.36</td>
<td>3.60</td>
<td>3.85</td>
<td>4.11</td>
</tr>
<tr>
<td>Set II</td>
<td>3.14</td>
<td>3.24</td>
<td>3.54</td>
<td>4.04</td>
<td>4.74</td>
</tr>
</tbody>
</table>

8. In 2009, 2012, and 2015, the number (in millions) of smart phones sold in the world was 172.4, 680.1, and 1423.9, respectively.
   (a) Let $t$ represent time in years since 2009, and let $S$ represent the number of smart phones sold in millions. Determine $M$, $A$, and $k$ for a logistic model, $S(t) = \frac{M}{1 + Ae^{-kt}}$, that fits the given data points.
   (b) What is the long-term expected maximum number of smart phones sold annually? That is, what is $\lim_{t \to \infty} S(t)$?
   (c) In what year does the model predict that smart-phones sales will reach $98\%$ of the expected maximum?

9. Sam was $28$ inches tall on her first birthday, $50$ inches tall on her eighth, and $62$ inches tall on her 14th.
   (a) Let $t$ represent Sam's age in years, and let $h$ represent her height in inches. Determine $M$, $A$, and $k$ for a logistic model, $h(t) = \frac{M}{1 + Ae^{-kt}}$, that fits the given height data.
   (b) What is Sam's long-term expected height? That is, what is $\lim_{t \to \infty} h(t)$?
   (c) At what age does the model predict that Sam will reach $95\%$ of her expected maximum height?

10. Suppose $500$ is deposited into an account paying interest at a rate of $7\%$, continuously compounded. Find a formula for the value of the account at time $t$. What is the value of the account after $3$ years?

11. How long will it take for $4000$ to double in value if it is deposited in an account bearing $7\%$ interest, continuously compounded?

12. How much must one invest today in order to receive $20,000$ after $5$ years if interest is compounded continuously at the rate $r = 9\%$?

13. A $10000$ investment increases in value at a continuously compounded rate of $9\%$. How large must the initial investment be in order to build up a value of $45000$ over a 7-year period?

14. Compute the PV of $5000$ received in $3$ years if the interest rate is (a) $6\%$ and (b) $11\%$. What is the PV in these two cases if the sum is instead received in $5$ years?

15. Is it better to receive $1000$ today or $1300$ in $4$ years? Consider $r = 0.08$ and $r = 0.03$.

16. Find the PV of an investment that pays out continuously at a rate of $800$ per year for $5$ years, assuming $r = 0.08$.

17. Find the PV of an income stream that pays out continuously at a rate $R(t) = 5000e^{0.1t}$ per year for $7$ years, assuming $r = 0.05$.

18. The decibel level for the intensity of a sound is a logarithmic scale defined by $D = 10 \log_{10} I + 120$, where $I$ is the intensity of the sound in watts per square meter.
   (a) Express $I$ as a function of $D$.
   (b) Show that when $D$ increases by $20$, the intensity increases by a factor of $100$.
   (c) Compute $dI/dD$.

19. Consider the equation $M_w = 9.3 \log_{10} E - 10.7$, relating the moment magnitude of an earthquake and the energy $E$ (in ergs) released by it.
   (a) Express $E$ as a function of $M_w$.
   (b) Show that when $M_w$ increases by $1$, the energy increases by a factor of approximately $31.6$.
   (c) Compute $dE/dM_w$.

20. Over rough uneven terrain, the log wind profile is expressed as $v = \frac{\ln(h/b_0)}{\ln(h/b_0)}$
   With $v_0 = 10$ m/s at $b_0 = 10$ m, determine $v$ and $dv/df$ at $h = 60$.

21. Over open water, the log wind profile is expressed as $v = \frac{\ln(h/b_0)}{\ln(h/b_0)}$
   With $v_0 = 10$ m/s at $b_0 = 10$ m, determine $v$ and $dv/df$ at $h = 60$.

22. The Palermo Technical Impact Hazard Scale $P$ is used to quantify the risk associated with the impact of an asteroid colliding with the earth:

   $P = \log_{10} \left( \frac{P_h}{0.017} \right)$

   where $P_h$ is the probability of impact, $T$ is the number of years until impact, and $E$ is the energy of impact (in megatons of TNT). The risk is greater than a random event of similar magnitude if $P > 0$.
   (a) Calculate $dP/dT$, assuming that $P_h = 2 \times 10^{-3}$ and $E = 2$ megatons.
   (b) Use the derivative to estimate the change in $P$ if $T$ increases from $8$ to $9$ years.

   In Exercises 23 and 24, a ball is launched straight up in the air and is acted on by air resistance and gravity as in Example 8. The function $H$ gives the maximum height of the ball as a function of the air-resistance parameter $k$. In each case, estimate the maximum height without air resistance by investigating $H(k)$ numerically.

23. A ball with a mass of $1$ kg is launched upward with an initial velocity of $60$ m/s, and

   $H(k) = \frac{60k - 9.8 \ln(\frac{200}{k} + 1)}{k^2}$
24. A ball with a mass of 500 g is launched upward with an initial velocity of 30 m/s, and
\[ H(k) = \frac{15k - 2.45\ln \left( \frac{308}{49} \right) + 1}{k^2} \]

25. The Beer–Lambert Law is used in spectroscopy to determine the molar absorptivity \( \alpha > 0 \) or the concentration \( c > 0 \) of a compound dissolved in a solution at low concentrations (Figure 7). The law states that the intensity \( I \) of light as it passes through the solution satisfies \( \ln(I/I_0) = \alpha x \), where \( I_0 \) is the initial intensity and \( x \) is the distance traveled by the light. Show that \( I \) decays exponentially as a function of \( x \).

\[ \text{Intensity } I \rightarrow \text{Distance } x \]

\[ I_0 \rightarrow \text{Solution} \]

\[ \text{FIGURE 7 Light of intensity } I_0 \text{ passing through a solution.} \]

26. Consider the logistic world population model from Example 2: \( P(t) = \frac{\mathfrak{M} \left( \frac{t}{t + 5} \right)}{1 + e^{-k(t-5)}} \).
   (a) Determine the time \( t \) when the population is half of the expected long-term limit of 17.4 billion.
   (b) There is a single point of inflection for \( P \). Determine it.
   [Note: You should find that the value of \( t \) is the same for both (a) and (b). This is not a coincidence; see the next exercise.]

27. Consider the general logistic function \( P(t) = \frac{\mathfrak{M} e^{kt}}{1 + Ae^{-kt}} \), with \( A, M, \) and \( k \) all positive. Show that
   (a) \( P'(t) = \frac{M \mathfrak{M} e^{kt}}{(1 + Ae^{-kt})^2} \) and \( P''(t) = \frac{M \mathfrak{M} e^{kt} A^2 e^{-kt}}{(1 + Ae^{-kt})^3} \).
   (b) \( \lim_{t \to -\infty} P(t) = 0 \) and \( \lim_{t \to \infty} P(t) = M \), and therefore \( P = 0 \) and \( P = M \) are horizontal asymptotes of \( P \).
   (c) \( P \) is increasing for all \( t \).
   (d) The only inflection point of \( P \) is at \( \left( \frac{\ln A}{k}, \frac{M}{2} \right) \). To the left of it, \( P \) is concave up, and to the right of it, \( P \) is concave down.

Further Insights and Challenges

Periodically Compounded Interest: When interest is compounded in an account \( n \) times in a year (rather than continuously), the annual interest rate \( r \) is divided by \( n \), and the amount paid each time that interest is compounded is \( A \left( 1 + \frac{r}{n} \right)^n \), where \( A \) is the amount that is in the account at the time the interest is compounded.

33. (a) Show that at an annual rate of \( r \), if interest is compounded \( n \) times in a year, then the amount in the account at the end of the year is \( A \left( 1 + \frac{r}{n} \right)^n \), where \( A \) was the amount in the account at the start of the year.
   (b) Show that if \( A_0 \) is initially invested in an account that pays annual interest \( r \), compounded \( n \) times yearly, then the amount in the account after \( t \) years is \( A(t) = A_0 \left( 1 + \frac{r}{n} \right)^{nt} \).

Exercises 28 and 29: The Gompertz differential equation
\[ \frac{dy}{dt} = ky \ln \left( \frac{y}{M} \right) \]
(where \( M \) and \( k \) are constants) was introduced in 1825 by the English mathematician Benjamin Gompertz and is still used today to model aging and mortality.

28. Show that \( y = Me^{ex} \) satisfies Eq. (2) for any constant \( a \).

29. To model mortality in a population of 200 laboratory rats, a scientist assumes that the number \( P(t) \) of rats alive at time \( t \) (in months) satisfies Eq. (2) with \( M = 204 \) and \( k = 0.15 \) (Figure 8). Find \( P(t) \) [note that \( P(0) = 200 \)] and determine the population after 20 months.

\[ \text{Rat population } P(t) \]

\[ \text{Time (months)} \]

30. A company can earn additional profits of $500,000/year for 5 years by investing $2 million to upgrade its factory. Is the investment worthwhile if the interest rate is 6%? (Assume the savings are received as a lump sum at the end of each year.)

31. A new computer system costing $25,000 will reduce labor costs by $7000/year for 5 years.
   (a) Is it a good investment if \( r = 8\% \) ?
   (b) How much money will the company actually save?

32. \( \square \) Banker's Rule of 70: If you earn an interest rate of \( R \) percent, continuously compounded, your money doubles after approximately \( 70/R \) years. For example, at \( R = 5\% \), your money doubles after 70/5 or 14 years. Use the concept of doubling time to justify the Banker's Rule. [Note: Sometimes, the rule 72/R is used. It is less accurate but easier to apply because 72 is divisible by more numbers than 70.)]
308  CHAPTER 7  EXPONENTIAL AND LOGARITHMIC FUNCTIONS

(c) Prove that
\[
\frac{r}{1 + r/n} \leq \ln((1 + r/n)^n) \leq r
\]

(d) Prove that \( \lim_{n \to \infty} \ln((1 + r/n)^n) = r \).

(e) Prove that \( \lim_{n \to \infty} (1 + r/n)^n = e^r \).

(f) Prove that for an interest rate \( r \), continuous compounding of interest is the limit as \( n \to \infty \) of \( n \)-times annual periodic compounding of interest.

7.5 L'Hôpital's Rule

L'Hôpital's Rule is a valuable tool for computing certain limits that are otherwise difficult to evaluate, and also for determining "asymptotic behavior" (limits at infinity). Consider the limit of a quotient:

\[
\lim_{x \to a} \frac{f(x)}{g(x)}
\]

Roughly speaking, L'Hôpital's Rule states that when \( f(x)/g(x) \) has an indeterminate form of type 0/0 or \( \infty/\infty \) at \( x = a \), then we can replace \( f(x)/g(x) \) by the quotient of the derivatives \( f'(x)/g'(x) \).

**THEOREM 1: L'Hôpital's Rule**

Assume that \( f \) and \( g \) are differentiable on an open interval containing \( a \) and that

\[
f(a) = g(a) = 0 \]

Also assume that \( g'(x) \neq 0 \) (except possibly at \( a \)). Then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

if the limit on the right exists or is infinite (\( \infty \) or \( -\infty \)). This conclusion also holds if \( f \) and \( g \) are differentiable for \( x \) near \( a \) (but not equal to) \( a \) and

\[
\lim_{x \to a} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm \infty
\]

Furthermore, this rule is valid for one-sided limits.

**EXAMPLE 1:** Use L'Hôpital's Rule to evaluate \( \lim_{x \to 2} \frac{x^3 - 8}{x^3 + 2x - 20} \).

Solution Let \( f(x) = x^3 - 8 \) and \( g(x) = x^3 + 2x - 20 \). Both \( f \) and \( g \) are differentiable and \( f(x)/g(x) \) is indeterminate of type 0/0 at \( x = 2 \) because \( f(2) = g(2) = 0 \).

Furthermore, \( g'(x) = 4x^3 + 2 \) is nonzero near \( x = 2 \), so L'Hôpital's Rule applies. We may replace the numerator and denominator by their derivatives to obtain

\[
\lim_{x \to 2} \frac{x^3 - 8}{x^3 + 2x - 20} = \lim_{x \to 2} \frac{3x^2}{4x^3 + 2} = \lim_{x \to 2} \frac{3x^2}{4x^3 + 2} = \frac{3(2^2)}{4(2^3) + 2} = \frac{12}{34} = \frac{6}{17}
\]

L'Hôpital's Rule

**EXAMPLE 2:** Evaluate \( \lim_{x \to \pi/2} \frac{\cos^2 x}{1 - \sin x} \).

Solution Again, the quotient is indeterminate of type 0/0 at \( x = \pi/2 \) since

\[
\cos^2 \left( \frac{\pi}{2} \right) = 0, \quad 1 - \sin \frac{\pi}{2} = 1 - 1 = 0
\]
The other hypotheses are satisfied, so we may apply L'Hôpital's Rule:

\[
\lim_{x \to \pi/2} \frac{\cos^2 x}{1 - \sin x} = \lim_{x \to \pi/2} \frac{(\cos^2 x)'}{1 - \sin x} = \lim_{x \to \pi/2} -\frac{2 \cos x \sin x}{-\cos x} = \lim_{x \to \pi/2} (2 \sin x) = 2
\]

L'Hôpital's Rule

Simplified

Note that the quotient \(\frac{-2 \cos x \sin x}{-\cos x}\) is also indeterminate at \(x = \pi/2\). We removed this indeterminacy by cancelling the factor \(-\cos x\).

**Example 3** The Form \(0 \cdot \infty\) Evaluate \(\lim_{x \to 0^+} x \ln x\).

**Solution** This limit is one-sided because \(f(x) = x \ln x\) is not defined for \(x \leq 0\). Furthermore, as \(x \to 0^+\),

- \(x\) approaches 0.
- \(\ln x\) approaches \(-\infty\).

So \(f(x)\) presents an indeterminate form of type \(0 \cdot \infty\). To apply L'Hôpital's Rule, we rewrite our function as \(f(x) = (\ln x)/x\) so that \(f(x)\) presents an indeterminate form of type \(-\infty/\infty\). Then L'Hôpital's Rule applies:

\[
\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{(\ln x)'}{x'} = \lim_{x \to 0^+} \left(-\frac{1}{x-1}\right) = \lim_{x \to 0^+} (-x) = 0
\]

L'Hôpital's Rule

Simplified

**Example 4** Using L'Hôpital's Rule Twice Evaluate \(\lim_{x \to 0} \frac{e^x - x - 1}{\cos x - 1}\).

**Solution** The limit is in the indeterminate form \(0/0\) since at \(x = 0\), we have

\[e^x - x - 1 = e^0 - 0 - 1 = 0, \quad \cos x - 1 = \cos 0 - 1 = 0\]

A first application of L'Hôpital's Rule gives

\[
\lim_{x \to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{(e^x - x - 1)'}{\cos x - 1} = \lim_{x \to 0} \frac{e^x - 1}{-\sin x} = \lim_{x \to 0} \frac{1 - e^x}{\sin x}
\]

This limit is again indeterminate of type \(0/0\), so we apply L'Hôpital's Rule a second time:

\[
\lim_{x \to 0} \frac{1 - e^x}{\sin x} = \lim_{x \to 0} \frac{-e^x}{\cos x} = \frac{-e^0}{\cos 0} = -1
\]

It follows that

\[
\lim_{x \to 0} \frac{e^x - x - 1}{\cos x - 1} = -1
\]

**Example 5** Maximum Height Under Air Resistance In Example 8 in the previous section, we introduced a function

\[H(k) = \frac{30k - 9.8 \ln \left(\frac{18.5}{49} + 1\right)}{k^3}\]

that gives the maximum height attained by a 1-kg ball launched upward at 30 m/s with gravity and air resistance acting on it. (The function is derived in Section 10.2.) The variable \(k\) reflects the strength of the air resistance. We investigated what happens to the maximum height as the air resistance approaches zero; that is, we investigated \(\lim_{k \to 0} H(k)\) numerically. Show this limit can be evaluated using L'Hôpital's Rule and find the limit.
Solution The quotient \[ \frac{30k - 9.8 \ln \left( \frac{150}{49} + 1 \right)}{k^2} \] has the indeterminate form 0/0. To evaluate the limit, we need to use L’Hôpital’s Rule twice:

\[
\lim_{k \to 0} \frac{30k - 9.8 \ln \left( \frac{150}{49} + 1 \right)}{k^2} = \lim_{k \to 0} \frac{30 - 9.8 \frac{1}{2k}}{2k} = \lim_{k \to 0} \frac{4500/49}{2} = \frac{2250}{49} \approx 45.92
\]

This value of 45.92 matches our numerical estimate in Example 8 in the previous section and the result we obtained separately in Example 7 in Section 3.4, where we considered the launched projectile’s height, ignoring air resistance altogether.

**EXAMPLE 6 Assumptions Matter** Can L’Hôpital’s Rule be applied to \( \lim_{x \to 1} \frac{x^2 + 1}{2x + 1} \)?

**Solution** The answer is no. The function does not have an indeterminate form because

\[
\left. \frac{x^2 + 1}{2x + 1} \right|_{x=1} = \frac{1^2 + 1}{2 \cdot 1 + 1} = \frac{2}{3}
\]

This limit can be evaluated directly by substitution: \( \lim_{x \to 1} \frac{x^2 + 1}{2x + 1} = \frac{2}{3} \). An incorrect application of L’Hôpital’s Rule gives the wrong answer:

\[
\lim_{x \to 1} \left( \frac{x^2 + 1}{2x + 1} \right)' = \lim_{x \to 1} \frac{2x}{2} = 1 \ (\text{not equal to original limit})
\]

**EXAMPLE 7 The Form \( \infty - \infty \)** Evaluate \( \lim_{x \to 0} \left( \frac{\sin x}{x} - \frac{1}{x} \right) \).

**Solution** Both \( 1/\sin x \) and \( 1/x \) become infinite at \( x = 0 \), so we have an indeterminate form of type \( \infty - \infty \). We rewrite the function as

\[
\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}
\]

to obtain an indeterminate form of type 0/0. Applying L’Hôpital’s Rule twice yields

\[
\lim_{x \to 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{x \cos x + \sin x}
\]

L’Hôpital’s Rule

\[
= \lim_{x \to 0} \frac{\sin x}{x \cos x + 2 \cos x} = \frac{0}{2} = 0
\]

L’Hôpital’s Rule again

This value of the limit is confirmed graphically in Figure 1.

Limits of functions of the form \( f(x)^{g(x)} \) can lead to the indeterminate forms \( 0^0 \), \( 1^\infty \), or \( \infty^0 \). These are indeterminate since the limit can take on a variety of values, depending on the relative rates at which the base and exponent approach their limits. In evaluating these limits, we use the change-of-base formula to write \( f(x)^{g(x)} = e^{g(x) \ln f(x)} \) and then we obtain

\[
\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x) \ln f(x)} = e^{\lim_{x \to a} g(x) \ln f(x)}
\]

The last equality is justified by the continuity of the exponential function.
EXAMPLE 8 The Form $0^0$  Evaluate $\lim_{x \to 0^+} x^x$. 

Solution With $x^x = e^{x \ln x}$ by the change-of-base formula, it will be enough to consider the limit of $x \ln x$. Example 3 showed $\lim_{x \to 0^+} x \ln x = 0$. Therefore,

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1$$

This value for the limit is confirmed graphically in Figure 2.

In Section 1 in this chapter, we pointed out that $e$ is the value that $(1 + x)^{1/x}$ approaches as $x$ approaches 0. This can be verified now by evaluating $\lim_{x \to 0} (1 + x)^{1/x}$ using L'Hôpital's Rule.

EXAMPLE 9 The Form $1^\infty$  Find $\lim_{x \to 0} (1 + x)^{1/x}$.

Solution This has the indeterminate form $1^\infty$. We take the approach used in Example 8. Thus, we write $(1 + x)^{1/x} = e^{1/x \ln(1 + x)}$ and consider $\lim_{x \to 0} \frac{1}{x} \ln(1 + x)$. We obtain (using L'Hôpital's Rule for the first equality)

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \frac{1}{1 + x} = 1$$

Therefore,

$$\lim_{x \to 0} (1 + x)^{1/x} = \lim_{x \to 0} e^{\frac{1}{x} \ln(1 + x)} = e^{\lim_{x \to 0} \frac{1}{x} \ln(1 + x)} = e^1 = e$$

Note that if we substitute $x = \frac{1}{r}$ into $\lim_{r \to \infty} \left(1 + \frac{1}{r}\right)^r$ we obtain the limit in the previous example. Therefore, $\lim_{r \to \infty} \left(1 + \frac{1}{r}\right)^r = e$. It is important to be familiar with these limits whose values are $e$:

$$e = \lim_{x \to 0} (1 + x)^{1/x} \quad \text{and} \quad e = \lim_{r \to \infty} \left(1 + \frac{1}{r}\right)^r$$

They arise in limit evaluations that we will see subsequently in the text.

CONCEPTUAL INSIGHT Exponential Limit Forms  Knowing that $0 \cdot \infty$ is an indeterminate form, and using the exponential identity $a^x = e^{x \ln a}$, we can see why $0^0$, $1^\infty$, and $\infty^0$ are indeterminate forms. A similar approach also shows why $0^\infty$ is not indeterminate and corresponds to a limit that equals 0.

The Form $0^0$: If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x)^{g(x)} = 0$. Therefore, in the limit, the equivalent exponential expression $e^{g(x) \ln f(x)}$ has an exponent in the indeterminate form $0 \cdot \infty$ since $g(x) \to 0$ and $\ln f(x) \to -\infty$ [because $f(x) \to 0$]. Therefore, $0^0$ is an indeterminate form.

Similar arguments can be made to demonstrate that $1^\infty$ and $\infty^0$ are indeterminate forms (see Exercise 57).

The Form $0^\infty$: If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} f(x)^{g(x)} = 0$. Therefore, in the limit, the equivalent exponential expression $e^{g(x) \ln f(x)}$ has an exponent in the form $\infty \cdot (-\infty)$. Since the limit of the exponent is $-\infty$, it follows that the limit of $e^{g(x) \ln f(x)}$ is 0, and therefore the limit of $f(x)^{g(x)}$ is as well. Thus, the form $0^\infty$ is not indeterminate but instead corresponds to a limit that is equal to 0.
Comparing Growth of Functions

Sometimes, we are interested in determining which of two given functions grows faster. For example, Quick Sort and Bubble Sort are two standard computer algorithms for sorting data (e.g., alphabetizing, ordering according to rank). The average time required to sort a list of size $n$ is approximately $n \log n$ for Quick Sort and $n^2$ for Bubble Sort. Which algorithm is faster when the size $n$ is large? This problem amounts to comparing the growth of $Q(x) = x \ln x$ and $B(x) = x^2$ as $x \to \infty$.

We say that $f(x)$ grows faster than $g(x)$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x \to \infty} g(x) = 0$$

To indicate that $f(x)$ grows faster than $g(x)$, we use the notation $g(x) \ll f(x)$. For example, $x \ll x^2$ because

$$\lim_{x \to \infty} \frac{x^2}{x} = \lim_{x \to \infty} x = \infty$$

To compare the growth of functions, we need a version of L'Hôpital's Rule that applies to limits at infinity.

**Theorem 2** L'Hôpital's Rule for Limits at Infinity

Assume that $f$ and $g$ are differentiable in an interval $(b, \infty)$ and that $g'(x) \neq 0$ for $x > b$. If $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} g(x)$ exist and either both are zero or both are infinite, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists. A similar result holds for limits as $x \to -\infty$.

**Example 10** The Form $\frac{\infty}{\infty}$

Which of $B(x) = x^2$ or $Q(x) = x \ln x$ grows faster as $x \to \infty$?

**Solution** Both $B(x)$ and $Q(x)$ approach infinity as $x \to \infty$, so L'Hôpital's Rule applies to the quotient:

$$\lim_{x \to \infty} \frac{B(x)}{Q(x)} = \lim_{x \to \infty} \frac{x^2}{x \ln x} = \lim_{x \to \infty} \frac{x}{\ln x} = \lim_{x \to \infty} \frac{1}{x^{-1}} = \lim_{x \to \infty} x = \infty$$

We conclude that $x \ln x \ll x^2$ (Figure 3).

Note that this example implies that Quick Sort is a much faster sorting algorithm than Bubble Sort for large $n$.

In Section 1 in this chapter, we asserted that exponential functions increase more rapidly than the power functions. We now prove this by showing that $x^n \ll e^t$ for every exponent $n$ (Figure 4).

**Theorem 3** Growth of $f(t) = e^t$

$$x^n \ll e^t \quad \text{for every exponent} \ n$$

In other words, $\lim_{t \to \infty} \frac{e^t}{x^n} = \infty$ for all $n$.  

---

The content is presented in a clear and structured manner, ensuring that the mathematical expressions and theorems are accurately transcribed. The use of L'Hôpital's Rule and the comparison of growth rates of functions are explained with clarity and precision.
**Proof** We first prove the theorem for positive integers $n$.

$$
\lim_{x \to \infty} \frac{e^x}{x^n} = \lim_{x \to \infty} \frac{e^x}{nx^{n-1}} = \lim_{x \to \infty} \frac{e^x}{n(n-1)x^{n-2}} = \ldots = \lim_{x \to \infty} \frac{e^x}{n!}
$$

We applied L'Hôpital's Rule $n$ times, each time obtaining an indeterminate form $\frac{\infty}{\infty}$, until the last stage shown. In $\lim_{x \to \infty} \frac{e^x}{x^n}$, the numerator goes to $\infty$ and the denominator is constant (relative to $x$). Therefore, that limit is infinite, implying that $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$ if $n$ is a positive integer.

If $n$ is any exponent, we can choose a natural number $k$ such that $k > n$. It is easy to see that $x^n \ll x^k$, and because we also have $x^k \ll e^n$, it follows that $x^n \ll e^n$ for all exponents $n$.

**Proof of L'Hôpital's Rule**

We prove L'Hôpital's Rule here only in the first case of Theorem 1—namely, in the case that $f(a) = g(a) = 0$. We also assume that $f'$ and $g'$ are continuous at $x = a$ and that $g'(a) \neq 0$. Then $g(x) \neq g(a)$ for $x$ near $a$, but not equal to $a$, and

$$
\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x) - f(a)}{x - a}.
$$

By the Quotient Law for Limits and the definition of the derivative,

$$
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x) - f(a)}{\lim_{x \to a} g(x) - g(a)} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.
$$

---

**7.5 SUMMARY**

- L'Hôpital's Rule: Assume that $f$ and $g$ are differentiable near $a$ and that $f(a) = g(a) = 0$.

Assume also that $g'(x) \neq 0$ (except possibly at $a$). Then

$$
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
$$

provided that the limit on the right exists or is infinite ($\infty$ or $-\infty$).

- L'Hôpital's Rule applies to indeterminate forms $0/0$ and $\pm \infty/\infty$. It can also apply to limits in any of the forms $0 \cdot \infty$, $\infty - \infty$, $0/0$, $1^\infty$, and $\infty^0$ by converting the expression to one in either the form $0/0$ or the form $\pm \infty/\infty$.

- L'Hôpital's Rule also applies to limits as $x \to \infty$ or $x \to -\infty$.

- In comparing the growth rates of functions, we say that $f(x)$ grows faster than $g(x)$, and we write $g \ll f$, if

$$
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty
$$
7.5 EXERCISES

Preliminary Questions

1. What is wrong with applying L'Hôpital's Rule to $\lim_{x \to 0} \frac{x^2 - 2x}{3x - 2}$?

2. Does L'Hôpital's Rule apply to $\lim_{x \to a} f(x)g(x)$ if $f(x)$ and $g(x)$ both approach 0 as $x \to a$?

3. What is wrong with saying, "To apply L'Hôpital's Rule to the limit $\lim_{x \to a} \frac{\ln(1 - x)}{x}$, use the Quotient Rule to differentiate $\ln(1 - x)$ and then take the limit."

4. What is wrong with applying L'Hôpital's Rule to $\lim_{x \to 0^+} x^{1/2}$?

5. What property of the function $f(x) = e^x$ allows us to say $\lim_{x \to a} e^{f(x)} = e^{f(a)}$?

Exercises

In Exercises 1–10, evaluate the limit, using L'Hôpital's Rule where it applies.

1. $\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 4}$

2. $\lim_{x \to 5} \frac{x^2 - 25}{5 - 4x - x^2}$

3. $\lim_{x \to 10} \frac{x^3 - 64}{x^2 - 16}$

4. $\lim_{x \to 1} \frac{x^3 + 2x + 1}{x^2 - 2x - 1}$

5. $\lim_{x \to 2} \frac{\sqrt{x + 1} - 2}{x^3 - 7x - 6}$

6. $\lim_{x \to 3} \frac{\sin 4x}{x^3 + 3x + 1}$

7. $\lim_{x \to x_0} \frac{\cos 2x - 1}{\sin 5x}$

8. $\lim_{x \to 0} \frac{\cos x - \sin x}{\tan x}$

9. $\lim_{x \to 0} \frac{\cos x}{x}$

10. $\lim_{x \to 0} \frac{\cos x - \sin x}{\tan x}$

11. $\lim_{x \to \infty} \frac{9x^4 + 4}{3 - 2x}$

12. $\lim_{x \to \infty} \frac{x}{\sin x}$

13. $\lim_{x \to \infty} \frac{\ln x}{x}$

14. $\lim_{x \to 0} \frac{\ln(1 + x)}{x}$

15. $\lim_{x \to \infty} \frac{\ln(x^2 + 1)}{x}$

16. $\lim_{x \to 0^+} \frac{x^2}{\ln x}$

In Exercises 11–16, use L'Hôpital's Rule to evaluate the limit.

17. $\lim_{x \to 1} \frac{\sqrt{8 + x - 3x^{1/3}}}{x^2 - 3x + 2}$

18. $\lim_{x \to 4} \frac{\left(\frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4}\right)}{x}$

19. $\lim_{x \to \infty} \frac{3x - 2}{x - 5}$

20. $\lim_{x \to 0^+} \frac{\sqrt[3]{3} + 3x}{x^{3/2} - x}$

21. $\lim_{x \to 9} \frac{7x^2 + 4x}{9 - 3x^2}$

22. $\lim_{x \to \infty} \frac{3x^3 + 4x^2}{4x^3 - 7x^2}$

23. $\lim_{x \to 1} \frac{(1 + 3x)^{1/2} - 2}{x - 1}$

24. $\lim_{x \to 1} \frac{x^{3/2} - 2x - 16}{x^{1/3} - 2}$

25. $\lim_{x \to 0^+} \frac{\sin 2x}{x}$

26. $\lim_{x \to 0^+} \frac{\tan 4x}{\tan 5x}$

27. $\lim_{x \to 0^+} \frac{x}{\sin x}$

28. $\lim_{x \to 0} \left(\cot x - \frac{1}{x}\right)$

29. $\lim_{x \to \pi/2} \frac{x - \pi/2}{\sin x}$

30. $\lim_{x \to \pi/2} \frac{\tan x}{\tan x}$

31. $\lim_{x \to 0^+} \frac{\cos x + \frac{x}{4}}{\sin x}$

32. $\lim_{x \to 0^+} \frac{1}{1 - \cos x}$

33. $\lim_{x \to 0} \frac{\cos x}{x}$

34. $\lim_{x \to 0} \frac{1}{x^2 - \tan^2 x}$

35. $\lim_{x \to 0} \frac{\sqrt{x} - x}{x^2 - 2}$

36. $\lim_{x \to 0} \frac{e^x - e^{-x}}{x^2}$

37. $\lim_{x \to 1} \frac{\ln x}{x^2 - 1} - \frac{\ln x}{x - 1}$

38. $\lim_{x \to 0} \frac{1}{x^2}$

39. $\lim_{x \to 1} \frac{e^x - 1}{x - 1}$

40. $\lim_{x \to 0} \frac{e^x - e}{x - 1}$

41. $\lim_{x \to 0} \frac{e^x - 1}{x^2}$

42. $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$

43. $\lim_{x \to 1} \frac{\sin x}{\ln x}$

44. $\lim_{x \to 0} \frac{e^x - x^3 + 9}{x^2}$

45. $\lim_{x \to 0} \frac{e^{\frac{1}{x^2}}}{e^{\frac{x}{x^2}}}$

46. $\lim_{x \to 0} \frac{e^{x^2}}{x}$

47. $\lim_{x \to 0} \frac{e^{x^2}}{x}$

48. $\lim_{x \to 0} \frac{x^2}{x}$

49. $\lim_{x \to 0} \frac{(\cos x)^{1/2}}{x}$

50. $\lim_{x \to 0} \frac{\cos x}{(x + 1)^x}$

51. Evaluate $\lim_{x \to -\pi/2} \cos mx$, where $m, n \neq 0$ are integers.

52. Evaluate $\lim_{x \to 0} \frac{\sin x}{x^m}$ for any numbers $m, n \neq 0$.

53. Evaluate each of the following limits.

(a) $\lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{x^2}$

(b) $\lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x$

54. Show that $\lim_{x \to \infty} (1 + \frac{\ln x}{x})^x = e^x$. 

Scanned with CamScanner
In Exercises 55–56, a ball is launched straight up in the air and is acted on by air resistance and gravity as in Example 5. The function \( H(t) \) gives the maximum height that the projectile attains as a function of the air resistance parameter \( k \). In each case, determine the maximum height as we let the air resistance term go to zero; that is, determine \( \lim_{k \to 0} H(k) \).

55. A ball with a mass of 1 kg is launched upward with an initial velocity of 10 m/s, and
\[
H(k) = \frac{60k - 9.8 \ln(\frac{\text{max}}{45} + 1)}{k^2}
\]
(Compare with Exercise 23 in the previous section and Exercise 29 in Section 3.4.)

56. A ball with a mass of 500 g is launched upward with an initial velocity of 30 m/s, and
\[
H(k) = \frac{15k - 2.45 \ln(\frac{\text{max}}{45} + 1)}{k^2}
\]
(Compare with Exercise 24 in the previous section.)

57. In each case, show that the form is indeterminate by showing that if \( \lim_{x \to 0} f(x)^{g(x)} \) has the form \( \infty^\infty \), then the limit in the exponent in \( \lim_{x \to 0} g(x) \) has a known indeterminate form.
(a) \( \frac{1}{\ln x} \)
(b) \( \infty^0 \)

58.GU Can L'Hôpital's Rule be applied to \( \lim_{x \to 0} x^{\ln(1/x)} \)? Does a graphical or numerical investigation suggest that the limit exists?

59. Let \( f(x) = x^{1/2} \) for \( x > 0 \).
(a) Calculate \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to 0^-} f(x) \).
(b) Find the maximum value of \( f \) and determine the intervals on which \( f \) is increasing or decreasing.

60. (a) Use the results of Exercise 59 to prove that \( x^{1/2} = c \) has a unique solution if \( 0 < c \leq 1 \) or \( c = e^{1/2} \), has two solutions if \( 1 < c < e^{1/2} \), and has no solutions if \( c > e^{1/2} \).
(b)GU Plot the graph of \( f(x) = x^{1/2} \) and verify that it confirms the conclusions of (a).

61. Determine whether \( f < g \) or \( g < f \) (or neither) for the functions \( f(x) = \log_{10} x \) and \( g(x) = 1 - \log_{10} x \).

62. Show that \( \lim_{x \to 0} x^{1/2} \) and \( \lim_{x \to 0} (1 - x)^{1/2} \) exist and are equal to zero.

63. Just as exponential functions are distinguished by their rapid rate of increase, the logarithm functions grow particularly slowly. Show that \( \lim_{x \to 0^+} x^a \) for all \( a > 0 \).

64. Show that \( \lim_{x \to 0^+} x^a \) for all \( N \) and all \( a > 0 \).

65. Determine whether \( \lim_{x \to 0^+} e^{\sqrt{x}} \) or \( \lim_{x \to 0^+} e^{1/x^2} \) exists. Hint: Use the substitution \( u = e^x \) instead of L'Hôpital's Rule.

66. Show that \( \lim_{x \to \infty} x e^{-x} = 0 \) for all whole numbers \( n > 0 \).

67. Assumptions Matter Suppose \( f(x) = x(2 + \sin x) \) and let \( g(x) = x^2 + 1 \).
(a) Show directly that \( \lim_{x \to \infty} f(x)/g(x) = 0 \).
(b) Show that \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty \), but \( \lim_{x \to \infty} f(x)/g(x) \) does not exist.
Do (a) and (b) contradict L'Hôpital's Rule? Explain.

68. Let \( H(b) = \lim_{x \to 0^+} \frac{\ln(1 + b^x)}{x} \) for \( b > 0 \).
(a) Show that \( H(b) = \ln b \) if \( b > 1 \).
(b) Determine \( H(b) \) for \( 0 < b < 1 \).

69. Let \( G(b) = \lim_{x \to 0^+} (1 + b^x)^{1/x} \).
(a) Use the result of Exercise 68 to evaluate \( G(b) \) for all \( b > 0 \).
(b)GU Verify your result graphically by plotting \( y = (1 + b^x)^{1/x} \) together with the horizontal line \( y = G(b) \) for the values \( b = 0.25, 0.5, 2, 3 \).

70. Show that \( \lim_{x \to 0^+} x^k e^{-x} = 0 \) for all \( k \). Hint: Compare with \( \lim_{x \to 0^+} x^k e^{-x} = 0 \).

In Exercises 71–73, let \( f(x) = \begin{cases} e^{1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \)

These exercises show that \( f \) has an unusual property: All of its derivatives at \( x = 0 \) exist and are equal to zero.

71. Show that \( \lim_{x \to 0^+} \frac{f(x)}{x^2} = 0 \) for all \( k \). Hint: Let \( t = x^{-1} \) and apply the result of Exercise 70.

72. Show that \( f''(0) \) exists and is equal to zero. Also, verify that \( f''(0) \) exists and is equal to zero.

73. Show that for \( k \geq 1 \) and \( x \neq 0 \),
\[
f^{(k)}(x) = \frac{P(x) e^{-1/x^2}}{x^r}
\]
for some polynomial \( P(x) \) and some exponent \( r \geq 1 \). Use the result of Exercise 71 to show that \( f^{(k)}(0) \) exists and is equal to zero for all \( k \geq 1 \).

Further Insights and Challenges

74. Show that L'Hôpital's Rule applies to \( \lim_{x \to \infty} \frac{x^a}{\sqrt{x^a + 1}} \) but that it does not apply to \( \lim_{x \to \infty} \frac{x^a}{\sqrt{x^a + 1}} \).

75. The Second Derivative Test for critical points fails if \( f''(c) = 0 \). This exercise develops a Higher Derivative Test based on the sign of the first nonzero derivative. Suppose that \( f''(c) = f'''(c) = \cdots = f^{(n-1)}(c) = 0 \), but \( f^{(n)}(c) \neq 0 \).
(a) Show, by applying L'Hôpital's Rule \( n \) times, that
\[
\lim_{x \to c} \frac{f(x) - f(c)}{(x - c)^n} = \frac{1}{n!} f^{(n)}(c)
\]
where \( n! = n(n-1)(n-2)\cdots2\cdot1 \).

(b) Use (a) to show that if \( n \) is even, then \( f^{(n)}(c) > 0 \) and is a local minimum if \( f^{(n)}(c) < 0 \). Hint: If \( n \) is even, then \( f^{(n)}(c) > 0 \) for \( x \neq c \), so \( f^{(n)}(c) > 0 \).
(c) Use (a) to show that if \( n \) is odd, then \( f^{(n)}(c) < 0 \) and is a local maximum if \( f^{(n)}(c) > 0 \). Hint: If \( f^{(n)}(c) > 0 \).
(c) Use (a) to show that if \( n \) is odd, then \( f^{(n)}(c) < 0 \) and is a local maximum if \( f^{(n)}(c) > 0 \).

76. When a spring with natural frequency \( \lambda/2\pi \) is driven with a sinusoidal force \( \sin(\omega t) \) with \( \omega \neq \lambda \), it oscillates according to
\[
y(t) = \frac{1}{\lambda^2 - \omega^2} (\lambda \sin(\lambda t) - \omega \sin(\omega t))
\]
Let \( y(0) = \lim_{t \to \infty} y(t) \).

Scanned with CamScanner
7.6 Inverse Trigonometric Functions

In this section, we introduce the inverse trigonometric functions and their derivatives, and we examine some integrals involving these functions. We have seen that an inverse function $f^{-1}$ exists if and only if $f$ is one-to-one on its domain. Because the trigonometric functions are not one-to-one, we must restrict their domains to define their inverses.

First, consider the sine function. Figure 1 shows that $f(\theta) = \sin \theta$ is one-to-one on $[-\pi/2, \pi/2]$. With this interval as domain, the inverse is called the arcsine function and is denoted $\theta = \sin^{-1} x$ or $\theta = \text{arc} \sin x$. By definition,

$$\theta = \sin^{-1} x \text{ is the unique angle in } [-\pi/2, \pi/2] \text{ such that } \sin \theta = x$$

The range of $f(x) = \sin x$ is $[-1, 1]$, so $f^{-1}(x) = \sin^{-1} x$ has domain $[-1, 1]$. Table 1 gives some values of $\theta = \sin^{-1} x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-1$</th>
<th>$-\sqrt{2}/2$</th>
<th>$-\sqrt{3}/2$</th>
<th>$-\pi/4$</th>
<th>$0$</th>
<th>$\pi/4$</th>
<th>$\sqrt{3}/2$</th>
<th>$\sqrt{2}/2$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = \sin^{-1} x$</td>
<td>$-\frac{\pi}{2}$</td>
<td>$-\frac{\pi}{4}$</td>
<td>$-\frac{\pi}{6}$</td>
<td>$0$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

**Example 1** (a) Show that $\sin^{-1}(\sin(\pi/4)) = \pi/4$.

(b) Show that $\sin^{-1}(\sin(\pi/3)) \neq \pi/3$.

Solution The equation $\sin^{-1}(\sin \theta) = \theta$ is valid if $\theta$ lies in $[-\pi/2, \pi/2]$. 

Do not confuse the inverse $\sin^{-1} x$ with the reciprocal $$(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$$

The inverse functions $\sin^{-1} x$, $\cos^{-1} x$, etc. are often denoted $\arcsin x$, $\arccos x$, and so on.

**FIGURE 1**

Summary of inverse relation between the sine and arcsine functions:

- $\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$
- $\sin^{-1}(\sin \theta) = \theta$ for $-\pi/2 \leq \theta \leq \pi/2$
(a) Because $\frac{\pi}{4}$ lies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$.

(b) Now consider $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$. Note that the point $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is associated with the angle $\frac{5\pi}{4}$, as shown in Figure 2. Therefore, $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$. By Table 1, $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$. So, $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = -\frac{\pi}{4} \neq \frac{5\pi}{4}.$

The cosine function is one-to-one on $[0, \pi]$ rather than $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (Figure 3). With this domain, the inverse is called the arccosine function and is denoted $\theta = \cos^{-1}x$ or $\theta = \arccos x$. It has domain $[-1, 1]$. By definition,

$$\theta = \cos^{-1}x$$

is the unique angle in $[0, \pi]$ such that $\cos \theta = x$.

To compute the derivatives of inverse trigonometric functions, we will need to simplify composite expressions such as $\cos(\sin^{-1}x)$ and $\tan(\sin^{-1}x)$. This can be done in two ways: by referring to the appropriate right triangle or by using trigonometric identities.

**EXAMPLE 2** Find an alternative form in terms of $x$ for each of $\cos(\sin^{-1}x)$ and $\tan(\sin^{-1}x)$.

**Solution** This problem asks for the values of $\cos \theta$ and $\tan \theta$ at the angle $\theta = \sin^{-1}x$. Consider a right triangle with hypotenuse of length 1 and angle $\theta$ such that $\sin \theta = x$, as in Figure 4. By the Pythagorean Theorem, the adjacent side has length $\sqrt{1-x^2}$. Now we can read off the values from Figure 4:

$$\cos(\sin^{-1}x) = \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{\sqrt{1-x^2}}$$

$$\tan(\sin^{-1}x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}$$

Alternatively, we may argue using trigonometric identities. Because $\sin \theta = x$,

$$\cos(\sin^{-1}x) = \cos \theta = \sqrt{1-\sin^2 \theta} = \sqrt{1-x^2}$$

We are justified in taking the positive square root in either approach because $\theta = \sin^{-1}x$ lies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\cos \theta$ is positive in this interval.

We now address the remaining trigonometric functions. The function $f(\theta) = \tan \theta$ is one-to-one on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and $f(\theta) = \cot \theta$ is one-to-one on $(0, \pi)$ (see Figure 10 in Section 1.4). We define their inverses by restricting them to these domains:
\[ \theta = \tan^{-1} x \] is the unique angle in \((-\frac{\pi}{2}, \frac{\pi}{2})\) such that \(\tan \theta = x\).

\[ \theta = \cot^{-1} x \] is the unique angle in \((0, \pi)\) such that \(\cot \theta = x\).

The range of both \(f(\theta) = \tan \theta\) and \(f(\theta) = \cot \theta\) is the set of all real numbers \(\mathbb{R}\). Therefore, \(\theta = \tan^{-1} x\) and \(\theta = \cot^{-1} x\) have domain \(\mathbb{R}\) (Figure 5).

The function \(f(\theta) = \sec \theta\) is not defined at \(\theta = \frac{\pi}{2}\), but we see in Figure 6 that it is one-to-one on \([0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\). Similarly, \(f(\theta) = \csc \theta\) is not defined at \(\theta = 0\), but it is one-to-one on \([-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\). We define the inverse functions as follows:

\[ \theta = \sec^{-1} x \] is the unique angle in \([0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\) such that \(\sec \theta = x\).

\[ \theta = \csc^{-1} x \] is the unique angle in \([\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\) such that \(\csc \theta = x\).

Figure 6 shows that the range of \(f(\theta) = \sec \theta\) is the set of real numbers \(x\) such that \(|x| \geq 1\). The same is true of \(f(\theta) = \csc \theta\). It follows that both \(\theta = \sec^{-1} x\) and \(\theta = \csc^{-1} x\) have domain \([x : |x| \geq 1]\).

![Figure 6](image)

**Derivatives of Inverse Trigonometric Functions**

We now apply implicit differentiation to determine the derivatives of the inverse trigonometric functions. An interesting feature of these functions is that their derivatives are not trigonometric. Rather, they involve quadratic expressions and their square roots. Keep in mind the restricted domains of these functions.

**Theorem 1** Derivatives of Arcsine and Arccosine

\[ \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \]

**Proof** For \(y = \sin^{-1} x\), our goal is to find \(\frac{dy}{dx}\). By applying sine to both sides, we have

\[ \sin y = x \]

Differentiating both sides of the equation with respect to \(x\), and treating \(y\) as a function of \(x\), we obtain

\[ \frac{\cos y}{\frac{dy}{dx}} = 1 \]
\[ \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos (\sin^{-1} x)} = \frac{1}{\sqrt{1 - x^2}} \]

where the last equality holds by Example 2.

The derivation of \( \frac{d}{dx} (\cos^{-1} x) \) is similar (see Exercise 49).

**EXAMPLE 3** Calculate \( f'(\frac{1}{2}) \), where \( f(x) = \arcsin(x^2) \).

Solution Recall that \( \arcsin x \) is another notation for \( \sin^{-1} x \). By the Chain Rule,

\[ \frac{d}{dx} \arcsin(x^2) = \frac{d}{dx} (\sin^{-1}(x^2)) = \frac{1}{\sqrt{1 -(x^2)^2}} \frac{d}{dx} (x^2) = \frac{2x}{\sqrt{1 - x^4}} \]

\[ f'(\frac{1}{2}) = \frac{2(\frac{1}{2})}{\sqrt{1 - (\frac{1}{2})^4}} = \frac{1}{\sqrt[4]{16}} = \frac{4}{\sqrt{15}} \]

**THEOREM 2** Derivatives of Inverse Trigonometric Functions

\[
\begin{align*}
\frac{d}{dx} \tan^{-1} x &= \frac{1}{x^2 + 1}, & \frac{d}{dx} \cot^{-1} x &= -\frac{1}{x^2 + 1} \\
\frac{d}{dx} \sec^{-1} x &= \frac{1}{|x|\sqrt{x^2-1}}, & \frac{d}{dx} \csc^{-1} x &= -\frac{1}{|x|\sqrt{x^2-1}}
\end{align*}
\]

**EXAMPLE 4** Calculate \( \frac{d}{dx} (\csc^{-1}(e^x + 1)) \bigg|_{x=0} \).

Solution Apply the Chain Rule using the formula \( \frac{d}{du} \csc^{-1} u = -\frac{1}{|u|\sqrt{u^2-1}} \):

\[ \frac{d}{dx} \csc^{-1}(e^x + 1) = -\frac{1}{|e^x + 1|\sqrt{(e^x + 1)^2-1}} \frac{d}{dx} (e^x + 1) = -\frac{e^x}{(e^x + 1)\sqrt{e^{2x} + 2e^x}} \]

We have replaced \( |e^x + 1| \) by \( e^x + 1 \) because this quantity is positive. Now, we have

\[ \frac{d}{dx} \csc^{-1}(e^x + 1) \bigg|_{x=0} = -\frac{e^0}{(e^0 + 1)\sqrt{e^0 + 2e^0}} = -\frac{1}{2\sqrt{3}} \]

The derivative formulas for the inverse trigonometric functions yield the following integration formulas:

**Integrals Involving Inverse Trigonometric Functions**

\[
\begin{align*}
\int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x + C \\
\int \frac{dx}{x^2+1} &= \tan^{-1} x + C \\
\int \frac{dx}{|x|\sqrt{x^2-1}} &= \sec^{-1} x + C
\end{align*}
\]
In this list, we omit the integral formulas corresponding to the derivatives of \( y = \cos^{-1} x \), \( y = \cot^{-1} x \), and \( y = \csc^{-1} x \) because their derivatives are the negative of the derivatives of \( y = \sin^{-1} x \), \( y = \tan^{-1} x \), and \( y = \sec^{-1} x \), respectively. The omitted derivatives do not result in antiderivative formulas for new functions, just alternate formulas for functions for which we have a formula above. For example,

\[
\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}, \quad \int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C
\]

**EXAMPLE 5** Evaluate \( \int_0^1 \frac{dx}{x^2+1} \).

**Solution** This integral is the area of the region in Figure 7. By Eq. (3),

\[
\int_0^1 \frac{dx}{x^2+1} = \tan^{-1} x \bigg|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}
\]

**EXAMPLE 6** Using Substitution Evaluate \( \int_{1/\sqrt{2}}^1 \frac{dx}{x \sqrt{4x^2-1}} \).

**Solution** Notice that \( \sqrt{4x^2-1} \) can be written as \( \sqrt{(2x)^2-1} \), so it makes sense to try the substitution \( u = 2x \). Thus, \( x = \frac{1}{2} u \) and \( dx = \frac{1}{2} du \). Furthermore,

\[
u^2 = 4x^2 \quad \text{and} \quad \sqrt{4x^2-1} = \sqrt{u^2-1}
\]

The new limits of integration are \( u(1/\sqrt{2}) = 2(1/\sqrt{2}) = \sqrt{2} \) and \( u(1) = 2 \). By Eq. (4),

\[
\int_{1/\sqrt{2}}^1 \frac{dx}{x \sqrt{4x^2-1}} = \int_{\sqrt{2}}^2 \frac{1}{2} \frac{du}{u \sqrt{u^2-1}} = \int_{\sqrt{2}}^2 \frac{du}{u \sqrt{u^2-1}}
\]

\[
= \sec^{-1} 2 - \sec^{-1} \sqrt{2}
\]

\[
= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}
\]

**EXAMPLE 7** Using Substitution Evaluate \( \int_0^{3/4} \frac{dx}{\sqrt{9-16x^2}} \).

**Solution** Let us first rewrite the integrand:

\[
\sqrt{9-16x^2} = \sqrt{9 \left( 1 - \frac{16x^2}{9} \right)} = 3 \sqrt{1 - \left( \frac{4x}{3} \right)^2}
\]

Thus, it makes sense to use the substitution \( u = \frac{3}{4} x \). Then \( du = \frac{3}{4} dx \) and

\[
x = \frac{3}{4} u, \quad dx = \frac{3}{4} du, \quad \sqrt{9-16x^2} = 3 \sqrt{1-u^2}
\]

The new limits of integration are \( u(0) = 0 \) and \( u(3/4) = 1 \):

\[
\int_0^{3/4} \frac{dx}{\sqrt{9-16x^2}} = \int_0^1 \frac{3}{4} \frac{du}{3 \sqrt{1-u^2}} = \frac{1}{4} \sin^{-1} u \bigg|_0^1 = \frac{1}{4} (\sin^{-1} 1 - \sin^{-1} 0)
\]

\[
= \frac{1}{4} \left( \frac{\pi}{2} \right) = \frac{\pi}{8}
\]
7.6 SUMMARY

- The *arcsine* and *arccosine* are defined for $-1 \leq x \leq 1$:
  \[ \theta = \sin^{-1} x \text{ is the unique angle in } \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \text{ such that } \sin \theta = x. \]
  \[ \theta = \cos^{-1} x \text{ is the unique angle in } [0, \pi] \text{ such that } \cos \theta = x. \]

- $\tan^{-1} x$ and $\cot^{-1} x$ are defined for all $x$:
  \[ \theta = \tan^{-1} x \text{ is the unique angle in } \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \text{ such that } \tan \theta = x. \]
  \[ \theta = \cot^{-1} x \text{ is the unique angle in } (0, \pi) \text{ such that } \cot \theta = x. \]

- $\sec^{-1} x$ and $\csc^{-1} x$ are defined for $|x| \geq 1$:
  \[ \theta = \sec^{-1} x \text{ is the unique angle in } \left[ 0, \frac{\pi}{2} \right) \cup \left( \frac{\pi}{2}, \pi \right] \text{ such that } \sec \theta = x. \]
  \[ \theta = \csc^{-1} x \text{ is the unique angle in } \left[ -\frac{\pi}{2}, 0 \right) \cup \left( 0, \frac{\pi}{2} \right] \text{ such that } \csc \theta = x. \]

- Derivative formulas:
  \[ \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \quad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}. \]
  \[ \frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}, \quad \frac{d}{dx} \cot^{-1} x = -\frac{1}{x^2 + 1}. \]
  \[ \frac{d}{dx} \sec^{-1} x = \frac{1}{|x| \sqrt{x^2 - 1}}, \quad \frac{d}{dx} \csc^{-1} x = -\frac{1}{|x| \sqrt{x^2 - 1}}. \]

- Integral formulas:
  \[ \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C \]
  \[ \int \frac{dx}{x^2 + 1} = \tan^{-1} x + C \]
  \[ \int \frac{dx}{|x| \sqrt{x^2 - 1}} = \sec^{-1} x + C \]

7.6 EXERCISES

**Preliminary Questions**

1. Which of the following quantities is undefined?
   (a) $\sin^{-1}(-\frac{1}{2})$
   (b) $\cos^{-1}(2)$
   (c) $\csc^{-1}(\frac{1}{2})$
   (d) $\csc^{-1}(2)$

2. Give an example of an angle $\theta$ such that $\cos^{-1}(\cos \theta) \neq \theta$. Does this contradict the definition of inverse function?

3. What is the geometric interpretation of the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

4. What does this identity tell us about the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$?

5. Find $b$ such that $\int_0^b \frac{dx}{1 + x^2} = \frac{\pi}{3}$.

6. Which relation between $x$ and $u$ yields $\sqrt{16 + x^2} = 4\sqrt{1 + u^2}$?

**Exercises**

In Exercises 1–6, evaluate without using a calculator.

1. $\cos^{-1} 1$
2. $\sin^{-1} \frac{1}{2}$
3. $\cot^{-1} 1$
4. $\sec^{-1} \frac{2}{\sqrt{3}}$
5. $\tan^{-1} \sqrt{3}$
6. $\sin^{-1}(-1)$
In Exercises 7–16, compute without using a calculator.

7. \( \sin^{-1} \left( \sin \frac{\pi}{3} \right) \)

8. \( \sin^{-1} \left( \sin \frac{4\pi}{3} \right) \)

9. \( \cos^{-1} \left( \cos \frac{3\pi}{2} \right) \)

10. \( \sin^{-1} \left( \sin \left( \frac{5\pi}{6} \right) \right) \)

11. \( \tan^{-1} \left( \tan \frac{3\pi}{4} \right) \)

12. \( \tan^{-1} \left( \tan \pi \right) \)

13. \( \sec^{-1} \left( \sec 3\pi \right) \)

14. \( \sec^{-1} \left( \sec \frac{3\pi}{2} \right) \)

15. \( \csc^{-1} \left( \csc(-\pi) \right) \)

16. \( \cot^{-1} \left( \cot \left( -\frac{\pi}{4} \right) \right) \)

In Exercises 17–20, simplify by referring to the appropriate triangle or trigonometric identity.

17. \( \tan^{-1} \left( x \right) \)

18. \( \cos \left( \tan^{-1} x \right) \)

19. \( \cot^{-1} \left( x \right) \)

20. \( \sin \left( \cot^{-1} x \right) \)

In Exercises 21–28, refer to the appropriate triangle or trigonometric identity to compute the given value.

21. \( \cos \left( \sin^{-1} \frac{3}{5} \right) \)

22. \( \tan \left( \cos^{-1} \frac{3}{5} \right) \)

23. \( \tan \left( \sin^{-1} 0.8 \right) \)

24. \( \cos \left( \cot^{-1} 1 \right) \)

25. \( \cot \left( \sec^{-1} 2 \right) \)

26. \( \tan \left( \sec^{-1} 2 \right) \)

27. \( \cot \left( \tan^{-1} 20 \right) \)

28. \( \sin \left( \csc^{-1} 20 \right) \)

In Exercises 29–32, compute the derivative at the point indicated without using a calculator.

29. \( y = \sin^{-1} x, \quad x = \frac{1}{2} \)

30. \( y = \tan^{-1} x, \quad x = \frac{1}{2} \)

31. \( y = \sec^{-1} x, \quad x = 4 \)

32. \( y = \arccos 4x, \quad x = \frac{1}{2} \)

In Exercises 33–48, find the derivative.

33. \( y = \sin^{-1} (2x) \)

34. \( y = \tan^{-1} \left( \frac{x}{2} \right) \)

35. \( y = \cos^{-1} \left( x^2 \right) \)

36. \( y = \sec^{-1} \left( x + 1 \right) \)

37. \( y = x \tan^{-1} x \)

38. \( y = e^{\cos^{-1} x} \)

39. \( y = \csc \left( \csc^{-1} x \right) \)

40. \( y = \sin \left( \sin^{-1} x \right) \)

41. \( y = \sqrt{1 - x^2} + \sin^{-1} x \)

42. \( y = \tan^{-1} \left( 1 + x \right) \)

43. \( y = \left( \tan^{-1} x \right)^3 \)

44. \( y = \cos^{-1} \left( \sin^{-1} x \right) \)

45. \( y = \cos^{-1} \left( 1 - \sec^{-1} x \right) \)

46. \( y = \cos^{-1} \left( x + \sin^{-1} x \right) \)

47. \( y = \arccos \left( \arccos x \right) \)

48. \( y = \ln \left( \arccos x \right) \)

49. Use Figure 8 to prove that \( (\cos^{-1} x)' = -\frac{1}{\sqrt{1 - x^2}} \).

In Exercises 53–56, evaluate the definite integral.

53. \( \int_{0}^{1} \frac{dx}{x^2 + 1} \)

54. \( \int_{0}^{1} \frac{x \, dx}{x^2 + 1} \)

55. \( \int_{0}^{1/2} \frac{dx}{\sqrt{1 - x^2}} \)

56. \( \int_{-2}^{2} \frac{dx}{\sqrt{x^2 + 1}} \)

57. Use the substitution \( u = x^3 \) to prove \( \int_{9}^{81} \frac{dx}{\sqrt{x^2 + 1}} = \frac{1}{3} \tan^{-1} \frac{x}{3} + C \)

58. Use the substitution \( u = 2x \) to evaluate \( \int_{0}^{4} \frac{dx}{x^2 + 1} \)

In Exercises 59–72, calculate the integral.

59. \( \int_{0}^{1} \frac{dx}{x^2 + 3} \)

60. \( \int_{0}^{1} \frac{dt}{t^2 + 9} \)

61. \( \int_{0}^{1} \frac{dx}{\sqrt{1 - 16x^2}} \)

62. \( \int_{0}^{1} \frac{dx}{\sqrt{4 - 5x^2}} \)

63. \( \int_{0}^{1/2} \frac{dx}{\sqrt{5 - 3x^2}} \)

64. \( \int_{0}^{1} \frac{dx}{x \sqrt{12x^2 - 3}} \)

65. \( \int_{0}^{1} \frac{dx}{x \sqrt{4x^2 + 3}} \)

66. \( \int_{0}^{1} \frac{dx}{x^4 + 1} \)

67. \( \int_{0}^{1} \frac{dx}{x \sqrt{x^2 + 1}} \)

68. \( \int_{0}^{1} \frac{dx}{\sqrt{1 - x^2}} \)

69. \( \int_{0}^{1} \frac{dx}{(\cos^{-1} x)'x^2 + 1} \)

70. \( \int_{0}^{1} \frac{dx}{x \tan^{-1} x} \)

71. \( \int_{0}^{1} \frac{dx}{(\tan^{-1} x)'(1 + x^2)} \)

72. \( \int_{0}^{1} \frac{dx}{x^2 + 1} \)

In Exercises 73–108, evaluate the integral using the methods covered in this section.

73. \( \int ye^x \, dy \)

74. \( \int \frac{dx}{3x + 5} \)

75. \( \int \frac{x \, dx}{\sqrt{4x^2 + 9}} \)

76. \( \int (x - \frac{x^2}{2}) \, dx \)

77. \( \int 7^x \, dx \)

78. \( \int e^{x^2 - 12x} \, dx \)
79. \int \frac{\sec^2 \theta \tan^7 \theta}{\theta} \, d\theta
80. \int \frac{\cos(\ln t)}{t} \, dt
81. \int \frac{t \, dt}{\sqrt{7 - t^2}}
82. \int 2^xe^x \, dx
83. \int \frac{(3x + 2) \, dx}{x^2 + 4}
84. \int \tan(4x + 1) \, dx
85. \int \frac{dx}{\sqrt{1 - 16x^2}}
86. \int e^x \sqrt{e^x + 1} \, dt
87. \int (e^{-x} - 4x) \, dx
88. \int (7 - e^{2x}) \, dx
89. \int \frac{e^{2x} - e^{4x}}{e^x} \, dx
90. \int \frac{dx}{x \sqrt{5x^2 - 1}}
91. \int \frac{(x + 5) \, dx}{\sqrt{4 - x^2}}
92. \int (t + 1) \sqrt{1 + t^2} \, dt
93. \int e^x \cos(e^x) \, dx
94. \int \frac{e^x + 1}{\sqrt{e^x + 1}} \, dx
95. \int \frac{dx}{\sqrt{9 - 16x^2}}
96. \int \frac{dx}{2x \ln x + 3}
97. \int e^{x^2} + 13 \, dx
98. \int \frac{dx}{x \ln x}^3
99. \int x^3 \, dx
100. \int \frac{(3x - 1) \, dx}{9 - 2x + 3x^2}
101. \int \cot x \, dx
102. \int \frac{\cos x}{2 \sin x + 3} \, dx
103. \int \frac{\ln x + 5}{x} \, dx
104. \int (\sec \theta \tan \theta) \sec^2 \theta \, d\theta
105. \int 3x^2 \, dx
106. \int \frac{\ln(\ln x)}{x \ln x} \, dx
107. \int \cot x \ln(\sin x) \, dx
108. \int \frac{t \, dt}{\sqrt{1 - t^4}}
109. Use Figure 10 to prove
\[ \int_0^t \sqrt{1 - t^2} \, dt = \frac{1}{2} t \sqrt{1 - t^2} + \frac{1}{2} \sin^{-1} t \]

**FIGURE 10**

110. Use the substitution \( u = \tan x \) to evaluate
\[ \int \frac{dx}{1 + \sin^2 x} \]

**Hint:** Show that
\[ \frac{dx}{1 + \sin^2 x} = \frac{du}{1 + 2u^2} \]

111. **Prove:**
\[ \int \sin^{-1} t \, dt = \sqrt{1 - t^2} + t \sin^{-1} t + C \]

---

**Further Insights and Challenges**

112. A cylindrical tank of radius \( R \) and length \( L \) lying horizontally as in Figure 11 is filled with oil to height \( h \).
   (a) Show that the volume \( V(h) \) of oil in the tank as a function of height \( h \) is
   \[ V(h) = L \left( R^2 \cos^{-1} \left( 1 - \frac{h}{R} \right) - (R - h) \sqrt{2hR - h^2} \right) \]
   (b) Show that \( \frac{dV}{dh} = 2L \sqrt{h(2R - h)} \).
   (c) Suppose that \( R = 2 \) m and \( L = 12 \) m, and that the tank is filled at a constant rate of 1.5 m³/min. How fast is the height \( h \) increasing when \( h = 3 \) m?

**FIGURE 11** Oil in the tank has level \( h \).

113. (a) Explain why the shaded region in Figure 12 has area \( \int_0^{\ln a} e^y \, dy \).
   (b) Prove the formula \( \int_0^a \ln x \, dx = a \ln a - \int_0^{\ln a} e^y \, dy \).
   (c) Conclude that \( \int_0^a \ln x \, dx = a \ln a - a + 1 \).
   (d) Use the result of (a) to find an antiderivative of \( \ln x \).

**FIGURE 12**

---

Scanned with CamScanner
7.7 Hyperbolic Functions

The hyperbolic functions are certain special combinations of $e^x$ and $e^{-x}$ that play a role in engineering and physics (see Figure 1 for a real-life example). The hyperbolic sine and cosine, pronounced "sinch" and "cosh," are shown in Figure 2 and defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

![Graph of hyperbolic functions](image)

**Figure 2** The hyperbolic sine and cosine functions.

As the terminology suggests, there are similarities between the hyperbolic and trigonometric functions. Here are some examples:

- **Parity:** The trigonometric functions and their hyperbolic analogs have the same parity. Thus, $f(x) = \sin x$ and $f(x) = \sinh x$ are both odd, and $f(x) = \cos x$ and $f(x) = \cosh x$ are both even (Figure 2):

  $$\sinh(-x) = -\sinh x, \quad \cosh(-x) = \cosh x$$

- **Identities:** The basic trigonometric identity $\sin^2 x + \cos^2 x = 1$ has a hyperbolic analog:

  $$\cosh^2 x - \sinh^2 x = 1$$

The addition formulas satisfied by $\sin x$ and $\cos x$ also have hyperbolic analogs:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

- **Hyperbola instead of the circle:** Because of the identity $\cosh^2 t - \sinh^2 t = 1$, the point $(\cosh t, \sinh t)$ lies on the hyperbola $x^2 - y^2 = 1$, just as $(\cos t, \sin t)$ lies on the unit circle $x^2 + y^2 = 1$ (Figure 3).

- **Other hyperbolic functions:** The hyperbolic tangent, cotangent, secant, and cosecant functions (see Figure 4) are defined like their trigonometric counterparts:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \text{csch } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$
**EXAMPLE 1** Verifying the Basic Identity  
 Verify Eq. (1): $\cosh^2 x - \sinh^2 x = 1$.

**Solution**  
 Because $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$, we have

\[
\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x}
\]

We obtain Eq. (1) by multiplying these two equations together:

\[
\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x) = e^x \cdot e^{-x} = 1
\]

---

**Derivatives of Hyperbolic Functions**

The formulas for the derivatives of the hyperbolic functions are similar to those for the corresponding trigonometric functions, differing at most by a sign.

Consider the hyperbolic sine and cosine:

\[
\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}
\]

Their derivatives are

\[
\frac{d}{dx} \sinh x = \cosh x, \quad \frac{d}{dx} \cosh x = \sinh x
\]
We can check this directly. For example,

\[
\frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) = \left( \frac{e^x - e^{-x}}{2} \right)' = \frac{e^x + e^{-x}}{2} = \cosh x
\]

Note the resemblance to the formulas \( \frac{d}{dx} \sin x = \cos x \), \( \frac{d}{dx} \cos x = -\sin x \). The derivatives of the other hyperbolic functions, which are computed in a similar fashion, also differ from their trigonometric counterparts by a sign at most.

**Derivatives of Hyperbolic and Trigonometric Functions**

\[
\begin{array}{ll}
\frac{d}{dx} \tanh x &= \text{sech}^2 x, & \frac{d}{dx} \tan x &= \sec^2 x, \\
\frac{d}{dx} \coth x &= -\text{csch}^2 x, & \frac{d}{dx} \cot x &= -\csc^2 x, \\
\frac{d}{dx} \text{sech} x &= -\text{sech} x \tanh x, & \frac{d}{dx} \sec x &= \sec x \tan x, \\
\frac{d}{dx} \text{csch} x &= -\text{csch} x \coth x, & \frac{d}{dx} \csc x &= -\csc x \cot x
\end{array}
\]

**EXAMPLE 2** Verify \( \frac{d}{dx} \coth x = -\text{csch}^2 x \).

**Solution** By the Quotient Rule and the identity \( \cosh^2 x - \sinh^2 x = 1 \),

\[
\frac{d}{dx} \coth x = \left( \frac{\cosh x}{\sinh x} \right)' = \frac{(\sinh x)(\cosh x)' - (\cosh x)(\sinh x)'}{\sinh^2 x}
\]

\[
= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\text{csch}^2 x
\]

**EXAMPLE 3** Calculate: (a) \( \frac{d}{dx} \cosh(3x^2 + 1) \) and (b) \( \frac{d}{dx} \sinh x \tanh x \).

**Solution**

(a) By the Chain Rule, \( \frac{d}{dx} \cosh(3x^2 + 1) = 6x \sinh(3x^2 + 1) \).

(b) By the Product Rule,

\[
\frac{d}{dx} (\sinh x \tanh x) = \sinh x \text{sech}^2 x + \tanh x \cosh x = \text{sech} x \tanh x + \sinh x
\]

**Inverse Hyperbolic Functions**

Each of the hyperbolic functions except \( f(x) = \cosh x \) and \( f(x) = \text{sech} x \) is one-to-one on its domain and therefore has a well-defined inverse. The functions \( f(x) = \cosh x \) and \( f(x) = \text{sech} x \) are one-to-one on the restricted domain \( \{ x : x \geq 0 \} \). We let \( f(x) = \cosh^{-1} x \) and \( f(x) = \text{sech}^{-1} x \) denote the corresponding inverses (Figure 5). In reading the following table, keep in mind that the domain of the inverse is equal to the range of the function.
The graphs of \( y = \cosh^{-1} x \) and \( y = \text{sech}^{-1} x \) have a vertical tangent at the endpoint \( x = 1 \) of their domains and therefore the derivative is undefined there.

### Inverse Hyperbolic Functions and Their Derivatives

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sinh^{-1} x )</td>
<td>all ( x )</td>
<td>( \frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}} )</td>
</tr>
<tr>
<td>( y = \cosh^{-1} x )</td>
<td>( x \geq 1 )</td>
<td>( \frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} )</td>
</tr>
<tr>
<td>( y = \tanh^{-1} x )</td>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>( y = \coth^{-1} x )</td>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>( y = \text{sech}^{-1} x )</td>
<td>( 0 &lt; x \leq 1 )</td>
<td>( \frac{d}{dx} \text{sech}^{-1} x = \frac{1}{x \sqrt{1 - x^2}} )</td>
</tr>
<tr>
<td>( y = \text{csch}^{-1} x )</td>
<td>( x \neq 0 )</td>
<td>( \frac{d}{dx} \text{csch}^{-1} x = \frac{1}{</td>
</tr>
</tbody>
</table>

#### EXAMPLE 4
Verify the formula \( \frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2} \).

**Solution** Recall that if \( g \) is the inverse of \( f \), then \( g'(x) = 1/f'(g(x)) \). Applying this to \( f(x) = \tanh x \), and using the formula \( (\tanh x)' = \text{sech}^2 x \), we have

\[
\frac{d}{dx} \tanh^{-1} x = \frac{1}{\text{sech}^2 (\tanh^{-1} x)}
\]

Now, let \( t = \tanh^{-1} x \). Then

\[
\begin{align*}
\cosh^2 t - \sinh^2 t &= 1, & \text{basic identity} \\
1 - \tanh^2 t &= \text{sech}^2 t, & \text{divide by } \cosh^2 t \\
1 - x^2 &= \text{sech}^2 (\tanh^{-1} x), & \text{because } x = \tanh t
\end{align*}
\]

This gives the desired result:

\[
\frac{d}{dx} \tanh^{-1} x = \frac{1}{\text{sech}^2 (\tanh^{-1} x)} = \frac{1}{1 - x^2}
\]

#### EXAMPLE 5
The functions \( y = \tanh^{-1} x \) and \( y = \coth^{-1} x \) appear to have the same derivative. Does this imply that they differ by a constant?

**Solution** According to the table above, \( y = \tanh^{-1} x \) and \( y = \coth^{-1} x \) both have derivative \( 1/(1 - x^2) \). Although functions with the same derivative differ by a constant, this is the case only if they are defined on the same domain. The functions \( y = \tanh^{-1} x \) and \( y = \coth^{-1} x \) have disjoint domains and therefore do not differ by a constant (Figure 6).

Integrals involving hyperbolic functions and inverse hyperbolic functions are covered in Section 8.4.

#### Einstein’s Law of Velocity Addition

The hyperbolic tangent plays a role in the Special Theory of Relativity, developed by Albert Einstein in 1905. One consequence of this theory is that no object can travel faster
than the speed of light \( c \approx 3 \times 10^8 \) m/s. Einstein realized that this contradicts a law stated by Galileo more than 250 years earlier, namely that velocities add. Imagine a train traveling at \( u = 50 \) m/s and a man walking down the aisle in the train at \( v = 2 \) m/s. According to Galileo, the man’s velocity relative to the ground is \( u + v = 52 \) m/s. This agrees with our everyday experience. But now imagine an (unrealistic) rocket traveling away from the earth at \( u = 2 \times 10^6 \) m/s and suppose that the rocket fires a missile with velocity \( v = 1.5 \times 10^8 \) m/s (relative to the rocket). If Galileo’s Law were correct, the velocity of the missile relative to the earth would be \( u + v = 3.5 \times 10^8 \) m/s, which exceeds Einstein’s maximum speed limit of \( c \approx 3 \times 10^8 \) m/s.

However, Einstein’s theory replaces Galileo’s Law with a new law stating that the inverse hyperbolic tangents of the velocities add. More precisely, if \( u \) is the rocket’s velocity relative to the earth and \( v \) is the missile’s velocity relative to the rocket, then the velocity of the missile relative to the earth (Figure 7) is \( w \), where

\[
\tanh^{-1} \left( \frac{u}{c} \right) = \tanh^{-1} \left( \frac{u}{c} \right) + \tanh^{-1} \left( \frac{v}{c} \right)
\]

**Example 6** A rocket travels away from the earth at a velocity of \( 2 \times 10^8 \) m/s. A missile is fired at a velocity of \( 1.5 \times 10^8 \) m/s (relative to the rocket) away from the earth. Use Einstein’s Law to find the velocity \( w \) of the missile relative to the earth.

**Solution** According to Eq. (2),

\[
\tanh^{-1} \left( \frac{w}{c} \right) = \tanh^{-1} \left( \frac{2 \times 10^8}{3 \times 10^8} \right) + \tanh^{-1} \left( \frac{1.5 \times 10^8}{3 \times 10^8} \right) \approx 0.805 + 0.549 \approx 1.354
\]

Therefore, \( \frac{w}{c} \approx \tanh(1.354) \approx 0.875 \), and \( w \approx 0.875c \approx 2.6 \times 10^8 \) m/s. This value obeys the Einstein speed limit of \( 3 \times 10^8 \) m/s.

**Example 7** Low Velocities A plane traveling at 300 m/s fires a missile at a velocity of 200 m/s. Calculate the missile’s velocity \( w \) relative to the earth (in meters per second) using both Einstein’s Law and Galileo’s Law.

**Solution** According to Einstein’s law,

\[
\tanh^{-1} \left( \frac{w}{c} \right) = \tanh^{-1} \left( \frac{300}{c} \right) + \tanh^{-1} \left( \frac{200}{c} \right)
\]

\[
w = c \cdot \tanh \left( \tanh^{-1} \left( \frac{300}{c} \right) + \tanh^{-1} \left( \frac{200}{c} \right) \right) \approx 499.99999999967
\]

This is practically indistinguishable from the value \( w = 300 + 200 = 500 \) m/s, obtained using Galileo’s Law.

7.7 SUMMARY

- The hyperbolic sine and cosine:
  \[
  \sinh x = \frac{e^x - e^{-x}}{2} \quad \text{(odd function)}, \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \text{(even function)}
  \]

- The remaining hyperbolic functions:
  \[
  \tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}
  \]

- Basic identity: \( \cosh^2 x - \sinh^2 x = 1 \).
• Derivative formulas for hyperbolic functions:
  \[
  \frac{d}{dx} \sinh x = \cosh x, \quad \frac{d}{dx} \cosh x = \sinh x
  \]
  \[
  \frac{d}{dx} \tanh x = \text{sech}^2 x, \quad \frac{d}{dx} \text{coth} x = -\text{csch}^2 x
  \]
  \[
  \frac{d}{dx} \text{sech} x = -\text{sech} x \tanh x, \quad \frac{d}{dx} \text{csch} x = -\text{csch} x \coth x
  \]

• Derivative formulas for inverse hyperbolic functions:
  \[
  \frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}, \quad \frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \quad (x > 1)
  \]
  \[
  \frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2}, \quad \frac{d}{dx} \coth^{-1} x = \frac{1}{1 - x^2} \quad (|x| > 1)
  \]
  \[
  \frac{d}{dx} \text{sech}^{-1} x = \frac{1}{x \sqrt{1 - x^2}}, \quad \frac{d}{dx} \text{csch}^{-1} x = -\frac{1}{|x| \sqrt{x^2 + 1}} \quad (x \neq 0)
  \]

### 7.7 EXERCISES

#### Preliminary Questions

1. Which hyperbolic functions take on only positive values?
2. Which hyperbolic functions are increasing on their domains?

#### Exercises

1. Use a calculator to compute \( \sinh x \) and \( \cosh x \) for \( x = -3, 0, 5 \).
2. Compute \( \sinh(\ln 5) \) and \( \tanh(3 \ln 5) \) without using a calculator.
3. For which values of \( x \) are \( y = \sinh x \) and \( y = \cosh x \) increasing and decreasing?
4. Show that \( y = \tanh x \) is an odd function.
5. Refer to the graphs to explain why the equation \( \sinh x = t \) has a unique solution for every \( t \) and why \( \cosh x = t \) has two solutions for every \( t > 1 \).
6. Compute \( \cosh x \) and \( \tanh x \), assuming that \( \sinh x = 0.8 \).
7. Prove the addition formula for \( \cosh x \).
8. Use the addition formulas to prove:
   \[
   \sinh(2x) = 2 \cosh x \sinh x, \quad \cosh(2x) = \cosh^2 x + \sinh^2 x
   \]
   In Exercises 9–32, calculate the derivative.
9. \( y = \sinh(9x) \)  
10. \( y = \sinh(x^2) \)
11. \( y = \cosh^2(9 - 3t) \)  
12. \( y = \tanh(i^2 + 1) \)
13. \( y = \sqrt{\cosh x + 1} \)  
14. \( y = \sinh x \tanh x \)
15. \( y = \frac{\coth x}{1 + \tanh x} \)  
16. \( y = (\ln(\cosh x))^3 \)
17. \( y = \sinh(\ln x) \)  
18. \( y = e^{\cosh x} \)
19. \( y = \tanh(e^x) \)  
20. \( y = \sinh(\cosh^3 x) \)
21. \( y = \sech(\sqrt{x}) \)  
22. \( y = \ln(\coth x) \)
23. \( y = \sech x \cosh x \)  
24. \( y = \sinh^3 x \)
25. \( y = \cosh^{-1}(3x) \)  
26. \( y = \tanh^{-1}(x^4 + 3x^2) \)
27. \( y = (\sinh^{-1}(x^2))^3 \)  
28. \( y = (\cosh^{-1}(3x))^4 \)
29. \( y = e^{\cosh^{-1} x} \)  
30. \( y = \sinh^{-1}(\sqrt{x^2 + 1}) \)
31. \( y = \tanh^{-1}(\ln t) \)  
32. \( y = \ln(\tanh(3x)) \)
33. Show that for any constants \( M, k, \) and \( a \), the function
   \[
   y(t) = \frac{1}{2} M \left( 1 + \tanh \left( \frac{k(t - a)}{2} \right) \right)
   \]
   satisfies the logistic equation: \( \frac{dy}{dt} = k \left( 1 - \frac{y}{M} \right) \).
34. Show that \( V(x) = 2 \ln(\tanh(x/2)) \) satisfies the Poisson–Boltzmann equation \( V''(x) = \sinh(V(x)) \), which is used to describe electrostatic forces in certain molecules.
   In Exercises 35–40, prove the formula.
35. \( \frac{d}{dx} \tanh x = \text{sech}^2 x \)
36. \( \frac{d}{dx} \text{sech} x = -\text{sech} x \tanh x \)
37. \( \cosh(\sinh^{-1} x) = \sqrt{x^2 + 1} \)
38. \( \sinh^{-1}(\cosh x) = \sqrt{t^2 - 1} \) for \( t \geq 1 \)
39. \( \frac{d}{dt} \sinh^{-1} t = \frac{1}{\sqrt{t^2 + 1}} \)
40. \( \frac{d}{dt} \cosh^{-1} t = \frac{1}{\sqrt{t^2 - 1}} \) for \( t > 1 \).

41. Prove that \( \sinh^{-1} t = \ln(t + \sqrt{t^2 + 1}) \). Hint: Let \( t = \sinh x \). Prove that \( \cosh x = \sqrt{t^2 + 1} \) and use the relation \( \sinh x + \cosh x = e^x \).
42. Prove that \( \cosh^{-1} t = \ln(t + \sqrt{t^2 - 1}) \) for \( t > 1 \).
43. Prove that \( \tanh^{-1} t = \frac{1}{2} \ln \left( \frac{1 + t}{1 - t} \right) \) for \( |t| < 1 \).

44. Use differentiation to prove
\[
\int \text{sech} x \, dx = \tan^{-1}(\sinh x) + C
\]

45. An imaginary train moves along the track at velocity \( v \). Bionica walks down the aisle of the train with velocity \( u \). Compute the velocity \( w \) of Bionica relative to the ground using the laws of both Galilean and Einstein in the following cases:
(a) \( v = 500 \text{ m/s} \) and \( u = 10 \text{ m/s} \). Is your calculator accurate enough to detect the difference between the two laws?
(b) \( v = 10^3 \text{ m/s} \) and \( u = 10^6 \text{ m/s} \).

Further Insights and Challenges

46. Show that the linearization of the function \( y = \tanh^{-1} x \) at \( x = 0 \) is \( \tanh^{-1} x \approx x \). Use this to explain the following assertion: Einstein’s Law of Velocity Addition [Eq. (2)] reduces to Galileo’s Law if the velocities are small relative to the speed of light.

47. (a) Use the addition formulas for \( \sinh x \) and \( \cosh x \) to prove
\[
\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}
\]
(b) Use (a) to show that Einstein’s Law of Velocity Addition [Eq. (2)] is equivalent to
\[
\frac{u}{v} = \frac{u + v}{1 + \tanh u \tanh v}
\]
48. Show that \( \int_{-a}^{a} \cosh x \sinh x \, dx = 0 \) for all \( a \).

49. (a) Show that \( y = \tanh \beta \) satisfies the differential equation \( dy/dx = 1 - y^2 \) with initial condition \( y(0) = 0 \).
(b) Show that for arbitrary constants \( A, B \), the function \( y = A \tanh(Bx) \) satisfies
\[
dy = \frac{dy}{dx} = ABx - Bx \tanh^2(Bx)
\]
49. (c) Let \( v(t) \) be the velocity of a falling object of mass \( m \). For large velocities, air resistance is proportional to the square of velocity \( v(t)^2 \). If we choose coordinates so that \( v(t) > 0 \) for a falling object, then by Newton’s Law of Motion, there is a constant \( k > 0 \) such that
\[
\frac{dv}{dt} = g - k \frac{v^2}{m}
\]
Solve for \( v(t) \) by applying the result of (b) with \( A = \sqrt{gm/k} \) and \( B = \sqrt{g/k} \).
(d) Calculate the terminal velocity \( \lim_{t \to \infty} v(t) \).
(e) Find \( k \) if \( m = 150 \text{ pounds} \) and the terminal velocity is 100 miles per hour.

50. (GU) Suppose that \( L = 120 \) and \( M = 50 \). Experiment with your calculator to find an approximate value of \( a \) satisfying \( L = 2a \sinh(M/a) \) for greater accuracy, use Newton’s method or a computer algebra system.

51. Let \( M \) be a fixed constant. Show that the sag is given by \( s = a \cosh(M/a) - a \).
(a) Calculate \( \frac{ds}{da} \).
(b) Calculate \( \frac{da}{dL} \) by implicit differentiation using the relation \( L = 2a \sinh(M/a) \).
(c) Use (a) and (b) and the Chain Rule to show that
\[
\frac{ds}{dL} = \frac{dx}{da} \frac{dx}{dL} = \frac{\cosh(M/a) - (M/a) \sinh(M/a) + 1}{2 \sinh(M/a) - (2M/a) \cosh(M/a)}
\]
52. Assume that \( M = 50 \) and \( L = 160 \). In this case, a CAS can be used to show that \( a \approx 28.46 \).
(a) Use Eq. (3) and the Linear Approximation to estimate the increase in sag if \( L \) is increased from \( L = 160 \) to \( L = 161 \) and from \( L = 160 \) to \( L = 165 \).
(b) (CAS) If you have a CAS, compute \( s(161) - s(160) \) and \( s(165) - s(160) \) directly and compare with your estimates in (a).

53. Prove that every function \( f \) is the sum of an even function \( f_e \) and an odd function \( f_o \). [Hint: \( f(x) = \frac{1}{2} (f(x) + f(-x)) \).] Express \( f(x) = 5x^2 + 8x^{-1} \) in terms of \( \sinh(x) \) and \( \cosh(x) \).

54. Use the method of the previous problem to express
\[
f(x) = 7e^{-3x} + 4e^{3x}
\]
in terms of \( \sinh(3x) \) and \( \cosh(3x) \).
1. Which of the following is equal to \( \frac{d}{dx} 2^x \)?
   (a) 2x  
   (b) \((\ln 2)2^x\) 
   (c) \(x2^{x-1}\) 
   (d) \(\frac{1}{\ln 2} 2^x\)

2. Find the inverse of \( f(x) = \sqrt{x^2 - 8} \) and determine its domain and range.

3. Find the inverse of \( f(x) = \frac{x - 2}{x - 1} \) and determine its domain and range.

4. Find a domain on which \( h(t) = (t - 3)^2 \) is one-to-one and determine the inverse on this domain.

5. Show that \( g(x) = \frac{x}{x - 1} \) is equal to its inverse on the domain \( x \neq 1 \).

6. Describe the graphical interpretation of the relation \( g'(x) = 1/ f'(g(x)) \), where \( f \) and \( g \) are inverses of each other.

7. Suppose that \( g \) is the inverse of \( f \). Match the functions (a)–(d) with their inverses (i)–(iv).
   (a) \( f(x) + 1 \)  
   (b) \( f(x) + 1 \)  
   (c) \( 4f(x) \)  
   (d) \( f(4x) \)  
   (i) \( g(x)/4 \)  
   (ii) \( g(x/4) \)  
   (iii) \( g(x - 1) \)  
   (iv) \( g(x) - 1 \)

8. Find \( g'(x) \) where \( g \) is the inverse of a differentiable function \( f \) such that \( f(-1) = 8 \) and \( f'(-1) = 12 \).

9. Suppose that \( f(g(x)) = e^{2x} \), where \( g(1) = 2 \) and \( g'(1) = 4 \). Find \( f'(2) \).

10. Show that if \( f \) is a function satisfying \( f'(x) = f(x)^2 \), then its inverse \( g(x) = x^{-1} \).

In Exercises 11–40, find the derivative.

11. \( f(x) = 9e^{-4x} \)
12. \( f(x) = \ln(4x^2 + 1) \)
13. \( f(x) = e^{-x} \)
14. \( f(x) = \ln(x + e^x) \)
15. \( g(x) = (\ln x)^2 \)
16. \( G(x) = \ln(x^2) \)
17. \( g(t) = e^{t^2} \)
18. \( g(t) = t^2 e^{\ln t} \)
19. \( f(\theta) = \ln(\sin \theta) \)
20. \( f(\theta) = \sin(\ln \theta) \)
21. \( f(x) = \ln(e^x - 4x) \)
22. \( h(t) = \sec(x + \ln 2) \)
23. \( f(x) = e^{x + \ln x} \)
24. \( f(x) = e^{\sin x} \)
25. \( h(y) = 2^{-y} \)
26. \( h(y) = \ln \frac{1 + e^y}{1 - e^y} \)
27. \( f(x) = 7^{-x} \)
28. \( g(x) = \tan^{-1}(\ln x) \)
29. \( G(x) = \cos^{-1}(x^{-1}) \)
30. \( G(x) = \tan^{-1}(\sqrt{x}) \)
31. \( f(x) = \ln(\sec^{-1} x) \)
32. \( f(x) = e^{\sec^{-1} x} \)
33. \( R(x) = s^{\ln s} \)
34. \( f(x) = (\cos^2 x)^{\cos x} \)
35. \( G(t) = (\sin^2 t)^t \)
36. \( h(t) = t^{(e^t)} \)
37. \( g(t) = \sinh(y^2) \)
38. \( h(y) = \gamma \tanh(4y) \)
39. \( g(x) = \tanh^{-1}(\ln x) \)
40. \( g(x) = \sqrt{2} - 1 \sinh^{-1} t \)

41. The tangent line to the graph of \( y = f(x) \) at \( x = 4 \) has equation \( y = -2x + 12 \). Find the equation of the tangent line to \( y = g(x) \) at \( x = 4 \), where \( g \) is the inverse of \( f \).

In Exercises 42–44, let \( f(x) = xe^{-x} \).

42. Plot \( f \) and use the zoom feature to find two solutions of \( f(x) = 0.3 \).

43. Show that \( f \) has an inverse on \([1, \infty)\). Let \( g \) be this inverse. Find the domain and range of \( g \) and compute \( g'(2e^{-2}) \).

44. Show that \( f(x) = e \) has two solutions if \( 0 < e < e^2 \).

45. Determine \( M, A, \) and \( k \), for a logistic function \( f(t) = \frac{M}{1 + Ae^{-kt}} \) satisfying \( f(0) = 1, f(1) = 8, \) and \( f(2) = 14 \). What are the horizontal asymptotes of \( f \)?

46. Determine \( M, A, \) and \( k \), for a logistic function \( f(t) = \frac{M}{1 + Ae^{-kt}} \) satisfying \( f(0) = 10, f(4) = 35, \) and \( f(10) = 60 \). What are the horizontal asymptotes of \( f \)?

47. Find the local extrema of \( f(x) = e^{2x} - 4e^x \).

48. Find the points of inflection of \( f(x) = \ln(x^2 + 1) \) and determine whether the concavity changes from up to down or vice versa.

In Exercises 49–52, find the local extrema and points of inflection, and sketch the graph over the interval specified. Use L'Hôpital's Rule to determine the limits as \( x \to 0^+ \) or \( x \to \pm \infty \) if necessary.

49. \( y = x \ln x, \quad x > 0 \)
50. \( y = xe^{-x^2/2} \)
51. \( y = x(\ln x)^2, \quad x > 0 \)
52. \( y = \tan^{-1} \left( \frac{x^2}{4} \right) \)

In Exercises 53–58, use logarithmic differentiation to find the derivative.

53. \( y = \frac{(x + 1)^3}{(4x - 2)^2} \)
54. \( y = \frac{(x + 1)(x + 2)^2}{(x + 3)(x + 4)} \)
55. \( y = e^{(x-1)^2/(x-3)^2} \)
56. \( y = \frac{e^{\sin^{-1} x}}{\ln x} \)
57. \( y = \frac{e^{(x-2)^2}}{(x + 1)^2} \)
58. \( y = x \sqrt{x}(\ln x) \)

59. Over hilly forested terrain, the log wind profile is expressed as

\[
    v = \frac{u_0}{\ln(h/0.4)} \frac{\ln(h/0.4)}{\ln(h_0/0.4)}
\]

With \( u_0 = 20 \text{ m/s} \) at \( h_0 = 15 \text{ m} \), determine \( v \) and \( dv/dh \) at \( h = 80 \).
60. Over a flat open desert, the log wind profile is expressed as

\[ v = v_0 \frac{\ln(h/0.0002)}{\ln(80/0.0002)} \]

With \( v_0 = 20 \text{ m/s} \) at \( h_0 = 15 \text{ m} \), determine \( v \) and \( dv/dh \) at \( h = 80 \).

61. The energy (in ergs) associated with an earthquake of moment magnitude \( M_w \) satisfies \( \log_{10} E = 16.1 + 1.5M_w \). Calculate \( dE/dM_w \) for \( M_w = 3 \) and for \( M_w = 7 \).

62. The decibel level \( D \) for the intensity of a sound is related to the sound intensity \( I \) (in watts per square meter) by \( \log_{10} I = 12 - 0.1D \). Calculate \( dI/dD \) for \( D = 40 \) and for \( D = 80 \).

63. Image Processing: The intensity of a pixel in a digital image is measured by a number \( u \) between 0 and 1. Often, images can be enhanced by rescaling intensities, where pixels of intensity \( u \) are displayed with intensity \( g(u) \) for a suitable function \( g(u) \). An example is shown with a photograph of Amelia Earhart in Figure 1. One common choice is the sigmoidal correction, defined for constants \( a, b \) by

\[ g(u) = \frac{f(u) - f(0)}{f(1) - f(0)} \quad \text{where} \quad f(u) = (1 + e^{b(a-u)})^{-1} \]

Figure 2 shows that \( g(u) \) reduces the intensity of low-intensity pixels (where \( g(u) < u \)) and increases the intensity of high-intensity pixels.
(a) Verify that \( f'(u) > 0 \) and use this to show that \( g(u) \) increases from 0 to 1 for 0 ≤ \( u \) ≤ 1.
(b) Where does \( g \) have a point of inflection?

64. Let \( N(t) \) be the size of a tumor (in units of \( 10^6 \) cells) at time \( t \) (in days). According to the Gompertz Model, \( dN/dt = N(a - b \ln N) \), where \( a, b \) are positive constants. Show that the maximum value of \( N \) is \( a/e^b \) and that the tumor increases most rapidly when \( N = e^b/a \).

In Exercises 65–70, use the given substitution to evaluate the integral.

65. \( \int \frac{\ln(x)^2}{x} \ dx, \quad u = \ln x \)
66. \( \int \frac{dx}{4x^2 + 9}, \quad u = \frac{x}{3} \)
67. \( \int \frac{dx}{\sqrt{e^x - 1}}, \quad u = e^x \)
68. \( \int \frac{\cos^{-1} t \ dt}{\sqrt{1 - t^2}}, \quad u = \cos^{-1} t \)
69. \( \int \frac{dt}{t(1 + (\ln t)^2)}, \quad u = \ln t \)
70. \( \int \frac{e^t dt}{1 + e^t}, \quad u = 1 + e^t \)

In Exercises 71–94, calculate the integral.

71. \( \int e^{-2x} \ dx \)
72. \( \int x^2 e^x \ dx \)
73. \( \int e^{-2x} \sin(e^{-2x}) \ dx \)
74. \( \int \cos(\ln x) \ dx \)
75. \( \int e^{x-3} \ dx \)
76. \( \int \frac{dx}{x \sqrt{x}} \)
77. \( \int \frac{\ln x \ dx}{x} \)
78. \( \int \frac{e^{-x} \ dx}{e-x} \)
79. \( \int \frac{2x}{\sqrt{1 - x^2}} \ dx \)
80. \( \int \frac{e^x \ dx}{x^2} \)
81. \( \int \frac{(e^{2t} + e^{-2t}) \ dt}{4t + 12} \)
82. \( \int \frac{dx}{x^2 + 9} \)
83. \( \int \frac{dx}{\sqrt{1 - x^2}} \)
84. \( \int \frac{dx}{x^2 + 1} \)
85. \( \int \frac{x \ dx}{e^{-x} + 2} \)
86. \( \int \frac{e^{4x} \ dx}{x^2 + 2} \)
87. \( \int \frac{\sin^2 \theta \ cos^2 \theta + 1 \ d\theta}{\tan 2\theta \ d\theta} \)
88. \( \int \frac{2\pi x}{x} \)
89. \( \int \frac{\sin^{-1} x \ dx}{\sqrt{1 - x^2}} \)
90. \( \int \frac{dx}{25x^2} \)
91. \( \int \frac{dx}{x^3} \)
92. \( \int \frac{dx}{x^2 - 12} \)
93. \( \int \frac{dx}{x^2 + 2} \)
94. \( \int \frac{dx}{x^2 + 12} \)

95. In a first-order chemical reaction, the quantity \( y(t) \) of reactant at time \( t \) satisfies \( y' = -ky \), where \( k > 0 \). The dependence of \( k \) on temperature \( T \) (in kelvins) is given by the Arrhenius equation \( k = Ae^{-E_a/(RT)} \), where \( E_a \) is the activation energy (J•mol⁻¹•K⁻¹), \( R = 8.314 \text{ J•mol}^{-1}•\text{K}^{-1} \), and \( A \) is a constant. Assume that \( A = 72 \times 10^{12} \text{ hour}^{-1} \) and \( E_a = 1.1 \times 10^9 \text{ J} \). Calculate \( dE_a/dT \) for \( T = 500 \) and use the Linear Approximation to estimate the change in \( k \) if \( T \) is raised from 500 to 510 K.
96. An investment pays out $5000 at the end of the year for 3 years. Compute the PV, assuming an interest rate of 8%.

97. An equipment upgrade costing $1 million will save a company $320,000 per year for 4 years. Is this a good investment if the interest rate is r = 5%? What is the largest interest rate that would make the investment worthwhile? Assume that the savings are received as a lump sum at the end of each year.

98. Find the PV of an income stream paying out continuously at a rate of $4000 + 0.3t$ dollars per year for 5 years, assuming an interest rate of r = 4%.

99. Show that the function $R(x) = \frac{M}{x} \left( \frac{x}{2} + \tan^{-1} x \right)$, with $M > 0$, has the following properties similar to the general logistic function (Exercise 27 in Section 7.4):

(a) $\lim_{x \to \infty} R(x) = 0$ and $\lim_{x \to \infty} R(x) = M$, and therefore $R = 0$ and $R = M$ are horizontal asymptotes of $R$.

(b) $R$ is increasing for all $x$.

(c) $R$ has a single inflection point. The value of $R$ at the inflection point is $M/2$. To the left of the inflection point $R$ is concave up, to the right $R$ is concave down.

In Exercises 100–111, verify that L'Hôpital's Rule applies and evaluate the limit.

100. $\lim_{x \to 0} \frac{4x^2 - 12}{x^2 + 3x + 5} = 0$.

101. $\lim_{x \to 2} \frac{x^3 + 2x^2 - x - 2}{x^3 + 3x^2 + 4x - 8} = 2$.

102. $\lim_{x \to 0^+} \frac{x^{1/2} \ln x}{x}$.

103. $\lim_{t \to \infty} \frac{\ln(t^2 + 1)}{t} = 0$.

104. $\lim_{\theta \to \frac{\pi}{2}} \frac{2 \sin \theta - \sin 2\theta}{\theta - \left( \theta - \theta \cos \theta \right)}$.

105. $\lim_{x \to \infty} \frac{\sqrt{4 + x^2} - 2 \sqrt{1 + x}}{x^2} = 0$.

106. $\lim_{t \to \infty} \frac{\ln(t + 2)}{\log_2 t}$.

107. $\lim_{x \to 0} \frac{\frac{e^x}{x} - 1}{x}$.

108. $\lim_{y \to 0} \frac{\sin^{-1} y - y}{y^3} = 0$.

109. $\lim_{x \to 1} \frac{\sqrt{1 - x^2}}{\cos^{-1} x} = 0$.

110. $\lim_{x \to 1} \frac{\sinh(x^2)}{x}$.

111. $\lim_{x \to 0} \frac{\sin x}{x} = 0$.

112. Explain why L'Hôpital's Rule gives no information about $\lim_{x \to \infty} \frac{2x - \sin x}{3x + \cos 2x}$. Evaluate the limit by another method.

113. Let $f$ be a differentiable function with inverse $g$ such that $f(0) = 0$ and $f'(0) \neq 0$. Prove that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = f'(0)^2$$

114. Calculate the limit

$$\lim_{n \to \infty} \left( 1 + \frac{3}{n} \right)^n$$

115. Calculate the limit

$$\lim_{n \to \infty} \left( 1 + \frac{4}{n} \right)^n$$

116. In this exercise, we prove that for all $x > 0$,

$$x - \frac{x^2}{2} \leq \ln(1 + x) \leq x$$

(a) Show that $\ln(1 + x) = \int_0^x \frac{dt}{1 + t}$ for $x > 0$.

(b) Verify that $1 - t \leq \frac{1}{1 + t} \leq 1$ for all $t > 0$.

(c) Use (b) to prove Eq. (1).

(d) Verify Eq. (1) for $x = 0.5, 0.1,$ and 0.01.

117. Let

$$F(x) = x\sqrt{x^2 - 1} - 2 \int_1^x \sqrt{t^2 - 1} \, dt$$

Prove that $F(x)$ and $\cosh^{-1} x$ differ by a constant by showing that they have the same derivative. Then prove they are equal by evaluating both at $x = 1$.

In Exercises 118–121, let $g(d) = \tan^{-1}(\sinh y)$ be the so-called gudermannian, which arises in cartography. In a map of the earth constructed by Mercator projection, points located $y$ radial units from the equator correspond to points on the globe of latitude $g(d)$.

118. Prove that $\frac{d}{dy} g(d) = \sec y$.

119. Let $f(y) = 2 \tan^{-1}(e^y) - \pi/2$. Prove that $g(d) = f(y)$. Hint: Show that $g'(d) = f'(y)$ and $f(0) = g(0)$.

120. Show that $t(y) = \sinh^{-1} \left( \tan(\pi/2) \right)$ is the inverse of $g(d)$ for $0 \leq y < \pi/2$.

121. Verify that $t(y)$ in Exercise 120 satisfies $t'(y) = \sec y$ and find a value of $a$ such that

$$t(y) = \int_a^y \frac{dt}{\cos t}$$

122. Use L'Hôpital's Rule to prove that for all $a > 0$ and $b > 0$,

$$\lim_{n \to \infty} \left( \frac{a^{1/n} + b^{1/n}}{2} \right)^n = \sqrt{ab}$$

123. Let

$$F(x) = \int_2^x \frac{dt}{\ln t} \quad \text{and} \quad G(x) = \frac{x}{\ln x}$$

Verify that L'Hôpital's Rule may be applied to the limit

$$L = \lim_{x \to \infty} \frac{F(x)}{G(x)} \quad \text{and} \quad L = \lim_{x \to \infty} \frac{F(x)}{G(x)}$$

124. Let $f(x) = e^{-A/2}$, where $A > 0$. Given any $n$ numbers $a_1, a_2, \ldots, a_n$, set

$$\Phi(x) = f(x - a_1)f(x - a_2) \cdots f(x - a_n)$$

(a) Assume $n = 2$ and prove that $\Phi(x)$ attains its maximum value at the average $x = \frac{1}{2}(a_1 + a_2)$. Hint: Show that $d/dx \ln \Phi(x) = -Ax$ and calculate $\Phi'(x)$ using logarithmic differentiation.

(b) Show that for any $n$, $\Phi(x)$ attains its maximum value at $x = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$. This fact is related to the role of $f(x)$ (whose graph is a bell-shaped curve) in statistics.
8 TECHNIQUES OF INTEGRATION

In Section 5.7, we introduced substitution, one of the most important techniques of integration. In this chapter, we develop a second fundamental technique, Integration by Parts, as well as several techniques for treating particular classes of functions such as trigonometric and rational functions. However, there is no surefire method, and in fact, many important antiderivatives cannot be expressed in elementary terms. Therefore, we discuss numerical integration in the last section. Every definite integral can be approximated numerically to any desired degree of accuracy.

8.1 Integration by Parts

In this section, we derive a formula that often allows us to convert an integral that we cannot immediately evaluate into one that we can. The Integration by Parts formula is derived from the Product Rule.

Let \( u \) and \( v \) be functions of \( x \):

\[
\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u
\]

According to this formula, \( uv \) is an antiderivative of the right-hand side, so

\[
uv = \int u \frac{dv}{dx} \, dx + \int v \frac{du}{dx} \, dx
\]

Moving the second integral on the right to the other side, we obtain

\[
\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx
\]

By letting \( du = \frac{du}{dx} \, dx \) and \( dv = \frac{dv}{dx} \, dx \), we find the following

Integration by Parts

\[
\int u \, dv = uv - \int v \, du
\]

Because Integration by Parts applies to a product, we should consider using it when the integrand is a product of two functions. It is not, however, a product rule for antidifferentiation, because we cannot always use Integration by Parts to find an antiderivative of a product. Sometimes it works, as with \( f(x) = x \cos x \) in Example 1; other times it does not, as with \( f(x) = x \tan x \) (see Exercises 37 and 38). Furthermore, sometimes when the integrand is not expressed directly as a product, a clever application of Integration by Parts enables us to find an antiderivative (such as with \( f(x) = \ln x \) in Example 3).

**EXAMPLE 1** Evaluate \( \int x \cos x \, dx \).

Solution The integrand is a product, so we try writing \( x \cos x \, dx = u \, dv \) with

\[
u = x \quad dv = \cos x \, dx
\]

Differentiating \( u \) and antidi\(differentiating \( dv \)\), we find

\[
u = \sin x
\]

By the Integration by Parts formula,

\[
\int \frac{x \cos x \, dx}{u} = x \sin x - \int \frac{\sin x \, dx}{v} = x \sin x + \cos x + C
\]
Let's check the answer by taking the derivative:
\[
\frac{d}{dx} (x \sin x + \cos x + C) = x \cos x + \sin x - \sin x = x \cos x
\]

The key step in Integration by Parts is deciding how to write the integral as a product \( u \, dv \). Keep in mind that Integration by Parts expresses \( \int u \, dv \) in terms of \( uv \) and \( \int v \, du \).

This is useful only if \( v \, du \) is easier to integrate than \( u \, dv \). Here are two guidelines:

- Choose \( dv \) so that \( v \) can be evaluated.
- Choose \( u \) so that \( \frac{du}{dx} \) is simpler than \( u \) itself.

The choices for \( u \) and \( dv \) that we made in Example 1 were good choices because they enabled us to transform the integral to a simpler one. A bad choice would be \( u = \cos x \) and \( dv = x \, dx \). Then, with \( du = -\sin x \, dx \) and \( v = \frac{1}{2} x^2 \), the Integration by Parts formula yields
\[
\int x \cos x \, dx = \frac{1}{2} x^2 \cos x - \int \left( \frac{1}{2} x^2 \right) (-\sin x) \, dx
\]

In this case, the resulting integral (essentially \( \int x^2 \sin x \, dx \)) is more complicated than the one we had initially.

**EXAMPLE 2** Integrating by Parts More Than Once Evaluate \( \int x^2 \cos x \, dx \).

**Solution** Apply Integration by Parts a first time with \( u = x^2 \) and \( dv = \cos x \, dx \):
\[
\int x^2 \cos x \, dx = \frac{x^2}{2} \sin x = \int \left( \frac{1}{2} x^2 \right) (-\sin x) \, dx
\]

Now apply it again to the integral on the right, this time with \( u = x \) and \( dv = \sin x \, dx \):
\[
\int x \sin x \, dx = \frac{x \cos x}{2} - \int \left( \frac{1}{2} x \right) (-\cos x) \, dx
\]

Using this result in Eq. (2), we obtain
\[
\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx = x^2 \sin x - 2(-x \cos x + \sin x) + C
\]

The function \( f(x) = \ln x \) is one of the basic functions for which we have not yet seen an antiderivative formula. Now, with Integration by Parts, we can obtain one.

**EXAMPLE 3** Taking \( dv = dx \) Evaluate \( \int \ln x \, dx \).

**Solution** The integrand is not a product, so at first glance, this integral does not look like a candidate for Integration by Parts. However, we can treat \( \ln x \, dx \) as a product of \( \ln x \) and \( dx \). Then
\[
u = \ln x \quad dv = dx
\]
\[
u = x \quad du = \frac{1}{x} \, dx
\]
\[
\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx
\]
\[
= x \ln x - x + C
\]
Let's check the result of the previous example by differentiation:
\[ \frac{d}{dx}(x \ln x - x) = x \left( \frac{1}{x} \right) + (1) \ln x - 1 = 1 + \ln x - 1 = \ln x \]

Now, to our table of integral formulas we can add the indefinite integral of \( f(x) = \ln x \):
\[ \int \ln x \, dx = x \ln x - x + C \]

There is a convenient definite-integral version of the Integration by Parts formula:
\[ \int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du \]

We employ this in the next example.

**EXAMPLE 4 Red Fox Dispersal** You are investigating how the red fox disperses from its place of birth and have developed a model for determining the probability that breeding occurs within a particular distance from the birthplace. Specifically, for any \( x \) between 0 and 10, your model gives the likelihood that a fox breeds its first offspring within \( x \) km of its birthplace as \( \int_0^x 0.2e^{-0.16t} \, dt \). With such a model, the average birth-to-breeding distance can be determined via the integral \( \int_0^{10} x (0.2e^{-0.16x}) \, dx \). Compute this average.

**Solution** To compute the average distance \( \int_0^{10} x (0.2e^{-0.16x}) \, dx \), we use Integration by Parts with
\[ u = x \quad dv = 0.2e^{-0.16x} \, dx \]
\[ du = dx \quad v = \frac{0.2}{-0.16}e^{-0.16x} = -1.25e^{-0.16x} \]

Using the definite integral version of Integration by Parts, we have
\[ \int_0^{10} x (0.2e^{-0.16x}) \, dx = \left. -1.25xe^{-0.16x} \right|_0^{10} + 1.25 \int_0^{10} e^{-0.16x} \, dx \]
\[ = -12.5e^{-1.6} + 0 + \frac{1.25}{-0.16}e^{-0.16x} \bigg|_0^{10} \]
\[ \approx -12.5e^{-1.6} - 7.81e^{-1.6} + 7.81 \approx 3.71 \]

Thus, in the model, the average birth-to-breeding dispersal distance for the red fox is approximately 3.71 km.

**EXAMPLE 5 Going in a Circle?** Evaluate \( \int e^x \cos x \, dx \).

**Solution** There are two reasonable ways of writing \( e^x \cos x \, dx \) as \( u \, dv \). Let's try setting \( u = \cos x \). Then we have
\[ u = \cos x \quad dv = e^x \, dx \]
\[ du = -\sin x \, dx \quad v = e^x \]

Thus,
\[ \int e^x \cos x \, dx = e^x \cos x - \int e^x (-\sin x) \, dx \]

In Example 5, the choice \( u = e^x \), \( dv = \cos x \, dx \) works equally well.
Now use Integration by Parts on the integral on the right with $u = \sin x$:

$$
u = e^x \\
du = \cos x \, dx \\

\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

Eq. (4) brings us back to our original integral of $e^x \cos x$, so it looks as if we're going in a circle. But we can substitute Eq. (4) in Eq. (3) and solve for the integral of $e^x \cos x$:

$$
\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx = e^x \cos x + \left( e^x \sin x - \int e^x \cos x \, dx \right)
$$

Now we can add $\int e^x \cos x \, dx$ to both sides. Note that we add a "+ C" to the right side since we no longer have an integral on the right side of the equation that will generate the necessary arbitrary constant:

$$
2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C
$$

$$
\int e^x \cos x \, dx = \frac{1}{2} e^x (\cos x + \sin x) + C
$$

**Example 6: A Reduction Formula** Derive the reduction formula

$$
\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx
$$

Then use the reduction formula to evaluate $\int \sin^3 x \, dx$.

**Solution** Although we do not know how to integrate $\sin^n x$, we do know how to integrate $\sin x$. So we apply Integration by Parts as follows:

$$
u = \sin^{n-1} x \\
dv = \sin x \, dx \\
\int \sin^n x \, dx = -\sin^{n-1} x \cos x - \int \cos x (n-1) \sin^{n-2} x \cos x \, dx
$$

Using the fact that $\cos^2 x = 1 - \sin^2 x$, we obtain

$$
\int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx
$$

Adding $(n-1) \int \sin^n x \, dx$ to both sides, and then dividing by $n$, results in the desired reduction formula:

$$
n \int \sin^n x \, dx = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx
$$

$$
\int \sin^n x \, dx = \frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx
$$
Now applying the reduction formula in the case \( n = 3 \), we have
\[
\int \sin^3 x \, dx = -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C
\]

Reduction formulas for \( \int \cos^n x \, dx \), \( \int x^n e^x \, dx \), and \( \int \ln x \, dx \) are introduced and investigated in the exercises.

### 8.1 SUMMARY

- Integration by Parts formula: \( \int u \, dv = uv - \int v \, du \).
- The key step is deciding how to write the integrand as a product \( u \, dv \). Keep in mind that Integration by Parts is useful when \( v \, du \) is easier (or, at least, not more difficult) to integrate than \( u \, dv \). Here are some guidelines:
  - Choose \( u \) so that \( \frac{du}{dx} \) is simpler than \( u \) itself.
  - Choose \( dv \) so that \( v = \int dv \) can be evaluated.
  - Sometimes, \( dv = dx \) is a good choice.
  - Good choices for \( u \) include \( x^n \), \( \ln x \), and inverse trigonometric functions.

### 8.1 EXERCISES

#### Preliminary Questions

1. Which derivative rule is used to derive the Integration by Parts formula?
   
2. For each of the following integrals, state whether substitution or Integration by Parts should be used:

#### Exercises

In Exercises 1–6, evaluate the integral using the Integration by Parts formula with the given choice of \( u \) and \( dv \).

1. \( \int x \sin x \, dx; \quad u = x, \ dv = \sin x \, dx \)
2. \( \int x e^{2x} \, dx; \quad u = x, \ dv = e^{2x} \, dx \)
3. \( \int (2x + 9) e^x \, dx; \quad u = 2x + 9, \ dv = e^x \, dx \)
4. \( \int x \cos 4x \, dx; \quad u = x, \ dv = \cos 4x \, dx \)
5. \( \int x^3 \ln x \, dx; \quad u = \ln x, \ dv = x^3 \, dx \)
6. \( \int \tan^{-1} x \, dx; \quad u = \tan^{-1} x, \ dv = dx \)

In Exercises 7–34, evaluate using Integration by Parts.

7. \( \int (4x - 3) e^{-x} \, dx \)
8. \( \int (2x + 1) e^x \, dx \)
9. \( \int x e^{x^2} \, dx \)
10. \( \int x^2 e^x \, dx \)
11. \( \int x \cos 2x \, dx \)
12. \( \int x \sin(3 - x) \, dx \)
13. \( \int x^2 \sin x \, dx \)
14. \( \int x^2 \cos 3x \, dx \)
15. \( \int e^{-x} \sin x \, dx \)
16. \( \int e^x \sin 2x \, dx \)
17. \( \int e^{-x^2} \sin x \, dx \)
18. \( \int e^{2x} \cos 4x \, dx \)
19. \( \int x \ln x \, dx \)
20. \( \int \frac{\ln x}{x^2} \, dx \)
21. \( \int x^2 \ln x \, dx \)
22. \( \int x^{-5} \ln x \, dx \)
23. \( \int (\ln x)^2 \, dx \)
24. \( \int x(\ln x)^2 \, dx \)
25. \( \int \cos^{-1} x \, dx \)
26. \( \int \sin^{-1} x \, dx \)
27. \( \int \sec^{-1} x \, dx \)
28. \( \int x^5 \, dx \)
29. \( \int 3^x \cos x \, dx \)
30. \( \int x \sinh x \, dx \)
31. \[ \int x^2 \cosh x \, dx \]
32. \[ \int \cos x \cosh x \, dx \]
33. \[ \int \tanh^{-1} 4x \, dx \]
34. \[ \int \sinh^{-1} x \, dx \]

In Exercises 35–36, evaluate using substitution and then integration by parts.

35. \[ \int e^{\sqrt{x}} \, dx \] Hint: Let \( u = x^{1/2} \).
36. \[ \int x^3 e^{-x^2} \, dx \]

37. For \( \int x \tan x \, dx \), try integration by parts with \( u = x \), \( du = \tan x \, dx \) and with \( u = \tan x \), \( dv = x \, dx \), and describe the difficulty that you encounter in each case, keeping you from finding an antiderivative. (Note: there is no antiderivative formula for \( x \tan x \) involving elementary functions.)

38. For \( \int x \sec x \, dx \), try integration by parts with \( u = x \), \( du = \sec x \, dx \) and with \( u = \sec x \), \( dv = x \, dx \), and describe the difficulty that you encounter in each case, keeping you from finding an antiderivative. (Note: there is no antiderivative formula for \( x \sec x \) involving elementary functions.)

In Exercises 39–48, evaluate using integration by parts, substitution, or both if necessary.

39. \[ \int x \cos 4x \, dx \]
40. \[ \int \frac{\ln(\ln x) \, dx}{x} \]
41. \[ \int \frac{x \, dx}{\sqrt{x^2 + 1}} \]
42. \[ \int x^2 (x^3 + 9)^{1/2} \, dx \]
43. \[ \int \cos x \ln(\sin x) \, dx \]
44. \[ \int \sin \sqrt{x} \, dx \]
45. \[ \int x e^{\sqrt{x}} \, dx \]
46. \[ \int \frac{\tan \sqrt{x} \, dx}{\sqrt{x}} \]
47. \[ \int \frac{\ln(\ln x) \ln x \, dx}{x} \]
48. \[ \int \sin(\ln x) \, dx \]

In Exercises 49–58, compute the definite integral.

49. \[ \int_0^2 x e^x \, dx \]
50. \[ \int_0^1 \frac{e^{x^4}}{x} \, dx \]
51. \[ \int_1^2 x \ln x \, dx \]
52. \[ \int_1^e \frac{\ln x \, dx}{x^2} \]
53. \[ \int_0^1 x e^{-x^2} \, dx \]
54. \[ \int_0^1 \frac{x^3}{\sqrt{9 + x^2}} \, dx \]
55. \[ \int_0^1 x^3 \, dx \]
56. \[ \int_0^1 x \cos(\pi x) \, dx \]
57. \[ \int_0^1 e^x \sin x \, dx \]
58. \[ \int_0^1 \tan^{-1} x \, dx \]

59. Robin has been tracking her archery accuracy. For \( 0 \leq x \leq 60 \) the probability that an arrow that she shoots hits the target within \( x \) centimeters of the center is given by \( \int_0^x 0.071e^{-0.07x} \, dx \). The average distance of her shots from the center is given by \( \int_0^{60} x (0.071e^{-0.07x}) \, dx \). Compute the average distance.

60. When Darius is shooting for a bull’s eye in darts, the probability that a throw lands within \( x \) millimeters of the center of the dart board is given by \( \int_0^x 0.066e^{-0.066t} \, dt \) for \( 0 \leq x \leq 40 \).

(a) The bull’s eye has a diameter of 12.5 mm. What is the probability that a throw is a bull’s eye?
(b) The average distance of his throws from the center of the dart board is given by \( \int_0^{40} t (0.066e^{-0.066t}) \, dt \). Compute the average distance.

61. Use Eq. (5) to find \( \int \sin^3 x \, dx \).
62. Derive the reduction formula \( \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \).
63. Use the reduction formula from Exercise 62 to find \( \int \cos^3 x \, dx \).
64. Derive the reduction formula \( \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx \).
65. Use the reduction formula from Exercise 64 to find \( \int x^3 e^x \, dx \).
66. Use substitution and the reduction formula from Exercise 64 to evaluate \( \int x^4 e^x \, dx \).
67. Find a reduction formula for \( \int x^n e^{-x} \, dx \) similar to the formula appearing in Exercise 64.
68. Evaluate \( \int x^3 \ln x \, dx \) for \( n \neq -1 \). Which method should be used to evaluate \( \int x^{-1} \ln x \, dx \)?
69. Find the volume of the solid of revolution that results when the region under the graph of \( f(x) = x - \sin x \) for \( 0 \leq x \leq \pi \) is revolved around the \( x \)-axis.
70. Find the volume of the solid of revolution that results when the region under the graph of \( f(x) = \ln x \) for \( 1 \leq x \leq e \) is revolved around the \( x \)-axis.
71. Find the volume of the solid of revolution that results when the region under the graph of \( f(x) = 3 \sin x \) for \( 0 \leq x \leq \pi \) is revolved around:
   (a) the \( y \)-axis.
   (b) the line \( x = 1 \).
   (c) the line \( y = 2 \).
   (d) the line \( x = 2 \).

In Exercises 73–80, indicate a good method for evaluating the integral (but do not evaluate). Your choices are algebraic manipulation, substitution (specify \( u \) and \( du \)), and integration by parts (specify \( u \) and \( dv \)). It is apparent that the techniques you have learned thus far are not sufficient, state this.

73. \[ \int \sqrt{x} \ln x \, dx \]
74. \[ \int \frac{x^2 - \sqrt{x}}{2x} \, dx \]
75. \[ \int x^3 \, dx \]
76. \[ \int \frac{dx}{\sqrt{x^2 + 2}} \]
77. \[ \int x + 2 \, dx \]
78. \[ \int \frac{dx}{x^2 + 2x + 3} \]
79. \[ \int x \sin(3x + 4) \, dx \]
80. \[ \int x \cos(3x^2) \, dx \]

81. Evaluate \( \int (\sin^{-1} x)^2 \, dx \). Hint: Use integration by parts first and then substitution.
82. Evaluate \( \int \frac{\ln(x)}{x^2} \, dx \). Hint: Use substitution first and then integration by parts.
83. Evaluate \( \int 3^x \cos(x^4) \, dx \).

84. Evaluate \( \int_0^1 x^3e^{-x^2} \, dx \).

85. Find the area of the region that lies under the graph of \( y = (5 - x) \ln x \) and above the \( x \)-axis.

86. Find the area enclosed by \( y = \ln x \) and \( y = (\ln x)^2 \).

87. The present value (PV) of an investment that provides income continuously at a rate \( R(t) \) \$/year for \( T \) years, and earns interest at rate \( r \), is

\[
\int_0^T R(t)e^{-rt} \, dt.
\]

We think of present value as the payment that we would need to receive at \( t = 0 \), instead of the investment income, so that at time \( T \) the payment's value (with accumulated interest) would be the same as the amount accumulated from the income stream (also accumulating interest). Find the PV if \( R(t) = 5000 + 100t \) \$/year, \( r = 0.05 \), and \( T = 10 \) years.

88. Derive the reduction formula

\[
\int (\ln x)^k \, dx = x(\ln x)^k - k \int (\ln x)^{k-1} \, dx
\]

89. Use Eq. (6) to calculate \( \int (\ln x)^k \, dx \) for \( k = 2, 3 \).

90. Derive the reduction formulas

\[
\begin{align*}
\int x^n \cos x \, dx &= x^n \sin x - n \int x^{n-1} \sin x \, dx \\
\int x^n \sin x \, dx &= -x^n \cos x + n \int x^{n-1} \cos x \, dx
\end{align*}
\]

91. Prove that \( \int x^2e^x \, dx = e^x \left( \frac{x}{\ln b} - \frac{1}{(\ln b)^2} \right) + C \).

92. Define \( P_n(x) \) by

\[
\int x^ne^x \, dx = P_n(x)e^x + C
\]

Use the reduction formula in Exercise 64 to prove that \( P_n(x) = x^n - nP_{n-1}(x) \). Use this recursion relation to find \( P_n(x) \) for \( n = 1, 2, 3, 4 \). Note that \( P_0(x) = 1 \).

Further Insights and Challenges

93. The integration by parts formula can be written

\[
\int uv \, dx = uV - \int V \, du
\]

where \( V(x) \) satisfies \( V'(x) = u(x) \).

(a) Show directly that the right-hand side of Eq. (7) does not change if \( V(x) \) is replaced by \( V(x) + C \), where \( C \) is a constant.

(b) Use \( u = \tan^{-1}x \) and \( v = x \) in Eq. (7) to calculate \( \int x \tan^{-1}x \, dx \), but carry out the calculation twice: first with \( V(x) = \frac{1}{2} x^2 \) and then with \( V(x) = \frac{1}{2} x^2 + \frac{1}{2} \). Which choice of \( V(x) \) results in a simpler calculation?

94. Prove in two ways that

\[
\int_0^a f(x) \, dx = a f(a) - \int_0^a xf'(x) \, dx
\]

First use Integration by Parts. Then assume \( f \) is increasing. Use the substitution \( u = f(x) \) to prove that \( \int_0^a xf'(x) \, dx \) equals the area of the shaded region in Figure 1 and derive Eq. (8) a second time.

95. Assume that \( f(0) = f(1) = 0 \) and that \( f'' \) exists. Prove

\[
\int_0^1 f''(x) f(x) \, dx = -\int_0^1 f'(x)^2 \, dx
\]

Use this to prove that if \( f(0) = f(1) = 0 \) and \( f''(x) = \lambda f(x) \) for some constant \( \lambda \), then \( \lambda < 0 \). Can you think of a function satisfying these conditions for some \( \lambda \)?

96. Set \( I(a, b) = \int_0^a x^2(1 - x)^b \, dx \), where \( a, b \) are whole numbers.

(a) Use substitution to show that \( I(a, b) = I(b, a) \).

(b) Show that \( I(a, 0) = I(0, a) = \frac{1}{a + 1} \).

(c) Prove that for \( a \geq 1 \) and \( b \geq 0 \),

\[
I(a, b) = \frac{a}{b + 1} I(a - 1, b + 1)
\]

(d) Use (b) and (c) to calculate \( I(1, 1) \) and \( I(3, 2) \).

(e) Show that \( I(a, b) = \frac{a! b!}{(a + b + 1)!} \).

97. Let \( I_n = \int x^n \cos(x^2) \, dx \) and \( J_n = \int x^n \sin(x^2) \, dx \).

(a) Find a reduction formula that expresses \( I_n \) in terms of \( J_{n-2} \). Hint: Write \( x^n \cos(x^2) \) as \( x^{n-2}(x \cos(x^2)) \).

(b) Use the result of (a) to show that \( I_n \) can be evaluated explicitly if \( n \) is odd.

(c) Evaluate \( I_3 \).

8.2 Trigonometric Integrals

In this section, we investigate integrals of various products of powers of trigonometric functions. We can often compute these integrals by combining substitution and Integration by Parts with trigonometric identities. In the section summary we expand on the
table of integrals we have built so far, adding integral formulas employed or derived in
this section and some other formulas similar to them. We begin with integrals of the form
\[ \int \sin^m x \cos^n x \, dx \]
where \( m, n \) are whole numbers. The easier case is when at least one of \( m, n \) is odd.

**EXAMPLE 1**  Odd Power of \( \sin x \)  Evaluate \( \int \sin^3 x \, dx \).

**Solution**  We did this integral in Example 6 of the last section. However, we will use
a different method that is more broadly applicable. Because \( \sin^3 x \) is an odd power, we
split off one power of \( \sin x \) and use the identity \( \sin^2 x = 1 - \cos^2 x \) to convert the rest of
the integrand into an expression in \( \cos x \):
\[ \sin^3 x = (\sin^2 x)(\sin x) = (1 - \cos^2 x) \sin x \]

We then use the substitution \( u = \cos x, \, du = -\sin x \, dx \):

\[ \int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx = -\int (1 - u^2) \, du \]
\[ = \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C \]

The strategy of the previous example works when \( \sin^m x \) appears with \( m \), odd, no
matter what power of \( \cos x \) is present. Similarly, if \( n \) is odd, we write \( \cos^n x \) as a power
of \( (1 - \sin^2 x) \) times \( \cos x \).

**EXAMPLE 2**  Odd Power of \( \cos x \)  Evaluate \( \int \sin^4 x \cos^5 x \, dx \).

**Solution**  We take advantage of the fact that \( \cos^5 x \) is an odd power to write
\[ \sin^4 x \cos^5 x = \sin^4 x \cos^4 x (\cos x) = \sin^4 x (1 - \sin^2 x)^2 (\cos x) \]
\[ = (\sin^4 x - 2 \sin^6 x + \sin^8 x) \cos x \]

This allows us to use the substitution \( u = \sin x, \, du = \cos x \, dx \):

\[ \int \sin^4 x \cos^5 x \, dx = \int (\sin^4 x - 2 \sin^6 x + \sin^8 x) \cos x \, dx \]
\[ = \int (u^4 - 2u^6 + u^8) \, du \]
\[ = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C \]

We will need a different strategy when neither \( \sin x \) nor \( \cos x \) appears with an odd
power.

**EXAMPLE 3**  Evaluate \( \int \sin^2 x \, dx \).

**Solution**  We utilize the trigonometric identity called the double angle formula \( \sin^2 x = \frac{1}{2}(1 - \cos 2x) \). Then
\[ \int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C \]
Using the trigonometric identities in the margin, we can also integrate $\cos^2 x$, obtaining the following:

\[
\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C = \frac{x}{2} - \frac{1}{2} \sin x \cos x + C
\]

\[
\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C = \frac{x}{2} + \frac{1}{2} \sin x \cos x + C
\]

**Example 4** Evaluate $\int \sin^4 x \, dx$.

**Solution** Double angle formulas will be of assistance to us here as well. Using \( \sin^2 x = \frac{1}{2} (1 - \cos 2x) \), we obtain

\[
\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx = \int \left( \frac{1}{2} (1 - \cos 2x) \right)^2 \, dx
\]

\[
= \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx
\]

Applying the double angle formula to $\cos^2 2x$, we get

\[
\int \sin^4 x \, dx = \frac{1}{4} \int (1 - 2 \cos 2x + \cos^2 2x) \, dx
\]

\[
= \frac{1}{4} \int \left( -2 \cos 2x + \frac{1 + \cos 4x}{2} \right) \, dx
\]

\[
= \frac{1}{4} \int \left( \frac{3}{2} - 2 \cos 2x + \frac{\cos 4x}{2} \right) \, dx
\]

By simple $u$-substitutions, this yields

\[
\int \sin^4 x \, dx = \frac{1}{4} \left( \frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8} \right) + C = \frac{3x}{8} - \sin 2x + \frac{\sin 4x}{32} + C \,
\]

As indicated in the above margin comment, the reduction formula for $\int \sin^n x \, dx$ can be used for the previous two examples. There is also a corresponding reduction formula for $\int \cos^n x \, dx$. These two reduction formulas are (5) and (6) in the integral table at the end of this section. We employ the reduction formula for $\int \cos^n x \, dx$ in the next example.

**Example 5** Even Powers of $\sin x$ and $\cos x$ Evaluate $\int \sin^2 x \cos^4 x \, dx$.

**Solution** Here, $m = 2$ and $n = 4$. Since $m < n$, we replace $\sin^2 x$ by $1 - \cos^2 x$:

\[
\int \sin^2 x \cos^4 x \, dx = \int (1 - \cos^2 x) \cos^4 x \, dx = \int \cos^4 x \, dx - \int \cos^6 x \, dx
\]

The reduction formula for $n = 6$ gives

\[
\int \cos^6 x \, dx = \frac{1}{6} \cos^5 x \sin x + \frac{5}{6} \int \cos^4 x \, dx
\]

Using this result in the right-hand side of the first equation in this solution, we obtain

\[
\int \sin^2 x \cos^4 x \, dx = \int \cos^4 x \, dx - \left( \frac{1}{6} \cos^5 x \sin x + \frac{5}{6} \int \cos^4 x \, dx \right)
\]

\[
= \frac{1}{6} \cos^5 x \sin x + \frac{1}{6} \int \cos^4 x \, dx
\]
Next, we evaluate $\int \cos^4 x \, dx$ using the reduction formulas for $n = 4$ and $n = 2$:

\[
\int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx \\
= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left( \frac{1}{2} \cos x \sin x + \frac{1}{2} x \right) + C \\
= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + C
\]

Altogether,

\[
\int \sin^2 x \cos^4 x \, dx = -\frac{1}{6} \cos^5 x \sin x + \frac{1}{6} \left( \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x \right) + C \\
= -\frac{1}{6} \cos^5 x \sin x + \frac{1}{24} \cos^3 x \sin x + \frac{1}{16} \cos x \sin x + \frac{1}{16} x + C
\]

So far, we have found antiderivative formulas for all of the standard trigonometric functions except $\sec x$ and $\csc x$. For $\sin x$ and $\cos x$, they were obtained directly from the corresponding derivative formulas. For $\tan x$, we used a substitution in Example 10 in Section 7.3 (and $\cot x$ can be handled similarly). In the next example, we obtain an antiderivative formula for $\sec x$. With a similar method we can find an antiderivative of $\csc x$ (see Exercise 68).

**EXAMPLE 6  ** Integral of Secant  

Derive the formula

\[
\int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

**Solution** To integrate $\sec x$, we multiply by an unusual fraction that equals one. This approach enables us to use a substitution that results in an antiderivative that we can compute.

Multiply the integrand by

\[
1 = \frac{\sec x + \tan x}{\sec x + \tan x}
\]

Then

\[
\int \sec x \, dx = \int \sec x \left( \frac{\sec x + \tan x}{\sec x + \tan x} \right) \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx
\]

Noting that the numerator is the derivative of the denominator, we let the denominator be $u$:

\[
u = \sec x + \tan x \quad \text{and} \quad du = (\sec x \tan x + \sec^2 x) \, dx
\]

Then our integral becomes

\[
\int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C
\]

**EXAMPLE 7  ** Using a Table of Integrals  

Evaluate $\int_0^{\pi/4} \tan^3 x \, dx$.

**Solution** We use Eq. (8) in the table at the end of the section. With $k = 3$,

\[
\int_0^{\pi/4} \tan^3 x \, dx = \tan^2 x \bigg|_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx = \left( \frac{1}{2} \tan^2 x - \ln |\sec x| \right) \bigg|_0^{\pi/4}
\]

\[
= \left( \frac{1}{2} \tan^2 \frac{\pi}{4} - \ln |\sec \frac{\pi}{4}| \right) - \left( \frac{1}{2} \tan^2 0 - \ln |\sec 0| \right)
\]

\[
= \left( \frac{1}{2} (1)^2 - \ln \sqrt{2} \right) - \left( \frac{1}{2} (0)^2 - \ln |1| \right) = \frac{1}{2} - \ln \sqrt{2}
\]

In the margin at the top of the next page, we describe a method for integrating $\tan^n x \sec^m x$. 

The results of trigonometric integrals can be expressed in more than one way. A computer algebra system provided the following:

\[
\int \sin^2 x \cos^4 x \, dx = \frac{1}{6} \cos^5 x \sin x + \frac{1}{6} \left( \frac{1}{4} \cos^3 x \sin x + \frac{3}{8} \cos x \sin x + \frac{3}{8} x \right) + C
\]

Trigonometric identities can be used to show that this result agrees with the one in Example 5.

The integral $\int_0^x \sec t \, dt$ was first computed numerically in the 1590s by the English mathematician Edward Wright, decades before the invention of calculus. Although he did not invent the concept of an integral, Wright realized that the sums that approximate the integral hold the key to understanding the Mercator map projection, of great importance in navigation because it enabled sailors to reach their destinations along lines of fixed compass direction. The formula for the integral was first proved by James Gregory in 1669.
**Example 8** Evaluate \( \int \tan^3 x \sec^5 x \, dx \).

**Solution** We note that we have a copy of \( \sec x \tan x \) in the integrand, and the remaining powers of \( \tan x \) are even. So we separate out one copy of \( \sec x \tan x \) and convert the rest of the integrand into powers of \( \sec x \). Then since the derivative of \( \sec x \) is \( \sec x \tan x \), we set up for a \( u \)-substitution.

The first step is to use the identity \( \tan^2 x = \sec^2 x - 1 \):  
\[
\int \tan^3 x \sec^5 x \, dx = \int \frac{(\sec^2 x - 1)(\sec^4 x)(\sec x \tan x) \, dx}{\sec^6 x - \sec^4 x} \sec x \tan x \, dx
\]

Letting \( u = \sec x \), so \( du = \sec x \tan x \, dx \), we have
\[
\int \tan^3 x \sec^4 x \, dx = \int (u^6 - u^4) \, du = \frac{u^7}{7} - \frac{u^5}{5} + C
\]
\[
= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C
\]

Note that the above method works whenever we integrate \( \tan^m x \sec^n x \), where \( m \) is odd and \( n \) is even.

**Example 9** Evaluate \( \int \tan^2 x \sec^4 x \, dx \).

**Solution** In this case, since the derivative of \( \tan x \) is \( \sec^2 x \), we separate out \( \sec^2 x \) and convert the rest of the integrand into powers of \( \tan x \). Using the fact \( \sec^2 x = \tan^2 x + 1 \) yields
\[
\int \tan^2 x \sec^4 x \, dx = \int (\tan^2 x)(\sec^2 x + 1) \sec^2 x \, dx = \int (\tan^4 x + \tan^2 x) \sec^2 x \, dx
\]

Setting \( u = \sec x \) and \( du = \sec^2 x \, dx \), we have
\[
\int \tan^2 x \sec^4 x \, dx = \int (u^6 + u^4) \, du = \frac{u^7}{7} + \frac{u^5}{5} + C = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C
\]

The above method works to integrate \( \tan^m x \sec^n x \) whenever \( n > 0 \) is even. The last case to consider is when \( m \) is even and \( n \) is odd. In that case we convert the integrand to be entirely in terms of powers of \( \sec x \) and then apply reduction formulas. Examples of this case can be found in the exercises.

**Conceptual Insight** We previously mentioned that different methods for evaluating an indefinite integral may yield different expressions. Using identities and simplification, we can show that the results are equivalent. Keep in mind that
\[
\int f(x) \, dx = F(x) + C \quad \text{and} \quad \int f(x) \, dx = G(x) + C
\]
do not imply that \( F(x) = G(x) \). Instead, they indicate that the functions \( F \) and \( G \) differ by a constant. To verify that two such antiderivative results are equivalent, you need to show that \( F(x) = G(x) + C \) for some constant \( C \).

For example, separately using the substitutions \( u = \sin x \) and \( u = \cos x \), we obtain the results
\[
\int 2 \sin x \cos x \, dx = \sin^2 x + C \quad \text{and} \quad \int 2 \sin x \cos x \, dx = -\cos^2 x + C
\]
Of course, \( \sin^2 x \neq -\cos^2 x \). However, since \( \sin^2 x = -\cos^2 x + 1 \) it follows that the results are equivalent. Specifically,
\[
\int 2 \sin x \cos x \, dx = \sin^2 x + C = -\cos^2 x + 1 + C = -\cos^2 x + C
\]
where, in the last equality, we express \( 1 + C \) as an arbitrary constant \( C \).
The formulas in Eqs. (15)–(17) in the integral table at the end of the section give the integrals of the products \( \sin m x \sin n x \), \( \cos m x \cos n x \), and \( \sin m x \cos n x \). These integrals appear in the theory of Fourier Series, which is a fundamental technique used extensively in engineering and physics.

**EXAMPLE 10  Integral of \( \sin m x \cos n x \)**  Evaluate \( \int_0^\pi \sin 4x \cos 3x \, dx \).

**Solution**  Apply Eq. (16), with \( m = 4 \) and \( n = 3 \):

\[
\int_0^\pi \sin 4x \cos 3x \, dx = \left[ -\frac{\cos(4 - 3)x}{2(4 - 3)} + \frac{\cos(4 + 3)x}{2(4 + 3)} \right]_0^\pi \\
= \left[ -\frac{\cos x}{2} + \frac{\cos 7x}{14} \right]_0^\pi \\
= \left( \frac{1}{2} + \frac{1}{14} \right) - \left( -\frac{1}{2} - \frac{1}{14} \right) = \frac{8}{7}
\]

The following table of trigonometric integrals summarizes some of the integral formulas we have seen in this chapter and includes some other related formulas.

**TABLE OF TRIGONOMETRIC INTEGRALS**

<table>
<thead>
<tr>
<th>Integral</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int \sin^2 x , dx = \frac{x}{2} - \frac{\sin 2x}{4} + C = \frac{x}{2} - \frac{1}{2} \sin x \cos x + C )</td>
<td>3</td>
</tr>
<tr>
<td>( \int \cos^2 x , dx = \frac{x}{2} + \frac{\sin 2x}{4} + C = \frac{x}{2} + \frac{1}{2} \sin x \cos x + C )</td>
<td>4</td>
</tr>
<tr>
<td>( \int \sin^n x , dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x , dx )</td>
<td>5</td>
</tr>
<tr>
<td>( \int \cos^n x , dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x , dx )</td>
<td>6</td>
</tr>
<tr>
<td>( \int \tan x , dx = \ln</td>
<td>\sec x</td>
</tr>
<tr>
<td>( \int \tan^m x , dx = -\ln</td>
<td>\tan x - \tan x</td>
</tr>
<tr>
<td>( \int \cot x , dx = -\ln</td>
<td>\csc x</td>
</tr>
<tr>
<td>( \int \cot^m x , dx = -\frac{\cot^{m-1} x}{m-1} - \int \cot^{m-2} x , dx )</td>
<td>10</td>
</tr>
<tr>
<td>( \int \sec x , dx = \ln</td>
<td>\sec x + \tan x</td>
</tr>
<tr>
<td>( \int \sec^m x , dx = \frac{\tan x \sec^{m-2} x}{m-1} + \frac{m-2}{m-1} \int \sec^{m-2} x , dx )</td>
<td>12</td>
</tr>
<tr>
<td>( \int \csc x , dx = \ln</td>
<td>\csc x - \cot x</td>
</tr>
<tr>
<td>( \int \csc^m x , dx = -\frac{\cot x \csc^{m-2} x}{m-1} + \frac{m-2}{m-1} \int \csc^{m-2} x , dx )</td>
<td>14</td>
</tr>
<tr>
<td>( \int \sin m x \sin n x , dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C \ (m \neq \pm n) )</td>
<td>15</td>
</tr>
<tr>
<td>( \int \sin m x \cos n x , dx = -\frac{\cos(m-n)x}{2(m-n)} + \frac{\cos(m+n)x}{2(m+n)} + C \ (m \neq \pm n) )</td>
<td>16</td>
</tr>
<tr>
<td>( \int \cos m x \cos n x , dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C \ (m \neq \pm n) )</td>
<td>17</td>
</tr>
</tbody>
</table>
8.2 SUMMARY

- The integral \( \int \sin^n x \cos^n x \, dx \) can be evaluated as in the marginal note on page 415.
- The integral \( \int \tan^n x \sec^n x \, dx \) can be evaluated as in the marginal note on page 417.

8.2 EXERCISES

Preliminary Questions

1. Describe the technique used to evaluate \( \int \sin^2 x \, dx \).
2. Describe a way of evaluating \( \int \sin^6 x \, dx \).
3. Are reduction formulas needed to evaluate \( \int \sin^7 x \cos^2 x \, dx \)? Why or why not?
4. Describe a way of evaluating \( \int \sin^4 x \cos^3 x \, dx \).
5. Which integral requires more work to evaluate?

\[
\int \sin^{20} x \cos x \, dx \quad \text{or} \quad \int \sin^4 x \cos^4 x \, dx
\]
Explain your answer.

Exercises

In Exercises 1–6, evaluate the integral.

1. \( \int \cos^3 x \, dx \)
2. \( \int \sin^3 x \, dx \)
3. \( \int \sin^3 \theta \cos^2 \theta \, d\theta \)
4. \( \int \sin^5 x \cos x \, dx \)
5. \( \int \sin^3 t \cos^3 t \, dt \)
6. \( \int \sin^5 x \cos^5 x \, dx \)

7. Compute the area under the graph of \( y = \cos^3 x \) from \( x = 0 \) to \( x = \frac{\pi}{2} \).

8. Compute the area under the graph of \( y = \sin^3 x \) from \( x = 0 \) to \( x = \pi \).

In Exercises 9–12, evaluate the integrals.

9. \( \int \tan^3 x \sec x \, dx \)
10. \( \int \tan x \sec^3 x \, dx \)
11. \( \int \tan^3 x \sec^4 x \, dx \)
12. \( \int \tan^3 \theta \sec^3 \theta \, d\theta \)

In Exercises 13–16, evaluate using methods similar to those that apply to the integrals of \( \tan^m x \sec^n x \).

13. \( \int \cos^3 x \, dx \)
14. \( \int \csc^3 x \, dx \)
15. \( \int \cos^5 x \csc^2 x \, dx \)
16. \( \int \cot^5 x \csc x \, dx \)

17. Compute the area under the graph of \( y = \tan^3 x \) from \( x = 0 \) to \( x = \frac{\pi}{4} \).

18. Compute the area under the graph of \( y = \sec^4 x \) from \( x = 0 \) to \( x = \frac{\pi}{3} \).

In Exercises 19–46, evaluate the integral.

19. \( \int \sin^6 x \, dx \)
20. \( \int \sin^2 x \cos^3 x \, dx \)
21. \( \int \cos^5 x \sin x \, dx \)
22. \( \int \cos^5 (2 - x) \sin (2 - x) \, dx \)
23. \( \int \cos^6 (3x + 2) \, dx \)
24. \( \int \cos^7 3x \, dx \)
25. \( \int \cos^5 (\pi \theta) \sin^4 (\pi \theta) \, d\theta \)
26. \( \int \cos^{10} y \sin^3 y \, dy \)
27. \( \int \cos^5 x \, dx \)
28. \( \int \sin^7 x \, dx \)
29. \( \int \sec^2 (3 - 2x) \, dx \)
30. \( \int \csc^2 x \, dx \)
31. \( \int \tan x \sec^2 x \, dx \)
32. \( \int \tan^3 \theta \sec^3 \theta \, d\theta \)
33. \( \int \tan^5 x \sec^4 x \, dx \)
34. \( \int \tan^4 x \sec x \, dx \)
35. \( \int \tan^6 x \sec^4 x \, dx \)
36. \( \int \tan^3 x \sec^3 x \, dx \)
37. \( \int \cos^5 x \csc^5 x \, dx \)
38. \( \int \cot^2 x \csc^4 x \, dx \)
39. \( \int \sin 2x \cos 2x \, dx \)
40. \( \int \cos 4x \cos 6x \, dx \)
41. \( \int \sin 2x \cos^3 x \, dx \)
42. \( \int \sin^2 x \sec^4 x \, dx \)
43. \( \int \sec^3 (t^3) \, dt \)
44. \( \int \tan^6 (\ln t) \, dt \)
45. \( \int \cos^3 (\sin t) \cos t \, dt \)
46. \( \int e^t \tan^2 (e^t) \, dx \)

In Exercises 47–60, evaluate the definite integral.

47. \( \int_0^{\pi/2} \sin^2 x \, dx \)
48. \( \int_0^{\pi/2} \cos^2 x \, dx \)
49. \( \int_0^{\pi/2} \sin^3 x \, dx \)
50. \( \int_0^{\pi/2} \sin^2 x \cos^3 x \, dx \)
51. \( \int_0^{\pi/4} \frac{dx}{\cos x} \)
52. \( \int_{\pi/4}^{\pi/2} \frac{dx}{\sin x} \)
53. \( \int_0^{\pi/4} \tan x \, dx \)
54. \( \int_0^{\pi/4} \frac{dx}{\tan x} \)
55. \( \int_{-\pi/4}^{\pi/4} \sec^4 x \, dx \)
56. \( \int_{-\pi/4}^{\pi/4} \cot^4 x \csc^2 x \, dx \)
57. \[ \int_0^\pi \sin 3x \cos 4x \, dx \]
58. \[ \int_0^\pi \sin x \sin 3x \, dx \]
59. \[ \int_0^{\pi/6} \sin 2x \cos 4x \, dx \]
60. \[ \int_0^{\pi/4} \sin 7x \cos 2x \, dx \]

61. For a positive integer, compute the area under the graph of \( y = \sin^n x \cos^m x \) for \( 0 \leq x \leq \frac{\pi}{2} \).
62. For a positive integer, compute the area under the graph of \( y = \tan^n x \cos^m x \) for \( 0 \leq x \leq \frac{\pi}{4} \).

63. Use the identities for \( \sin 2x \) and \( \cos 2x \) on page 415 to verify that the following formulas are equivalent:

\[ \int \sin^4 x \, dx = \frac{1}{32} (12x - 8\sin 2x + \sin 4x) + C \]

\[ \int \sin^6 x \, dx = \frac{1}{40} (7 + 3\cos 2x) \sin^3 x + C \]

64. Evaluate \( \int \sin^2 x \cos x \, dx \) using the method described in the text and verify that your result is equivalent to the following result produced by a computer algebra system:

\[ \int \sin^2 x \cos x \, dx = \frac{1}{30} (7 + 3\cos 2x) \sin^3 x + C \]

65. Find the volume of the solid obtained by revolving \( y = \sin x \) for \( 0 \leq x \leq \frac{\pi}{2} \) about the \( x \)-axis.
66. Find the volume of the solid obtained by revolving \( y = \tan x \) for \( 0 \leq x \leq \frac{\pi}{4} \) about the \( x \)-axis.

67. Prove the reduction formula

\[ \int \tan^k x \, dx = \frac{\tan^{k-1} x}{k-1} - \int \tan^{k-2} x \, dx \]

\( \text{Hint: } \tan^k x = (\sec^2 x - 1) \tan^{k-2} x \).

68. Use the substitution \( u = \cos x - \sin x \) to evaluate \( \int \cos x \, dx \) (see Example 6).

69. Let \( I_m = \int_0^{\pi/2} \sin^m x \, dx \).

(a) Show that \( I_0 = \frac{\pi}{2} \) and \( I_1 = 1 \).

(b) Prove that, for \( m \geq 2 \),

\[ I_m = \frac{m - 1}{m} I_{m-1} \]

(c) Use (a) and (b) to compute \( I_m \) for \( m = 2, 3, 4, 5 \).

70. Evaluate \( \int_0^\pi \sin^2 mx \, dx \) for \( m \) an arbitrary integer.

71. Evaluate \( \int \sin x \ln(\sin x) \, dx \). \( \text{Hint: } \) Use Integration by Parts as a first step.

72. Total Energy A 100-watt (W) light bulb has resistance \( R = 144 \) ohms (\( \Omega \)) when attached to household current, where the voltage varies as \( V = V_0 \sin(2\pi f t) \) (\( V_0 = 110 \) V, \( f = 60 \) Hz). The energy (in joules) expended by the bulb over a period of \( T \) seconds is

\[ U = \int_0^T P(t) \, dt \]

where \( P = V^2/R \) (\( \text{W} \)) is the power. Compute \( U \) if the bulb remains on for 5 hours.

73. Let \( m, n \) be integers with \( m \neq \pm n \). Use Eqs. (15)-(17) to prove the so-called orthogonality relations that play a basic role in the theory of Fourier Series (Figure 1):

\[ \int_0^\pi \sin mx \sin nx \, dx = 0 \]

\[ \int_0^\pi \cos mx \cos nx \, dx = 0 \]

\[ \int_0^\pi \sin mx \cos nx \, dx = 0 \]

74. Use the trigonometric identity

\[ \sin mx \cos nx = \frac{1}{2} (\sin((m-n)x) + \sin((m+n)x)) \]

to prove Eq. (16) in the table of integrals on page 418.

75. Use Integration by Parts to prove that (for \( m \neq 1 \))

\[ \int \sec^m x \, dx = \frac{\tan x \sec^{m-1} x}{m-1} - \frac{m-2}{m-1} \int \sec^{m-2} x \, dx \]

76. Set \( I_m = \int_0^{\pi/2} \sin^m x \, dx \). Use Exercise 69 to prove that

\[ I_{2m} = \frac{2m - 1}{2m} I_{2m-1} \]

\[ I_{2m+1} = \frac{2m}{2m + 1} I_{2m} \]

Conclude that

\[ \frac{\pi}{2} = 1 \cdot \frac{2}{1} \cdot \frac{2 \cdot 4}{3 \cdot 5} \cdots \frac{2m}{2m + 1} \cdot \frac{2m}{2m + 1} I_{2m+1} \]

77. This is a continuation of Exercise 76.

(a) Prove that \( I_{2m+1} \leq I_{2m} \leq I_{2m-1} \). \( \text{Hint: } \)

\[ \sin^{2m+1} x \leq \sin^{2m} x \leq \sin^{2m-1} x \] for \( 0 \leq x \leq \frac{\pi}{2} \)

(b) Show that \( I_{2m-1} = \frac{1}{2m} \).

(c) Show that \( I_{2m} \leq \frac{1}{2m+1} \).

(d) Prove that \( \lim_{m \to \infty} \frac{I_{2m}}{I_{2m+1}} = 1 \).

(e) Finally, deduce the infinite product for \( \pi \) discovered by English mathematician John Wallis (1616-1703):

\[ \frac{\pi}{2} = \lim_{m \to \infty} \frac{2 \cdot 4 \cdot 6 \cdots 2m}{3 \cdot 5 \cdot 7 \cdots (2m+1)} \cdot \frac{2m}{2m+1} I_{2m+1} \]
8.3 Trigonometric Substitution

Our next goal is to integrate functions involving one of the square root expressions:

\[ \sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \sqrt{x^2 - a^2} \]

More generally, we will see that we can also integrate functions involving \( \sqrt{ax^2 + bx + c} \).

In each case, a substitution transforms the integral into a trigonometric integral. For example, if we let \( x = a \sin \theta \) in the first case (and assume \( a > 0 \) and \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\)) we can use the fact \( \sin^2 \theta + \cos^2 \theta = 1 \) to obtain

\[
\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = |a \cos \theta| = a \cos \theta
\]

where the last equality holds since \( a > 0 \) and since \( \cos \theta \geq 0 \) (because \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\)).

Therefore, using this substitution the term \( \sqrt{a^2 - x^2} \) is changed to \( a \cos \theta \). Integrals involving the other root functions listed above can also be simplified by applying a substitution and appropriate trigonometric identities as shown in the following examples.

**EXAMPLE 1** Evaluate \( \int \frac{1}{\sqrt{1 - x^2}} \, dx \).

Solution

**Step 1. Substitute to eliminate the square root.**

The integrand is defined for \(-1 < x < 1\), so we may set \( x = \sin \theta \), where \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\).

Then, with this substitution and the identity \( \sin^2 x + \cos^2 x = 1 \), we obtain

\[
\sqrt{1 - x^2} = \sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = |\cos \theta| = \cos \theta
\]

\[ \blacksquare \]

**Step 2. Evaluate the trigonometric integral.**

Since \( x = \sin \theta \), we have \( dx = \cos \theta \, d\theta \), and

\[
\frac{1}{\sqrt{1 - x^2}} \, dx = \frac{1}{\cos \theta} (\cos \theta \, d\theta) = d\theta.
\]

Thus,

\[
\int \frac{1}{\sqrt{1 - x^2}} \, dx = \int d\theta = \theta + C
\]

**Step 3. Convert back to the original variable.**

Since \( x = \sin \theta \) for \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\), it follows that \( \theta = \sin^{-1} x \), and

\[
\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C
\]

\[ \blacksquare \]

**EXAMPLE 2** Evaluate \( \int \sqrt{1 - x^2} \, dx \).

Solution

**Step 1. Substitute to eliminate the square root.**

As in the previous example, we set \( x = \sin \theta \), where \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\), and we have

\[
\sqrt{1 - x^2} = \sqrt{1 - \sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta
\]

**Step 2. Evaluate the trigonometric integral.**

Since \( x = \sin \theta \), we have \( dx = \cos \theta \, d\theta \), and \( \sqrt{1 - x^2} \, dx = \cos \theta (\cos \theta \, d\theta) \). Thus, \( \int \sqrt{1 - x^2} \, dx = \int \cos^2 \theta \, d\theta \).

Thus,

\[
\int \sqrt{1 - x^2} \, dx = \int \cos^2 \theta \, d\theta = \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C
\]
Step 3. Convert back to the original variable.

It remains to express the answer in terms of $x$. We have $x = \sin \theta$ and $\theta = \sin^{-1} x$, but how do we convert the $\cos \theta$ term? We could express it as $\cos (\sin^{-1} x)$, but there is a better, clearer way to express that term using right-triangle trigonometry. In Figure 1, we have a right triangle for which our substitution relationship $x = \sin \theta$ holds. In that triangle, $\cos \theta = \sqrt{1 - x^2}$, and thus,

$$
\int \sqrt{1 - x^2} \, dx = \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1 - x^2} + C \quad \blacksquare
$$

The following summarizes the substitution approach we employed in the first two examples:

**Integrals involving $\sqrt{a^2 - x^2}$**

If $\sqrt{a^2 - x^2}$ occurs in an integral where $a > 0$, try the substitution (assuming $-\frac{a}{2} \leq \theta \leq \frac{a}{2}$)

$$
x = a \sin \theta, \quad dx = a \cos \theta \, d\theta, \quad \sqrt{a^2 - x^2} = a \cos \theta
$$

This substitution approach can be used with integrands involving $(a^2 - x^2)^{n/2}$ for any integer $n$. The following example demonstrates a case with $n = 3$.

**Example 3**

**Integrand involving $(a^2 - x^2)^{3/2}$**

The graph of $f(x) = \frac{1}{(4 - x^2)^{3/2}}$ for $0 \leq x < 2$ is shown in Figure 2. It has a vertical asymptote at $x = 2$. What is the area under the graph over $[0, 1.999]$?

**Solution**

We need to compute $\int_{0}^{1.999} \frac{1}{(4 - x^2)^{3/2}} \, dx$. We will compute the corresponding indefinite integral first, and then evaluate the definite integral.

**Step 1. Substitute to eliminate the square root.**

In this case, $a = 2$ since $\sqrt{4 - x^2} = \sqrt{2^2 - x^2}$. Therefore, we use

$$
x = 2 \sin \theta, \quad dx = 2 \cos \theta \, d\theta, \quad \sqrt{4 - x^2} = 2 \cos \theta
$$

$$
\int \frac{1}{(4 - x^2)^{3/2}} \, dx = \int \frac{1}{2^3 \cos^3 \theta} \, 2 \cos \theta \, d\theta = \int \frac{1}{4 \cos^2 \theta} \, d\theta = \frac{1}{4} \int \sec^2 \theta \, d\theta
$$

**Step 2. Evaluate the trigonometric integral.**

$$
\frac{1}{4} \int \sec^2 \theta \, d\theta = \frac{1}{4} \tan \theta + C
$$

**Step 3. Convert back to the original variable.**

We must write $\tan \theta$ in terms of $x$. Since $x = 2 \sin \theta$, we have $\sin \theta = x/2$. Using the triangle in Figure 3 to determine an expression for $\tan \theta$, we obtain

$$
\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{4 - x^2}}
$$

Thus, we have

$$
\int \frac{1}{(4 - x^2)^{3/2}} \, dx = \frac{1}{4} \tan \theta + C = \frac{x}{4 \sqrt{4 - x^2}} + C
$$

Now, it follows that

$$
\int_{0}^{1.999} \frac{1}{(4 - x^2)^{3/2}} \, dx = \frac{x}{4 \sqrt{4 - x^2}} \bigg|_{0}^{1.999} \approx 7.903
$$

So the area under $f(x) = \frac{1}{(4 - x^2)^{3/2}}$ from $x = 0$ to $x = 1.999$ is approximately 7.903.

What happens to the area in the previous example if we change the upper limit of the integration, letting the right side of the region approach the asymptote at $x = 2$? It turns
out that as the upper limit approaches 2, the area gets larger and larger without bound (see Exercises 33 and 34). Interestingly, in contrast to the function in Example 3, there are functions that become infinite over an interval but the areas under the graphs of these functions are finite. We explore these ideas further in Section 8.7, investigating what are known as improper integrals.

When the integrand involves \( \sqrt{x^2 + a^2} \), try the substitution \( x = a \tan \theta \), assuming \( a > 0 \) and \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \). Then

\[
\sqrt{x^2 + a^2} = \sqrt{a^2 \tan^2 \theta + a^2} = \sqrt{a^2 (\tan^2 \theta + 1)} = a \sec \theta
\]

where the last equality holds since \( a > 0 \) and since \( \sec \theta \geq 0 \) for the values of \( \theta \) involved.

**Integrals Involving** \( \sqrt{x^2 + a^2} \) **If** \( \sqrt{x^2 + a^2} \) **occurs in an integral where** \( a > 0 \), **try the substitution (assuming** \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \))

\[
x = a \tan \theta, \quad dx = a \sec^2 \theta \, d\theta, \quad \sqrt{x^2 + a^2} = a \sec \theta
\]

**Example 4** Evaluate \( \int \frac{1}{\sqrt{x^2 + 9}} \, dx \).

**Solution** We have the form \( \sqrt{x^2 + a^2} \) with \( a = 3 \).

**Step 1.** Substitute to eliminate the square root.

\[
x = 3 \tan \theta, \quad dx = 3 \sec^2 \theta \, d\theta, \quad \sqrt{x^2 + 9} = 3 \sec \theta
\]

\[
\int \frac{1}{\sqrt{x^2 + 9}} \, dx = \int \left( \frac{1}{3 \sec \theta} \right) 3 \sec^2 \theta \, d\theta = \int \sec \theta \, d\theta
\]

**Step 2.** Evaluate the trigonometric integral.

\[
\int \frac{1}{\sqrt{x^2 + 9}} \, dx = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C
\]

**Step 3.** Convert back to the original variable.

Since \( x = 3 \tan \theta \), we have \( \tan \theta = \frac{x}{3} \). The triangle in Figure 4 can be used to determine an expression for \( \sec \theta \). It follows that

\[
\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\sqrt{x^2 + 9}}{3}
\]

\[
\int \frac{1}{\sqrt{x^2 + 9}} \, dx = \ln \left| \frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3} \right| + C
\]

Notice that we can rewrite this answer as follows, absorbing the constant term \((- \ln 3)\) into the arbitrary constant:

\[
\ln \left| \frac{\sqrt{x^2 + 9}}{3} + \frac{x}{3} \right| + C = \ln |\sqrt{x^2 + 9} + x| - \ln 3 + C = \ln |\sqrt{x^2 + 9} + x| + C
\]

Our last substitution addresses the case where the integrand involves \( \sqrt{x^2 - a^2} \). In this case, try the substitution \( x = a \sec \theta \). We assume that \( a > 0 \). Furthermore, to ensure the square root is defined, we must have either \( x \geq a \) or \( x \leq -a \). In the first case, we assume \( 0 \leq \theta < \frac{\pi}{2} \) in the substitution, and in the second case, we assume \( \pi < \theta < \frac{3\pi}{2} \). These choices for \( \theta \) guarantee that \( \tan \theta \) is positive in either case, and therefore,

\[
\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = \sqrt{a^2 (\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta}
\]

\[
= |a \tan \theta| = a \tan \theta
\]
Integrals Involving $\sqrt{x^2 - a^2}$: If $\sqrt{x^2 - a^2}$ occurs in an integral where $a > 0$, try the substitution (assuming $0 \leq \theta < \frac{\pi}{2}$ with $x \geq a$, and $\pi \leq \theta < \frac{3\pi}{2}$ with $x \leq -a$)

$$x = a \sec \theta, \quad dx = a \sec \theta \tan \theta d\theta, \quad \sqrt{x^2 - a^2} = a \tan \theta$$

**EXAMPLE 5** Evaluate $\int \frac{dx}{x^2 \sqrt{x^2 - 9}}$.

**Solution** In this case, make the substitution

$$x = 3 \sec \theta, \quad dx = 3 \sec \theta \tan \theta d\theta, \quad \sqrt{x^2 - 9} = 3 \tan \theta$$

$$\int \frac{dx}{x^2 \sqrt{x^2 - 9}} = \int \frac{3 \sec \theta \tan \theta d\theta}{9 \sec^2 \theta (3 \tan \theta)} = \frac{1}{9} \int \cos \theta d\theta = \frac{1}{9} \sin \theta + C$$

Since $x = 3 \sec \theta$, we have $\sec \theta = x/3$, and the triangle in Figure 5 shows that

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{x^2 - 9}}{x}$$

Therefore,

$$\int \frac{dx}{x^2 \sqrt{x^2 - 9}} = \frac{1}{9} \sin \theta + C = \frac{\sqrt{x^2 - 9}}{9x} + C$$

**Square Roots of General Quadratic Functions**

So far, we have dealt with the expressions $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$. By completing the square (Section 1.2), we can treat the more general form $\sqrt{ax^2 + bx + c}$.

**EXAMPLE 6** Completing the Square

Evaluate $\int \frac{dx}{(x^2 + 2x + 3)^{3/2}}$.

**Solution**

**Step 1.** Complete the square.

$$x^2 + 2x + 3 = (x^2 + 2x + 1) + 2 = (x + 1)^2 + 2$$

**Step 2.** Use substitution.

Let $u = x + 1, \ du = dx$:

$$\int \frac{dx}{(x^2 + 2x + 3)^{3/2}} = \int \frac{du}{(u^2 + 2)^{3/2}}$$

**Step 3.** Trigonometric substitution.

Evaluate the $u$-integral using trigonometric substitution:

$$u = \sqrt{2} \tan \theta, \quad u^2 + 2 = 2 \sec^2 \theta, \quad du = \sqrt{2} \sec^2 \theta d\theta$$

$$\int \frac{du}{(u^2 + 2)^{3/2}} = \int \frac{\sqrt{2} \sec^2 \theta d\theta}{2^{3/2} \sec^3 \theta} = \frac{1}{2} \int \cos \theta d\theta = \frac{1}{2} \sin \theta + C$$

Since $\tan \theta = \frac{u}{\sqrt{2}}$, we use the right triangle in Figure 6 to obtain

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{u}{\sqrt{u^2 + 2}}$$

Thus, Eq. (3) becomes

$$\int \frac{du}{(u^2 + 2)^{3/2}} = \frac{1}{2} \frac{u}{\sqrt{u^2 + 2}} + C$$
Step 4. Convert to the original variable.

Since \( u = x + 1 \) and \( u^2 + 2 = x^2 + 2x + 3 \), Eq. (3) becomes

\[
\int \frac{du}{(u^2 + 2)^{3/2}} = \frac{x + 1}{2\sqrt{x^2 + 2x + 3}} + C
\]

Therefore, by Eq. (2):

\[
\int \frac{dx}{(x^2 + 2x + 3)^{3/2}} = \frac{x + 1}{2\sqrt{x^2 + 2x + 3}} + C
\]

### 8.3 SUMMARY

- Trigonometric substitution:

<table>
<thead>
<tr>
<th>Square root form in integrand</th>
<th>Trigonometric substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = a \sin \theta ), ( dx = a \cos \theta , d\theta ), ( \sqrt{a^2 - x^2} = a \cos \theta )</td>
</tr>
<tr>
<td>( \sqrt{x^2 + a^2} )</td>
<td>( x = a \tan \theta ), ( dx = a \sec^2 \theta , d\theta ), ( \sqrt{x^2 + a^2} = a \sec \theta )</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec \theta ), ( dx = a \sec \theta \tan \theta , d\theta ), ( \sqrt{x^2 - a^2} = a \tan \theta )</td>
</tr>
</tbody>
</table>

Step 1. Substitute to eliminate the square root.

Step 2. Evaluate the trigonometric integral.

Step 3. Convert back to the original variable.

- The three trigonometric substitutions correspond to three right triangles (Figure 7) that we use to express the trigonometric functions of \( \theta \) in terms of \( x \).

**Figure 7** Right triangles used in trigonometric substitution.

- Integrands involving \( \sqrt{x^2 + bx + c} \) are treated by completing the square (see Example 6).

### 8.3 EXERCISES

**Preliminary Questions**

1. State the trigonometric substitution appropriate to the given integral:
   (a) \( \int \sqrt{9 - x^2} \, dx \)  
   (b) \( \int x^2(x^2 - 16)^{3/2} \, dx \)  
   (c) \( \int x^2(x^2 + 16)^{3/2} \, dx \)  
   (d) \( \int (x^2 - 5)^{-2} \, dx \)

2. Is trigonometric substitution needed to evaluate \( \int x \sqrt{9 - x^2} \, dx \)?

3. Express \( \sin 2\theta \) in terms of \( x = \sin \theta \).

4. Draw a triangle that would be used together with the substitution \( x = 3 \sec \theta \).

**Exercises**

In Exercises 1–4, evaluate the integral by following the steps given.

1. \( I = \int \frac{dx}{\sqrt{9 - x^2}} \)
   (a) Show that the substitution \( x = 3 \sin \theta \) transforms \( I \) into \( \int d\theta \), and evaluate \( I \) in terms of \( \theta \).
   (b) Evaluate \( I \) in terms of \( x \).

2. \( I = \int \frac{dx}{x^2 - x + 2} \)
   (a) Show that the substitution \( x = \sqrt{2} \sec \theta \) transforms the integral \( I \) into \( \frac{1}{2} \int \cos \theta \, d\theta \), and evaluate \( I \) in terms of \( \theta \).
(b) Use a right triangle to show that with the above substitution, \( \sin \theta = \frac{\sqrt{x^2 - 4}}{x} \).

(c) Evaluate \( I \) in terms of \( x \).

3. \( I = \int \frac{dx}{\sqrt{x^2 + 9}} \)
   
   (a) Show that the substitution \( x = \frac{1}{2} \tan \theta \) transforms \( I \) into \( \frac{1}{2} \int \sec \theta \, d\theta \).
   
   (b) Evaluate \( I \) in terms of \( x \) (refer to the table of integrals on page 418 in Section 8.2 if necessary).

4. \( I = \int \frac{dx}{(x^2 + 4)^{1/2}} \)
   
   (a) Show that the substitution \( x = 2 \tan \theta \) transforms the integral \( I \) into \( \frac{1}{8} \int \cos^3 \theta \, d\theta \).
   
   (b) Use the formula \( \int \cos^3 \theta \, d\theta = \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta \) to evaluate \( I \) in terms of \( \theta \).
   
   (c) Show that \( \sin \theta = \frac{x}{\sqrt{x^2 + 4}} \) and \( \cos \theta = \frac{2}{\sqrt{x^2 + 4}} \).

(d) Express \( I \) in terms of \( x \).

In Exercises 5–10, use the indicated substitution to evaluate the integral.

5. \( \int \sqrt{1 - 5x^2} \, dx \), \( x = \frac{4}{\sqrt{5}} \sin \theta \)

6. \( \int_{0}^{\pi/4} \frac{x^2}{x^2 + 4} \, dx \), \( x = \sin \theta \)

7. \( \int \frac{dx}{x^2 + 9} \), \( x = 3 \sec \theta \)

8. \( \int \frac{dx}{(x^2 + 4)^{1/2}} \), \( x = 2 \tan \theta \)

9. \( \int \frac{dx}{(x^2 - 4)^{1/2}} \), \( x = 2 \sec \theta \)

10. \( \int \frac{dx}{(4 + x^2)^{1/2}} \), \( x = \tan \theta \)

11. Evaluate \( \int \frac{xdx}{\sqrt{x^2 - 4}} \) in two ways: using the direct substitution \( u = x^2 - 4 \) and by trigonometric substitution.

12. Is the substitution \( u = x^2 - 4 \) effective for evaluating the integral \( \int \frac{x^2 \, dx}{\sqrt{x^2 - 4}} \)? If not, evaluate using trigonometric substitution.

13. Evaluate using the substitution \( u = 1 - x^2 \) or trigonometric substitution.
   
   (a) \( \int \frac{x \, dx}{\sqrt{1 - x^2}} \)

   (b) \( \int x^2 \sqrt{1 - x^2} \, dx \)

   (c) \( \int x^2 \sqrt{1 - x^2} \, dx \)

   (d) \( \int \frac{x \, dx}{\sqrt{1 - x}} \)

14. Evaluate:
   
   (a) \( \int \frac{dt}{(t^2 + 1)^{1/2}} \)

   (b) \( \int \frac{t \, dt}{(t^2 + 1)^{1/2}} \)

In Exercises 15–32, evaluate using trigonometric substitution. Refer to the table of trigonometric integrals as necessary.

15. \( \int \frac{x \, dx}{\sqrt{9 - x^2}} \)

16. \( \int \frac{dx}{(16 - x^2)^{3/2}} \)

17. \( \int \frac{dx}{x \sqrt{x^2 + 4}} \)

18. \( \int \sqrt{12 + 4x^2} \, dx \)

19. \( \int \frac{dx}{\sqrt{x^2 - 9}} \)

20. \( \int \frac{3 \, dx}{2\sqrt{t^2 - 25}} \)

21. \( \int \frac{dy}{y^2 \sqrt{5 - y^2}} \)

22. \( \int \frac{dy}{y^2 \sqrt{9 - y^2}} \)

23. \( \int \frac{dx}{\sqrt{25x^2 + 2}} \)

24. \( \int \frac{dt}{(9t^2 + 4)^2} \)

25. \( \int \frac{dx}{x^2 \sqrt{2x^2 - 4}} \)

26. \( \int \frac{dy}{\sqrt{x^2 - 9}} \)

27. \( \int \frac{x^2 \, dx}{(6x^2 - 49)^{1/2}} \)

28. \( \int \frac{dx}{(x^2 - 4)^{1/2}} \)

29. \( \int \frac{1 \, dt}{(t^2 + 9)^{3/2}} \)

30. \( \int \frac{dx}{(x^2 + 1)^{3/2}} \)

31. \( \int \frac{x^2 \, dx}{(x^2 - 1)^{1/2}} \)

32. \( \int \frac{x \, dx}{(x^2 - 1)^{1/2}} \)

33. Compute \( \int_{0}^{1} \frac{1}{(4 - x^2)^{1/2}} \, dx \).

34. Compute \( \int_{0}^{1} \frac{1}{(4 - x^2)^{1/2}} \, dx \), expressing the result in terms of \( r \).

Then, discuss what happens to the value of the definite integral as \( r \) approaches 2 from the left.

35. (a) Using a trigonometric substitution, compute the integral and show that for \( a > 0 \)
   
   \[ \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \]

(b) Verify the formula via differentiation.

36. (a) Using a trigonometric substitution, compute the integral and show that for \( a > 0 \)
   
   \[ \int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{1}{2a^2} \left( \frac{x}{x^2 + a^2} + \frac{1}{a} \tan^{-1} \frac{x}{a} \right) + C \]

(b) Verify the formula via differentiation.

37. Let \( I = \int \frac{dx}{\sqrt{x^2 - 4x + 8}} \).

(a) Complete the square to show that \( x^2 - 4x + 8 = (x - 2)^2 + 4 \).

(b) Use the substitution \( u = x - 2 \) to show that \( I = \int \frac{du}{\sqrt{u^2 + 4}} \).

Evaluate the \( u \)-integral.

38. Evaluate \( \int \frac{dx}{\sqrt{x^2 + 36}} \).

First, complete the square to write \( 12 = x^2 + 36 = (x + 6)^2 \).

In Exercises 39–44, evaluate the integral by completing the square and using trigonometric substitution.

39. \( \int \frac{dx}{\sqrt{x^2 + 4x + 13}} \)

40. \( \int \frac{dx}{\sqrt{x^2 + x - 3}} \)

41. \( \int \frac{dx}{\sqrt{x^2 + 6x}} \)

42. \( \int \sqrt{x^2 - 4x + 7} \, dx \)

43. \( \int \frac{dx}{\sqrt{x^2 - 4x + 3}} \)

44. \( \int \frac{dx}{(x^2 + 6x + 6)^{1/2}} \)

In Exercises 45–48, evaluate using integration by Parts as a first step.

45. \( \int \sec^{-1} x \, dx \)

46. \( \int \frac{dy}{x^2} \)

47. \( \int \ln(x^2 + 1) \, dx \)

48. \( \int x^2 \ln(x^2 + 1) \, dx \)

49. Find the average height of a point on the semicircle \( y = \sqrt{1 - x^2} \) for \( -1 \leq x \leq 1 \).

50. Find the volume of the solid obtained by revolving the graph of \( y = x^2 \) over \([0, 1]\) about the \( y \)-axis.
51. Find the volume of the solid obtained by revolving the region between the graph of \( y^2 - x^2 = 1 \) and the line \( y = 2 \) about the line \( y = 2 \).

52. Find the volume of revolution for the region in Exercise 51, but revolve around \( y = 3 \).

53. Compute \( \int \frac{dx}{x^2 - 1} \) in two ways and verify that the answers agree: first via trigonometric substitution and then using the identity
\[
\frac{1}{x^2 - 1} = \frac{1}{2} \left( \frac{1}{x - 1} - \frac{1}{x + 1} \right)
\]

54. (CAS) You want to divide an 18-in. pizza equally among three friends using vertical slices at \( \pm x \) as in Figure 8. Find an equation satisfied by \( x \), and find the approximate value of \( x \) using a computer algebra system.

![Figure 8: Dividing a pizza into three equal parts.](image)

55. A charged wire creates an electric field at a point \( P \) located at a distance \( D \) from the wire (Figure 9). The component \( E_x \) of the field perpendicular to the wire is given by:
\[
E_x = \int_{x_1}^{x_2} \frac{k \lambda D}{(x^2 + D^2)^{3/2}} \, dx
\]
where \( \lambda \) is the charge density (coulombs per meter), \( k = 8.99 \times 10^9 \) N-m^2/C^2 (Coulomb constant), and \( x_1, x_2 \) are as in the figure. Suppose that \( \lambda = 6 \times 10^{-6} \) C/m, and \( D = 3 \) m. Find \( E_x \) if (a) \( x_1 = 0 \) and \( x_2 = 30 \) m, and (b) \( x_1 = -15 \) m and \( x_2 = 15 \) m.

![Figure 9: Electric field due to a charged wire.](image)

Further Insights and Challenges

56. Let \( J_n = \int \frac{dx}{(x^2 + 1)^n} \).  
   (a) Compute \( J_1 \).
   (b) Use Integration by Parts to prove
   \[
   J_{n+1} = \left( 1 - \frac{1}{2n} \right) J_n + \left( \frac{1}{2n} \right) \frac{x}{(x^2 + 1)^n}
   \]
   (c) Use this recursion relation to calculate \( J_2 \) and \( J_3 \).

57. The area function \( F(x) = \int_0^x \sqrt{1-t^2} \, dt \) is an antiderivative of \( f(x) = \sqrt{1-x^2} \). Prove the formula
\[
\int_0^x \sqrt{1-t^2} \, dt = \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1-x^2} + C
\]
using geometry by interpreting the integral as the area of part of the unit circle.

8.4 Integrals Involving Hyperbolic and Inverse Hyperbolic Functions

In Section 7.7, we noted the similarities between hyperbolic and trigonometric functions. We also saw in Section 7.7 that the formulas for their derivatives resemble each other, differing in at most a sign. The derivative formulas for the hyperbolic functions are equivalent to the following integral formulas.

**Hyperbolic Integral Formulas**

\[
\begin{align*}
\int \sinh x \, dx &= \cosh x + C, \\
\int \cosh x \, dx &= \sinh x + C, \\
\int \sech^2 x \, dx &= \tanh x + C, \\
\int \csch^2 x \, dx &= -\coth x + C, \\
\int \text{sech} x \tanh x \, dx &= -\text{sech} x + C, \\
\int \text{csch} x \coth x \, dx &= -\text{csch} x + C
\end{align*}
\]
EXAMPLE 1 Calculate \( \int x \cosh(x^2) \, dx \).

Solution The substitution \( u = x^2, \, du = 2x \, dx \) yields
\[
\int x \cosh(x^2) \, dx = \frac{1}{2} \int \cosh u \, du = \frac{1}{2} \sinh u + C = \frac{1}{2} \sinh(x^2) + C \quad \square
\]

The techniques for computing trigonometric integrals discussed in Section 8.2 apply with little change to hyperbolic integrals. In place of trigonometric identities, we use the corresponding hyperbolic identities (see margin).

EXAMPLE 2 Powers of \( \sinh x \) and \( \cosh x \) Calculate: (a) \( \int \sinh^4 x \cosh^5 x \, dx \)

and (b) \( \int \cosh^2 x \, dx \).

Solution

(a) Since \( \cosh x \) appears to an odd power, use \( \cosh^2 x = 1 + \sinh^2 x \) to write
\[
\cosh^5 x = \cosh^4 x \cdot \cosh x = (\sinh^2 x + 1)^2 \cosh x
\]
Then use the substitution \( u = \sinh x, \, du = \cosh x \, dx \):
\[
\int \sinh^4 x \cosh^5 x \, dx = \int \frac{\sinh^4 x (\sinh^2 x + 1)^2 \cosh x \, dx}{u^4} = \int \frac{u^9}{(u^2 + 1)^4} \, du = \int (u^8 + 2u^6 + u^4) \, du = \frac{u^9}{9} + 2u^7 + u^5 + C = \frac{\sinh^9 x}{9} + \frac{2 \sinh^7 x}{7} + \frac{\sinh^5 x}{5} + C
\]

(b) For \( \int \cosh^2 x \, dx \), we use the identity \( \cosh^2 x = \frac{1}{2} (\cosh 2x + 1) \):
\[
\int \cosh^2 x \, dx = \frac{1}{2} \int (\cosh 2x + 1) \, dx = \frac{1}{2} \left( \frac{\sinh 2x}{2} + x \right) + C = \frac{1}{4} \sinh 2x + \frac{1}{2} x + C \quad \square
\]

Hyperbolic substitution may be used as an alternative to trigonometric substitution to integrate functions involving the following square root expressions:

<table>
<thead>
<tr>
<th>Square root form</th>
<th>Hyperbolic substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{x^2 + a^2} )</td>
<td>( x = a \tan \theta ) and ( \sqrt{x^2 - a^2} ) using ( x = a \sec \theta ). Identities can be used to show that the results coincide with those obtained from hyperbolic substitution (see Exercises 31–35).</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \cosh u ), ( dx = \sinh u , du \cdot \sqrt{x^2 - a^2} = \cosh u )</td>
</tr>
</tbody>
</table>

EXAMPLE 3 Hyperbolic Substitution Calculate \( \int \sqrt{x^2 + 16} \, dx \).

Solution

Step 1. Substitute to eliminate the square root.

Use the hyperbolic substitution \( x = 4 \sinh u, \, dx = 4 \cosh u \, du \). Then
\[
x^2 + 16 = 16(\sinh^2 u + 1) = (4 \cosh u)^2
\]
Furthermore, \( 4 \cosh u > 0 \), so \( \sqrt{x^2 + 16} = 4 \cosh u \), and thus,
\[
\int \sqrt{x^2 + 16} \, dx = \int (4 \cosh u) \, 4 \cosh u \, du = 16 \int \cosh^2 u \, du
\]
Step 2. Evaluate the hyperbolic integral.
We evaluated the integral of $\cosh^2 u$ in Example 2(b):

$$\int \sqrt{x^2 + 16} \, dx = 16 \int \cosh^2 u \, du = 16 \left( \frac{1}{4} \sinh 2u + \frac{1}{2} u + C \right)$$

$$= 4 \sinh 2u + 8u + C$$

Step 3. Convert back to the original variable.
To write the answer in terms of the original variable $x$, we note that

$$\sinh u = \frac{x}{4}, \quad u = \sinh^{-1} \frac{x}{4}$$

Use the identities recalled in the margin to write

$$4 \sinh 2u = 4(2 \sinh u \cosh u) = 8 \sinh u \sqrt{\sinh^2 u + 1}$$

$$= 8 \left( \frac{x}{4} \right) \sqrt{\left( \frac{x}{4} \right)^2 + 1} = 2x \sqrt{x^2 + 16} = \frac{1}{2} x \sqrt{x^2 + 16}$$

Now via Eq. (1), we have

$$\int \sqrt{x^2 + 16} \, dx = \frac{1}{2} x \sqrt{x^2 + 16} + 8 \sinh^{-1} \frac{x}{4} + C$$

In a similar manner, we could derive each of the following integral formulas. Alternatively, we can realize them from the corresponding derivative formulas for the inverse hyperbolic functions given in Section 7.7. Each formula is valid on the domain where the integrand and inverse hyperbolic function are defined.

**Theorem 1** Integrals Involving Inverse Hyperbolic Functions

\[
\int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x + C
\]

\[
\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x + C \quad \text{(for } x > 1)\]

\[
\int \frac{dx}{1 - x^2} = \tanh^{-1} x + C \quad \text{(for } |x| < 1)\]

\[
\int \frac{dx}{1 + x^2} = \coth^{-1} x + C \quad \text{(for } |x| > 1)\]

\[
\int \frac{dx}{x \sqrt{1 - x^2}} = -\text{sech}^{-1} x + C \quad \text{(for } 0 < x < 1)\]

\[
\int \frac{dx}{|x| \sqrt{1 + x^2}} = -\text{csch}^{-1} x + C \quad \text{(for } x \neq 0)\]

**Example 4** Evaluate: (a) $\int_2^4 \frac{dx}{\sqrt{x^2 - 1}}$ and (b) $\int_{0.2}^{0.6} \frac{x \, dx}{1 - x^4}$.

**Solution**

(a) By Theorem 1,

$$\int_2^4 \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} 4 \bigg|_2^4 = \cosh^{-1} 4 - \cosh^{-1} 2 \approx 0.75$$
(b) First, use the substitution \( u = x^2, \, du = 2x \, dx \). The new limits of integration become \( u = (0.2)^2 = 0.04 \) and \( u = (0.6)^2 = 0.36 \), so

\[
\int_{0.2}^{0.6} \frac{x \, dx}{1 - x^4} = \int_{0.04}^{0.36} \frac{du}{1 - u^2} = \frac{1}{2} \int_{0.04}^{0.36} \frac{du}{1 - u^2}
\]

By Theorem 1, both \( \tanh^{-1} u \) and \( \coth^{-1} u \) are antiderivatives of \( f(u) = (1 - u^2)^{-1} \). We use \( \tanh^{-1} u \) because the interval of integration \([0.04, 0.36]\) is contained in the domain \((-1, 1)\) of \( f(u) = \tanh^{-1} u \). If the limits of integration were contained in \((1, \infty)\) or \((-\infty, -1)\), we would use \( \coth^{-1} u \). The result is

\[
\frac{1}{2} \int_{0.04}^{0.36} \frac{du}{1 - u^2} = \frac{1}{2} (\tanh^{-1}(0.36) - \tanh^{-1}(0.04)) \approx 0.1684
\]

8.4 SUMMARY

- Integrals of hyperbolic functions:
  \[
  \int \sinh x \, dx = \cosh x + C, \quad \int \cosh x \, dx = \sinh x + C
  \]
  \[
  \int \sech^2 x \, dx = \tanh x + C, \quad \int \csch^2 x \, dx = -\coth x + C
  \]
  \[
  \int \sech x \tanh x \, dx = -\sech x + C, \quad \int \csch x \coth x \, dx = -\csch x + C
  \]

- Integrals involving inverse hyperbolic functions:
  \[
  \int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x + C
  \]
  \[
  \int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x + C \quad \text{(for } x > 1)\]
  \[
  \int \frac{dx}{1 - x^2} = \tanh^{-1} x + C \quad \text{(for } |x| < 1)\]
  \[
  \int \frac{dx}{1 + x^2} = \coth^{-1} x + C \quad \text{(for } |x| > 1)\]
  \[
  \int \frac{dx}{x\sqrt{1 - x^2}} = -\sech^{-1} x + C \quad \text{(for } 0 < x < 1)\]
  \[
  \int \frac{dx}{|x|\sqrt{1 + x^2}} = -\csch^{-1} x + C \quad \text{(for } x \neq 0)\]

8.4 EXERCISES

Preliminary Questions

1. Which hyperbolic substitution can be used to evaluate the following integrals?
   (a) \( \int \frac{dx}{\sqrt{x^2 + 1}} \)  \hspace{1cm} (b) \( \int \frac{dx}{\sqrt{x^2 + 9}} \)  \hspace{1cm} (c) \( \int \frac{dx}{\sqrt{9x^2 + 1}} \)

2. Which of the hyperbolic integration formulas differ from their trigonometric counterparts by a minus sign?

3. Which antiderivative of \( y = (1 - x^2)^{-1} \) should we use to evaluate the integral \( \int (1 - x^2)^{-1} \, dx \)?
Exercises

In Exercises 1–16, calculate the integral.

1. \( \int \cosh(3x) \, dx \)
2. \( \int x \sinh x \, dx \)
3. \( \int x \sinh x \, dx \)
4. \( \int \sinh^2 x \cosh x \, dx \)
5. \( \int \sech^2 (1 - 2x) \, dx \)
6. \( \int \tanh(3x) \sech(3x) \, dx \)
7. \( \int \tanh x \sech^2 x \, dx \)
8. \( \int \frac{\cosh x}{3 \sinh x + 4} \, dx \)
9. \( \int \tanh x \, dx \)
10. \( \int \frac{x \cosh(x^2)}{\sinh x} \, dx \)
11. \( \int \frac{\cosh x}{\sinh x} \, dx \)
12. \( \int \frac{x \cosh^3 x}{\sinh^2 x} \, dx \)
13. \( \int \sinh^3 (4x - 9) \, dx \)
14. \( \int \sinh^3 x \cosh^6 x \, dx \)
15. \( \int \sinh^3 x \cosh^3 x \, dx \)
16. \( \int \tanh^3 x \, dx \)

In Exercises 17–30, calculate the integral in terms of the inverse hyperbolic functions.

17. \( \int \frac{dx}{\sqrt{x^2 - 1}} \)
18. \( \int \frac{dx}{\sqrt{x^2 - 4}} \)
19. \( \int \frac{dx}{\sqrt{x^2 + 4}} \)
20. \( \int \frac{dx}{\sqrt{x^2 + 3x^2}} \)
21. \( \int \frac{x \, dx}{\sqrt{x^2 - 1}} \)
22. \( \int \frac{x^2 \, dx}{\sqrt{x^2 + 1}} \)
23. \( \int \frac{x \, dx}{\sqrt{\frac{1}{2} - 1-x^2}} \)
24. \( \int \frac{x^5 \, dx}{1-x^2} \)
25. \( \int \frac{dx}{\sqrt{\frac{1}{4} + x}} \)
26. \( \int \frac{dx}{\sqrt{\frac{1}{5} + x}} \)
27. \( \int \frac{dx}{\sqrt{3 - x^2}} \)
28. \( \int \frac{dx}{\sqrt{x^2 + 1}} \)
29. \( \int \frac{dx}{\sqrt{x^2 - 16}} \)
30. \( \int \frac{dx}{x^2 + 1} \)

31. Verify the formulas
   \( \sinh^{-1} x = \ln |x + \sqrt{x^2 + 1}| \)
   \( \cosh^{-1} x = \ln |x + \sqrt{x^2 - 1}| \) (for \( x \geq 1 \))

32. Verify that \( \tanh^{-1} x = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| \) for \( |x| < 1 \).

33. Evaluate \( \int \sqrt{x^2 + 9} \, dx \) using trigonometric substitution. Then use Exercise 31 to verify that your answer agrees with the answer in Example 3.

34. Evaluate \( \int \sqrt{x^2 - 9} \, dx \) in two ways: using trigonometric substitution and using hyperbolic substitution. Then use Exercise 31 to verify that the two answers agree.

35. Prove the reduction formula for \( n \geq 2 \):
   \( \int \cosh^n x \, dx = \frac{1}{n} \cosh^{n-1} x \sinh x + \frac{n-1}{n} \int \cosh^{n-2} x \, dx \)

36. Use Eq. (2) to evaluate \( \int \cosh^2 x \, dx \).

In Exercises 37–40, evaluate the integral.

37. \( \int \frac{x \, dx}{\sqrt{x^2 - 1}} \)
38. \( \int \frac{x \, dx}{\sqrt{x^2 - 1}} \)
39. \( \int \frac{x \, dx}{\tan^{-1} x} \)
40. \( \int \frac{x \, dx}{\tan^{-1} x} \)

41. (a) Compute the area under the graph of \( y = \sinh x \) for \( 0 \leq x \leq 5 \).
   (b) Compute the area under the graph of \( y = \sinh^{-1} x \) for \( 0 \leq x \leq \sinh 5 \).
   (c) Show that the sum of the areas in (a) and (b) is equal to \( 5 \sinh 5 \).
   (d) Refer to Figure 1 and explain why the sum of the areas in (a) and (b) is equal to \( 5 \sinh 5 \).

42. (a) Compute the area under the graph of \( y = \tanh x \) for \( 0 \leq x \leq 4 \).
   (b) Compute the area under the graph of \( y = \tanh^{-1} x \) for \( 0 \leq x \leq \tanh 4 \).
   (c) Show that the sum of the areas in (a) and (b) is equal to \( 4 \tanh 4 \).
   (d) Similar to Exercise 41(d), explain graphically why the sum of the areas in (a) and (b) is equal to \( 4 \tanh 4 \).

Further Insights and Challenges

43. Show that if \( u = \tanh(\frac{x}{2}) \), then
   \( \cosh x = \frac{1 + u^2}{1 - u^2}, \quad \sinh x = \frac{2u}{1 - u^2}, \quad dx = \frac{2du}{1 - u^2} \)
   Hint: For the first relation, use the identities
   \( \sinh^2 \left( \frac{x}{2} \right) = \frac{1}{2} (\cosh x - 1), \quad \cosh^2 \left( \frac{x}{2} \right) = \frac{1}{2} (\cosh x + 1) \)

44. \( \int \sec \theta \, d\theta \)
45. \( \int \frac{dx}{1 + \cosh x} \)
46. \( \int \frac{dx}{1 - \cosh x} \)
Exercises 47–50 refer to the function \( gd(y) = \tan^{-1}(\sinh y) \), called the Gudermannian. In a map of the earth constructed by Mercator projection, points located \( y \) radial units from the equator correspond to points on the globe of latitude \( gd(y) \).

47. Prove that \( \frac{d}{dy} gd(y) = \sech y \).

48. Let \( f(y) = 2 \tan^{-1}(e^y) - \pi / 2 \). Prove that \( gd(y) = f(y) \). Hint: Show that \( gd(y) = f'(y) \) and \( f(0) = g(0) \).

49. Let \( r(y) = \sinh^{-1}(\tan y) \). Show that \( r(y) \) is the inverse of \( gd(y) \) for \( 0 \leq y < \pi / 2 \).

50. Verify that \( r(y) \) in Exercise 49 satisfies \( r'(y) = \sec y \), and find a value of \( a \) such that

\[
r(y) = \int_a^y \frac{dt}{\cos t}
\]

8.5 The Method of Partial Fractions

The Method of Partial Fractions is used to integrate rational functions:

\[
f(x) = \frac{P(x)}{Q(x)}
\]

where \( P \) and \( Q \) are polynomials. The idea is to write \( f \) as a sum of simpler rational functions that can be integrated directly. For example, expressing \( \frac{1}{x^2 - 1} \) as a sum

\[
\frac{1}{x^2 - 1} = \frac{1}{x - 1} - \frac{1}{x + 1}
\]

enables us to evaluate the integral

\[
\int \frac{dx}{x^2 - 1} = \frac{1}{2} \int \frac{dx}{x - 1} - \frac{1}{2} \int \frac{dx}{x + 1} = \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1|
\]

A rational function \( P/Q \) is called proper if the degree of \( P \) [denoted \( \deg(P) \)] is less than the degree of \( Q \). For example,

\[
\frac{x^2 - 3x + 7}{x - 16}, \quad \frac{2x^2 + 7}{x - 5}, \quad \frac{x - 2}{x - 5}
\]

Proper

Not proper

Suppose first that \( P/Q \) is proper and that the denominator \( Q(x) \) factors as a product of distinct linear factors. In other words,

\[
P(x)
\]

\[
Q(x) = (x - a_1)(x - a_2) \cdots (x - a_n)
\]

where the values \( a_1, a_2, \ldots, a_n \) are all distinct and \( \deg(P) < n \). Then there is a partial fraction decomposition:

\[
\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}
\]

for suitable constants \( A_1, \ldots, A_n \). For example,

\[
\frac{5x^2 + x - 28}{(x + 1)(x - 2)(x - 3)} = \frac{2}{x + 1} + \frac{2}{x - 2} + \frac{5}{x - 3}
\]

To obtain a partial fraction decomposition, we must find the constants \( A_1, \ldots, A_n \). There are two different methods that we will employ: Undetermined Coefficients and Value Substitution. Ultimately, the process involves solving a system of linear equations in the constants.

In the examples that follow, given a fraction \( \frac{P(x)}{Q(x)} \), we will see how these methods can be used to obtain a partial fraction decomposition. Once we have found the partial fraction decomposition, we can integrate the individual terms using techniques that we have previously seen.
EXAMPLE 1  Finding the Constants  
Evaluate \( \int \frac{dx}{x^2 - 7x + 10} \)

Solution

Step 1. Determine the form of the partial fraction decomposition.

The denominator factors as \( x^2 - 7x + 10 = (x-2)(x-5) \), so we look for a partial fraction decomposition:

\[
\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}
\]

Step 2. Determine the constants.

To find \( A \) and \( B \), first multiply by \((x-2)(x-5)\) to clear denominators:

\[
1 = (x-2)(x-5) \left( \frac{A}{x-2} + \frac{B}{x-5} \right)
\]

\[
1 = A(x-5) + B(x-2)
\]

\[
1 = (A+B)x + (-5A-2B) \tag{2}
\]

We will use the method of Undetermined Coefficients to determine the constants. Note that on the left side of Eq. (2), there are no \( x \) terms, and on the right side the coefficient of \( x \) is \( A+B \). Furthermore, the constant terms are 1 on the left and \(-5A-2B\) on the right. Thus, equating coefficients of like powers of \( x \) we obtain

\[
0 = A + B \quad \text{(coefficient of } x) \tag{1}
\]

\[
1 = -5A - 2B \quad \text{(constant terms)}
\]

The first of these equations implies \( B = -A \). Substituting \(-A\) for \( B \) in the second equation and solving for \( A \), we find \( A = -\frac{1}{3} \). It follows that \( B = \frac{1}{3} \). The resulting partial fraction decomposition is

\[
\frac{1}{(x-2)(x-5)} = \frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5} \tag{3}
\]

Step 3. Carry out the integration.

\[
\int \frac{dx}{(x-2)(x-5)} = -\frac{1}{3} \int \frac{dx}{x-2} + \frac{1}{3} \int \frac{dx}{x-5}
\]

\[= -\frac{1}{3} \ln |x-2| + \frac{1}{3} \ln |x-5| + C \]

EXAMPLE 2  Evaluate \( \int \frac{x^2 + 2}{(x-1)(2x-8)(x+2)} \, dx \).

Solution

Step 1. Determine the form of the partial fraction decomposition.

The decomposition has the form

\[
\frac{x^2 + 2}{(x-1)(2x-8)(x+2)} = \frac{A}{x-1} + \frac{B}{2x-8} + \frac{C}{x+2} \tag{3}
\]

Step 2. Determine the constants.

As before, multiply by \((x-1)(2x-8)(x+2)\) to clear denominators:

\[x^2 + 2 = A(2x-8)(x+2) + B(x-1)(x+2) + C(x-1)(2x-8) \tag{4}\]
In the method of Value Substitution, we strategically substitute values for \( x \) that provide simple equations to solve—either individually or as a system—for the constants.

You can check your partial fraction decomposition by adding together the resulting fractions and verifying that the result is the fraction that you had to start.

In this case, we will use Value Substitution to determine the constants. Note that in Eq. (4) the factor \( x - 1 \) appears in two terms on the right-hand side. Thus, if we substitute 1 for \( x \) in the equation, those terms drop out, leaving us with a simple equation to solve for \( A \). Specifically, substituting \( x = 1 \), we obtain

\[
1^2 + 2 = A(-6)(3) + 0 + 0
\]

Therefore, \( 3 = -18A \), yielding \( A = -\frac{1}{6} \). Similarly, substituting 4 for \( x \), the two \( 2x - 8 \) terms become 0, providing us with

\[
4^2 + 2 = 0 + B(3)(6) + 0
\]

From this, we obtain \( B = 1 \). Finally, \( C \) is determined by setting \( x = -2 \) in Eq. (4):

\[
(-2)^2 + 2 = 0 + 0 + C(-3)(-12)
\]

Thus, \( C = \frac{1}{6} \), and the resulting partial fraction decomposition is

\[
\frac{x^2 + 2}{(x - 1)(2x - 8)(x + 2)} = -\frac{1}{6} \frac{1}{x - 1} + \frac{1}{2x - 8} + \frac{1}{6} \frac{1}{x + 2}
\]

**Step 3. Carry out the integration.**

\[
\int \frac{x^2 + 2}{(x - 1)(2x - 8)(x + 2)} \, dx = -\frac{1}{6} \int \frac{dx}{x - 1} + \frac{1}{2} \int \frac{dx}{2x - 8} + \frac{1}{6} \int \frac{dx}{x + 2}
\]

\[
= -\frac{1}{6} \ln |x - 1| + \frac{1}{2} \ln |2x - 8| + \frac{1}{6} \ln |x + 2| + C
\]

When using Value Substitution to determine the constants, look for values of \( x \) that result in equations that are as simple as possible when the values are substituted for \( x \). In the previous example, each constant was obtained directly by substituting an appropriate value for \( x \). This is not always possible, but we can always obtain a system of equations in the constants that can be solved.

Now, what do we do if the denominator has a repeated linear factor? For instance, in the next example, \( (x + 2)^2 \) is a factor of the denominator. In the decomposition, we need to include terms \( \frac{A}{x + 2} \) and \( \frac{B}{(x + 2)^2} \).

In general, each factor \((x - a)^n\) contributes the following sum of terms to the partial fraction decomposition:

\[
\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_n}{(x - a)^n}
\]

**EXAMPLE 3  Repeated Linear Factors**

Evaluate \( \int \frac{3x - 9}{(x + 2)^2(x - 1)} \, dx \).

Solution

**Step 1. Determine the form of the partial fraction expansion.**

We are looking for a partial fraction decomposition of the form

\[
\frac{3x - 9}{(x + 2)^2(x - 1)} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{C}{x - 1}
\]

**Step 2. Determine the constants.**

Let’s clear denominators to obtain

\[
3x - 9 = A(x - 1)(x + 2) + B(x - 1) + C(x + 2)^2
\]

We use Value Substitution in this case. As in the previous example, by substituting appropriately, we can obtain single equations that yield values for \( B \) and \( C \). We cannot obtain \( A \) directly in this manner, but a further substitution provides an equation that can be used to determine \( A \), given the values of \( B \) and \( C \).
We substitute into Eq. (5) as follows:

- Set \( x = 1 \). This gives \( -6 = 9C \).
- Set \( x = -2 \). This gives \( -15 = -3B \).
- Set \( x = 0 \). This gives \( -9 = -2A - B + 4C \)

We now have three equations that we can easily solve for \( A \), \( B \), and \( C \). The first two equations yield \( C = \frac{1}{3} \) and \( B = 5 \), respectively. With those solutions, we then obtain \( A = \frac{3}{2} \) from the third. Therefore, we have

\[
\frac{3x - 9}{(x - 1)(x + 2)^2} = \frac{\frac{3}{2}}{x + 2} + \frac{5}{(x + 2)^2} - \frac{\frac{3}{2}}{x - 1}
\]

**Step 3.** Carry out the integration.

\[
\int \frac{3x - 9}{(x - 1)(x + 2)^2} \, dx = \frac{2}{3} \int \frac{dx}{x - 1} + \frac{2}{3} \int \frac{dx}{x + 2} + 5 \int \frac{dx}{(x + 2)^2}
\]

\[
= \frac{2}{3} \ln |x - 1| + \frac{2}{3} \ln |x + 2| - \frac{5}{x + 2} + C
\]

If \( P/Q \) is not proper—that is, if \( \deg(P) \geq \deg(Q) \)—we use long division to write

\[
\frac{P(x)}{Q(x)} = g(x) + \frac{R(x)}{Q(x)}
\]

where \( g \) is a polynomial and \( R/Q \) is proper. Then, integrating \( P(x)/Q(x) \) involves evaluating an integral of the polynomial \( g(x) \) (which is straightforward) and integrating \( R(x)/Q(x) \) via a partial fraction decomposition, if possible.

**EXAMPLE 4 Long Division Necessary** Evaluate \( \int \frac{x^3 + 1}{x^2 - 4} \, dx \).

**Solution** Using long division, we write

\[
\frac{x^3 + 1}{x^2 - 4} = x + \frac{4x + 1}{x^2 - 4} = x + \frac{4x + 1}{(x - 2)(x + 2)}
\]

It is not difficult to show that the second term has a partial fraction decomposition:

\[
\frac{4x + 1}{x^2 - 4} = \frac{9}{x - 2} + \frac{7}{x + 2}
\]

Therefore,

\[
\int \frac{(x^3 + 1) \, dx}{x^2 - 4} = \int x \, dx + \int \frac{9 \, dx}{x - 2} + \frac{7}{4} \int \frac{dx}{x + 2}
\]

\[
= \frac{1}{2} x^2 + \frac{9}{4} \ln |x - 2| + \frac{7}{4} \ln |x + 2| + C
\]

**Quadratic Factors**

A quadratic polynomial \( x^2 + ax + b \) is called irreducible if it cannot be written as a product of two linear factors (without using complex numbers). If the denominator of a proper rational function has an irreducible quadratic factor \((x^2 + ax + b)^M\), then it contributes a sum of the following type to a partial fraction decomposition:

\[
\frac{A_1 x + B_1}{x^2 + ax + b} + \frac{A_2 x + B_2}{(x^2 + ax + b)^2} + \cdots + \frac{A_M x + B_M}{(x^2 + ax + b)^M}
\]
For example,
\[
\frac{4 - 12x}{(x + 1)(x^2 + x + 4)^2} = \frac{1}{x + 1} - \frac{x}{x^2 + x + 4} - \frac{4x + 12}{(x^2 + x + 4)^2}
\]

**EXAMPLE 5** Irreducible Versus Reducible Quadratic Factors

Evaluate:

(a) \[\int \frac{18}{(x + 3)(x^2 + 9)} \, dx\]

(b) \[\int \frac{18}{(x + 3)(x^2 - 9)} \, dx\]

**Solution**

(a) The quadratic factor \(x^2 + 9\) is irreducible, so the partial fraction decomposition has the form

\[
\frac{18}{(x + 3)(x^2 + 9)} = \frac{A}{x + 3} + \frac{Bx + C}{x^2 + 9}
\]

Clear the denominators to obtain

\[18 = A(x^2 + 9) + (Bx + C)(x + 3)\]

We will use Undetermined Coefficients, so we multiply out the right side and then combine terms in like powers of \(x\):

\[18 = Ax^2 + 9A + Bx^2 + 3Bx + Cx + 3C\]

\[18 = (A + B)x^2 + (3B + C)x + 9A + 3C\]

Now, equating coefficients of like powers of \(x\):

\[0 = A + B \quad \text{(coefficient of } x^2\text{)}\]

\[0 = 3B + C \quad \text{(coefficient of } x\text{)}\]

\[18 = 9A + 3C \quad \text{(constant terms)}\]

Into the third equation, we can substitute \(A = -B\) from the first equation and \(C = -3B\) from the second equation to obtain

\[18 = -9B - 9B\]

This yields \(B = -1\). Now, knowing \(B = -1\), we obtain \(A = -B = 1\) and \(C = -3B = 3\). We can then compute the integral:

\[
\int \frac{18 \, dx}{(x + 3)(x^2 + 9)} = \int \frac{dx}{x + 3} + \int \frac{(-x + 3) \, dx}{x^2 + 9}
\]

\[
= \int \frac{dx}{x + 3} - \int \frac{x \, dx}{x^2 + 9} + \int \frac{3 \, dx}{x^2 + 9}
\]

\[= \ln |x + 3| - \frac{1}{2} \ln(x^2 + 9) + \tan^{-1} \frac{x}{3} + C\]

The last line comes from applying the formulas in the margin.

(b) The polynomial \(x^2 - 9\) is reducible because \(x^2 - 9 = (x - 3)(x + 3)\). Therefore, the partial fraction decomposition has the form

\[
\frac{18}{(x + 3)(x^2 - 9)} = \frac{A}{x - 3} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2}
\]

Clear the denominators:

\[18 = A(x + 3)^2 + B(x + 3)(x - 3) + C(x - 3)\]

We use Value Substitution. The substitutions \(x = 3\) and \(x = -3\) provide equations that yield values for \(A\) and \(C\), respectively. An additional substitution (we will use \(x = 0\)) results in a third equation that will be used to determine \(B\):
Set $x = 3$. This gives $18 = 36A$.
Set $x = -3$. This gives $18 = -6C$.
Set $x = 0$. This gives $18 = 9A - 9B - 3C$.

The solutions are $A = \frac{1}{3}$, $C = -3$, and $B = -\frac{1}{3}$. Determining the integral, we find

\[
\int \frac{18}{(x+3)(x^2-9)} \, dx = \frac{1}{2} \int \frac{dx}{x-3} - \frac{1}{2} \int \frac{dx}{x+3} - 3 \int \frac{dx}{(x+3)^2} = \frac{1}{2} \ln |x-3| - \frac{1}{2} \ln |x+3| + 3(x+3)^{-1} + C
\]

In the next example, we use both Value Substitution and Undetermined Coefficients to find the partial fraction decomposition. Due to the complexity of the polynomials involved, after Value Substitution determines one constant, Undetermined Coefficients is the best option for determining the remaining ones.

**Example 6: Repeated Quadratic Factor**

Evaluate \( \int \frac{4-x}{x(x^2+2)^2} \, dx \).

**Solution**

The partial fraction decomposition has the form

\[
\frac{4-x}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}
\]

Clear the denominators by multiplying through by \(x(x^2+2)^2\):

\[
4-x = A(x^2+2)^2 + (Bx+C)(x^2+2) + (Dx+E)x
\]

We compute \(A\) directly by setting \(x = 0\). Then Eq. (6) reduces to \(4 = 4A\), or \(A = 1\).

We cannot find the remaining constants as simply as we determined \(A\). The best option is to use Undetermined Coefficients to set up a system of four equations in the four unknowns \(B, C, D,\) and \(E\). To begin, we substitute the known value \(A = 1\) in Eq. (6) and expand:

\[
4-x = (x^4+4x^2+4) + (Bx^4+2Bx^2+Cx^3+2Cx) + (Dx^3+Ex)
\]

\[
-1 = (1+B)x^4 + (C+4+2B+D)x^3 + (2C+E)x
\]

Now equate the coefficients on the two sides of the equation:

\[
0 = 1 + B \quad \text{(coefficient of } x^4)\]
\[
0 = C \quad \text{(coefficient of } x^3)\]
\[
0 = 4 + 2B + D \quad \text{(coefficient of } x^3)\]
\[
-1 = 2C + E \quad \text{(coefficient of } x)\]

These equations yield \(B = -1, C = 0, D = -2,\) and \(E = -1\). Thus,

\[
\int \frac{(4-x) \, dx}{x(x^2+2)^2} = \int \frac{dx}{x} - \int \frac{x \, dx}{x^2+2} - \int \frac{(2x+1) \, dx}{(x^2+2)^2} = \ln |x| - \frac{1}{2} \ln (x^2+2) - \int \frac{(2x+1) \, dx}{(x^2+2)^2}
\]
The middle integral was evaluated using the substitution \( u = x^2 + 2 \), \( du = 2x \, dx \). The third integral separates as a sum:

\[
\int \frac{(2x + 1) \, dx}{(x^2 + 2)^2} = \int \frac{2x \, dx}{(x^2 + 2)^2} + \int \frac{dx}{(x^2 + 2)^2}
\]

Use substitution \( u = x^2 + 2 \).

\[
= -\frac{1}{u} + \int \frac{du}{u^2}
\]

To evaluate the integral in Eq. (7), we use the trigonometric substitution

\[
x = \sqrt{2} \tan \theta, \quad dx = \sqrt{2} \sec^2 \theta \, d\theta, \quad x^2 + 2 = 2 \tan^2 \theta + 2 = 2 \sec^2 \theta
\]

Referring to Figure 1, we obtain

\[
\int \frac{dx}{(x^2 + 2)^2} = \int \frac{\sqrt{2} \sec^2 \theta \, d\theta}{(2 \tan^2 \theta + 2)^2} = \int \frac{\sqrt{2} \sec^2 \theta \, d\theta}{4 \sec^4 \theta}
\]

\[
= \frac{\sqrt{2}}{8} \tan^{-1} \frac{x}{\sqrt{2}} + \frac{\sqrt{2}}{8} \ln \left( \frac{x}{\sqrt{2}} + \sqrt{x^2 + 2} \right) + C
\]

Collecting all the terms, we have

\[
\int \frac{4 - x}{x(x^2 + 2)^2} \, dx = \ln |x| + \frac{1}{2} \ln(x^2 + 2) + \frac{1 - x}{x^2 + 2} - \frac{1}{4\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + C
\]

**CONCEPTUAL INSIGHT** The examples in this section illustrate a general fact: The integral of a proper rational function can be expressed as a sum of rational functions, arctangents of linear or quadratic polynomials, and logarithms of linear or quadratic polynomials. Other types of functions, such as exponential and trigonometric functions, do not appear.

At the beginning of this section, we explained in a marginal comment that the Fundamental Theorem of Algebra guarantees that every polynomial can be factored into linear and quadratic terms. Thus, every proper rational function can be written as a sum of terms of the form \( \frac{A}{x - c} \) and \( \frac{Ax + B}{(x^2 + bx + c)^n} \) for real numbers \( a, b, c, A, B, \) and \( n \). This property is simple in theory, but in practice, finding a sum can be difficult. Our challenge is to identify the factors of the denominator. Indeed, sometimes it is not possible to determine the exact factors and, at best, we can only approximate them. To find the numerators in the partial fraction decomposition once the denominator is factored, we can set up and solve a system of linear equations with the constants as unknowns.

**Using a Computer Algebra System**

Finding a partial fraction decomposition of a rational function \( P/Q \) often requires laborious computation. Fortunately, most computer algebra systems are able to compute partial fraction decompositions whenever the factors of \( Q \) can be determined exactly. Even though, in theory, all polynomials can be factored into a product of linear and quadratic terms, it is not always possible to determine the factors exactly. The rational function

\[
f(x) = \frac{1}{x^4 + 2x^2 + 1}
\]

is an example where an exact partial fraction decomposition cannot be found. Try it on a computer algebra system to see what results.
8.5 SUMMARY

The Method of Partial Fractions enables us to separate a complicated rational function into a sum of simpler rational terms that are easier to integrate than the original function. Assume first that \( P/Q \) is a proper rational function [i.e., \( \text{deg}(P) < \text{deg}(Q) \)] and that \( Q(x) \) can be factored explicitly as a product of linear and irreducible quadratic terms.

- If \( Q(x) \) is equal to a product of powers of linear factors \((x - a)^M\) and irreducible quadratic factors \((x^2 + ax + b)^N\), then the partial fraction decomposition of the rational function \( P(x)/Q(x) \) is a sum of terms of the following type:
  \[
  (x - a)^M \text{ contributes } \frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_M}{(x - a)^M} \\
  (x^2 + ax + b)^N \text{ contributes } \frac{A_1x + B_1}{x^2 + ax + b} + \frac{A_2x + B_2}{(x^2 + ax + b)^2} + \cdots + \frac{A_Nx + B_N}{(x^2 + ax + b)^N}
  \]

- The methods of Value Substitution and Undetermined Coefficients can be used to determine the constants \( A_j \) and \( B_j \) in the partial fraction decomposition.
- The individual terms in the decomposition then can be integrated. Substitution, completing the square, and trigonometric substitution may be needed to integrate the terms corresponding to \((x^2 + ax + b)^N\) (see Example 6).
- If \( P/Q \) is improper, use long division (see Example 4) to express \( P/Q \) as a sum of a polynomial and a proper rational function. The Method of Partial Fractions can then be applied to the proper rational function.

8.5 EXERCISES

Preliminary Questions

1. Suppose that \( \int f(x) dx = \ln x + \sqrt{x - 1} + C \). Can \( f \) be a rational function? Explain.

2. Which of the following are proper rational functions?
   (a) \( \frac{x}{x + 3} \)  
   (b) \( \frac{x^2}{x^2} \)  
   (c) \( \frac{x^2}{x + 2} \)  
   (d) \( \frac{x}{x^2 - 3} \)

3. Which of the following quadratic polynomials are irreducible? To check, complete the square if necessary.
   (a) \( x^2 + 5 \)  
   (b) \( x^2 - 5 \)  
   (c) \( x^2 + 4x + 6 \)  
   (d) \( x^2 + 4x + 2 \)

4. Let \( P/Q \) be a proper rational function where \( Q(x) \) factors as a product of distinct linear factors \((x - a_i)\). Then
   \[
   \int \frac{P(x)}{Q(x)} dx = \frac{P(x)}{Q(x)} + C
   \]
   (choose the correct answer):
   (a) is a sum of logarithmic terms \( A_i \ln(x - a_i) \) for some constants \( A_i \).
   (b) may contain a term involving the arctangent.

Exercises

1. Match the rational functions (a)-(d) with the corresponding partial fraction decompositions (i)-(iv).
   (a) \( \frac{x^2 + 4x + 12}{(x + 2)(x^2 + 4)} \)  
   (b) \( \frac{2x^2 + 8x + 24}{(x + 2)(x^2 + 4)} \)  
   (c) \( \frac{x^2 - 4x + 8}{(x - 1)^2(x - 2)^2} \)  
   (d) \( \frac{x^2 - 4x + 8}{(x + 2)(x^2 + 4)} \)

   (i) \( x - 2 + \frac{4}{x + 2} \)  
   (ii) \( -8 + \frac{4}{x^2 + 4} \)  
   (iii) \( \frac{1}{x + 2} + \frac{2}{x + 2} \)  
   (iv) \( \frac{1}{x + 2} + \frac{4}{x^2 + 4} \)

2. Determine the constants \( A, B \):
   \[
   \frac{2x - 3}{(x - 3)(x - 4)} = \frac{A}{x - 3} + \frac{B}{x - 4}
   \]

3. Clear the denominators in the following partial fraction decomposition and determine the constant \( B \) (substitute a value of \( x \) or use the method of undetermined coefficients).
   \[
   \frac{3x^2 + 11x + 12}{(x + 1)(x + 3)^2} = \frac{1}{x + 1} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2}
   \]

4. Find the constants in the partial fraction decomposition
   \[
   \frac{2x + 4}{(x - 2)(x^2 + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 4}
   \]
In Exercises 5–8, evaluate using long division first to write \( f(x) \) as the sum of a polynomial and a proper rational function.

5. \( \int \frac{x \, dx}{3x - 4} \)

6. \( \int \frac{(x^2 + 2) \, dx}{x + 3} \)

7. \( \int \frac{(x^3 + 2x^2 + 1) \, dx}{x + 2} \)

8. \( \int \frac{(x^3 + 1) \, dx}{x^2 + 1} \)

In Exercises 9–46, evaluate the integral.

9. \( \int \frac{dx}{x(x - 2)(x - 4)} \)

10. \( \int \frac{(2 - x) \, dx}{x + 4} \)

11. \( \int \frac{dx}{x(3x + 1)} \)

12. \( \int \frac{dx}{x^2 - 5x + 6} \)

13. \( \int \frac{x^2 \, dx}{x^2 + 9} \)

14. \( \int \frac{dx}{(x - 2)(x - 3)(x + 2)} \)

15. \( \int \frac{(x^2 + 3x - 44) \, dx}{x + 3 + (x + 5)(x - 2)} \)

16. \( \int \frac{3 \, dx}{x + 3 + (x + 4)(x + 2)} \)

17. \( \int \frac{(x^2 + 11x) \, dx}{x(2x + 1)} \)

18. \( \int \frac{(2x - 21x)(dx)}{x - 3)(2x + 3)} \)

19. \( \int \frac{(2x - 3x \, dx}{x - 2x - 2} \)

20. \( \int \frac{(x^2 - 1) \, dx}{x + 1} \)

21. \( \int \frac{dx}{x(2x + 2)} \)

22. \( \int \frac{dx}{x^2 + 3} \)

23. \( \int \frac{dx}{2x^2 - 3} \)

24. \( \int \frac{dx}{x^2 - 4(x - 1)} \)

25. \( \int \frac{dx}{x^3 + x^2 - x - 1} \)

26. \( \int \frac{dx}{x^3 - 3x^2 + 4} \)

27. \( \int \frac{dx}{x(x - 1)(x - 3)} \)

28. \( \int \frac{dx}{x^2 + 6(x - 1)(x - 3)} \)

29. \( \int \frac{dx}{x^2 - 1(x - 3)} \)

30. \( \int \frac{dx}{x^2 - 1(x - 3)} \)

31. \( \int \frac{dx}{x(x - 1)} \)

32. \( \int \frac{dx}{x^2 - 1} \)

33. \( \int \frac{dx}{x^2 + 2x + 5} \)

34. \( \int \frac{dx}{x^2 + 2x + 5} \)

35. \( \int \frac{dx}{x^2 + 2x + 5} \)

36. \( \int \frac{dx}{x^2 + 2x + 5} \)

37. \( \int \frac{dx}{x^2 + 2x + 3} \)

38. \( \int \frac{dx}{x^2 + 2x + 3} \)

39. \( \int \frac{dx}{x^2 + 2x + 3} \)

40. \( \int \frac{dx}{x^2 + 2x + 3} \)

41. \( \int \frac{dx}{x(x^2 + 8)^2} \)

42. \( \int \frac{100x \, dx}{x^2 + 1} \)

43. \( \int \frac{dx}{(x + 2)(x^2 + 4x + 10)} \)

44. \( \int \frac{9 \, dx}{x + 1(x + 2x + 6)} \)

45. \( \int \frac{25 \, dx}{x^2 + 2x + 5} \)

46. \( \int \frac{(x^2 + 3) \, dx}{x^2 + 2x + 5} \)

In Exercises 47–50, evaluate by using first substitution and then partial fractions if necessary.

47. \( \int \frac{x \, dx}{x + 2} \)

48. \( \int \frac{x \, dx}{x + 2} \)

49. \( \int \frac{e^x \, dx}{e^{2x} - 1} \)

50. \( \int \frac{sec \theta \, d\theta}{tan^2 \theta - 1} \)

51. Evaluate \( \int \frac{dx}{x - 1}. \) Hint: Use the substitution \( u = \sqrt{x} \) (sometimes called a rationalizing substitution).

52. Evaluate \( \int \frac{dx}{x^{1/2} - x^{1/3}}. \) Hint: Use the substitution \( u = x^{1/6} \).

53. Evaluate \( \int \frac{dx}{x^{3/4} - 4x^{3/4}}. \)

54. Evaluate \( \int \frac{dx}{x^{1/4} + 2x^{1/3}}. \)

55. Evaluate \( \int \frac{dx}{x^3 - 1} \) in two ways: using partial fractions and using trigonometric substitution. Verify that the two answers agree.

56. [GU] Graph the equation \((x - 40)^2 = 10x(x - 30)\) and find the volume of the solid obtained by revolving the region between the graph and the x-axis for \(0 \leq x \leq 30\) around the x-axis.

57. Show that the substitution \( u = 2 \tan^{-1} t \) (Figure 2) yields the formulas

\[
\cos \theta = \frac{1 - t^2}{1 + t^2}, \quad \sin \theta = \frac{2t}{1 + t^2}, \quad d\theta = \frac{2 \, dt}{1 + t^2}
\]

This substitution transforms the integral of any rational function of \(\cos \theta\) and \(\sin \theta\) into an integral of a rational function of \(t\) (which can then be evaluated using partial fractions). Use it to evaluate

\[\int \frac{d\theta}{\cos \theta + \frac{1}{2} \sin \theta}\]

**Figure 2**

58. Use the substitution of Exercise 57 to evaluate \( \int \frac{d\theta}{\cos \theta + \sin \theta} \).

Further Insights and Challenges

59. Prove the general formula

\[\int \frac{dx}{(x - a)(x - b)} = \frac{1}{a - b} \ln |x - a| - \frac{1}{2} \ln |x + b| + C\]

where \(a, b\) are constants such that \(a \neq b\).

60. The method of partial fractions shows that

\[\int \frac{dx}{x^2 - 1} = \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C\]

A computer algebra system evaluates this integral as \(-\tanh^{-1} x\), where \(\tanh^{-1} x\) is the inverse hyperbolic tangent function. Can you reconcile the two answers?

61. Suppose that \(Q(x) = (x - a)(x - b)\), where \(a \neq b\), and let \(P/Q\) be a proper rational function so that

\[
P(x) \quad Q(x) = \frac{A}{(x - a)} + \frac{B}{(x - b)}
\]
8.6 Strategies for Integration

In Chapter 5, and in the preceding sections of this chapter, we have seen a variety of techniques for evaluating various integrals. In a general setting, when confronted with a given integral, it will not appear in a particular section devoted to a particular technique of integration. Hence, it is important to be able to recognize which technique of integration is likely to apply. This section is devoted to that topic. In addition to considering how to recognize what technique to apply, we also discuss how tables of integrals and how computer algebra systems can be utilized to find an integral.

In general, there are no hard and fast rules for evaluating a given indefinite integral. But there are various heuristics that help us to determine which techniques are likely to apply. Here is an overview of the different approaches that we can employ to evaluate an integral.

**Simplification:** Do any algebraic simplification possible. Cancel terms in fractions when possible. For instance,

1. \[ \int \frac{x^3 - 1}{x - 1} \, dx = \int \frac{(x - 1)(x^2 + x + 1)}{x - 1} \, dx = \int (x^2 + x + 1) \, dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C \]

2. \[ \int \frac{x - x^3}{\sqrt{x}} \, dx = \int (x^{1/2} - x^{3/2}) \, dx = \frac{2}{3}x^{3/2} - \frac{2}{7}x^{7/2} + C \]

3. \[ \int \frac{\csc^2 x}{x} \, dx = \int \csc^2 x \, dx = -\cot x + C \]

4. \[ \int e^x \sin x \, dx. \] It is tempting to try to apply Integration by Parts to this integral, but this method does not work. Instead, we replace \( \sin x \) with its expression in terms of exponential functions to obtain

\[ \int e^x \sin x \, dx = \int e^x \left(\frac{e^x - e^{-x}}{2}\right) \, dx = \frac{1}{2} \int (e^{2x} - 1) \, dx = \frac{e^{2x}}{4} - \frac{x}{2} + C \]

**Substitution:** Any time we recognize that our integral is of the form \( \int f(g(x))g'(x) \, dx \), we can use substitution. The key to successfully substituting is that if we want to replace \( g(x) \) by \( u \), we need a \( g'(x) \) to appear in the integrand in such a way that we can pair it with \( dx \) to obtain the necessary \( du \).

So, for example, \( \int x^2 \sin(x^3) \, dx \) is set up for substitution since if we let \( u = x^3 \), then \( du = 3x^2 \, dx \), and we have the requisite \( x^2 \, dx \) to make this work. We simply substitute \( (1/3)du \) for \( x^2 \, dx \), and we will obtain an integral that we then can compute.

In the case of \( \int e^{\sin x} \cos x \, dx \), we can use \( u = \sin x \) and \( du = \cos x \, dx \). In the case of \( \int e^{\sqrt{e^x + 1}} \, dx \), we can let \( u = e^x + 1 \) and then \( du = e^x \, dx \), which we have.

Sometimes, some extra algebraic work is needed to carry out a substitution so that all of the \( x \) terms are changed to terms in the new variable \( u \).
EXAMPLE 1 Find \( \int x^3 \sqrt{1 + x^2} \, dx \).

Solution Our first inclination is to try the substitution \( u = 1 + x^2 \) since that expression is inside the square root. But then \( du = 2x \, dx \), and we have an extra factor of \( x^2 \) left over. Instead of turning to another method of integration, though, we can note that since \( u = 1 + x^2, x^2 = u - 1 \). Thus, we have

\[
\int x^3 \sqrt{1 + x^2} \, dx = \int x^2 \sqrt{1 + x^2} \, dx = \int (u - 1) \sqrt{u} \frac{du}{2}
\]

\[
= \frac{1}{2} \int \left( u^{3/2} - u^{1/2} \right) \, du = \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3} + C
\]

\[
= \frac{(1 + x^2)^{5/2}}{5} - \frac{(1 + x^2)^{3/2}}{3} + C
\]

This integral could also be evaluated using Integration by Parts, but that would be a more involved process.

Integration by Parts: As we have seen, if there is a product in the integrand, we can split the integral into two pieces \( u \) and \( dv \) and then apply Integration by Parts to obtain

\[
\int u \, dv = uv - \int v \, du
\]

For example, look at \( \int x e^x \, dx \). In this case, it is obvious that Integration by Parts applies. If the \( x \) were not present, we could easily integrate \( e^x \). So choosing \( u = x \) and \( dv = e^x \, dx \) will yield \( \int x e^x \, dx = x e^x - \int e^x \, dx \). The remaining integral is now one we can compute.

In particular, we are looking for integrands that are products such that differentiating one term in the product and integrating the other yields an integral that is easier to evaluate. Here are some cases to look for:

1. \( \int x^n f(x) \, dx \). Assuming \( f(x) \) can be repeatedly integrated, we may use repeated applications of Integration by Parts, always choosing \( u \) equal to the remaining power of \( x \), until we eliminate the powers of \( x \), leaving something we can integrate. Candidates for \( f(x) \) include \( \sin x, \cos x, e^x, ax, \sinh x, \cosh x, \sec^2 x, \), and \( \csc^2 x \), among others. Note that the same technique applies when we replace \( x^n \) in the integrand with a more complicated polynomial. Repeated application of Integration by Parts can be used to eliminate the polynomial.

2. \( \int e^x \sin x \, dx \). Two applications of Integration by Parts, choosing \( u = e^x \) both times, yields

\[
\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx
\]

Instead of obtaining a simpler integral on the right, we obtained the integral we started with. Adding it to both sides, we get

\[
2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + C
\]

\[
\int e^x \sin x \, dx = \frac{1}{2}(-e^x \cos x + e^x \sin x) + C
\]
3. In some special cases, we can use Integration by Parts for \(\int f(x)\,dx\) by setting \(u = f(x)\) and \(dv = dx\). For example, this works with the integrals \(\int \ln x\,dx\), \(\int \tan^{-1} x\,dx\), and \(\int \sin^{-1} x\,dx\).

4. Sometimes integration by parts is easier if we integrate, rather than differentiate, the \(x^n\) factor, as in the next example.

**EXAMPLE 2** Compute \(\int x^2 \ln x\,dx\).

**Solution** We proceed as follows:

\[
\begin{align*}
u & = \ln x \\
du & = \frac{1}{x}\,dx \\
v & = \frac{1}{3}x^3
\end{align*}
\]

\[
\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3\right) \frac{1}{x}\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{3} \int x^2\,dx
\]

\[
= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C
\]

For the previous example, using \(u = x^2\) and \(dv = \ln x\,dx\) also works, but this alternative is more complicated than the choice we used (see Exercise 60).

**Special Techniques:** Consider the form of the integral. Does it fall into one of the categories for which we have a special technique?

1. Is it a trigonometric integral? If it is of the form

\[
\int \sin^n x \cos^n x\,dx, \quad \text{or} \quad \int \tan^n x \sec^n x\,dx
\]

then we have specific rules that can be applied (as outlined in Section 8.2) to simplify the integral into one that can be computed.

2. Does the integrand contain

\[
\sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \text{or} \quad \sqrt{x^2 - a^2}
\]

or more generally,

\[
(a^2 - x^2)^{n/2}, \quad (x^2 + a^2)^{n/2}, \quad \text{or} \quad (x^2 - a^2)^{n/2}
\]

for an integer \(n\)? If so, then try a trigonometric substitution, \(x = a \sin \theta, x = a \tan \theta, \) or \(x = a \sec \theta,\) respectively, as outlined in Section 8.3.

**EXAMPLE 3** Evaluate \(\int (x^2 + 16)^{3/2}\,dx\).

**Solution** Let \(x = 4 \tan \theta\). Then \((x^2 + 16)^{3/2} = (16 \tan^2 \theta + 16)^{3/2} = (16 \sec^2 \theta)^{3/2} = 64 \sec^3 \theta\) and \(dx = 4 \sec^2 \theta\,d\theta\). Hence, we have

\[
\int (x^2 + 16)^{3/2}\,dx = \int 64 \sec^3 \theta (4 \sec^2 \theta)\,d\theta = 256 \int \sec^5 \theta\,d\theta
\]
Now, we can use the reduction formula in Eq. (12) from Section 8.2 to obtain

\[
256 \int \sec^5 \theta \, d\theta = 256 \left( \frac{\tan \theta \sec^3 \theta}{4} + \frac{3}{4} \int \sec^3 \theta \, d\theta \right)
\]

\[
= 256 \left( \frac{\tan \theta \sec^3 \theta}{4} + \frac{3}{4} \left( \frac{\tan \theta \sec \theta}{2} + \frac{1}{2} \int \sec \theta \, d\theta \right) \right)
\]

\[
= 256 \left( \frac{\tan \theta \sec^3 \theta}{4} + \frac{3}{4} \left( \frac{\tan \theta \sec \theta}{2} + \frac{1}{2} \ln |\sec \theta + \tan \theta| \right) \right) + C
\]

Since \( x = 4 \tan \theta \), we know \( \tan \theta = \frac{x}{4} \), and by the triangle in Figure 1, \( \sec \theta = \sqrt{x^2 + 16} \). Thus, we have

\[
\int (x^2 + 16)^{3/2} \, dx
\]

\[
= 64 \frac{x}{4} (x^2 + 16)^{3/2} + 96 \frac{x}{4} (x^2 + 16)^{1/2} + 96 \ln \left| (x^2 + 16)^{1/2} + \frac{x}{4} \right| + C
\]

If the integral contains an expression of the form \( \sqrt{ax^2 + bx + c} \), we can complete the square inside the square root in order to obtain one of the cases discussed above.

3. Is it a rational function \( \frac{P}{Q} \) for polynomials \( P \) and \( Q \)? If so, we can often apply the Method of Partial Fractions. First, if \( P \) has a degree that is at least as large as the degree of \( Q \), we divide \( P(x) \) by \( Q(x) \) to obtain a polynomial (which is easily integrated) together with a remainder term \( \frac{R(x)}{Q(x)} \), to which the Method of Partial Fractions can be applied.

**EXAMPLE 4** Find \( \int \frac{x^2 + 2x + 10}{x^2 + x - 6} \, dx \).

**Solution** Noting that both the numerator and the denominator have degree 2, we divide to obtain

\[
\frac{x^2 + 2x + 10}{x^2 + x - 6} = 1 + \frac{x + 16}{x^2 + x - 6}
\]

Then we can write

\[
\frac{x + 16}{x^2 + x - 6} = \frac{x + 16}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2}
\]

Clearing the denominator, this yields

\[
x + 16 = A(x - 2) + B(x + 3)
\]

We use Value Substitution to determine \( A \) and \( B \). Substituting \( x = 2 \) yields \( 18 = 5B \), and therefore, \( B = \frac{18}{5} \). Furthermore, substituting \( x = -3 \) results in \( 13 = -5A \), implying \( A = -\frac{13}{5} \). Thus,

\[
\int \frac{x^2 + 2x + 10}{x^2 + x - 6} \, dx = \int \left( 1 + \frac{-13}{5(x + 3)} + \frac{18}{5(x - 2)} \right) \, dx
\]

\[
= x - \frac{13}{5} \ln |x + 3| + \frac{18}{5} \ln |x - 2| + C
\]
**Integral Table:** Integral tables (such as the one on the final three pages of this text) contain a list of forms of many common integrals. Given a particular integral that we may want to evaluate, if we can get it into the form of one of the integrals in a table, then we can apply the given formula to evaluate the integral. Let's consider a couple of examples.

**EXAMPLE 5** Evaluate \( \int \frac{\sqrt{9 - 4x^2}}{x} \, dx \).

**Solution** Looking down the list of integrals given at the end of the text, we see the formula that appears to apply here is #69:

\[
\int \frac{\sqrt{a^2 - u^2}}{u} \, du = \sqrt{a^2 - u^2} - a \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C
\]

We can put our integral in the correct form by factoring the 4 out of the square root:

\[
\sqrt{9 - 4x^2} = 2\sqrt{9/4 - x^2}
\]

Now we use formula #69 with \( a = 3/2 \):

\[
\int \frac{\sqrt{9 - 4x^2}}{x} \, dx = 2 \int \frac{\sqrt{9/4 - x^2}}{x} \, dx = 2 \sqrt{9/4 - x^2} - 3 \ln \left| \frac{3/2 + \sqrt{9/4 - x^2}}{x} \right| + C
\]

Notice that the integral in the previous example could also be evaluated by applying a trigonometric substitution. In fact, many of the formulas in the integral table result from integral techniques and formulas that we have developed.

**EXAMPLE 6** Evaluate \( \int 6x \tan^2(x^2) \, dx \).

**Solution** The integral table formula that seems applicable is #36:

\[
\int \tan^2 u \, du = \tan u - u + C
\]

Although the integrand does not have the form in this formula, observe that a substitution \( u = x^2 \) will put us in a position to use formula #36. With this substitution, we have \( du = 2x \, dx \), so \( 3u = 6x \, dx \). Therefore,

\[
\int 6x \tan^2(x^2) \, dx = 3 \int \tan^2 u \, du = 3 \tan u - 3u + C
\]

\[
= 3 \tan(x^2) - 3x^2 + C
\]

**Computer Algebra System:** In order to find these last integrals using an integral table, we needed to do some algebra or apply some integration techniques to get them to match the pattern for one of the formulas at the back of the book. Computers are particularly adept at attempting various rearrangements and integration techniques and then matching the resulting integrals. For this reason, computer algebra systems are very good at evaluating integrals. However, keep in mind that the output that they generate may not be the form that is the most convenient to use. For example, if we want to find \( \int x(x - 3)^{15} \, dx \), we do a substitution for \( u = x - 3 \) (and therefore \( x = u + 3 \)) and find

\[
\int x(x - 3)^{15} \, dx = \int (u + 3)u^{15} \, du = \int u^{16} + 3u^{15} \, du = \frac{1}{17}u^{17} + \frac{3}{16}u^{16} + C
\]

\[
= \frac{1}{17}(x - 3)^{17} + \frac{3}{16}(x - 3)^{16} + C
\]
On the other hand, the following answer was obtained from a computer algebra system:

\[
\int (x-3)^5 \, dx = \frac{x^{17}}{17} - \frac{45x^{16}}{16} + 63x^{15} - \frac{1755x^{14}}{2} + 8505x^{13} - \frac{243,243x^{12}}{4} \\
+ 331,695x^{11} - \frac{2,814,669x^{10}}{2} + 4,691,115x^9 - \frac{98,513,415x^8}{8} \\
+ 25,332,021x^7 - \frac{80,601,885x^6}{2} + 48,361,131x^5 - \frac{167,403,915x^4}{4} \\
+ 23,914,845x^3 - \frac{14,348,907x^2}{2} + C
\]

As you can see, the answer that we obtained by substitution is simpler to state and would be more convenient to use in computations.

**Numerical Approximation:** If the integral is a definite integral, and finding an antiderivative is difficult or impossible, then the integral can be approximated numerically. One approach would be to use an approximating Riemann sum. Other numerical methods are introduced in Section 8.8.

**EXAMPLE 7** The Fresnel function \( S(x) = \int_0^x \sin(t^2) \, dt \) (Figure 2) is important in the field of optics. It has a maximum at \( x = \sqrt{\pi} \) (the least positive \( x \) at which \( S(x) = 0 \). Approximate the maximum value of \( S \).

**Solution** We need to approximate \( \int_0^{\sqrt{\pi}} \sin(t^2) \, dt \). Note, we cannot evaluate this definite integral via antidifferentiation since there is no antiderivative of \( f(t) = \sin(t^2) \) that can be expressed in terms of elementary functions. Using technology to compute an approximating Riemann sum with 1000 subintervals, we obtain

\[ S(\sqrt{\pi}) = \int_0^{\sqrt{\pi}} \sin(t^2) \, dt \approx 0.895 \]

### 8.6 SUMMARY

Strategy for integration:

- **Simplification:** Do any algebraic simplification possible.
- **Substitution:** Consider possible substitutions, \( u = g(x) \), keeping in mind that we will need the presence of \( g'(x) \) since \( du = g'(x) \, dx \).
- **Integration by Parts:** Consider Integration by Parts, choosing \( u \) to be a function that can be differentiated, \( dv \) to be something that can be integrated, and such that \( \int u \, dv \) is an easier integral than the original. Keep in mind that Integration by Parts can work when the integrand is not an obvious product, as long as \( u \) and \( dv \) are chosen properly.
- **Trigonometric Integral:** If the integral is a trigonometric integral of the form \( \int \sin^n x \cos^m x \, dx \) or \( \int \tan^n x \sec^m x \, dx \), then trigonometric identities and reduction formulas can be applied.
- **Trigonometric Substitution:** Does the integral contain \( \sqrt{a^2 - x^2} \), \( \sqrt{x^2 - a^2} \), or \( \sqrt{x^2 + a^2} \)? If so, then try a trigonometric substitution.
- **Partial Fractions:** Is the integrand a rational function \( \frac{P}{Q} \) for polynomials \( P \) and \( Q \)? If so, the Method of Partial Fractions might enable you to separate the function into a sum of simpler rational terms that can be integrated.
- **Integral Table:** Determine whether the integral, after suitable manipulation, matches an integral in an integral table.
SECTION 8.6 Strategies for Integration 447

- Computer Algebra System: Consider using a computer algebra system to determine the integral.
- Numerical Approximation: If the integral is a definite integral, and none of the above methods work, approximate the integral numerically.

8.6 EXERCISES

Preliminary Questions

For each of the following, state what method applies and how one applies it, but do not evaluate the integral.

1. \( \int x \sin x \, dx \)
2. \( \int \sqrt{1 + x^2} \, dx \)
3. \( \int \frac{1 + x^2}{1 - x^2} \, dx \)
4. \( \int \cos^2 x \, \sin x \, dx \)
5. \( \int x \ln x \, dx \)
6. \( \int \sqrt{1 - x^2} \, dx \)
7. \( \int \sin^3 x \, \cos^2 x \, dx \)

For each of the following, find the formula in the integral table at the back of the text that can be applied to find the integral.

8. \( \int \frac{3x^2 \, dx}{5x + 2} \)
9. \( \int \frac{\sqrt{25 + 16x^2} \, dx}{x^2} \)
10. \( \int \sec^2 (4x) \, dx \)
11. \( \int \frac{x^2}{\sqrt{x^2 + 2x + 3}} \, dx \)

Exercises

In Exercises 1–10, indicate a good method for evaluating the integral (but do not evaluate). Your choices are substitution (specify u and du), integration by parts (specify u and dv), a trigonometric method, or trigonometric substitution (specify). If it appears that these techniques are not sufficient, state this.

1. \( \int \frac{x \, dx}{\sqrt{12 - 6x - x^2}} \)
2. \( \int \frac{x^2 - 1 \, dx}{x} \)
3. \( \int \sin^3 x \, \cos^3 x \, dx \)
4. \( \int x \sec^2 x \, dx \)
5. \( \int \frac{dx}{\sqrt{9 - x^2}} \)
6. \( \int \frac{1 - x^2 \, dx}{x} \)
7. \( \int \sin^{3/2} x \, dx \)
8. \( \int x^3 \sqrt{x + 1} \, dx \)
9. \( \int \frac{dx}{x + 1}\left(x + 2\right)^3 \)
10. \( \int \frac{dx}{x^2 + 4x + 5} \)

In Exercises 11–59, evaluate the integral using the appropriate method or combination of methods covered thus far in the text. You may use the integral tables at the end of the text, but do not use a computer algebra system.

11. \( \int \frac{dx}{x^2 \sqrt{4 - x^2}} \)
12. \( \int \frac{dx}{x^2 - 1} \)
13. \( \int \cos^2 4x \, dx \)
14. \( \int x \sec x \, \tan x \, dx \)
15. \( \int x \sin x \, \tan x \, dx \)
16. \( \int \frac{dx}{x^2 + 9} \)
17. \( \int \frac{dx}{x^2 + 9} \)
18. \( \int \tan^2 \theta \, d\theta \)
19. \( \int \tan^2 x \, \sec^2 x \, dx \)
20. \( \int \frac{(3x^2 - 1) \, dx}{x(x^2 - 1)} \)
21. \( \int \frac{x \, dx}{(x^2 - 1)^{3/2}} \)
22. \( \int \frac{dx}{x^2 - 1} \)
23. \( \int \frac{dx}{x^2 + 1} \)
24. \( \int \frac{dx}{x^2 + 1} \)
25. \( \int \frac{(x + 1) \, dx}{(x^2 + 4x + 8)^2} \)
26. \( \int \frac{dx}{x^3} \)
27. \( \int \frac{x^{1/2} \, dx}{x^{1/3} + 1} \)
28. \( \int \frac{dx}{\sqrt{16 + x^2}} \)
29. \( \int \frac{x^2 \, dx}{1 + e^x} \)
30. \( \int \frac{dt}{(1 + 4t^2)^{3/2}} \)
31. \( \int x^3 \, dx \)
32. \( \int \frac{x^2 + 1 \, dx}{x^2 + 1} \)
33. \( \int \frac{x^2 + 1 \, dx}{x^2 + 1} \)
34. \( \int x^2 + 6 \, dx \)
35. \( \int \frac{x^2 + 1 \, dx}{x^2 + 1} \)
36. \( \int \frac{x^2 + 1 \, dx}{x^2 + 1} \)
37. \( \int \frac{dx}{x^2 + 1} \)
38. \( \int \frac{x^2 + 1 \, dx}{x^2 + 1} \)
39. \( \int \frac{x \, dx}{\sqrt{x - 1}} \)
40. \( \int \frac{x \, dx}{\sqrt{x + 2}} \)
41. \( \int \frac{x^2 \, dx}{\sqrt{x + 2}} \)
42. \( \int \frac{dx}{\sqrt{x^2 - 16}} \)
43. \( \int \frac{dx}{\sqrt{1 - x^2}} \)
44. \( \int \ln(x^2 - 9) \, dx \)
45. \( \int \frac{dx}{x^2 - 6x + 7} \)
46. \( \int e^x \sqrt{x^2 - 1} \, dx \)
47. \( \int \frac{dx}{x^2 - 1} \)
48. \( \int \frac{dx}{x^2 - 1} \)
49. \( \int \frac{x \, dx}{x^2 - 1} \)
50. \( \int \frac{dx}{x^2 - 1} \)
51. \( \int \frac{dx}{x^2 - 1} \)
52. \( \int \frac{dx}{x^2 - 1} \)
53. \( \int \frac{dx}{x^2 - 1} \)
54. \( \int \tan x \, \sec^{5/2} x \, dx \)
55. \[ \int (3 \sec x - \cos x)^2 \, dx \]
56. \[ \int x^3 \ln x \, dx \]
57. \[ \int \frac{(1 + \ln x)^2}{x} \, dx \]
58. \[ \int \frac{e^x}{e^{2x} - 1} \, dx \]
59. \[ \int \frac{dx}{\sqrt{x^2 - 36}} \]
60. Use Integration by Parts to compute \[ \int x^2 \ln x \, dx \], setting \( u = x^2 \) and \( dv = \ln x \, dx \).

### 8.7 Improper Integrals

The integrals we have studied so far represent signed areas of bounded regions. However, we also wish to consider unbounded regions, such as the region under the graph of \( y = \frac{1}{1 + x^2} \), for \(-\infty < x < \infty\), shown in Figure 1. Integrals over such regions arise in applications and are represented by what are known as **improper integrals**.

There are two ways an integral can be improper: (1) The interval of integration is infinite, or (2) the integrand tends to infinity on a finite interval and therefore the graph of the function has a vertical asymptote. We deal first with improper integrals over infinite intervals. One or both endpoints may be infinite:

\[
\int_{-\infty}^{-} f(x) \, dx, \quad \int_{a}^{\infty} f(x) \, dx, \quad \int_{-\infty}^{\infty} f(x) \, dx
\]

How can an unbounded region have finite area? To answer this question, we must specify what we mean by the area of an unbounded region. Consider the area [Figure 2(A)] under the graph of \( f(x) = e^{-x} \) over the finite interval \([0, R] \):

\[
\int_{0}^{R} e^{-x} \, dx = \left. -e^{-x} \right|_{0}^{R} = -e^{-R} + e^{0} = 1 - e^{-R}
\]

As \( R \to \infty \), this area approaches a finite value [Figure 2(B)]:

\[
\int_{0}^{\infty} e^{-x} \, dx = \lim_{R \to \infty} \int_{0}^{R} e^{-x} \, dx = \lim_{R \to \infty} (1 - e^{-R}) = 1
\]

It seems reasonable to take this limit as the definition of the area under the graph over the infinite interval \([0, \infty)\). Thus, the unbounded region in Figure 2(C) has area 1.

(A) Bounded region has area \( 1 - e^{-R} \)  
(B) Area approaches 1 as \( R \to \infty \)  
(C) Unbounded region has area 1

**Definition** **Improper Integral**: Fix a number \( a \) and assume that \( f \) is integrable over \([a, b]\) for all \( b > a \). The **improper integral of \( f \) over \([a, \infty)\)** is defined as the following limit (if it exists):

\[
\int_{a}^{\infty} f(x) \, dx = \lim_{R \to \infty} \int_{a}^{R} f(x) \, dx
\]

We say that the improper integral **converges** if the limit exists and that it **diverges** if the limit does not exist.

Similarly, we define

\[
\int_{-\infty}^{a} f(x) \, dx = \lim_{R \to -\infty} \int_{R}^{a} f(x) \, dx
\]
**EXAMPLE 1** Show that \( \int_2^\infty \frac{dx}{x^3} \) converges and compute its value.

Solution

\[
\int_2^\infty \frac{dx}{x^3} = \lim_{R \to \infty} \int_2^R \frac{dx}{x^3} = \lim_{R \to \infty} \left[ \frac{-1}{2} x^{-2} \right]_2^R = \lim_{R \to \infty} \left( \frac{-1}{2} (R^2) + \frac{1}{2} (2^2) \right) = \lim_{R \to \infty} \left( \frac{1}{8} - \frac{1}{2R^2} \right) = \frac{1}{8}
\]

We conclude that the unbounded shaded region in Figure 3 has area \( \frac{1}{8} \).

**EXAMPLE 2** Determine whether \( \int_{1000}^\infty \frac{dx}{x} \) converges.

Solution Since the integration is from 1000 to \( \infty \), we take \( R > 1000 \) and compute the limit as \( R \to \infty \):

\[
\int_{1000}^R \frac{dx}{x} = \lim_{R \to \infty} \int_{1000}^R \frac{dx}{x} = \lim_{R \to \infty} \ln |x| \bigg|_{1000}^R = \lim_{R \to \infty} \ln R - \ln 1000 = \infty
\]

The limit is infinite, so the improper integral diverges. We conclude that the area of the unbounded region in Figure 4 is infinite.

Note that in the previous example, even though \( f(x) = 1/x \) is less than 0.001 in \((1000, \infty)\) and is decreasing to 0, the area under the graph is infinite. Such is the somewhat perplexing nature of improper integrals.

**CONCEPTUAL INSIGHT** If you compare the unbounded shaded regions in Figures 3 and 4, you might wonder why one has finite area and the other has infinite area. Convergence of an improper integral depends on how rapidly \( f(x) \) tends to zero as \( x \to \infty \) (or \( x \to -\infty \)). The previous two examples show that \( f(x) = x^{-3} \) tends to zero quickly enough that the integral converges, whereas \( f(x) = x^{-1} \) does not.

An improper integral of a power function \( f(x) = x^{-p} \) is called a \( p \)-integral. Note that \( f(x) = x^{-p} \) decreases more rapidly as \( p \) gets larger. Interestingly, our next theorem shows that the exponent \( p = -1 \) is the dividing line between convergence and divergence.

**THEOREM 1** The \( p \)-Integral over \([a, \infty)\) For \( a > 0 \),

\[
\int_a^\infty \frac{dx}{x^p} = \begin{cases} 
\frac{a^{1-p}}{1-p} & \text{if } p > 1 \\
\text{diverges} & \text{if } p \leq 1
\end{cases}
\]

**Proof** Denote the \( p \)-integral by \( J \). Then

\[
J = \lim_{R \to \infty} \int_a^R x^{-p} \, dx = \lim_{R \to \infty} \left[ \frac{x^{1-p}}{1-p} \right]_a^R = \lim_{R \to \infty} \left( \frac{R^{1-p}}{1-p} - \frac{a^{1-p}}{1-p} \right)
\]

If \( p > 1 \), then \( 1-p < 0 \) and \( R^{1-p} \) tends to zero as \( R \to \infty \). In this case, \( J = \frac{a^{1-p}}{p-1} \).

If \( p < 1 \), then \( 1-p > 0 \) and \( R^{1-p} \) tends to \( \infty \). In this case, \( J \) diverges. If \( p = 1 \), then \( J \) diverges because

\[
\lim_{R \to \infty} \int_a^R x^{-1} \, dx = \lim_{R \to \infty} (\ln R - \ln a) = \infty
\]
EXAMPLE 3  Gabriel’s Horn is the surface obtained by rotating the graph of \( f(x) = \frac{1}{x} \), for \( x \geq 1 \), about the \( x \)-axis (Figure 5). This surface is interesting because it contains a finite volume but has an infinite surface area. (We verify the latter fact in Section 9.2.) Compute the volume contained in Gabriel’s Horn.

Solution  The volume contained in the horn is the volume obtained by rotating the area under the graph of \( f(x) = \frac{1}{x} \), for \( x > 1 \), about the \( x \)-axis. We find the volume by the Disk Method from Section 6.3 where the radius of the disk is \( r = f(x) = \frac{1}{x} \):

\[
\text{volume} = \pi \int_1^\infty r^2 \, dx = \pi \int_1^\infty \left( \frac{1}{x} \right)^2 \, dx = \lim_{R \to \infty} \pi \int_1^R x^{-2} \, dx
\]

\[
= \lim_{R \to \infty} -\pi x^{-1} \bigg|_1^R = -\pi \lim_{R \to \infty} \left( \frac{1}{R} - 1 \right) = \pi
\]

Therefore, the volume contained in Gabriel’s Horn is \( \pi \). □

A doubly infinite improper integral is defined as a sum (provided that both integrals on the right converge):

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{0} f(x) \, dx + \int_{0}^{\infty} f(x) \, dx
\]

2

We can use some number other than \( 0 \) as a choice of where to split the integral, if it is more convenient to do so.

EXAMPLE 4  Determine if \( \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \) converges and, if so, compute its value.

Solution  

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1 + x^2} \, dx + \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx
\]

assuming both of these integrals converge. For the second of these,

\[
\int_{0}^{\infty} \frac{1}{1 + x^2} \, dx = \lim_{R \to \infty} \int_{0}^{R} \frac{1}{1 + x^2} \, dx = \lim_{R \to \infty} \tan^{-1} x \bigg|_{0}^{R} = \lim_{R \to \infty} \tan^{-1} R - 0 = \frac{\pi}{2}
\]

Similarly,

\[
\int_{-\infty}^{0} \frac{1}{1 + x^2} \, dx = \lim_{R \to \infty} \int_{R}^{0} \frac{1}{1 + x^2} \, dx = \lim_{R \to \infty} \tan^{-1} x \bigg|_{R}^{0} = 0 - \lim_{R \to \infty} \tan^{-1} R = \frac{\pi}{2}
\]

Thus, since both integrals converge,

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1 + x^2} \, dx + \int_{0}^{\infty} \frac{1}{1 + x^2} \, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi
\]

Sometimes it is necessary to use L’Hôpital’s Rule to determine the limits that arise in improper integrals.

EXAMPLE 5  Using L’Hôpital’s Rule  Calculate \( \int_{0}^{\infty} xe^{-x} \, dx \).

Solution  The integral corresponds to the area in Figure 6. First, we compute the associated indefinite integral using Integration by Parts with \( u = x \) and \( dv = e^{-x} \, dx \):

\[
\int xe^{-x} \, dx = -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C
\]

\[
\int_{0}^{R} xe^{-x} \, dx = -(x+1)e^{-x} \bigg|_{0}^{R} = -(R+1)e^{-R} + 1 = 1 - \frac{R+1}{e^R}
\]

\[
\int_{0}^{\infty} xe^{-x} \, dx = \lim_{R \to \infty} \int_{0}^{R} xe^{-x} \, dx = \lim_{R \to \infty} \left( 1 - \frac{R+1}{e^R} \right) = \lim_{R \to \infty} \left( 1 - 1 + \frac{R+1}{e^R} \right) = 1
\]

2
Now, compute the improper integral as a limit using L'Hôpital's Rule:

$$\int_{0}^{\infty} xe^{-x} \, dx = 1 - \lim_{R \to \infty} \frac{R + 1}{e^R} = 1 - \lim_{R \to \infty} \frac{1}{e^R} = 1 - 0 = 1$$

Improper integrals arise in applications when it makes sense to treat certain large quantities as if they were infinite. In the next example, we determine the escape velocity of an object launched from Earth by assuming that the velocity is sufficient to take it "infinitely far" into space.

**EXAMPLE 6 Escape Velocity** The earth exerts a gravitational force of magnitude $F(r) = \frac{GM_em}{r^2}$ on an object of mass $m$ at distance $r$ from the center of the earth.

(a) Find the work required to move the object infinitely far from the earth.

(b) Calculate the escape velocity $v_{esc}$ on the earth's surface.

**Solution** This amounts to computing a $p$-integral with $p = 2$. Recall that work is the integral of force as a function of distance (Section 6.5).

(a) The work required to move an object from the earth's surface ($r = r_e$) to a distance $R$ from the center is

$$\int_{r_e}^{R} \frac{GM_em}{r^2} \, dr = \frac{GM_em}{r_e} |_{r_e}^{R} = GM_em \left(1 - \frac{1}{R} \right) \text{joules}$$

The work moving the object "infinitely far away" is the improper integral

$$GM_em \int_{r_e}^{\infty} \frac{dr}{r^2} = \lim_{R \to \infty} GM_em \left(1 - \frac{1}{R} \right) = \frac{GM_em}{r_e} \text{joules}$$

(b) By the principle of Conservation of Energy, an object launched with velocity $v_0$ will escape the earth's gravitational field if its kinetic energy $\frac{1}{2}mv_0^2$ is at least as large as the work required to move the object to infinity—that is, if

$$\frac{1}{2}mv_0^2 \geq \frac{GM_em}{r_e} \quad \Rightarrow \quad v_0 \geq \left(\frac{2GM_em}{r_e}\right)^{1/2}$$

Using the values recalled in the marginal note, we find that $v_0 \geq 11,200 \text{ m/s}$. The minimal velocity is the escape velocity $v_{esc} = 11,200 \text{ m/s}$.

**EXAMPLE 7 Present Value of Future Income** If an investment pays a dividend continuously at a rate of $R(t)$ $$/year and earns interest at rate $r$ (in decimal form), then the present value of the dividend income, for the period from $t = 0$ to $t = T$, is given by

$$PV = \int_{0}^{T} R(t)e^{-rt} \, dt.$$ 

We think of present value as the payment that we would need to receive at $t = 0$, instead of the dividend income, so that at time $T$ the payment's value (with accumulated interest) would be the same as the amount accumulated from the dividend income (with its accumulated interest). It is essentially the present worth of the income we are about to receive up to time $T$.

Assuming that the dividend rate is $6000$/year, and the interest rate is 4%, compute the present value if the dividends continue forever.

**Solution** Over an infinite time interval,

$$PV = \int_{0}^{\infty} 6000e^{-0.04t} \, dt = \lim_{R \to \infty} \left[ \frac{6000e^{-0.04t}}{-0.04} \right]_{0}^{T} = \frac{6000}{0.04} = \$150,000$$

Although an infinite amount of money is paid out during the infinite time interval, the total present value is finite.
Unbounded Functions

An integral over a finite interval \([a, b]\) is improper if the integrand is unbounded. In this case, the region in question is unbounded in the vertical direction. For example, \(\int_0^b \frac{dx}{\sqrt{x}}\) is improper because the integrand \(f(x) = x^{-1/2}\) tends to \(\infty\) as \(x \to 0^+\) (Figure 7). Improper integrals of this type are defined as one-sided limits.

**Definition Unbounded Integrands** If \(f\) is continuous on \([a, b]\) and \(\lim_{x \to a^+} f(x) = \pm \infty\), we define

\[
\int_a^b f(x) \, dx = \lim_{R \to b^-} \int_a^R f(x) \, dx
\]

Similarly, if \(f\) is continuous on \([a, b]\) and \(\lim_{x \to b^-} f(x) = \pm \infty\),

\[
\int_a^b f(x) \, dx = \lim_{R \to a^+} \int_R^b f(x) \, dx
\]

In both cases, we say that the improper integral converges if the limit exists and that it diverges otherwise.

Note that if there is a single point \(c\) in the interval \([a, b]\) such that \(\lim_{x \to c^-} f(x) = \pm \infty\),

or \(\lim_{x \to c^+} f(x) = \pm \infty\), and if \(\int_a^c f(x) \, dx\) and \(\int_c^b f(x) \, dx\) both converge, then we define

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

**Example 8** Calculate: (a) \(\int_0^9 \frac{dx}{\sqrt{x}}\) and (b) \(\int_0^{1/2} \frac{dx}{x}\).

**Solution** Both integrals are improper because the integrands have infinite discontinuities at \(x = 0\). The first integral converges:

\[
\int_0^9 \frac{dx}{\sqrt{x}} = \lim_{R \to 0^+} \int_R^9 x^{-1/2} \, dx = \lim_{R \to 0^+} 2x^{1/2}\bigg|_R^9 = \lim_{R \to 0^+} (6 - 2R^{1/2}) = 6
\]

The second integral diverges:

\[
\int_0^{1/2} \frac{dx}{x} = \lim_{R \to 0^+} \int_R^{1/2} \frac{dx}{x} = \lim_{R \to 0^+} \left( \ln \frac{1}{2} - \ln R \right)
\]

\[= \ln \frac{1}{2} - \lim_{R \to 0^+} \ln R = \infty\]

**Example 9** Calculate \(\int_0^2 \frac{dx}{(x - 1)^{3/2}}\).

**Solution** This integral is improper with an infinite discontinuity at \(x = 1\) (Figure 8). Therefore, we write

\[
\int_0^2 \frac{dx}{(x - 1)^{3/2}} = \int_0^1 \frac{dx}{(x - 1)^{3/2}} + \int_1^2 \frac{dx}{(x - 1)^{3/2}}
\]
We consider each integral individually:

\[
\int_0^1 \frac{dx}{(x-1)^{\frac{3}{2}}} = \lim_{R \to 1^-} \int_0^R \frac{dx}{(x-1)^{\frac{3}{2}}} = \lim_{R \to 1^-} 3(x-1)^{\frac{1}{2}} \bigg|_0^R \\
= \lim_{R \to 1^-} 3(R-1)^{\frac{1}{2}} - 3(-1)^{\frac{1}{2}} = 3
\]

\[
\int_1^2 \frac{dx}{(x-1)^{\frac{3}{2}}} = \lim_{R \to 1^-} \int_1^R \frac{dx}{(x-1)^{\frac{3}{2}}} = \lim_{R \to 1^-} 3(x-1)^{\frac{1}{2}} \bigg|_1^R \\
= 3(1)^{\frac{1}{2}} - \lim_{R \to 1^-} 3(R-1)^{\frac{1}{2}} = 3
\]

Therefore, we obtain

\[
\int_0^2 \frac{dx}{(x-1)^{\frac{3}{2}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{3}{2}}} + \int_1^2 \frac{dx}{(x-1)^{\frac{3}{2}}} = 3 + 3 = 6
\]

The proof of the next theorem is similar to the proof of Theorem 1 (see Exercise 52).

**Theorem 2** The $p$-Integral over $[0, a]$ For $a > 0$,

\[
\int_0^a \frac{dx}{x^p} = \begin{cases} 
\frac{a^{1-p}}{1-p} & \text{if } p < 1 \\
\text{diverges} & \text{if } p \geq 1
\end{cases}
\]

**Graphical Insight** The $p$-integrals \( \int_1^\infty x^{-p} \, dx \) and \( \int_0^1 x^{-p} \, dx \) have opposite behavior for $p \neq 1$. The first converges only for $p > 1$, and the second converges only for $p < 1$ (both diverge for $p = 1$). This is reflected in the graphs of $y = x^{-p}$ (with $p > 1$) and $y = x^{-q}$ (with $0 < q < 1$) in Figure 9.

Since $0 < q < 1$, the values of $f(x) = x^{-q}$ are arbitrarily large near $x = 0$ and decrease rapidly as $x$ increases to 1, thereby resulting in the divergence of \( \int_1^\infty x^{-q} \, dx \). However, $f(x) = x^{-q}$ decreases slowly to 0 as $x \to \infty$, resulting in the divergence of \( \int_0^1 x^{-q} \, dx \).

The graphs for $q < 1$ and $p > 1$ switch relative positions and behaviors at the point of intersection $(1, 1)$.

Since $p > 1$, the values of $f(x) = x^{-p}$ are arbitrarily large near $x = 0$, but decrease slowly as $x \to 1$, resulting in the divergence of \( \int_0^1 x^{-p} \, dx \). On the other hand, with $p > 1$, $f(x) = x^{-p}$ decreases rapidly to 0 as $x \to \infty$, resulting in the convergence of \( \int_1^\infty x^{-p} \, dx \).

**Comparing Integrals**

Sometimes we are interested in determining whether an improper integral converges, even if we cannot find its exact value. For instance, the integral

\[
\int_1^\infty \frac{e^{-x}}{x} \, dx
\]
cannot be evaluated explicitly. However, if \( x \geq 1 \), then

\[
0 \leq \frac{1}{x} \leq 1 \quad \Rightarrow \quad 0 \leq \frac{e^{-x}}{x} \leq e^{-x}
\]

In other words, the graph of \( y = e^{-x}/x \) lies underneath the graph of \( y = e^{-x} \) for \( x \geq 1 \) (Figure 10). Therefore,

\[
0 \leq \int_{1}^{\infty} \frac{e^{-x}}{x} \, dx \leq \int_{1}^{\infty} e^{-x} \, dx = e^{-1}
\]

Converges by direct computation.

Since the larger integral converges, we can expect that the smaller integral also converges (and that its value is some positive number less than \( e^{-1} \)). This type of conclusion is stated in the next theorem. A proof is provided in a supplement on the text's Web site.

**THEOREM 3 Comparison Test for Improper Integrals**

Assume that \( f \) and \( g \) are continuous functions such that \( f(x) \geq g(x) \geq 0 \) for \( x \geq a \):

- If \( \int_{a}^{\infty} f(x) \, dx \) converges, then \( \int_{a}^{\infty} g(x) \, dx \) also converges.
- If \( \int_{a}^{\infty} g(x) \, dx \) diverges, then \( \int_{a}^{\infty} f(x) \, dx \) also diverges.

The Comparison Test is also valid for improper integrals of unbounded functions on a finite interval.

**EXAMPLE 10** Show that \( \int_{1}^{\infty} \frac{dx}{\sqrt{x^3 + 1}} \) converges.

**Solution** We cannot evaluate this integral, but we can use the Comparison Test. To show convergence, we must compare the integrand \((x^3 + 1)^{-1/2}\) with a larger function whose integral we can compute.

It makes sense to compare with \( x^{-3/2} \) because \( \sqrt{x^3} \leq \sqrt{x^3 + 1} \). Therefore,

\[
\frac{1}{\sqrt{x^3 + 1}} \leq \frac{1}{\sqrt{x^3}} = x^{-3/2}
\]

The integral of the larger function converges, so the integral of the smaller function also converges:

\[
\int_{1}^{\infty} \frac{dx}{x^{3/2}} \quad \text{converges} \quad \Rightarrow \quad \int_{1}^{\infty} \frac{dx}{\sqrt{x^3 + 1}} \quad \text{converges}
\]

**EXAMPLE 11 Choosing the Right Comparison** Does \( \int_{1}^{\infty} \frac{dx}{\sqrt{x + e^{3x}}} \) converge?

**Solution** Since \( \sqrt{x} \geq 0 \), we have \( \sqrt{x + e^{3x}} \geq e^{3x} \), and therefore,

\[
\frac{1}{\sqrt{x + e^{3x}}} \leq \frac{1}{e^{3x}}
\]

Furthermore,

\[
\int_{1}^{\infty} \frac{dx}{e^{3x}} = \lim_{R \to \infty} \frac{-1}{3} e^{-3x} \Big|_{1}^{R} = \lim_{R \to \infty} \frac{1}{3} (e^{-3} - e^{-3R}) = \frac{1}{3} e^{-3} \quad \text{(converges)}
\]
Our integral converges by the Comparison Test:

\[
\int_1^\infty \frac{dx}{e^{3x}} \text{ converges } \Rightarrow \int_1^\infty \frac{dx}{\sqrt{x} + e^{3x}} \text{ also converges}
\]

Integral of larger function

Integral of smaller function

Had we not been thinking, we might have tried to use the inequality

\[
\frac{1}{\sqrt{x} + e^{3x}} \leq \frac{1}{\sqrt{x}}
\]

However, \( \int_1^\infty \frac{dx}{\sqrt{x}} \) diverges (\( p \)-integral with \( p < 1 \)), and this says nothing about the integral of the smaller function (Figure 11).

**EXAMPLE 12 Endpoint Discontinuity**

Does \( \int_0^{0.5} \frac{dx}{x^8 + x^2} \) converge?

**Solution**

This integrand has a discontinuity at \( x = 0 \), since \( \lim_{x \to 0^+} \frac{1}{x^8 + x^2} = +\infty \).

We might try the comparison

\[
x^8 + x^2 > x^2 \quad \Rightarrow \quad \frac{1}{x^8 + x^2} < \frac{1}{x^2}
\]

However, the \( p \)-integral \( \int_0^{0.5} \frac{dx}{x^2} \) diverges, so this says nothing about the integral involving the smaller function. But notice that if \( 0 < x < 0.5 \), then \( x^8 < x^2 \), and therefore,

\[
x^8 + x^2 < 2x^2 \quad \Rightarrow \quad \frac{1}{x^8 + x^2} > \frac{1}{2x^2}
\]

Since \( \int_0^{0.5} \frac{dx}{2x^2} \) diverges, \( \int_0^{0.5} \frac{dx}{x^8 + x^2} \) also diverges.

**8.7 SUMMARY**

- An improper integral is defined as the limit of definite integrals:

\[
\int_a^\infty f(x) \, dx = \lim_{R \to \infty} \int_a^R f(x) \, dx
\]

The improper integral converges if this limit exists, and it diverges otherwise.

- If \( f \) is continuous on \([a, b] \) and \( \lim_{x \to b^-} f(x) = \pm\infty \), then

\[
\int_a^b f(x) \, dx = \lim_{R \to b^-} \int_a^R f(x) \, dx
\]

- If \( f \) is continuous on \([a, b] \) and \( \lim_{x \to c^-} f(x) = \pm\infty \) or \( \lim_{x \to c^+} f(x) = \pm\infty \), where \( a < c < b \) and if \( \int_a^c f(x) \, dx \) and \( \int_c^b f(x) \, dx \) converge, then

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

- An improper integral of \( x^{-p} \) is called a \( p \)-integral. For \( a > 0 \),
The Comparison Test: Assume that $f$ and $g$ are continuous functions such that $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then

- If $\int_a^\infty f(x) \, dx$ converges, then $\int_a^\infty g(x) \, dx$ converges.

- If $\int_a^\infty g(x) \, dx$ diverges, then $\int_a^\infty f(x) \, dx$ diverges.

- Remember that the Comparison Test provides no information if the integral of the larger function diverges or if the integral of the smaller function converges.

- The Comparison Test is also valid for improper integrals of functions with infinite discontinuities at an endpoint of the integral.

### 8.7 Exercises

#### Preliminary Questions

1. State whether each of the following integrals converges or diverges:
   
   (a) $\int_1^\infty \frac{1}{x^3} \, dx$
   
   (b) $\int_0^1 x^{-3} \, dx$
   
   (c) $\int_1^\infty x^{-2/3} \, dx$
   
   (d) $\int_1^\infty x^{-2/3} \, dx$

2. Is $\int_0^{\pi/2} \cot x \, dx$ an improper integral? Explain.

3. Find a value of $b > 0$ that makes $\int_0^b \frac{1}{x^{1/4}} \, dx$ an improper integral.

4. Which comparison would show that $\int_0^\infty \frac{dx}{x + e^x}$ converges?

5. Explain why it is not possible to draw any conclusions about the convergence of $\int_1^\infty \frac{e^{-x}}{x} \, dx$ by comparing with the integral $\int_1^\infty \frac{dx}{x}$.

#### Exercises

1. Which of the following integrals is improper? Explain your answer, but do not evaluate the integral.
   
   (a) $\int_0^1 \frac{dx}{x^{1/3}}$
   
   (b) $\int_1^\infty \frac{dx}{x^{1/3}}$
   
   (c) $\int_1^\infty e^{-x} \, dx$
   
   (d) $\int_0^1 \frac{dx}{x^{1/3}}$
   
   (e) $\int_0^\infty \sin x \, dx$
   
   (f) $\int_0^1 \frac{dx}{x^{1/3}}$
   
   (g) $\int_0^\infty \ln x \, dx$
   
   (h) $\int_0^\infty \frac{dx}{x^{1/3}}$

2. Let $f(x) = x^{-1/3}$.
   
   (a) Evaluate $\int_1^R f(x) \, dx$.
   
   (b) Evaluate $\int_1^\infty f(x) \, dx$ by computing the limit

   $$\lim_{R \to \infty} \int_1^R f(x) \, dx$$

3. Prove that $\int_1^\infty x^{-2/3} \, dx$ diverges by showing that

   $$\lim_{R \to \infty} \int_1^R x^{-2/3} \, dx = \infty$$

4. Determine whether $\int_1^R \frac{dx}{(3-x)^{3/2}}$ converges by computing

   $$\lim_{R \to 3} \int_0^R \frac{dx}{(3-x)^{3/2}}$$

In Exercises 5–40, determine whether the improper integral converges and, if so, evaluate it.

5. $\int_1^\infty \frac{dx}{x^{1/3}}$

6. $\int_1^\infty \frac{dx}{x^{2/3}}$

7. $\int_{-\infty}^1 e^{-0.001u} \, du$

8. $\int_0^\infty \frac{dt}{t}$
9. \[ \int_{0}^{4} \frac{dx}{\sqrt{4-x}} \]
10. \[ \int_{0}^{5} \frac{dx}{x^{1/2} + 20} \]
11. \[ \int_{0}^{1} \frac{dx}{x^{1/2}} \]
12. \[ \int_{0}^{6} \frac{dx}{x - 5} \]
13. \[ \int_{3}^{\infty} \frac{dx}{x^{1/2}} \]
14. \[ \int_{0}^{\infty} \frac{dx}{(x+1)^{3}} \]
15. \[ \int_{-3}^{\infty} \frac{dx}{x + 4} \]
16. \[ \int_{2}^{\infty} \frac{e^{-3x} \, dx}{x} \]
17. \[ \int_{1}^{\infty} \frac{dx}{x^{3/2}} \]
18. \[ \int_{2}^{\infty} \frac{x^{1/2} \, dx}{x^{1/2}} \]
19. \[ \int_{2}^{\infty} \frac{e^{-3} \, dx}{x} \]
20. \[ \int_{0}^{\infty} e^{\frac{1}{3}x} \, dx \]
21. \[ \int_{0}^{\infty} e^{-\frac{1}{3}x} \, dx \]
22. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2}} \]
23. \[ \int_{0}^{\infty} \frac{dx}{x^{3} - x} \]
24. \[ \int_{0}^{\infty} \frac{dx}{x + 2} \]
25. \[ \int_{0}^{\infty} \frac{dx}{x + x} \]
26. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} - x^{2}} \]
27. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} - x} \]
28. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} - x^{2}} \]
29. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
30. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
31. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
32. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
33. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
34. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
35. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
36. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
37. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
38. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
39. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]
40. \[ \int_{0}^{\infty} \frac{dx}{x^{1/2} + 1} \]

In Exercises 45–48, determine whether the doubly infinite improper integral converges and, if so, evaluate it. Use definition (2).

45. \[ \int_{-\infty}^{\infty} \frac{x \, dx}{1 + x^2} \]
46. \[ \int_{-\infty}^{\infty} \frac{e^{-x^2} \, dx}{1 + x^2} \]
47. \[ \int_{-\infty}^{\infty} \frac{e^{-x^2} \, dx}{1 + x^2} \]
48. \[ \int_{-\infty}^{\infty} \frac{dx}{1 + x^2 + 1} \]

49. Determine whether \[ \int_{1}^{\infty} \frac{dx}{x^{1/2}} \] converges and, if so, to what.

50. Consider the integral \[ \int_{0}^{\infty} x \, dx. \]
(a) Show that it diverges.
(b) Show that \[ \lim_{R \to \infty} \int_{0}^{R} x \, dx \] converges, thereby demonstrating that the definition of \[ \int_{0}^{\infty} f(x) \, dx \] needs to be adhered to carefully.

51. For which values of \( a \) does \[ \int_{0}^{\infty} e^{ax} \, dx \] converge?

52. Show that \[ \int_{1}^{\infty} \frac{dx}{x^p} \] converges if \( p < 1 \) and diverges if \( p \geq 1 \).

53. Sketch the region under the graph of the function \( f(x) = \frac{1}{1 + x^2} \) for \( -\infty < x < \infty \), and show that its area is \( \pi \).

54. Show that \[ \frac{1}{\sqrt{x^2 + 1}} \leq 1 \] for all \( x \), and use this to prove that \[ \int_{1}^{\infty} \frac{dx}{\sqrt{x^2 + 1}} \] converges.

55. Show that \[ \int_{1}^{\infty} \frac{dx}{x^2 + 4} \] converges by comparing with \[ \int_{1}^{\infty} \frac{dx}{x^2} \].

56. Show that \[ \int_{1}^{\infty} \frac{dx}{x^2 + 4} \] converges by comparing with \[ \int_{1}^{\infty} \frac{dx}{x^2} \].

57. Show that \( 0 \leq e^{-x^2} \leq e^{-x^2} \) for \( x \geq 1 \) (Figure 12). Use the Comparison Test to show that \[ \int_{0}^{\infty} e^{-x^2} \, dx \] converges. Hint: It suffices (why?) to make the comparison for \( x \geq 1 \) because \[ \int_{0}^{\infty} e^{-x^2} \, dx = \int_{1}^{\infty} e^{-x^2} \, dx + \int_{1}^{\infty} e^{-x^2} \, dx \]

![Figure 12 Comparison of y = e^{-|x|} and y = e^{-x^2}](image)

58. Prove that \[ \int_{-\infty}^{\infty} e^{-x^2} \, dx \] converges by comparing with \[ \int_{-\infty}^{\infty} e^{-|x|} \, dx \] (Figure 12).

59. Show that \[ \int_{1}^{\infty} \frac{1 - \sin x}{x^2} \, dx \] converges.
60. Let \( a > 0 \). Recall that \( \lim_{x \to \infty} \frac{x^a}{\ln x} = \infty \) (by Exercise 63 in Section 7.5).
(a) Show that \( x^a > 2 \ln x \) for all \( x \) sufficiently large.
(b) Show that \( e^{-x^a} < x^{-2} \) for all \( x \) sufficiently large.
(c) Show that \( \int_0^\infty e^{-x^a} \, dx \) converges.

In Exercises 61–75, use the Comparison Test to determine whether or not the integral converges.

61. \( \int_1^\infty \frac{1}{\sqrt{x^3 + 2}} \, dx \)
62. \( \int_0^\infty \frac{dx}{(x^3 + 2x + 4)^{1/2}} \)
63. \( \int_3^\infty \frac{dx}{\sqrt{x - 1}} \)
64. \( \int_0^\infty \frac{dx}{x^{1/3} + x^3} \)
65. \( \int_1^\infty e^{-(x^a + 1)} \, dx \)
66. \( \int_0^\infty \frac{\sin x}{\sqrt{x}} \, dx \)
67. \( \int_0^\infty e^x \, dx \)
68. \( \int_0^\infty \frac{dx}{x^4 + e^x} \)
69. \( \int_0^\infty \frac{1}{x + 1} \, dx \)
70. \( \oint_0^\infty \frac{dx}{\sinh x} \)
71. \( \int_0^\infty \frac{1}{\sqrt{x^3 + x^2}} \, dx \)
72. \( \int_0^\infty \frac{dx}{\sqrt{x^3 + x^2}} \)
73. \( \int_0^1 \frac{dx}{(x^2 + x^4)^{1/3}} \)
74. \( \int_0^\infty \frac{dx}{(x + x^3)^{1/3}} \)
75. \( \int_0^1 \frac{dx}{x e^x + x^2} \)

Hint for Exercise 74: Show that for \( x \geq 1 \),
\[ \frac{1}{(x + x^2)^{1/3}} \leq \frac{1}{21/2x^{2/3}} \]

Hint for Exercise 75: Show that for \( 0 \leq x \leq 1 \),
\[ \frac{1}{xe^x + x^2} \geq \frac{1}{(e+1)x} \]

76. Use the Comparison Test to determine for what values of \( p \) this integral converges:
\[ \int_1^\infty \frac{dx}{x^p \ln x} \]

77. Consider \( \int_0^\infty \frac{dx}{x^{1/2}(x + 1)} \) as the sum of the two improper integrals
\[ \int_0^1 \frac{dx}{x^{1/2}(x + 1)} + \int_1^\infty \frac{dx}{x^{1/2}(x + 1)} \]
Use the Comparison Test to show that the integral converges.

78. Determine whether \( \int_0^\infty \frac{dx}{x^{1/3}(x + 1)} \) (defined as in Exercise 77) converges.

79. An investment pays a dividend of $250/year continuously forever. If the interest rate is 7%, what is the present value of the entire income stream generated by the investment?

80. An investment is expected to earn profits at a rate of \( 10,000e^{0.02t} \) dollars per year forever. Find the present value of the income stream if the interest rate is 4%.

81. Compute the present value of an investment that generates income at a rate of \( 3000e^{0.03t} \) dollars per year forever, assuming an interest rate of 6%.

82. Find the volume of the solid obtained by rotating the region below the graph of \( y = e^{-x^2} \) about the x-axis for \( 0 \leq x < \infty \).

83. When a capacitor of capacitance \( C \) is charged by a source of voltage \( V \), the power expended at time \( t \) is
\[ P(t) = \frac{V^2}{R} (e^{-t/RC} - e^{-2t/RC}) \]
where \( R \) is the resistance in the circuit. The total energy stored in the capacitor is
\[ W = \int_0^\infty P(t) \, dt \]
Show that \( W = \frac{1}{2} CV^2 \).

84. Let \( f(x) = e^{-0.05x} (1 + \sin x) \).
(a) Obtain a plot of \( f(x) \) for \( 0 \leq x \leq 20 \), and discuss the behavior of the function for positive and increasing \( x \).
(b) \( \int_0^\infty f(x) \, dx \) is the area above the positive x-axis and under the infinitely many “humps” of the graph of \( f \). Compute this area.

85. Compute the volume of the solid obtained by rotating the region below the graph of \( y = e^{-x^2/2} \) about the x-axis for \( -\infty < x < \infty \).

86. For which integers \( p \) does \( \int_0^\infty \frac{dx}{x^{1/2} \ln x^p} \) converge?

87. Conservation of Energy can be used to show that when a mass \( m \) oscillates at the end of a spring with spring constant \( k \), the period of oscillation is
\[ T = \frac{4\sqrt{m}}{\sqrt{E}} \int_0^{2\pi} \frac{dx}{\sqrt{2E - kx^2}} \]
where \( E \) is the total energy of the mass. Show that this is an improper integral with value \( T = 2\pi \sqrt{m/k} \).

In Exercises 88–91, the Laplace transform of a function \( f \) is the function \( \mathcal{L}f(s) \) of the variable \( s \) defined by the improper integral (if it converges):
\[ \mathcal{L}f(s) = \int_0^\infty f(x)e^{-sx} \, dx \]
Laplace transforms are widely used in physics and engineering.

88. Show that if \( f(x) = C \), where \( C \) is a constant, then \( \mathcal{L}f(s) = C/s \) for \( s > 0 \).
89. Show that if \( f(x) = \sin \alpha x \), then \( \mathcal{L}f(s) = \frac{\alpha}{s^2 + \alpha^2} \).
90. Compute \( \mathcal{L}f(s) \), where \( f(x) = e^{-ax} \) and \( s > 0 \).
91. Compute \( \mathcal{L}(f(x)) \), where \( f(x) = \cos \alpha x \) and \( s > 0 \).

92. When a radioactive substance decays, the fraction of atoms present at time \( t \) is \( f(t) = e^{-kt} \), where \( k > 0 \) is the decay constant. It can be shown that the average life of an atom (until it decays) is \( A = \frac{1}{k} \int_0^\infty t f(t) \, dt \). Use Integration by Parts to show that \( A = \frac{1}{k} \int_0^\infty f(t) \, dt \) and compute \( A \). What is the average decay time of radon-222, whose half-life is 3.825 days?

93. Let \( J_n = \int_0^\infty x^n e^{-sx} \, dx \), where \( n \geq 1 \) is an integer and \( s > 0 \). Prove that
\[ J_n = \frac{n}{s} J_{n-1} \]
and \( J_0 = 1/s \). Use this to compute \( J_4 \). Show that \( J_n = n!/s^{n+1} \).
94. Let \( a > 0 \) and \( n > 1 \). Define \( f(x) = \frac{x^n}{e^{ax} - 1} \) for \( x \neq 0 \) and \( f(0) = 0 \).

(a) Use L'Hôpital's Rule to show that \( f \) is continuous at \( x = 0 \).

(b) Show that \( \int_0^\infty f(x) \, dx \) converges. Hints: Show that \( f(x) \leq 2x^n e^{-ax} \) if \( x \) is large enough. Then use the Comparison Test and Exercise 93.

95. According to Planck's Radiation Law, the amount of electromagnetic energy with frequency between \( v \) and \( v + \Delta v \) that is radiated by a so-called black body at temperature \( T \) is proportional to \( F(v) \Delta v \), where

\[
F(v) = \left( \frac{8\pi \hbar}{c^2} \right) \frac{v^3}{e^{\hbar c/vT} - 1}
\]

where \( \hbar, c, h \) are physical constants. Use Exercise 94 to show that the total radiated energy

\[
E = \int_0^\infty F(v) \, dv
\]

is finite. To derive his law, Planck introduced the quantum hypothesis in 1900, which marked the birth of quantum mechanics.

Further Insights and Challenges

96. Consider \( \int_a^b x^n \ln x \, dx \).

(a) Show that the integral diverges for \( p = -1 \).

(b) Show that if \( p \neq -1 \), then

\[
\int_a^b x^n \ln x \, dx = \frac{x^{n+1}}{n+1} \left( \ln x - \frac{x^{n+1}}{n+1} \right) + C
\]

(c) Use L'Hôpital's Rule to show that the integral converges if \( p > -1 \) and diverges if \( p < -1 \).

97. Let

\[
F(x) = \int_2^x \frac{dt}{\ln t} \quad \text{and} \quad G(x) = \frac{x}{\ln x}
\]

Verify that L'Hôpital's Rule applies to the limit \( L = \lim_{x \to \infty} \frac{F(x)}{G(x)} \) and evaluate \( L \).

In Exercises 98–100, an improper integral \( \int_a^b f(x) \, dx \) is called absolutely convergent if \( \int_a^b |f(x)| \, dx \) converges. It can be shown that if an integral is absolutely convergent, then it is convergent.

98. Show that \( \int_1^\infty \sin x \, dx \) is absolutely convergent.

99. Show that \( \int_1^\infty e^{-x} \, dx \) is absolutely convergent.

100. Let \( f(x) = \sin x / x \) and consider \( \int_0^\infty f(x) \, dx \). We define \( f(0) = 1 \). Then \( f \) is continuous and the integral is not improper at \( x = 0 \).

(a) Show that

\[
\int_0^\infty \sin x / x \, dx = -\cos x \bigg|_0^\infty - \int_0^\infty \cos x / x \, dx
\]

(b) Show that \( \int_0^\infty (\cos x / x^2) \, dx \) converges. Conclude that the limit as \( R \to \infty \) of the integral in (a) exists and is finite.

(c) Show that \( \int_0^\infty f(x) \, dx \) converges.

FIGURE 1 Areas under the graph of \( y = e^{-x^2/2} \) are approximated using numerical integration.

\[
F(v) = \left( \frac{8\pi \hbar}{c^2} \right) \frac{v^3}{e^{\hbar c/vT} - 1}
\]

It is known that \( f = \frac{1}{2} \). However, the integral is not absolutely convergent. The convergence depends on cancellation, as shown in Figure 13.

101. The gamma function, which plays an important role in advanced applications, is defined for \( n \geq 1 \) by

\[
\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} \, dt
\]

(a) Show that the integral defining \( \Gamma(n) \) converges for \( n \geq 1 \) (it actually converges for all \( n > 0 \)). Hints: Show that \( t^{n-1} e^{-t} < t^{-2} \) for \( t \) sufficiently large.

(b) Show that \( \Gamma(n + 1) = n \Gamma(n) \) using Integration by Parts.

(c) Show that \( \Gamma(n + 1) = n! \) if \( n \geq 1 \) is an integer. Hints: Use (b) repeatedly. Thus, \( \Gamma(n) \) provides a way of defining \( n \)-factorial when \( n \) is not an integer.

102. Use the results of Exercise 101 to show that the Laplace transform (see Exercises 88–91 above) of \( x^n \) is \( \frac{n!}{s^{n+1}} \).

8.8 Numerical Integration

Numerical integration is the process of approximating a definite integral using well-chosen sums of function values. It is needed when we cannot find an antiderivative explicitly, as in the case of the Gaussian function \( f(x) = e^{-x^2/2} \) (Figure 1).

In Section 5.1, we saw that we can approximate a definite integral by splitting the interval of integration \([a, b]\) into \( N \) subintervals, each of size \( \Delta x \). Then we take the value of the function at each left-hand endpoint, multiply that by the width of the interval \( \Delta x \), and sum over the intervals. This approximation is known as the left-endpoint approximation. Similarly, we saw a right-hand approximation. The third method that we introduced
in that section, which we reconsider here, uses the midpoints of the intervals, and usually gives a better approximation.

The Midpoint Rule

To approximate the definite integral \( \int_{a}^{b} f(x) \, dx \), we fix a whole number \( N \) and divide \([a, b]\) into \( N \) subintervals of length \( \Delta x = (b - a) / N \). The endpoints of the subintervals are

\[
x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_N = b
\]

The midpoint approximation \( M_N \) is the sum of the areas of the rectangles of height \( f(c_j) \) and base \( \Delta x \), where \( c_j \) is the midpoint of the interval \([x_{j-1}, x_j]\) [Figure 2(A)]. Note that the midpoint of \([x_{j-1}, x_j]\) is \( \frac{x_{j-1} + x_j}{2} = a + (j - \frac{1}{2}) \Delta x \), and therefore \( c_j = a + (j - \frac{1}{2}) \Delta x \).

**Midpoint Rule**

The \( N \)th midpoint approximation to \( \int_{a}^{b} f(x) \, dx \) is

\[
M_N = \Delta x \left( f(c_1) + f(c_2) + \cdots + f(c_N) \right)
\]

where \( \Delta x = \frac{b - a}{N} \) and \( c_j = a + (j - \frac{1}{2}) \Delta x \) is the midpoint of \([x_{j-1}, x_j]\).

**GRAPHICAL INSIGHT** \( M_N \) has a second interpretation as the sum of the areas of tangential trapezoids—that is, trapezoids whose top edges are tangent to the graph of \( f \) at the midpoints \( c_j \) [Figure 2(B)]. The trapezoids have the same area as the rectangles because the top edge of the trapezoid passes through the midpoint of the top edge of the rectangle, as shown in Figure 3.

With subinterval endpoints \( a = x_0, x_1, x_2, \ldots, x_N = b \), for a function \( f \), we denote the function value \( f(x_j) \) by \( y_j \) throughout the remainder of the section.

The Trapezoidal Rule

The **Trapezoidal Rule** \( T_N \) approximates \( \int_{a}^{b} f(x) \, dx \) by the area of the trapezoids obtained by joining the points \((x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)\) with line segments, as in Figure 4. The area of the \( j \)th trapezoid is \( \frac{1}{2} \Delta x (y_{j-1} + y_j) \), and therefore,
\[ T_N = \frac{1}{2} \Delta x(y_0 + y_1) + \frac{1}{2} \Delta x(y_1 + y_2) + \cdots + \frac{1}{2} \Delta x(y_{N-1} + y_N) \]
\[ = \frac{1}{2} \Delta x \left( (y_0 + y_1) + (y_1 + y_2) + \cdots + (y_{N-1} + y_N) \right) \]

Note that each value \( y_i \) occurs twice except for \( y_0 \) and \( y_N \), so we obtain
\[ T_N = \frac{1}{2} \Delta x \left( y_0 + 2y_1 + 2y_2 + \cdots + 2y_{N-1} + y_N \right) \]

\( T_N \) approximates the integral for any \( f \) that is integrable over \([a, b] \), so we have the following integral-approximation rule:

**Trapezoidal Rule** The \( N \)th trapezoidal approximation to \( \int_a^b f(x) \, dx \) is
\[ T_N = \frac{1}{2} \Delta x \left( y_0 + 2y_1 + \cdots + 2y_{N-1} + y_N \right) \]

where \( \Delta x = \frac{b-a}{N} \) and \( y_j = f(x_j) \).

**CONCEPTUAL INSIGHT** We see in Figure 5 that the area of the \( j \)th trapezoid is equal to the average of the areas of the endpoint rectangles with heights \( y_{j-1} \) and \( y_j \). It follows that \( T_N \) is equal to the average of the right- and left-endpoint approximations \( R_N \) and \( L_N \) introduced in Section 5.1:
\[ T_N = \frac{1}{2} (R_N + L_N) \]

In general, this average is a better approximation than either \( R_N \) alone or \( L_N \) alone.

**EXAMPLE 1** **CAS** Calculate \( T_8 \) for the integral \( \int_1^3 \sin(x^2) \, dx \). Then use a computer algebra system to calculate \( T_N \) for \( N = 50, 100, 500, 1000, \) and \( 10,000 \).

**Solution** Divide \([1, 3]\) into \( N = 8 \) subintervals of length \( \Delta x = \frac{3-1}{8} = \frac{1}{4} \). Then sum the function values at the endpoints (Figure 6) with the appropriate coefficients:
\[ T_8 = \frac{1}{2} \left( \frac{1}{4} \right) \left[ \sin(1^2) + 2 \sin(1.25^2) + 2 \sin(1.5^2) + 2 \sin(1.75^2) \right. \]
\[ + 2 \sin(2^2) + 2 \sin(2.25^2) + 2 \sin(2.5^2) + 2 \sin(2.75^2) + \sin(3^2) \left. \right] \]
\[ \approx 0.4281 \]

In general, \( \Delta x = (3-1)/N = 2/N \) and \( x_j = 1 + 2j/N \). In summation notation,
\[ T_N = \frac{1}{2} \left( \frac{2}{N} \right) \left[ \sin(1^2) + 2 \sum_{j=1}^{N-1} \sin \left( \left( 1 + \frac{2j}{N} \right)^2 \right) + \sin(3^2) \right] \]

We evaluate the inner sum on a CAS. The results in Table 1 suggest that \( \int_1^3 \sin(x^2) \, dx \) is approximately 0.4633.

**TABLE 1**

| \( N \) | \( T_N \)  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.4624205</td>
</tr>
<tr>
<td>100</td>
<td>0.4630759</td>
</tr>
<tr>
<td>500</td>
<td>0.4632855</td>
</tr>
<tr>
<td>1000</td>
<td>0.4632920</td>
</tr>
<tr>
<td>10,000</td>
<td>0.4632942</td>
</tr>
</tbody>
</table>
Error Bounds

In applications, it is important to know the accuracy of a numerical approximation. We define the error in $M_N$ and $T_N$ by

$$\text{error}(M_N) = \left| \int_a^b f(x) \, dx - M_N \right|, \quad \text{error}(T_N) = \left| \int_a^b f(x) \, dx - T_N \right|$$

According to the next theorem, the magnitudes of these errors are related to the size of the second derivative $f''(x)$. A proof of Theorem 1 is provided in a supplement on the text's Web site.

**THEOREM 1** Error Bound for $M_N$ and $T_N$. Assume $f''$ exists and is continuous. Let $K_2$ be a number such that $|f''(x)| \leq K_2$ for all $x$ in $[a, b]$. Then

$$\text{error}(M_N) \leq \frac{K_2(b - a)^3}{24N^2}, \quad \text{error}(T_N) \leq \frac{K_2(b - a)^3}{12N^2}$$

**GRAPHICAL INSIGHT** Note that the Error Bound for $M_N$ is one-half of the Error Bound for $T_N$, suggesting that $M_N$ is generally more accurate than $T_N$. Why do both Error Bounds depend on $f''(x)$? The second derivative measures concavity, so if $f''(x)$ is large, then the graph of $f$ bends a lot and trapezoids do a poor job of approximating the region under the graph. Under such a circumstance, the errors in both $T_N$ and $M_N$ (which uses tangential trapezoids) are likely to be large (Figure 7).

**EXAMPLE 2** Checking the Error Bound

Consider the integral $\int_1^4 \sqrt{x} \, dx$.

(a) Calculate $M_6$ and $T_6$ for the integral.
(b) Calculate the Error Bounds.
(c) Calculate the integral exactly and verify that the Error Bounds are satisfied.

**Solution**

(a) Divide $[1, 4]$ into six subintervals of width $\Delta x = \frac{4 - 1}{6} = \frac{1}{2}$. Using the endpoints and midpoints shown in Figure 8, we obtain

$$M_6 = \frac{1}{2} \left( \sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75} \right) \approx 4.669245$$

$$T_6 = \frac{1}{2} \left( \frac{1}{2} \right) \left( \sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4} \right) \approx 4.661488$$

**FIGURE 7** $M_N$ and $T_N$ are more accurate when $|f''(x)|$ is small.

**FIGURE 8** Interval $[1, 4]$ divided into $N = 6$ subintervals.
(b) Let \( f(x) = \sqrt{x} \). We must find a number \( K_2 \) such that \(|f''(x)| \leq K_2\) for \( 1 \leq x \leq 4 \). We have \( f''(x) = -\frac{1}{4}x^{-3/2} \). The absolute value \(|f''(x)| = \frac{1}{4}x^{-3/2}\) is decreasing on \([1, 4]\), so its maximum occurs at \( x = 1 \) (Figure 9). Thus, we may take \( K_2 = |f''(1)| = \frac{1}{4} \).

By Theorem 1,

\[
\text{error}(M_6) \leq \frac{K_2(b-a)^3}{24N^2} = \frac{\frac{1}{4}(4-1)^3}{24(6)^2} = \frac{1}{128} \approx 0.0078
\]

\[
\text{error}(T_6) \leq \frac{K_2(b-a)^3}{12N^2} = \frac{\frac{1}{4}(4-1)^3}{12(6)^2} = \frac{1}{64} \approx 0.0156
\]

(c) The exact value is

\[
\int_1^4 \sqrt{x} \, dx = \left. \frac{2}{3}x^{3/2} \right|_1^4 = \frac{14}{3},
\]

so the actual errors are

\[
\text{error}(M_6) \approx \left| \frac{14}{3} - 4.669245 \right| = 0.00258 \quad \text{(less than Error Bound 0.0078)}
\]

\[
\text{error}(T_6) \approx \left| \frac{14}{3} - 4.661488 \right| = 0.00518 \quad \text{(less than Error Bound 0.0156)}
\]

The actual errors are less than the Error Bound, so Theorem 1 is verified.

The Error Bound can be used to determine values of \( N \) so that \( M_N \) or \( T_N \) approximates an integral to a given accuracy.

**EXAMPLE 3 Obtaining the Desired Accuracy** Find \( N \) such that \( T_N \) approximates \( \int_0^3 e^{-x^2} \, dx \) with an error of at most \( 10^{-4} \). Then, for this value of \( N \), use technology to determine \( T_N \) and discuss the resulting integral approximation.

**Solution** Let \( f(x) = e^{-x^2} \). To apply the Error Bound, we must find a number \( K_2 \) such that \(|f''(x)| \leq K_2\) for all \( x \in [0, 3] \). We have \( f'(x) = -2xe^{-x^2} \) and \( f''(x) = (4x^2 - 2)e^{-x^2} \).

A graphing utility was used to plot \( f'' \) (Figure 10). The graph shows that the maximum value of \(|f''(x)|\) on \([0, 3]\) is \(|f''(0)| = |2| = 2\), so we take \( K_2 = 2 \) in the Error Bound:

\[
\text{error}(T_N) \leq \frac{K_2(b-a)^3}{12N^2} = \frac{2(3-0)^3}{12N^2} = \frac{9}{2N^2}
\]

The error is at most \( 10^{-4} \) if

\[
\frac{9}{2N^2} \leq 10^{-4} \quad \Rightarrow \quad N^2 \geq 9 \times 10^4 \quad \Rightarrow \quad N \geq \frac{300}{\sqrt{2}} \approx 212.1
\]

We conclude that \( T_{213} \) has an error of at most \( 10^{-4} \).

A CAS shows that \( T_{213} \approx 0.8862 \). Since the error is at most \( 10^{-4} \), we can infer that the actual value lies between 0.8861 and 0.8863, and therefore, \( \int_0^3 e^{-x^2} \, dx \approx 0.886 \), accurate to three decimal places.

**Simpson's Rule**

As we have seen, the Midpoint Rule uses trapezoids that are tangent to the curve to approximate the area under the curve. The Trapezoidal Rule uses trapezoids with vertices on the curve to approximate the area. In both cases, the top edge of each trapezoid is a line segment. One might wonder whether we could do better using some other curve at the top of each region. In Simpson's Rule, we replace the line segments with parabolas, allowing us to obtain an approximation that is usually substantially more accurate.

To begin, we again subdivide \([a, b]\) into \( N \) subintervals, each of length \( \Delta x = \frac{b-a}{N} \). However, we require \( N \) to be even. Then we pair up the resulting intervals, \([x_0, x_1]\) with
[x_1, x_2], [x_2, x_3] with [x_3, x_4], and so on. For each pair of intervals, we find a parabola that passes through the three points on the curve associated with the endpoints of the two intervals, as in Figure 11. Then we take the sum of the areas that are under the parabolas and over the corresponding intervals to approximate the area under the curve.

![Figure 11](image)

**Figure 11** Approximating the area under the curve using parabolas (blue, through P_0, P_1, P_2; green, through P_2, P_3, P_4; orange, through P_4, P_5, P_6).

Our goal is to develop an approximation formula that, like the Trapezoidal Rule, expresses the approximation in terms of the function values, y_i = f(x_i), at the subinterval endpoints. We begin with the case of a pair of intervals [−Δx, 0] and [0, Δx] centered around the origin. We assume that the corresponding three points on the curve are P_0(−Δx, y_0), P_1(0, y_1), and P_2(Δx, y_2). See Figure 12.

Recall that the general equation for a parabola is y = Cx^2 + Dx + E for constants C, D, and E. The area that is under the parabola and above the two intervals [−Δx, 0] and [0, Δx] is obtained by integrating:

\[
\text{area} = \int_{−Δx}^{Δx} Cx^2 + Dx + E \, dx = \frac{Cx^3}{3} + \frac{Dx^2}{2} + Ex \bigg|_{−Δx}^{Δx}
\]

\[\text{area} = 2 \left( \frac{C(Δx)^3}{3} + EΔx \right) = \frac{Δx}{3} (2C(Δx)^2 + 6E)\]

We want to express this area in terms of the values y_0, y_1, and y_2. Because the parabola must pass through the three points P_0, P_1, and P_2, we know the coordinates of each must satisfy the equation of the parabola. Hence, we obtain the three equations:

\[y_0 = C(Δx)^2 - DΔx + E\]
\[y_1 = E\]
\[y_2 = C(Δx)^2 + DΔx + E\]

Multiplying the middle equation through by 4 and then adding the three equations yields

\[y_0 + 4y_1 + y_2 = 2C(Δx)^2 + 6E\]

Thus, from Eq. (2) it follows that the area under the parabola is \(\frac{Δx}{3}(y_0 + 4y_1 + y_2)\).

This area depends only on the y-coordinates of the three points, so we obtain a similar expression if we site the parabola over any of the subsequent pairs of adjacent subintervals. Therefore, we can approximate the area under the curve by

\[\frac{Δx}{3}(y_0 + 4y_1 + y_2) + \frac{Δx}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{Δx}{3}(y_{N−2} + 4y_{N−1} + y_N)\]

Simplifying, we obtain the following approximation formula that is valid for any function f that is integrable over [a, b]:

\[\int_{a}^{b} f(x) \, dx \approx \frac{Δx}{3}(y_0 + 4y_1 + y_2) + \frac{Δx}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{Δx}{3}(y_{N−2} + 4y_{N−1} + y_N)\]
**Simpson’s Rule** For \( N \) even, the \( N \)th approximation to \( \int_a^b f(x) \, dx \) by Simpson’s Rule is

\[
S_N = \frac{1}{3} \Delta x \left[ y_0 + 4y_1 + 2y_2 + \cdots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N \right]
\]

where \( \Delta x = \frac{b-a}{N} \) and \( y_j = f(x_j) \).

As mentioned previously, we derived the approximation rules in this section in the case \( f(x) \geq 0 \) in order to make it easier to picture the computations in terms of area. The Midpoint Rule, the Trapezoidal Rule, and Simpson’s Rule apply to any integrable function.

**CONCEPTUAL INSIGHT** Comparing Simpson’s Rule to the Midpoint Rule and the Trapezoidal Rule, we see that Simpson’s Rule is a linear combination of the other two rules. That is to say, Simpson’s Rule is given by \( S_N = \frac{1}{3} M_{N/2} + \frac{1}{2} T_{N/2} \). When a function is always concave up or always concave down, the value of the actual integral is sandwiched between \( M_{N/2} \) and \( T_{N/2} \). So a linear combination of the two should do better in this and many other cases. That \( M_{N/2} \) is more heavily weighted in the linear combination is advantageous, as we have seen its Error Bound is half that of \( T_{N/2} \).

**EXAMPLE 4** Use Simpson’s Rule with \( N = 8 \) to approximate \( \int_2^4 \sqrt{1 + x^3} \, dx \).

**Solution** We have \( \Delta x = \frac{4-2}{8} = \frac{1}{4} \). Figure 13 shows the endpoints and coefficients needed to compute \( S_8 \) using Eq. (3):

\[
S_8 = \frac{1}{3} \left( \frac{1}{4} \right) \left[ \sqrt{1 + 2^3} + 4\sqrt{1 + 2.5^3} + 2\sqrt{1 + 3^3} + 4\sqrt{1 + 3.75^3} + 2\sqrt{1 + 4^3} \\
+ 4\sqrt{1 + 3.25^3} + 2\sqrt{1 + 3.5^3} + 4\sqrt{1 + 3.75^3} + 2\sqrt{1 + 4^3} \right] \\
\approx \frac{1}{12} \left[ 3 + 4(3.52003) + 2(4.07738) + 4(4.66871) + 2(5.2915) \\
+ 4(5.94375) + 2(6.62382) + 4(7.33037) + 8.06226 \right] \approx 10.74159
\]

**EXAMPLE 5** **Estimating Integrals from Numerical Data** The velocity (in kilometers per hour) of a Piper Cub aircraft traveling due west is recorded every minute during the first 10 minutes after takeoff. Use Simpson’s Rule to estimate the distance traveled.

<table>
<thead>
<tr>
<th>( t ) (min)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ) (km/h)</td>
<td>0</td>
<td>80</td>
<td>100</td>
<td>128</td>
<td>144</td>
<td>160</td>
<td>152</td>
<td>136</td>
<td>128</td>
<td>120</td>
<td>136</td>
</tr>
</tbody>
</table>

**Solution** The distance traveled is the integral of velocity. We convert from minutes to hours because velocity is given in kilometers per hour, and thus we apply Simpson’s Rule, where the number of intervals is \( N = 10 \) and each interval has length \( \Delta t = \frac{10}{60} \) h:

\[
S_{10} = \left( \frac{1}{3} \right) \left( \frac{1}{60} \right) \left[ 0 + 4(80) + 2(100) + 4(128) + 2(144) + 4(160) \\
+ 2(152) + 4(136) + 2(128) + 4(120) + 136 \right] \approx 20.4 \text{ km}
\]

The distance traveled is approximately 20.4 km (Figure 14).

We now state (without proof) the Error Bound for Simpson’s Rule. Set

\[
\text{error}(S_N) = \left| \int_a^b f(x) \, dx - S_N(f) \right|
\]
The error involves the fourth derivative, which we assume exists and is continuous.

**Theorem 2** Error Bound for $S_N$ 
Let $K_4$ be a number such that $|f^{(4)}(x)| \leq K_4$ for all $x \in [a, b]$. Then

$$\text{error}(S_N) \leq \frac{K_4(b - a)^5}{180N^4}$$

**Example 6** Calculate $S_8$ for $\int_1^3 \frac{1}{x^4} \, dx$.

(a) Find a bound for the error in $S_8$.
(b) Find $N$ such that $S_N$ has an error of at most $10^{-6}$. Then, for this value of $N$, use technology to determine $S_N$ and discuss the resulting integral approximation.

**Solution** The width is $\Delta x = \frac{3 - 1}{8} = \frac{1}{4}$ and the endpoints in the partition of $[1, 3]$ are $1, 1.25, 1.5, \ldots, 2.75, 3$. Using Eq. (3) with $f(x) = x^{-4}$, we obtain

$$S_8 = \frac{1}{3} \left( \frac{1}{4} \right) \left[ \frac{1}{1} + \frac{4}{1.25} + \frac{4}{1.75} + \frac{2}{2.25} + \frac{4}{2.75} + \frac{1}{3} \right]$$

$$\approx 1.09873$$

(a) The fourth derivative $f^{(4)}(x) = 24x^{-5}$ is decreasing, so the max of $|f^{(4)}(x)|$ on $[1, 3]$ is $|f^{(4)}(1)| = 24$. Therefore, we use the Error Bound with $K_4 = 24$:

$$\text{error}(S_N) \leq \frac{K_4(b - a)^5}{180N^4} = \frac{24(3 - 1)^5}{180N^4} = \frac{64}{15N^4}$$

(b) The error will be at most $10^{-6}$ if $N$ satisfies

$$\text{error}(S_N) = \frac{64}{15N^4} \leq 10^{-6}$$

In other words,

$$N^4 \geq 10^6 \left( \frac{64}{15} \right) \quad \text{or} \quad N \geq \left( \frac{10^6 - 64}{15} \right)^{1/4} \approx 45.45$$

Thus, we may take $N = 46$. Using technology, we find that $S_{46} \approx 1.098612$, and therefore, we can conclude that $\int_1^3 \frac{1}{x^4} \, dx \approx 1.098612$ with an error less than $10^{-6}$.

The value of the integral is $\ln 3$, and therefore, we have determined $\ln 3 \approx 1.098612$ with an error less than $10^{-6}$.

**8.8 Summary**

- We consider three numerical approximations to $\int_a^b f(x) \, dx$: the Midpoint Rule $M_N$, the Trapezoidal Rule $T_N$, and Simpson's Rule $S_N$ (the latter for $N$ even).

$$M_N = \Delta x (f(e_1) + f(e_2) + \cdots + f(e_N)) \quad (e_j = a + \left( j - \frac{1}{2} \right) \Delta x)$$

$$T_N = \frac{1}{2} \Delta x \left[ y_0 + 2 y_1 + 2 y_2 + \cdots + 2 y_{N-1} + y_N \right]$$

$$S_N = \frac{1}{3} \Delta x \left[ y_0 + 4 y_1 + 2 y_2 + \cdots + 4 y_{N-3} + 2 y_{N-2} + 4 y_{N-1} + y_N \right]$$

where $\Delta x = (b - a)/N$ and $y_j = f(a + j \Delta x)$. 

- Sources: Simpson's Rule provides good approximations, more sophisticated techniques are implemented in computer algebra systems. These techniques are studied in the area of mathematics called numerical analysis.
8.8 EXERCISES

Preliminary Questions
1. What are $T_1$ and $T_2$ for a function on $[0, 2]$ such that $f(0) = 3, f(1) = 4$, and $f(2) = 3$?
2. For which graph in Figure 15 will $T_N$ overestimate the integral? What about $M_N$?

3. How large is the error when the Trapezoidal Rule is applied to a linear function? Explain graphically.
4. What is the maximum possible error if $T_N$ is used to approximate
   \[ \int_a^b f(x) \, dx \]
   where $|f''(x)| \leq 2$ for all $x$.
5. What are the two graphical interpretations of the Midpoint Rule?

Exercises

In Exercises 1–4, calculate $M_N$ and $T_N$ for the value of $N$ indicated and compare with the actual value of the integral.

1. $\int_1^3 x^2 \, dx$, $N = 4$
2. $\int_0^4 \sqrt{x} \, dx$, $N = 4$
3. $\int_0^3 x^3 \, dx$, $N = 6$
4. $\int_0^2 e^x \, dx$, $N = 6$

In Exercises 5–12, calculate $M_N$ and $T_N$ for the value of $N$ indicated.

5. $\int_1^2 \frac{dx}{x}$, $N = 6$
6. $\int_1^2 \sqrt{x^4 + 1} \, dx$, $N = 5$
7. $\int_0^{\pi/2} \sqrt{\sin x} \, dx$, $N = 6$
8. $\int_0^{3/4} \sec^2 x \, dx$, $N = 6$
9. $\int_0^1 \ln x \, dx$, $N = 5$
10. $\int_1^2 \frac{dx}{\ln x}$, $N = 5$
11. $\int_0^1 e^{-x} \, dx$, $N = 5$
12. $\int_{-2}^1 e^{3x} \, dx$, $N = 6$

In Exercises 13–16, calculate $S_N$ given by Simpson's Rule for the value of $N$ indicated and compare with the actual value of the integral.

13. $\int_0^1 x^2 \, dx$, $N = 4$
14. $\int_0^4 \sqrt{x} \, dx$, $N = 4$
15. $\int_0^3 e^{-x} \, dx$, $N = 6$
16. $\int_0^{\pi/2} \sin x \, dx$, $N = 6$

In Exercises 17–24, calculate $S_N$ given by Simpson's Rule for the value of $N$ indicated.

17. $\int_0^4 \sin(x^2) \, dx$, $N = 0$
18. $\int_0^4 \cos(x^2) \, dx$, $N = 0$
19. $\int_0^4 e^{-x^2} \, dx$, $N = 0$
20. $\int_0^4 e^{-x^2} \, dx$, $N = 0$
21. $\int_0^4 \ln x \, dx$, $N = 0$
22. $\int_0^4 \sqrt{x^4 + 1} \, dx$, $N = 0$
23. $\int_0^4 \tan 3 \theta d\theta$, $N = 0$
24. $\int_0^4 (x^2 + 1)^{1/2} \, dx$, $N = 0$

In Exercises 25–28, calculate the approximation to the volume of the solid obtained by rotating the graph around the given axis.

25. $y = \cos x$; $[0, \frac{\pi}{2}]$; $x$-axis; $M_8$
26. $y = \cos x$; $[0, \frac{\pi}{2}]$; $y$-axis; $S_8$
27. $y = e^{-x^2}$; $[0, 1]$; $x$-axis; $T_8$
28. $y = e^{-x^2}$; $[0, 1]$; $y$-axis; $S_8$

The back of Jon's guitar (Figure 16) is 19 in. long. Jon measured the width at 1-in. intervals, beginning and ending $\frac{1}{2}$ in. from the ends, obtaining the results

6, 9, 10.25, 10.75, 10.75, 10.25, 9.75, 9.5, 10, 11.25, 12.75, 13.75, 14.25, 14.5, 14.5, 14, 13.25, 11.25, 9
Use the Midpoint Rule to estimate the area of the back.

**FIGURE 16** Back of guitar.

30. Use Simpson’s Rule to determine the average temperature in a museum over a 3-hour period if the temperatures (in degrees Celsius), recorded at 15-minute intervals, are:


31. **Tsunami Arrival Times** Scientists estimate the arrival times of tsunamis (seismic ocean waves) based on the point of origin, \( P \), and ocean depths. The speed \( s \) of a tsunami in miles per hour is approximately \( s = \sqrt{15d} \), where \( d \) is the ocean depth in feet.

(a) Let \( f(x) \) be the ocean depth \( x \) miles from \( P \) (in the direction of the coast). Argue using Riemann sums that the time \( T \) required for the tsunami to travel \( M \) miles toward the coast is:

\[
T = \int_0^M \frac{dx}{\sqrt{15f(x)}},
\]

(b) Use Simpson’s Rule to estimate \( T \) if \( M = 1000 \) and the ocean depths (in feet), measured at 100-mile intervals starting from \( P \), are:

- 13000, 11500, 10500, 9000, 8500, 7000, 6000, 4400, 3800, 3200, 2000

32. Use \( S_8 \) to estimate \( \int_0^{\pi/2} \sin x \cos x \, dx \), taking the value of \( \sin x \) at \( x = 0 \) to be 1.

33. Calculate \( T_9 \) for the integral \( I = \int_0^2 x^3 \, dx \).

(a) Is \( T_9 \) too large or too small? Explain graphically.

(b) Show that \( K_2 = |f''(2)| \) may be used in the Error Bound and find a bound for the error.

(c) Evaluate \( I \) and check that the actual error is less than the bound computed in (b).

34. Calculate \( M_9 \) for the integral \( I = \int_0^1 x \sin(x^2) \, dx \).

(a) [CAS] Use a plot of \( f'' \) to show that \( K_2 = 3.2 \) may be used in the Error Bound and find a bound for the error.

(b) [CAS] Evaluate \( I \) numerically and check that the actual error is less than the bound computed in (a).

In Exercises 35–38, state whether \( T_9 \) or \( M_9 \) underestimates or overestimates the integral and find a bound for the error (but do not calculate \( T_9 \) or \( M_9 \)).

35. \( \int_0^1 \frac{1}{x} \, dx \), \( T_{10} \) and \( M_{10} \).

36. \( \int_0^2 e^{-x^4} \, dx \), \( T_{20} \)

37. \( \int_1^a \ln x \, dx \), \( M_{10} \)

38. \( \int_0^{\pi/4} \cos x \, dx \), \( M_{20} \)

39. \( \int_0^1 x^4 \, dx \)

40. \( \int_0^3 (5x^4 - x^5) \, dx \)

41. \( \int_2^3 \frac{1}{x} \, dx \)

42. \( \int_0^3 e^{-x} \, dx \)

43. Compute the Error Bound for the approximations \( T_{10} \) and \( M_{10} \) to \( \int_0^3 (x^3 + 1)^{-1/2} \, dx \), using Figure 17 to determine a value of \( K_2 \). Then find a value of \( N \) such that the error in \( M_N \) is at most \( 10^{-6} \).

**FIGURE 17** Graph of \( f'' \), where \( f(x) = (x^3 + 1)^{-1/2} \).

44. (a) Compute \( S_8 \) for the integral \( I = \int_0^1 e^{-2x} \, dx \).

(b) Show that \( K_4 = 16 \) may be used in the Error Bound and compute the Error Bound.

(c) Evaluate \( I \) and check that the actual error is less than the bound for the error computed in (b).

45. Calculate \( S_6 \) for \( \int_1^5 \ln x \, dx \) and calculate the Error Bound. Then find a value of \( N \) such that \( S_N \) has an error of at most \( 10^{-6} \).

46. Find a bound for the error in the approximation \( S_{10} \) to \( \int_0^3 e^{-x^2} \, dx \) (use Figure 18 to determine a value of \( K_4 \)). Then find a value of \( N \) such that \( S_N \) has an error of at most \( 10^{-6} \).

**FIGURE 18** Graph of \( f^{(4)} \), where \( f(x) = e^{-x^2} \).

47. [CAS] Use a computer algebra system to compute and graph \( f^{(4)} \) for \( f(x) = \sqrt{1 + x^2} \), and find a bound for the error in the approximation \( S_5 \) to \( \int_0^3 f(x) \, dx \).

48. [CAS] Use a computer algebra system to compute and graph \( f^{(4)} \) for \( f(x) = \tan x - \sec x \), and find a bound for the error in the approximation \( S_{10} \) to \( \int_0^{\pi/4} f(x) \, dx \).
In Exercises 49-52, use the Error Bound to find a value of $N$ for which $\text{Error}(S_N) \leq 10^{-3}$.

49. $\int_0^1 x^{4/3} \, dx$

50. $\int_0^4 x^4 \, dx$

51. $\int_0^1 e^{x^2} \, dx$

52. $\int_0^4 \sin(x) \, dx$

53. (CAS) Show that $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$ (use Eq. (3) in Section 7.6).

(a) Use a computer algebra system to graph the function $f(x) = (1 + x^2)^{-1}$ and find its maximum on $[0, 1]$.

(b) Find a value of $N$ such that $S_N$ approximates the integral with an error of at most $10^{-5}$. Calculate the corresponding approximation and confirm that you have computed $\frac{\pi}{4}$ to at least four decimal places.

4.44 Let $J = \int_0^1 e^{-x^2} \, dx$ and $J_N = \int_0^1 e^{-x^2} \, dx$. Although $e^{-x^2}$ has no elementary antiderivative, it is known that $J = \sqrt{\pi}/2$. Let $T_N$ be the $N$th trapezoidal approximation to $J$. Calculate $T_N$ and show that $T_N$ approximates $J$ to three decimal places.

55. Let $f(x) = \sin(x^2)$ and $I = \int_0^1 f(x) \, dx$.

(a) Check that $f''(x) = 2\cos(x^2) - 4x^2 \sin(x^2)$. Then show that $|f''(x)| \leq 6$ for $x \in [0, 1]$. Hint: Note that $|2\cos(x^2)| \leq 2$ and $|4x^2 \sin(x^2)| \leq 4$ for $x \in [0, 1]$.

Further Insights and Challenges

61. Show that if $f(x) = rx + s$ is a linear function ($r, s$ constants), then $T_N = \int_a^b f(x) \, dx$ for all $N$ and all endpoints $a, b$.

62. Show that if $f(x) = px^2 + qx + r$ is a quadratic polynomial, then $S_2 = \int_a^b f(x) \, dx$. In other words, show that

\[
\int_a^b f(x) \, dx = \frac{b-a}{6} (30 + 4y_1 + y_2),
\]

where $y_0 = f(a)$, $y_1 = f\left(\frac{a+b}{2}\right)$, and $y_2 = f(b)$. Hint: Show this first for $f(x) = 1, x, x^2$ and use linearity.

63. For $N$ even, divide $[a, b]$ into $N$ subintervals of width $\Delta x = \frac{b-a}{N}$.

Set $x_j = a + j \Delta x$, $y_j = f(x_j)$, and

\[
S_{2N} = \frac{b-a}{3N} (y_0 + 4y_1 + 4y_2 + \cdots + 4y_{2N-2} + y_{2N}).
\]

(a) Show that $S_N$ is the sum of the approximations on the intervals $[x_j, x_{j+1}]-$that is, $S_N = S_2 + S_4 + \cdots + S_{2N-2}$.

(b) Show that $\text{Error}(S_N)$ is at most $\frac{1}{4N^2}$.

(c) Find an $N$ such that $|I - M_N| \leq 10^{-3}$.

56. (CAS) The Error Bound for $M_N$ is proportional to $1/N^2$, so the Error Bound decreases by $\frac{1}{4}$ if $N$ is increased to $2N$. Compute the actual error in $M_N$ for $\int_0^1 \sin x \, dx$ for $N = 4, 8, 16, 32, 64$. Does the actual error seem to decrease by $\frac{1}{4}$ as $N$ is doubled?

57. (CAS) Observe that the Error Bound for $T_N$ (which has 12 in the denominator) is twice as large as the Error Bound for $M_N$ (which has 24 in the denominator). Compute the actual error in $T_N$ for $\int_0^1 \sin x \, dx$ for $N = 4, 8, 16, 32, 64$ and compare it with the calculations of Exercise 56. Does the actual error in $T_N$ seem to be roughly twice as large as the error in $M_N$ in this case?

58. (CAS) Explain why the Error Bound for $S_N$ decreases by $\frac{1}{4}$ if $N$ is increased to $2N$. Compute the actual error in $S_N$ for $\int_0^1 \sin x \, dx$ for $N = 4, 8, 16, 32, 64$. Does the actual error seem to decrease by $\frac{1}{4}$ as $N$ is doubled?

59. Verify that $S_2$ yields the exact value of $\int_0^1 (x - x^2) \, dx$.

60. Verify that $S_2$ yields the exact value of $\int_a^b (x - x^2) \, dx$ for all $a < b$.

61. Show that if $f(x) = ax + b$ is a linear function ($a, b$ constants), then $S_N = \int_a^b f(x) \, dx$ for all $N$ and all endpoints $a, b$.

62. Show that if $f(x) = px^2 + qx + r$ is a quadratic polynomial, then $S_2 = \int_a^b f(x) \, dx$. In other words, show that

\[
\int_a^b f(x) \, dx = \frac{b-a}{6} (30 + 4y_1 + y_2),
\]

where $y_0 = f(a)$, $y_1 = f\left(\frac{a+b}{2}\right)$, and $y_2 = f(b)$. Hint: Show this first for $f(x) = 1, x, x^2$ and use linearity.

63. For $N$ even, divide $[a, b]$ into $N$ subintervals of width $\Delta x = \frac{b-a}{N}$.

Set $x_j = a + j \Delta x$, $y_j = f(x_j)$, and

\[
S_{2N} = \frac{b-a}{3N} (y_0 + 4y_1 + 4y_2 + \cdots + 4y_{2N-2} + y_{2N}).
\]

(a) Show that $S_N$ is the sum of the approximations on the intervals $[x_j, x_{j+1}]-$that is, $S_N = S_2 + S_4 + \cdots + S_{2N-2}$.

(b) By Exercise 62, $S_2 = \int_a^b f(x) \, dx$ if $f$ is a quadratic polynomial. Use (a) to show that $S_N$ is exact for all $N$ if $f$ is a quadratic polynomial.

64. Show that $S_2$ also gives the exact value for $\int_a^b x^3 \, dx$ and conclude, as in Exercise 63, that $S_N$ is exact for all cubic polynomials. Show by counterexample that $S_N$ is not exact for integrals of $x^4$.

65. Use the Error Bound for $S_N$ to obtain another proof that Simpson’s Rule is exact for all cubic polynomials.

66. Sometimes Simpson’s Rule Performs Poorly Calculate $M_0$ and $S_0$ for the integral $\int_0^1 \sqrt{1-x^2} \, dx$, whose value we know to be $\frac{\pi}{4}$ (one-quarter of the area of the unit circle).

(a) We usually expect $S_N$ to be more accurate than $M_N$. Which of $M_0$ and $S_0$ is more accurate in this case?

(b) How do you explain the result of part (a)? Hint: The Error Bounds are not valid because $|f''(x)|$ and $|f^{(4)}(x)|$ tend to $\infty$ as $x \to 1$, but $|f^{(4)}(x)|$ goes to infinity more quickly.

CHAPTER REVIEW EXERCISES

1. Match the integrals (a)-(e) with their antiderivatives (i)-(v) on the basis of the general form (do not evaluate the integrals).

(a) $\int \frac{x \, dx}{\sqrt{x^2 - 4}}$

(b) $\int \frac{(2x + 9) \, dx}{x^2 + 4}$

(c) $\int \sin^3 x \cos^2 x \, dx$

(d) $\int \frac{dx}{x \sqrt{16x^2 - 1}}$

(e) $\int \frac{16 \, dx}{x(x - 4)^2}$
(i) \( \sec^{-1} 4x + C \)
(ii) \( \log |x| - \log |x - 4| - \frac{4}{x - 4} + C \)
(iii) \( \frac{30}{7} (3 \cos^5 x - 3 \cos^3 x \sin^2 x - 7 \cos^3 x) + C \)
(iv) \( \frac{9}{2} \tan^{-1} \frac{x}{2} + \ln(x^2 + 4) + C \) \quad (v) \( \sqrt{x^2 - 4} + C \)

2. Evaluate \( \int \frac{xdx}{x^2 + 2} \) in two ways: using substitution and using the Method of Partial Fractions.

In Exercises 3–12, evaluate using the suggested method.

3. \( \int \cos^3 \theta \sin^5 \theta d\theta \) \quad [Write \( \cos^3 \theta \) as \( \cos \theta (1 - \sin^2 \theta) \).

4. \( \int x e^{-12x} dx \) \quad (Integration by Parts)

5. \( \int \sec^3 \theta \tan^4 \theta d\theta \) \quad (trigonometric identity, reduction formula)

6. \( \int \frac{4x + 4}{x - 5}(x + 3) dx \) \quad (partial fractions)

7. \( \int \frac{1}{x(x^2 - 1)^{3/2}} dx \) \quad (trigonometric substitution)

8. \( \int \frac{dx}{1 + x^2 - 3/2} \) \quad (trigonometric substitution)

9. \( \int \frac{dx}{x^{3/2} + x^{1/2}} \) \quad (substitution)

10. \( \int \frac{dx}{x + x^{-1}} \) \quad (rewrite integrand)

11. \( \int \frac{dx}{x^2 + 4x - 5} \) \quad (Integration by Parts)

12. \( \int \frac{dx}{x^2 + 4x - 5} \) \quad (complete the square, substitution, partial fractions)

In Exercises 13–64, evaluate using the appropriate method or combination of methods.

13. \( \int_{0}^{1} x^2 e^{4x} dx \)
14. \( \int \frac{x^2}{\sqrt{9 - x^2}} dx \)

15. \( \int \cos^9 \theta \sin^3 \theta d\theta \)
16. \( \int \sec^2 \theta \tan^4 \theta d\theta \)

17. \( \int \frac{(6x + 4)dx}{x^2 - 1} \)
18. \( \int_{0}^{9} \frac{dt}{(t^2 - 1)^3} \)

19. \( \int \frac{d\theta}{\cos^9 \theta} \)
20. \( \int \sin 2\theta \sin^2 \theta d\theta \)

21. \( \int_{0}^{1} \ln(4 - 2x) dx \)
22. \( \int (\ln(x + 1))^3 dx \)

23. \( \int \sin^5 \theta d\theta \)
24. \( \int \cos^6(9x - 2) dx \)

25. \( \int_{0}^{\pi/4} \sin 3x \cos 5x dx \)
26. \( \int \sin 2x \cos^2 x dx \)

27. \( \int \sqrt{x} \sec^2 x dx \)
28. \( \int (\sec x + \tan x)^2 dx \)

29. \( \int \sin^3 \theta \cos \theta \sin \theta d\theta \)
30. \( \int \cot^3 x \csc x dx \)

31. \( \int \cos^2 x \csc^2 x dx \)
32. \( \int_{\pi/2}^{\pi} \cot^2 \theta d\theta \)

33. \( \int_{\pi/4}^{\pi/2} \cos^2 x \csc^3 x dx \)
34. \( \int_{\pi/4}^{\pi/2} \csc^2 x \cot^2 x dx \)

35. \( \int \frac{dt}{(t - 3)(t + 4)} \)
36. \( \int \sqrt{x^2 + 9} dx \)

37. \( \int \frac{dx}{x^2 + x + 3/3} \)
38. \( \int \frac{dx}{x + x^{2/3}} \)

39. \( \int \frac{dx}{x^{3/2} + ax^{1/2}} \)
40. \( \int \frac{dx}{(x - b)^2 + 4} \)

41. \( \int \frac{dx}{(x^2 - x)(x + 2)^3} \)
42. \( \int \frac{7x^2 + x}{(x - 2)(x + 1)(x + 1)} dx \)

43. \( \int \frac{16dx}{(x - 2)(x + 1)} \)
44. \( \int \frac{dx}{x^2 + 25} \)

45. \( \int \frac{dx}{x^2 + 8x + 25} \)
46. \( \int \frac{dx}{x^2 + 25} \)

47. \( \int \frac{dx}{x^2 + 8x + 25} \)
48. \( \int_{0}^{1} \frac{dx}{t^2 + 1} \)

49. \( \int \frac{dx}{x^2 + 1} \)
50. \( \int \frac{dx}{(x^2 + 5)^{3/2}} \)

51. \( \int (x + 1)e^{x - 2x} dx \)
52. \( \int \frac{dx}{x + x - 1} \)

53. \( \int x^3 \cos(x^2) dx \)
54. \( \int x^3(lnx)^2 dx \)

55. \( \int x \tan^{-1} x dx \)
56. \( \int \tan^{-1} x dx \)

57. \( \int \ln(x^2 + 9) dx \)
58. \( \int (\sin x)(\cosh x) dx \)

59. \( \int \cosh 2t dt \)
60. \( \int \sinh^3 x \cosh x dx \)

61. \( \int \cosh^2 (1 - 4t) dt \)
62. \( \int_{0}^{\pi/4} \frac{dx}{\sin^3 x} \)

63. \( \int_{0}^{1} \frac{dx}{\sin^3 x} \)
64. \( \int \frac{dx}{\tan^2 x + 1} \)

65. Use the substitution \( u = \tan^{-1} t \) to evaluate \( \int \frac{du}{\cos^2 t + \sin^2 t} \).

66. Find the volume obtained by rotating the region enclosed by \( y = \ln x \) and \( y = (\ln x)^2 \) about the \( y \)-axis.
67. Let \( I_n = \int \frac{x^n}{x^2 + 1} \, dx \).
(a) Prove that \( I_n = \frac{x^{n-1}}{n-1} - I_{n-2} \).
(b) Use (a) to calculate \( I_n \) for \( 0 \leq n \leq 5 \).
(c) Show that, in general,
\[
I_{2n+1} = \frac{x^{2n}}{2n} - \frac{x^{2n-2}}{2n-2} + \cdots
\]
\( + (-1)^{n-1} \frac{x^2}{2} + (-1)^n \frac{1}{2} \ln(x^2 + 1) + C \)
\( I_{2n} = \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3} + \cdots
\]
\( + (-1)^{n-1} x + (-1)^n \tan^{-1} x + C \).

68. Let \( J_n = \int x^n e^{-x^2} \, dx \).
(a) Show that \( J_1 = -e^{-x^2/2} \).
(b) Prove that \( J_n = -x^{n-1} e^{-x^2/2} + (n - 1)J_{n-2} \).
(c) Use (a) and (b) to compute \( J_3 \) and \( J_5 \).

In Exercises 69–78, determine whether the improper integral converges and, if so, evaluate it.

69. \( \int_0^\infty \frac{dx}{(x + 2)^2} \)
70. \( \int_0^\infty \frac{dx}{x^{3/2}} \)

71. \( \int_0^1 \frac{dx}{x^{3/2}} \)
72. \( \int_0^\infty \frac{dx}{x^{12/5}} \)
73. \( \int_0^{\pi/2} \frac{dx}{x^2 + 1} \)
74. \( \int_0^\infty e^{4x} \, dx \)
75. \( \int_0^{\pi/2} \cot \theta \, d\theta \)
76. \( \int_1^\infty \frac{dx}{x + 2(2x + 3)} \)
77. \( \int_0^\infty (5 + x)^{-1/3} \, dx \)
78. \( \int_2^5 (5 - x)^{-1/3} \, dx \)

79. \( \int_0^\infty \frac{dx}{x^2 - 4} \)
80. \( \int_0^\infty (\sin^2 x)e^{-x} \, dx \)
81. \( \int_0^3 \frac{dx}{x^4 + \cos^2 x} \)
82. \( \int_1^\infty \frac{dx}{x^{1/3} + x^{2/3}} \)
83. \( \int_0^1 \frac{dx}{x^{1/3} + x^{2/3}} \)
84. \( \int_0^\infty e^{-x^2} \, dx \)

85. Calculate the volume of the infinite solid obtained by rotating the region under \( y = (x^2 + 1)^{-2} \) for \( 0 \leq x < \infty \) about the \( y \)-axis.

86. Let \( R \) be the region under the graph of \( y = (x + 1)^{-1} \) for \( 0 \leq x < \infty \). Which of the following quantities is finite?
(a) The area of \( R \)
(b) The volume of the solid obtained by rotating \( R \) about the \( x \)-axis
(c) The volume of the solid obtained by rotating \( R \) about the \( y \)-axis

87. Show that \( \int_0^\infty x^n e^{-x^2} \, dx \) converges for all \( n > 0 \). Hint: First observe that \( x^n e^{-x^2} < x^n e^{-x} \) for \( x > 1 \). Then show that \( x^n e^{-x} < x^{-2} \) for \( x \) sufficiently large.

88. Compute the Laplace transform \( \mathcal{L}\{f(x)\} \) of the function \( f(x) = x \) for \( s > 0 \). See Exercises 88–91 in Section 8.7 for the definition of \( \mathcal{L}\{f(x)\} \).

89. Compute the Laplace transform \( \mathcal{L}\{f(x)\} \) of the function \( f(x) = x^2 e^{ax} \) for \( s > a \).

90. Estimate \( \int_0^5 f(x) \, dx \) by computing \( T_3 \), \( M_3 \), \( T_5 \), and \( S_5 \) for a function \( f \) taking on the values in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>( \frac{1}{2} )</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>(-\frac{3}{2})</td>
<td>(-4)</td>
<td>(-2)</td>
</tr>
</tbody>
</table>

91. State whether the approximation \( M_N \) or \( T_N \) is larger or smaller than the integral.
(a) \( \int_0^\pi \sin x \, dx \)
(b) \( \int_0^\pi \cos x \, dx \)
(c) \( \int_1^2 \frac{dx}{x^2} \)
(d) \( \int_0^1 \sin x \, dx \)

92. The rainfall rate (in inches per hour) was measured hourly during a 10-hour thunderstorm with the following results:

\[ 0, 0.41, 0.49, 0.32, 0.3, 0.23, 0.09, 0.08, 0.05, 0.11, 0.12 \]

Use Simpson's Rule to estimate the total rainfall during the 10-h period.

In Exercises 93–98, compute the given approximation to the integral.

93. \( \int_0^1 e^{-x^2} \, dx \), \( M_5 \)  
94. \( \int_2^4 \sqrt{6x^3 + 1} \, dx \), \( T_3 \)
95. \( \int_\pi/4^\pi \sqrt{\sin \theta} \, d\theta \), \( M_4 \)  
96. \( \int_1^4 \frac{dx}{x^{1/3}} \), \( T_6 \)
97. \( \int_0^1 e^{-x^2} \, dx \), \( S_4 \)  
98. \( \int_0^\pi \cos(x^2) \, dx \), \( S_8 \)

99. The following table gives the area \( A(h) \) of a horizontal cross section of a pond at depth \( h \). Use the Trapezoidal Rule to estimate the volume \( V \) of the pond (Figure 1).

<table>
<thead>
<tr>
<th>( h ) (ft)</th>
<th>( A(h) ) (acres)</th>
<th>( h ) (ft)</th>
<th>( A(h) ) (acres)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.8</td>
<td>10</td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>2.4</td>
<td>12</td>
<td>0.6</td>
</tr>
<tr>
<td>4</td>
<td>1.8</td>
<td>14</td>
<td>0.2</td>
</tr>
<tr>
<td>6</td>
<td>1.5</td>
<td>16</td>
<td>0.1</td>
</tr>
<tr>
<td>8</td>
<td>1.2</td>
<td>18</td>
<td>0</td>
</tr>
</tbody>
</table>
100. Suppose that the second derivative of the function $A$ in Exercise 99 satisfies $|A''(h)| \leq 1.5$. Use the Error Bound to find the maximum possible error in your estimate of the volume $V$ of the pond.

101. Find a bound for the error $|M_{16} - \int_1^3 x^3 \, dx|$.

102. (GU) Let $f(x) = \sin(x^3)$. Find a bound for the error

$$\left| T_{24} - \int_0^{\pi/2} f(x) \, dx \right|$$

Hint: Find a bound $K_2$ for $|f''(x)|$ by plotting $f''$ with a graphing utility.

103. Find a value of $N$ such that

$$\left| M_N - \int_0^{\pi/4} \tan x \, dx \right| \leq 10^{-4}$$

104. Find a value of $N$ such that $S_N$ approximates $\int_2^5 x^{-1/4} \, dx$ with an error of at most $10^{-2}$ (but do not calculate $S_N$).
9 FURTHER APPLICATIONS OF THE INTEGRAL

In this chapter, we examine a few more applications of the integral. In the first section we take a brief look at the importance of integration in probability theory. Following that, we use the integral to define and compute the length of curves and the area of surfaces of revolution. The last two sections address important physical applications involving pressure, force, and center of mass.

9.1 Probability and Integration

What is the probability that a customer will arrive at the Rogadzo Pizza Parlor in the next 45 seconds? What is the probability of scoring above 90% on the NICE (National Integral Competency Exam)? Probabilities such as these are given by a number between 0 and 1, where 0 means there is no probability the event will occur and 1 means that the event is sure to happen. These probabilities are best described as areas under the graph of a function \( y = p(x) \) called a probability density function (Figure 1). The methods of integration developed in this chapter are used extensively in the study of such functions.

In probability theory, the variable \( X \) that represents the phenomenon we are analyzing (time to arrival, exam score, etc.) is called a random variable. The probability that \( X \) lies in a given range \([a, b]\) is denoted

\[
P(a \leq X \leq b)
\]

For example, the probability of a customer arriving within the next 30 to 45 seconds is denoted \( P(30 \leq X \leq 45) \).

We say that \( p \) is a probability density function for \( X \) if it is a continuous function such that

\[
P(a \leq X \leq b) = \int_a^b p(x) \, dx
\]

We assume that a probability density function \( p \) has a domain \( J \) that is an interval, and that \( J \) contains all of the possible values of the random variable \( X \). We allow the possibility that \( J \) is an infinite interval. Furthermore, \( p \) must satisfy two important conditions. First, \( p(x) \geq 0 \) for all \( x \) in the domain, because a probability cannot be negative. Second,

\[
\int_J p(x) \, dx = 1
\]

Notice that the integral in Eq. (1) is evaluated over the whole domain and represents the probability that the value of \( X \) is in the domain. This integral must equal 1 because it is certain (the probability is 1) that the value of \( X \) lies in \( J \) by assumption. That the integral of \( p \) over \( J \) is 1 ensures that \( P(a \leq X \leq b) \) is a number in the interval \([0, 1]\) for any \( a \) and \( b \) in the domain.

**EXAMPLE 1** Find a constant \( C \) for which \( p(x) = \frac{C}{x^2 + 1} \) is a probability density function with domain \((-\infty, \infty)\). Then compute \( P(1 \leq X \leq 4) \).

**Solution** We must choose \( C \) so that Eq. (1) is satisfied. The improper integral is a sum of two integrals (see Example 4 of Section 8.7):

\[
\int_{-\infty}^{\infty} p(x) \, dx = C \int_{-\infty}^{0} \frac{dx}{x^2 + 1} + C \int_{0}^{\infty} \frac{dx}{x^2 + 1} = C \frac{\pi}{2} + C \frac{\pi}{2} = C\pi
\]
Therefore, Eq. (1) is satisfied if \( C\pi = 1 \) or \( C = \pi^{-1} \). We have
\[
P(1 < X < 4) = \int_{1}^{4} p(x) \, dx = \int_{1}^{4} \pi^{-1} \frac{dx}{x^2 + 1} = \pi^{-1}(\tan^{-1} 4 - \tan^{-1} 1) \approx 0.17
\]
Thus, \( X \) lies between 1 and 4 with probability approximately 0.17, or a 17% chance (Figure 2).

**CONCEPTUAL INSIGHT** If \( X \) is a random variable with probability density function \( p \), then the probability of \( X \) taking on any specific value \( a \) is zero because \( \int_{a}^{a} p(x) \, dx = 0 \). So what is the meaning of \( p(a) \)? We must think of it this way: The probability that \( X \) lies in a small interval \( [a, a + \Delta x] \) is approximately \( p(a)\Delta x \):
\[
P(a \leq X \leq a + \Delta x) = \int_{a}^{a+\Delta x} p(x) \, dx \approx p(a)\Delta x
\]
A probability density is similar to a linear mass density \( \rho(x) \). The mass of a small segment \( [a, a + \Delta x] \) is approximately \( p(a)\Delta x \), but the mass of any particular point \( x = a \) is zero.

The mean or average value of a random variable is the quantity
\[
\mu = \mu(X) = \int_{f}^{x} xp(x) \, dx
\]
if this integral exists. The symbol \( \mu \) is a lowercase Greek letter mu.

In the next example, we consider the **exponential probability density** with parameter \( r > 0 \), defined on \((0, \infty)\) by
\[
p(t) = \frac{1}{r} e^{-t/r}
\]
This density function is often used to model "waiting times" between events that occur randomly. Exercise 12 asks you to verify that \( p(t) \) satisfies Eq. (1).

**EXAMPLE 2** **Mean of a Random Variable with Exponential Density** Let \( r > 0 \). Calculate the mean of a random variable with the exponential probability density 
\( p(t) = \frac{1}{r} e^{-t/r} \) on \([0, \infty)\).

**Solution** The mean is the integral of \( tp(t) \) over \([0, \infty)\). Using Integration by Parts with
\[
u = t/r \quad \text{and} \quad dv = e^{-t/r} \, dt,
\]
we have \( du = dt/r, \quad v = -e^{-t/r}, \) and
\[
\int tp(t) \, dt = \int \left( \frac{1}{r} e^{-t/r} \right) \, dt = -te^{-t/r} + \int e^{-t/r} \, dt = -(r + t)e^{-t/r}
\]
Thus the mean \( \mu \) is given by
\[
\mu = \int_{0}^{\infty} tp(t) \, dt = \left[ \frac{1}{r} e^{-t/r} \right]_{0}^{\infty} = \lim_{R \to \infty} -e^{-r/R} \left[ (r + t)e^{-t/r} \right]_{0}^{R}
\]
\[
= \lim_{R \to \infty} -e^{-R/r} \left( r - (r + R)e^{-R/r} \right) = r
\]
The last equality holds since \((r + R)e^{-R/r} \to 0\) as \( R \to \infty \). It follows that the mean of a random variable with the probability density function \( p(t) = \frac{1}{r} e^{-t/r} \), over \([0, \infty)\), is \( r \).
EXAMPLE 3 Waiting Time. The waiting time $T$ between customer arrivals in a drive-through fast-food restaurant is a random variable with exponential probability density. If the average waiting time is 60 seconds, what is the probability that a customer will arrive within 30 to 45 s after another customer?

Solution If the average waiting time is $60$ s, then $r = 60$ and $p(t) = \frac{1}{60} e^{-t/60}$ because the mean of $\frac{1}{r} e^{-t/r}$ is $r$ by the previous example. Therefore, the probability of waiting between 30 and 45 s for the next customer is

$$P(30 \leq T \leq 45) = \int_{30}^{45} \frac{1}{60} e^{-t/60} \, dt = -e^{-t/60}\bigg|_{30}^{45} = -e^{-3/4} + e^{-1/2} \approx 0.134$$

This probability is the area of the shaded region in Figure 3.

The normal density functions, whose graphs are the familiar bell-shaped curves, appear in a surprisingly wide range of applications. The standard normal density is defined over $(-\infty, \infty)$ by

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

That $p(x)$ satisfies Eq. (1) is not easy to show. One problem is that $p$ does not have an elementary antiderivative. In Exercise 57 in Section 16.4 we will see an approach using multivariable calculus.

More generally, we define the normal density function with mean $\mu$ and standard deviation $\sigma$:

$$p(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

The standard deviation $\sigma$ measures the spread; for larger values of $\sigma$, the graph is more spread out about the mean $\mu$ (Figure 4). The standard normal density in Eq. (3) has $\mu = 0$ and $\sigma = 1$. A random variable with a normal density function is said to have a normal or Gaussian distribution. Examples of data whose distribution is modeled well by a normal distribution include current sale prices for houses in Denver, heights of female children of age 11 in Egypt, and systolic blood pressure readings for adults in Frigento, Italy. The normal distribution is ubiquitous in everyday life. For example, Figure 5 shows a bar graph of data from a survey on the time of day that workers in the United States leave for work. Note the approximate bell-shaped curve generated by the data.

One difficulty with normal density functions is that they do not have elementary antiderivatives. As a result, we cannot evaluate the probabilities

$$P(a \leq X \leq b) = \frac{1}{\sigma \sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/(2\sigma^2)} \, dx$$

explicitly. However, the next theorem shows that these probabilities can all be expressed in terms of a single function called the standard normal cumulative distribution function:

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} \, dx$$

Observe that $F(z)$ is equal to the shaded area under the graph in Figure 6. The function $F$ is not an elementary function. Numerical values of $F(z)$, obtained by integral approximation, are typically available on scientific calculators and computer algebra systems.
THEOREM 1  If $X$ has a normal distribution with mean $\mu$ and standard deviation $\sigma$, then for all $a \leq b$,

$$P(X \leq b) = F \left( \frac{b - \mu}{\sigma} \right)$$

$$P(a \leq X \leq b) = F \left( \frac{b - \mu}{\sigma} \right) - F \left( \frac{a - \mu}{\sigma} \right)$$

**Proof**  We use two changes of variables, first $u = x - \mu$ and then $t = u/\sigma$:

$$P(X \leq b) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{b-\mu} e^{-u^2/(2\sigma^2)} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(b-\mu)/\sigma} e^{-t^2/2} dt = F \left( \frac{b - \mu}{\sigma} \right)$$

This proves Eq. (4). Equation (5) follows because $P(a \leq X \leq b)$ is the area under the graph between $a$ and $b$, and this is equal to the area to the left of $b$ minus the area to the left of $a$ (Figure 7).

**CONCEPTUAL INSIGHT**  Why have we defined the mean of a continuous random variable $X$ as the integral $\mu = \int_{-\infty}^{\infty} xp(x) \, dx$? Suppose first we are given $N$ numbers $a_1, a_2, \ldots, a_N$, and for each value $x$, let $N(x)$ be the number of times $x$ occurs among the $a_j$. Then a randomly chosen $a_j$ has value $x$ with probability $p(x) = N(x)/N$. For example, given the numbers 4, 4, 5, 5, 5, 8, we have $N = 6$ and $N(5) = 3$. The probability of choosing a 5 is $p(5) = N(5)/N = \frac{3}{6} = \frac{1}{2}$. Now observe that we can write the mean (average value) of the $a_j$ in terms of the probabilities $p(x)$:

$$\frac{a_1 + a_2 + \cdots + a_N}{N} = \frac{1}{N} \sum_{x} N(x) x = \sum_{x} x p(x)$$

For example,

$$\frac{4 \cdot 4 + 5 \cdot 5 + 5 \cdot 5 + 8}{6} = \frac{1}{6} (2 \cdot 4 + 3 \cdot 5 + 1 \cdot 8) = 4p(4) + 5p(5) + 8p(8)$$

In defining the mean of a continuous random variable $X$, we replace the sum $\sum_{x} x p(x)$ with the integral $\mu = \int_{-\infty}^{\infty} x p(x) \, dx$. This makes sense because the integral is the limit of sums $\sum_{x} p(x) \Delta x$, and as we have seen, $p(x) \Delta x$ is the approximate probability that $X$ lies in $[x, x + \Delta x]$.

**EXAMPLE 4**  Assume that the scores $X$ on a standardized test are normally distributed with mean $\mu = 500$ and standard deviation $\sigma = 100$. Find the probability that a test chosen at random has score

(a) at most 600.

(b) between 450 and 650.

**Solution**  We use a computer algebra system to evaluate $F(c)$ numerically.

(a) Apply Eq. (4) with $\mu = 500$ and $\sigma = 100$:

$$P(x \leq 600) = F \left( \frac{600 - 500}{100} \right) = F(1) \approx 0.84$$

(b) Apply Eq. (5):

$$P(450 \leq X \leq 650) = F \left( \frac{650 - 500}{100} \right) - F \left( \frac{450 - 500}{100} \right) = F(1.5) - F(-0.5)$$

$$= F(1.5) - (1 - F(0.5)) \approx 0.068$$
Thus, a randomly chosen score is 600 or less with a probability of 0.84, or 84%.
(b) Applying Eq. (5), we find that a randomly chosen score lies between 450 and 650 with a probability of 62.5%:
\[ P(450 \leq x \leq 650) = F(1.5) - F(-0.5) = 0.933 - 0.308 = 0.625 \]

9.1 SUMMARY

- If \( X \) is a continuous random variable with probability density function \( p \), then
  \[ P(a \leq X \leq b) = \int_a^b p(x) \, dx \]
- Probability densities with domain \( J \) satisfy two conditions: \( p(x) \geq 0 \) for \( x \) in \( J \), and \( \int_J p(x) \, dx = 1 \).
- Mean (or average) value of \( X \): \( \mu = \int_J x p(x) \, dx \)
- Exponential density function of mean \( r \): \( p(x) = \frac{1}{r} e^{-x/r} \)
- Normal density of mean \( \mu \) and standard deviation \( \sigma \): \( p(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-(x-\mu)^2/(2\sigma^2)} \)
- Standard cumulative normal distribution function: \( F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} \, dt \)
- If \( X \) has a normal distribution of mean \( \mu \) and standard deviation \( \sigma \), then
  \[ P(X \leq b) = F \left( \frac{b - \mu}{\sigma} \right) \]
  \[ P(a \leq X \leq b) = F \left( \frac{b - \mu}{\sigma} \right) - F \left( \frac{a - \mu}{\sigma} \right) \]

9.1 EXERCISES

Preliminary Questions
1. The function \( p(x) = \cos x \) satisfies \( \int_{-\pi/2}^{\pi/2} p(x) \, dx = 1 \). Is \( p \) a probability density function on \([−\pi/2, \pi/2]\)?
2. Estimate \( P(2 \leq X \leq 2.1) \), assuming that the probability density function of \( X \) satisfies \( p(2) = 0.2 \).
3. Which exponential probability density has mean \( \mu = \frac{1}{2} \)?
4. \( p(x) = Ce^{-x} \) on \((−\infty, \infty)\). \( P(−4 \leq X \leq 4) \)
5. This function, called the Gumbel density, is used to model extreme events such as floods and earthquakes.
6. Verify that \( p(x) = 3x^{-4} \) is a probability density function on \([1, \infty)\) and calculate its mean value.
7. Show that the density function \( p(x) = \frac{2}{\pi(x^2 + 1)} \) on \([0, \infty)\) has infinite mean.
8. Verify that \( p(r) = \frac{1}{3\pi} r^{-1/2} \) satisfies the condition
   \[ \int_0^\infty p(r) \, dr = 1 \]
9. Verify that for all \( r > 0 \), the exponential density function \( p(r) = \frac{1}{r} e^{-r/2} \) satisfies the condition
   \[ \int_0^\infty p(r) \, dr = 1 \]
13. The life $X$ (in hours) of a battery in constant use is a random variable with exponential density. What is the probability that the battery will last more than 12 h if the average life is 8 h?

14. The time between incoming phone calls at a call center is a random variable with exponential density. There is a 30% probability of waiting 20 seconds or more between calls. What is the average time between calls?

15. The distance $r$ between the electron and the nucleus in a hydrogen atom (in its lowest energy state) is a random variable with probability density $p(r) = 4a_0^{-3}e^{-2r/a_0}$ for $r \geq 0$, where $a_0$ is the Bohr radius (Figure 8). Calculate the probability $P$ that the electron is within one Bohr radius of the nucleus. The value of $a_0$ is approximately $5.29 \times 10^{-11}$ m, but this value is not needed to compute $P$.

![Figure 8: Probability density function $p(r) = 4a_0^{-3}e^{-2r/a_0}$](image)

16. Show that the distance $r$ between the electron and the nucleus in Exercise 15 has mean $\mu = 3a_0/2$.

In Exercises 17–22, $F(z)$ denotes the cumulative normal distribution function. Refer to a calculator or computer algebra system to obtain values of $F(z)$.

17. Express the area of region $A$ in Figure 9 in terms of $F(z)$ and compute its value.

![Figure 9: Normal density function with $\mu = 120$ and $\sigma = 30$.](image)

18. Show that the area of region $B$ in Figure 9 is equal to $1 - F(1.5)$ and compute its value. Verify numerically that this area is also equal to $F(1.5)$ and explain why graphically.

19. Assume $X$ has a standard normal distribution ($\mu = 0$, $\sigma = 1$). Express each of the following probabilities in terms of $F(z)$ and determine the value of each.

   (a) $P(X \leq 1.2)$ 
   (b) $P(X \geq -0.4)$

20. Assume $X$ has a normal distribution with $\mu = 0$ and $\sigma = 5$. Express each of the following probabilities in terms of $F(z)$ and determine the value of each.

   (a) $P(X \leq 1.2)$ 
   (b) $P(X \geq -0.4)$

21. Use a graph to show that $F(-z) = 1 - F(z)$ for all $z$. Then show that if $p(x)$ is a normal density function with mean $\mu$ and standard deviation $\sigma$, then for all $r \geq 0$, $P(-\infty < X < \mu + r\sigma) = 2F(r) - 1$.

22. The average September rainfall in Erie, Pennsylvania, is a random variable $X$ with mean $\mu = 102$ mm. Assume that the amount of rainfall is normally distributed with standard deviation $\sigma = 48$.

   (a) Express $P(128 \leq X \leq 150)$ in terms of $F(z)$ and compute its value numerically.

   (b) Let $P'$ be the probability that September rainfall will be at least 120 mm. Express $P'$ as an integral of an appropriate density function and compute its value numerically.

23. A bottling company produces bottles of fruit juice that are filled, on average, with 32 ounces of juice. Due to random fluctuations in the machinery, the actual volume of juice is normally distributed with a standard deviation of 0.4 oz. Let $P'$ be the probability of a bottle having less than 31 oz. Express $P'$ as an integral of an appropriate density function and compute its value numerically.

24. According to Maxwell's Distribution Law, in a gas of molecular mass $m$, the speed $v$ of a molecule in a gas at temperature $T$ (kelvins) is a random variable with density $\rho(v) = \frac{m}{2\pi kT}^{3/2}v^2e^{-m^2v^2/(2kT)}$ ($v \geq 0$), where $k$ is Boltzmann's constant. Show that the average molecular speed is equal to $(8kT/m)^{1/2}$. The average speed of oxygen molecules at room temperature is around 450 m/s.

25. Define the median of a probability distribution to be that value $a$ such that $\int_{-\infty}^{a} p(x)dx = \int_{a}^{\infty}p(x)dx = \frac{1}{2}$. Show that if a probability function is symmetric about the line $x = m$, then $m$ is both the mean and the median.

26. Define the quartiles of a probability function to be those values $a_1, a_2,$ and $a_3$ such that $P(-\infty < x \leq a_1) = P(a_1 < x \leq a_2) = P(a_2 < x \leq a_3) = P(a_3 < x < \infty) = \frac{1}{4}$. Find the quartile values for the probability function $p(x) = \frac{1}{1+x^2}$.

In Exercises 27–30, calculate $\mu$ and $\sigma$, where $\sigma$ is the standard deviation, defined by $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)dx$.

The smaller the value of $\sigma$, the more tightly clustered are the values of the random variable $X$ about the mean $\mu$. (The limit of integration need not be $\pm \infty$ if $p$ is defined over a smaller domain.)

27. $p(x) = \frac{5}{2\pi}e^{-x^2/2}$ on $[0, \infty)$

28. $p(x) = \frac{1}{\pi\sqrt{1-x^2}}$ on $(-1, 1)$

29. $p(x) = \frac{1}{3}e^{-x^2/3}$ on $[0, \infty)$

30. $p(x) = \frac{1}{r}e^{-x/r}$ on $[0, \infty)$, where $r > 0$
Further Insights and Challenges

31. (a) The time to decay of an atom in a radioactive substance is a random variable $X$. The law of radioactive decay states that if $N$ atoms are present at time $t = 0$, then $Nf(t)$ atoms will be present at time $t$, where $f(t) = e^{-kt}$ ($k > 0$ is the decay constant). Explain the following statements:

(b) The probability density function of $X$ is $y = -f'(t)$.
(c) The average time to decay is $1/k$.

32. The half-life of radon-222 is 3.825 days. Use Exercise 31 to compute:

(a) the average time to decay of a radon-222 atom.
(b) the probability that a given atom will decay in the next 24 hours.

9.2 Arc Length and Surface Area

We have seen that integrals are used to compute total amounts (such as distance traveled, total mass, total cost). Another such quantity is the length of a curve (also called arc length). We derive a formula for arc length using our standard procedure: approximation followed by passage to a limit. In this case, we approximate the curve by a path made up of line segments connecting points on the curve. It is easy to find the length of a collection of line segments. We improve the approximation by using more, but smaller, segments. Then we take the limit of the sum of their lengths as the number of line segments grows.

To make this precise, consider the graph of $y = f(x)$ over an interval $[a, b]$. Choose a partition $P$ of $[a, b]$ into $N$ subintervals with endpoints

$$P : a = x_0 < x_1 < \cdots < x_N = b$$

Recall that the norm of the partition, $\|P\|$, is the length of the largest subinterval in the partition; that is, the largest of the distances $x_i - x_{i-1}$. Let $P_i = (x_i, f(x_i))$ be the point on the graph corresponding to $x_i$, and join the points $P_{i-1}$ and $P_i$ by a line segment $L_i$. The curve $L$, made up of the segments $L_i$, is called a polygonal approximation (Figure 1). The length of $L$, which we denote $|L|$, is the sum of the lengths $|L_i|$ of the segments:

$$|L| = |L_1| + |L_2| + \cdots + |L_N| = \sum_{i=1}^{N} |L_i|$$

The polygonal approximations improve as the norm of the partition decreases.

As may be expected, the polygonal approximations $L$ approximate the curve more and more closely as the norm of the partition $P$ decreases, as illustrated in Figure 2. Based on this idea, we define the arc length $s$ of the graph to be the limit of the polygonal approximation lengths $|L|$ as $\|P\| \to 0$:

$$\text{arc length } s = \lim_{\|P\| \to 0} \sum_{i=1}^{N} |L_i|$$

To compute the arc length $s$, we express the limit of the polygonal approximations as an integral. Figure 3 shows that the segment $L_i$ is the hypotenuse of a right triangle of base $\Delta x_i = x_i - x_{i-1}$ and height $|f(x_i) - f(x_{i-1})|$. By the Pythagorean Theorem,

$$|L_i| = \sqrt{(\Delta x_i)^2 + (f(x_i) - f(x_{i-1}))^2}$$
We shall assume that \( f' \) exists and is continuous. Then, by the Mean Value Theorem, there is a value \( c_i \) in \([x_{i-1}, x_i]\) such that

\[
f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}) = f'(c_i)\Delta x_i
\]

and therefore,

\[
|L_i| = \sqrt{(\Delta x_i)^2 + (f'(c_i)\Delta x_i)^2} = \sqrt{(\Delta x_i)^2(1 + f'(c_i)^2)} = \sqrt{1 + f'(c_i)^2} \Delta x_i
\]

We find that the length \( |L| \) is a Riemann sum for \( \sqrt{1 + f'(x)^2} \):

\[
|L| = |L_1| + |L_2| + \cdots + |L_N| = \sum_{i=1}^{N} \sqrt{1 + f'(c_i)^2} \Delta x_i
\]

This function is continuous, and hence integrable, so the Riemann sums approach

\[
\int_a^b \sqrt{1 + f'(x)^2} \, dx
\]

as \( N \) becomes infinite.

**Formula for Arc Length**  Assume that \( f' \) exists and is continuous on the interval \([a, b]\). Then the arc length \( s \) of \( y = f(x) \) over \([a, b]\) is equal to

\[
s = \int_a^b \sqrt{1 + f'(x)^2} \, dx
\]

**EXAMPLE 1** Find the arc length \( s \) of the graph of \( f(x) = \frac{1}{12}x^3 + x^{-1} \) over the interval \([1, 3]\) (Figure 4).

**Solution** First, let's calculate \( 1 + f'(x)^2 \). Since \( f'(x) = \frac{1}{4}x^2 - x^{-2} \),

\[
1 + f'(x)^2 = 1 + \left( \frac{1}{4}x^2 - x^{-2} \right)^2 = 1 + \left( \frac{1}{16}x^4 - \frac{1}{2}x^2 + x^{-4} \right)
\]

\[
= \frac{1}{16}x^4 + \frac{1}{2}x^2 + x^{-4} = \left( \frac{1}{4}x^2 + x^{-2} \right)^2
\]

Fortunately, since this expression for \( 1 + f'(x)^2 \) is a square, the arc-length integral simplifies nicely and is easily computed:

\[
s = \int_1^3 \sqrt{1 + f'(x)^2} \, dx = \int_1^3 \left( \frac{1}{4}x^2 + x^{-2} \right) \, dx = \left( \frac{1}{12}x^3 - x^{-1} \right)_1^3
\]

\[
= \left( \frac{9}{4} - \frac{1}{3} \right) - \left( \frac{1}{12} - 1 \right) = \frac{17}{6}
\]

**EXAMPLE 2** Length of a Hanging Cable  Based on physical principles, the height of a cable hanging under its own weight is modeled well using the hyperbolic cosine function. Suppose we have a cable hanging from two poles that are 100 ft apart (located at \( x = -50 \) and \( x = 50 \)) and such that its height is given by \( h(x) = 250 \cosh(0.004x) - 225 \) (Figure 5). Note that in the middle, the height of the cable is \( h(0) = 25 \) ft, and at the poles,
the height is \( h(-50) = h(50) = 250 \cosh(0.2) - 225 \approx 30.0 \) ft. So the cable hangs 5 ft lower in the middle than the height at the poles. What is the length of the cable?

**Solution** We have \( h'(x) = (0.004)(250 \sinh(0.004x)) = \sinh(0.004x) \). Therefore,

\[
1 + h'(x)^2 = 1 + \sinh^2(0.004x) = \cosh^2(0.004x)
\]

Because hyperbolic cosine is a positive function, it follows that

\[
\sqrt{1 + h'(x)^2} = \cosh(0.004x)
\]

So, the length of the curve is

\[
\int_{-50}^{50} \sqrt{1 + h'(x)^2} \, dx = \int_{-50}^{50} \cosh(0.004x) \, dx = \frac{1}{0.004} \sinh(0.004x) \bigg|_{-50}^{50} = 250(\sinh(0.2) - \sinh(-0.2)) \approx 100.67 \text{ ft}
\]

It is interesting, and perhaps surprising, that with just 8 in. of length beyond the 100-ft direct distance from pole to pole, the cable hangs 5 ft lower at the middle than at the ends.

In Examples 1 and 2, we were able to compute the arc length exactly because \( 1 + f'(x)^2 \) could be expressed as a square that enabled us to simplify the integrand. Usually, \( \sqrt{1 + f'(x)^2} \) does not have an elementary antiderivative and there is no explicit formula for the arc length. However, we can always approximate arc length by using numerical integration.

**EXAMPLE 3** No Exact Formula for Arc Length  
Approximate the length \( s \) of \( y = \sin x \) over \([0, \pi]\) using Simpson’s Rule, computing \( S_N \) for \( N = 6 \).

**Solution** We have \( y' = \cos x \) and \( \sqrt{1 + (y')^2} = \sqrt{1 + \cos^2 x} \). The arc length is

\[
s = \int_0^\pi \sqrt{1 + \cos^2 x} \, dx
\]

This integral cannot be evaluated explicitly, so we approximate it by applying Simpson’s Rule (Section 8.8) with \( N = 6 \). Divide \([0, \pi]\) into subintervals of width \( \Delta x = \pi/6 \). Then

\[
S_6 = \frac{\Delta x}{3} \left( g(0) + 4g\left(\frac{\pi}{6}\right) + 2g\left(\frac{2\pi}{6}\right) + 4g\left(\frac{3\pi}{6}\right) + 2g\left(\frac{4\pi}{6}\right) + 4g\left(\frac{5\pi}{6}\right) + g(\pi) \right)
\]

\[
\approx \frac{\pi}{18} (1.1412 + 5.2915 + 2.2361 + 4 + 2.2361 + 5.2915 + 1.1412) \approx 3.82
\]

Thus, \( s \approx 3.82 \) (Figure 6).

**Surface Area**

The surface area \( S \) of a surface of revolution (Figure 7) can be computed by an integral that is similar to the arc length integral. Suppose that \( f(x) \geq 0 \), so the graph lies above the \( x \)-axis. We revolve the graph around the \( x \)-axis to obtain a surface of revolution \( R \). Our goal is to determine the surface area \( S \) of \( R \). To do so, we start by creating another surface of revolution \( R^* \) by rotating a polygonal approximation to \( y = f(x) \) about the \( x \)-axis. The result is a surface \( R^* \) that lies very close to \( R \) and whose surface area approximates \( S \) (Figure 8).
We will set up a Riemann sum that approximates the surface area of $R^*$ and therefore that also approximates the surface area of $R$. Taking a limit results in a definite-integral formula for $S$, the surface area of $R$.

The surface $R^*$ is made up of slanted bands, as shown in Figure 8. Consider the slanted band corresponding to the interval $[x_{i-1}, x_i]$. The segment $L_i$ along the slanted band is a segment in the polygonal approximation for $y = f(x)$. As in the derivation of the arc length formula, the length of $L_i$ can be expressed as

$$|L_i| = \sqrt{1 + f'(c_i)^2 \Delta x_i}$$

for some $c_i$ in $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$.

Now, as Figure 9 illustrates, we can approximate the surface area of the single slanted band with the surface area of a cylinder of width $|L_i|$ and radius $f(b_i)$ for any $b_i$ in $[x_{i-1}, x_i]$. Since we can use any $b_i$ in $[x_{i-1}, x_i]$ for this approximation, we will use $b_i = c_i$ to match the value used in the expression for $|L_i|$.

The surface area of a cylinder of radius $r$ and width $w$ is $2\pi rw$; therefore, if we let $b_i = c_i$, the surface area of the cylinder in Figure 9 is

$$2\pi f(c_i)|L_i| = 2\pi f(c_i) \sqrt{1 + f'(c_i)^2} \Delta x_i$$

The surface area of each slanted band in $R^*$ is approximated by the surface area of such a cylinder. We add up the surface areas of these cylinders to obtain an approximation to the surface area of $R^*$:

$$\text{surface area of } R^* \approx 2\pi \sum_{i=1}^{N} f(c_i) \sqrt{1 + f'(c_i)^2} \Delta x_i$$

As the norm of the partition goes to zero, the error in this approximation of the surface area of $R^*$ also goes to zero. Furthermore, the surface area of $R^*$ approaches the surface area of $R$. Therefore, the sum on the right-hand side of the approximation approaches $S$ in the limit. That sum is a Riemann sum that converges to the integral in the following definition:

**Area of a Surface of Revolution** Assume that $f(x) \geq 0$ and that $f'$ exists and is continuous on the interval $[a, b]$. The surface area $S$ of the surface obtained by rotating the graph of $f$ about the $x$-axis for $a \leq x \leq b$ is equal to

$$S = 2\pi \int_{a}^{b} f(x) \sqrt{1 + [f'(x)]^2} \, dx$$
EXAMPLE 4 Calculate the surface area of a sphere of radius $R$.

Solution The graph of $f(x) = \sqrt{R^2 - x^2}$ is a semicircle of radius $R$ (Figure 10). We obtain a sphere by rotating it about the $x$-axis. We have

\[
f'(x) = -\frac{x}{\sqrt{R^2 - x^2}}, \quad 1 + f'(x)^{2} = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}
\]

The surface area integral gives us the usual formula for the surface area of a sphere:

\[
S = 2\pi \int_{-R}^{R} f(x)\sqrt{1 + f'(x)^{2}} \, dx = 2\pi \int_{-R}^{R} \frac{R}{\sqrt{R^2 - x^2}} \, dx
\]

\[
= 2\pi R \int_{-R}^{R} dx = 2\pi R \left(2R\right) = 4\pi R^2
\]

EXAMPLE 5 Find the surface area of the surface, called a paraboloid, that is obtained by rotating the graph of $f(x) = \sqrt{x}$ about the $x$-axis for $0 \leq x \leq 1$.

Solution The graph of $f(x) = \sqrt{x}$ is the top half of a parabola opening along the $x$-axis, which becomes a paraboloid when rotated about the $x$-axis (Figure 11). Then $f'(x) = \frac{1}{2\sqrt{x}}$ and hence, we obtain

\[
S = 2\pi \int_{0}^{1} f(x)\sqrt{1 + f'(x)^{2}} \, dx = 2\pi \int_{0}^{1} \sqrt{x} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^{2}} \, dx
\]

\[
= 2\pi \int_{0}^{1} \frac{\sqrt{x}}{2\sqrt{x}} \sqrt{4x + 1} \, dx = \pi \int_{0}^{1} \sqrt{4x + 1} \, dx
\]

\[
= \frac{\pi}{6} (4x + 1)^{3/2} \bigg|_{0}^{1} = \frac{\pi}{6} (5^{3/2} - 1) \approx 5.3304
\]

EXAMPLE 6 A Corrugated Pipe A corrugated pipe is obtained by rotating the graph of $f(x) = 1 + 0.1 \sin(10x)$, for $0 \leq x \leq 10$, around the $x$-axis (Figure 12). What is the surface area of the pipe [assuming that $x$ and $f(x)$ are in meters]?

Solution Note that $f'(x) = 10(0.1 \cos(10x)) = \cos(10x)$. Substituting into the surface area formula, we have

\[
S = 2\pi \int_{0}^{10} (1 + 0.1 \sin(10x))\sqrt{1 + \cos^2(10x)} \, dx
\]

We use numerical approximation to obtain a result here. Using Simpson's Rule with $N = 100$, a computer algebra system yields $S \approx 76.4$ m².

EXAMPLE 7 Gabriel's Horn In Example 3 in Section 8.7, we introduced Gabriel's Horn (Figure 13), the surface obtained by rotating the graph of $f(x) = \frac{1}{x}$, for $x \geq 1$, about the $x$-axis. There we saw that the volume enclosed in the horn is $\pi$, and therefore is finite. Prove that the surface area of Gabriel's Horn is infinite.

Solution With $f(x) = \frac{1}{x} = x^{-1}$, we have $f'(x) = -x^{-2}$ and $f'(x)^{2} = x^{-4}$. The surface area of Gabriel's Horn is

\[
S = 2\pi \int_{1}^{\infty} x^{-1}\sqrt{1 + x^{-4}} \, dx
\]

Now, $x^{-1}\sqrt{1 + x^{-4}} > x^{-1}$ over $[1, \infty)$, and $\int_{1}^{\infty} x^{-1} \, dx$ diverges (to infinity) by Theorem 1 in Section 8.7. By the Comparison Test for Improper Integrals (Theorem 3 in Section 8.7), it follows that $S$, the surface area of Gabriel's Horn, is infinite.
9.2 SUMMARY

- The arc length of \( y = f(x) \) over the interval \([a, b]\) is
  \[
  s = \int_a^b \sqrt{1 + f'(x)^2} \, dx.
  \]
- Assume that \( f(x) \geq 0 \). The surface area of the surface obtained by rotating the graph of \( f \) about the \( x \)-axis for \( a \leq x \leq b \) is
  \[
  \text{surface area} = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} \, dx
  \]
- Use numerical integration to approximate arc length or surface area when the integral cannot be evaluated explicitly.

9.2 EXERCISES

**Preliminary Questions**

1. Which integral represents the length of the curve \( y = \cos x \) between 0 and \( \frac{\pi}{2} \)?
   \[
   \int_0^{\frac{\pi}{2}} \sqrt{1 + \cos^2 x} \, dx,
   \]
   \[
   \int_0^{\frac{\pi}{2}} \sqrt{1 + \sin^2 x} \, dx
   \]
2. By rotating the line \( y = r \) about the \( x \)-axis, for \( x \) in the interval \([0, h]\), and applying the surface area formula, obtain the well-known fact that the surface area of a cylinder of radius \( r \) and length \( h \) is given by \( 2\pi rh \).
3. If \( 0 \leq f(x) \leq g(x) \) for \( x \) in the interval \([a, b]\), can the surface obtained by rotating the graph of \( y = g(x) \) about the \( x \)-axis over the interval have less surface area than the surface obtained by rotating the graph of \( y = f(x) \) around the \( x \)-axis over the same interval?
4. Use the formula for arc length to show that for any constant \( C \), the graphs \( y = f(x) \) and \( y = f(x) + C \) have the same length over every interval \([a, b]\). Explain geometrically.
5. Use the formula for arc length to show that the length of a graph over \([1, 4]\) cannot be less than 3.

**Exercises**

1. Express the arc length of the curve \( y = x^4 \) between \( x = 2 \) and \( x = 6 \) as an integral (but do not evaluate).
2. Express the arc length of the curve \( y = \tan x \) for \( 0 \leq x \leq \frac{\pi}{4} \) as an integral (but do not evaluate).
3. Find the arc length of \( y = \sqrt{x} + x^{-1} \) for \( 1 \leq x \leq 2 \). **Hint:** Show that \( 1 + (y')^2 = \left( \frac{1}{2} x^{-\frac{1}{2}} + x^{-1} \right)^2 \).
4. Find the arc length of \( y = \left( \frac{1}{2} \right)^x + \frac{1}{2x} \) over \([1, 4]\). **Hint:** Show that \( 1 + (y')^2 \) is a perfect square.
5. In Exercises 5–10, calculate the arc length over the given interval.
   
   5. \( y = 3x + 1 \), \([0, 3]\)
   
   6. \( y = 9 - 3x \), \([1, 3]\)
   
   7. \( y = x^{3/2} \), \([1, 2]\)
   
   8. \( y = \frac{1}{4} x^{3/2} - x^{1/2} \), \([2, 8]\)
   
   9. \( y = \frac{1}{2} x^2 - \frac{1}{2} \ln x \), \([1, 2e]\)
   
   10. \( y = \ln(\cos x) \), \([0, \frac{\pi}{2}]\)
6. In Exercises 11–18, approximate the arc length of the curve over the interval using the Trapezoidal Rule \( T_n \), the Midpoint Rule \( M_n \), or Simpson’s Rule \( S_n \) as indicated.
   
   11. \( y = \frac{1}{4} x^4 \), \([1, 2]\), \( T_5 \)
   
   12. \( y = \sin x \), \([0, \frac{\pi}{2}] \), \( M_8 \)
   
   13. \( y = x^{-1} \), \([1, 2]\), \( S_8 \)
   
   14. \( y = e^{-x^2} \), \([0, 2]\), \( S_8 \)
   
   15. \( y = \ln x \), \([1, 3]\), \( M_6 \)
   
   16. \( y = \cos x \), \([0, 2]\), \( T_6 \)
   
   17. \( \text{CAS} \) \( y = x \sin x \), \([0, 10\pi]\), \( T_{100} \)
   
   18. \( \text{CAS} \) \( y = \frac{1}{1-x} \), \([0, 0.99] \), \( S_{100} \)
19. Calculate the length of the astroid \( x^{2/3} + y^{2/3} = 1 \) (Figure 14).

![Graph of x^{2/3} + y^{2/3} = 1.](image)

20. Show that the arc length of the astroid \( x^{2/3} + y^{2/3} = a^{2/3} \) (for \( a > 0 \)) is proportional to \( a \).

21. Find the length of the arc of the curve \( y = (y - 2)^3 \) from \( P(1, 3) \) to \( Q(8, 6) \).

22. Find the arc length of the curve shown in Figure 15.

![Graph of 9y^2 = x(x - 3)^2.](image)
23. Find the value of $a$ such that the arc length of the catenary $y = \cosh x$ for $-a \leq x \leq a$ equals 10.

24. Calculate the arc length of the graph of $f(x) = mx + r$ over $[a, b]$ in two ways: using the Pythagorean theorem (Figure 16) and using the arc length integral.

25. Show that the circumference of the unit circle is equal to

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

(an improper integral)

Evaluate, thus verifying that the circumference is $2\pi$.

26. Generalize the result of Exercise 25 to show that the circumference of the circle of radius $r$ is $2\pi r$.

27. Calculate the arc length of $y = x^2$ over $[0,a]$. Hint: Use trigonometric substitution. Evaluate for $a = 1$.

28. Express the arc length of $g(x) = \sqrt{x}$ over $[0,1]$ as a definite integral. Then use the substitution $u = x^{1/2}$ to show that this arc length is equal to the arc length of $y = x^{1/2}$ over $[0,1]$ (but do not evaluate the integrals). Explain this result graphically.

29. Find the arc length of $y = e^x$ over $[0,a]$. Hint: Try the substitution $u = e^x$ followed by partial fractions.

30. Show that the arc length of $y = \ln(f(x))$ for $a \leq x \leq b$ is

$$\int_a^b \sqrt{f'(x)^2 + 1} dx$$

31. Use Eq. (3) to compute the arc length of $y = \ln(\sin x)$ for $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$.

32. Use Eq. (3) to compute the arc length of $y = \ln\left(\frac{e^{x^2} + 1}{x-1}\right)$ over $[1,3]$.

33. Show that if $0 \leq f'(x) \leq 1$ for all $x$, then the arc length of $y = f(x)$ over $[a, b]$ is at most $\sqrt{2}(b - a)$. Show that for $f(x) = x$, the arc length equals $\sqrt{2}(b - a)$.

34. Use the Comparison Theorem (Section 5.2) to prove that the arc length of $y = x^{1/2}$ over $[1,2]$ is not less than $\frac{3}{2}$.

35. Approximate the arc length of one-quarter of the unit circle (which we know is $\frac{\pi}{2}$) by computing the length of the polygonal approximation with $N = 4$ segments (Figure 17).

36. **CAS** A merchant intends to produce specially carpets in the shape of the region in Figure 18, bounded by the axes and graph of $y = 1 - x^2$ (units in yards). Assume that material costs $50/y^2$ and that it costs $50L$ dollars to cut the carpet, where $L$ is the length of the curved side of the carpet. The carpet can be sold for $150A$ dollars, where $A$ is the carpet's area. Using numerical integration with a computer algebra system, find the whole number $n$ for which the merchant's profits are maximal.

In Exercises 37–46, compute the surface area of revolution about the $x$-axis over the interval.

37. $y = x$, $[0,4]$ 38. $y = 4x + 3$, $[0,1]$ 39. $y = x^3$, $[0,2]$ 40. $y = x^2$, $[0,10]$ 41. $y = x^2$, $[0,2]$ 42. $y = x^2$, $[0,10]$ 43. $y = (4 - x^2)^{1/2}$, $[0,8]$ 44. $y = e^{-x}$, $[0,1]$ 45. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $[1,e]$ 46. $y = \sin x$, $[0,\pi]$ 47. $y = x^n$, $[1,3]$ 48. $y = x^4$, $[0,1]$ 49. $y = e^{-x^2/2}$, $[0,2]$ 50. $y = \tan x$, $[0,\frac{\pi}{4}]$

51. Find the area of the surface obtained by rotating $y = \cosh x$ over $[1-\ln 2, \ln 2]$ around the $x$-axis.

52. Show that a spherical cap of height $h$ and radius $R$ (Figure 19) has surface area $2\pi RH$.

53. Find the surface area of the torus obtained by rotating the circle $x^2 + (y-b)^2 = r^2$ about the $x$-axis (Figure 20).
In Exercises 54-58, the graph of $y = f(x)$, for $a \leq x \leq b$, is rotated about the $y$-axis. In this situation, the surface area of the resulting surface is

$$ S = 2\pi \int_a^b x \sqrt{1 + f'(x)^2} \, dx $$

Determine the surface area for each surface of revolution. If the surface area cannot be computed exactly, find an approximate value.

**Further Insights and Challenges**

59. Find the surface area of the ellipsoid obtained by rotating the ellipse

$$ \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 $$

about the $x$-axis.

60. Show that if the arc length of $y = f(x)$ over $[0, a]$ is proportional to $a$, then $y = f(x)$ must be a linear function.

61. **CAS** Let $L$ be the arc length of the upper half of the ellipse with equation

$$ y = \frac{b}{a} \sqrt{a^2 - x^2} $$

(Figure 21) and let $\eta = \sqrt{1 - (b^2/a^2)}$. Use substitution to show that

$$ L = a \int_{-\pi/2}^{\pi/2} \sqrt{1 - \eta^2 \sin^2 \theta} \, d\theta $$

Use a computer algebra system to approximate $L$ for $a = 2, b = 1$.

**Figure 21** Graph of the ellipse $y = \frac{1}{2} \sqrt{4 - x^2}$.

62. Prove that the portion of a sphere of radius $R$ seen by an observer located at a distance $d$ above the North Pole has area $A = 2\pi d R^2 / (d + R)$. Hint: According to Exercise 52, the cap has surface area $2\pi Rh$. Show that $h = d R / (d + R)$ by applying the Pythagorean Theorem to the three right triangles in Figure 22.

**Figure 22** Spherical cap observed from a distance $d$ above the North Pole.

63. **CAS** Suppose that the observer in Exercise 62 moves off to infinity—that is, $d \to \infty$. What do you expect the limiting value of the observed area to be? Check your guess by using the formula for the area in the previous exercise to calculate the limit.

64. **CAS** Let $M$ be the total mass of a metal rod in the shape of the curve $y = f(x)$ over $[a, b]$ whose mass density $\rho(x)$ varies as a function of $x$. Use Riemann sums to justify the formula

$$ M = \int_a^b \rho(x) \sqrt{1 + f'(x)^2} \, dx $$

65. **CAS** Let $f$ be an increasing function on $[a, b]$ and let $g$ be its inverse. Argue on the basis of arc length that the following equality holds:

$$ \int_a^b \sqrt{1 + f'(x)^2} \, dx = \int_{f(a)}^{f(b)} \sqrt{1 + g'(y)^2} \, dy $$

Then use the substitution $u = f(x)$ to prove Eq. (4).

**9.3 Fluid Pressure and Force**

Fluid force is the force on an object submerged in a fluid. Divers feel this force as they descend below the water surface (Figure 1). Our calculation of fluid force is based on two laws that determine the pressure exerted by a fluid:

- Fluid pressure $p$ is proportional to depth.
- Fluid pressure does not act in a specific direction. Rather, a fluid exerts pressure on each side of an object in the perpendicular direction (Figure 2).

This second fact, known as Pascal's principle, points to an important difference between fluid pressure and the pressure exerted by one solid object on another.

**Fluid Pressure** The pressure $p$ at depth $h$ in a fluid of mass density $\rho$ is

$$ p = \rho gh $$

The pressure acts at each point on an object in the direction perpendicular to the object's surface at that point.
Pressure, by definition, is force per unit area.
- The SI unit of pressure is the pascal (Pa) (1 Pa = 1 N/m² = 1 kg/m³).
- Mass density (mass per unit volume) is denoted \( \rho \) (Greek rho).
- The factor \( \rho g \) is the density by weight, where \( g = 9.8 \text{ m/s}^2 \) is the acceleration due to gravity.

**EXAMPLE 1** Calculate the fluid force on the top and bottom of a box of dimensions 2 \( \times \) 2 \( \times \) 5 m, submerged in a pool of water with its top 3 m below the water surface (Figure 2). The density of water is \( \rho = 10^3 \text{ kg/m}^3 \).

Solution The top of the box is located at depth \( h = 3 \) m, so by Eq. (1) with \( g = 9.8 \text{ m/s}^2 \),

\[
\text{pressure on top } p = \rho gh = 10^3(9.8)(3) = 29,400 \text{ pascals}
\]

The top has area \( A = 4 \text{ m}^2 \) and the pressure is constant, so

\[
\text{downward force on top } F = pA = 29,400 \times 4 = 117,600 \text{ newtons}
\]

The bottom of the box is at depth \( h = 8 \) m, so the total force on the bottom is

\[
\text{upward force on bottom } = pA = \rho g A = 10^3(9.8)(8) \times 4 = 313,600 \text{ newtons}
\]

In the next example, the pressure varies with depth, and it is necessary to calculate the force as an integral.

**EXAMPLE 2** Calculating Force Using Integration Calculate the fluid force \( F \) on the side of the box in Example 1.

Solution Since the pressure varies with depth, we divide the side of the box into \( N \) thin horizontal strips (Figure 3). Let \( F_j \) be the force on the \( j \)th strip. The total force \( F \) is equal to the sum of the forces on the strips:

\[
F = F_1 + F_2 + \cdots + F_N
\]

**Step 1.** Approximate the force on a strip.

We’ll use the variable \( y \) to denote depth, where \( y = 0 \) at the water level and \( y \) is positive in the downward direction. Thus, a larger value of \( y \) denotes greater depth. Each strip is a rectangle of height \( \Delta y = 5/N \) and length 2, so the area of a strip is \( 2\Delta y \). The bottom edge of the \( j \)th strip has depth \( y_j = 3 + j\Delta y \).

If \( \Delta y \) is small, the pressure on the \( j \)th strip is nearly constant with value \( \rho g y_j \) (because all points on the strip lie at nearly the same depth \( y_j \)), so we can approximate the force on the \( j \)th strip:

\[
F_j \approx \rho g y_j \times (2\Delta y) = (\rho g) 2 y_j \Delta y
\]

**Step 2.** Approximate total force as a Riemann sum.

\[
F = F_1 + F_2 + \cdots + F_N \approx \rho g \sum_{j=1}^{N} 2 y_j \Delta y
\]

The sum on the right is a Riemann sum that converges to the integral \( \rho g \int_3^8 2y \, dy \).

The interval of integration is [3, 8] because the box extends from \( y = 3 \) to \( y = 8 \) (the Riemann sum has been set up with \( y_0 = 3 \) and \( y_N = 8 \)).

**Step 3.** Evaluate total force as an integral.

We obtain

\[
F = \rho g \int_3^8 2y \, dy = (\rho g)y^2 \Big|_3^8 = (10^3)(9.8)(8^2 - 3^2) = 539,000 \text{ newtons}
\]
Now, we'll add another complication: allowing the widths of the horizontal strips to vary with depth (Figure 4). Denote the width at depth \( y \) by \( f(y) \):

\[
f(y) = \text{width of the side at depth } y
\]

As before, assume that the object extends from \( y = a \) to \( y = b \). Divide the flat side of the object into \( N \) horizontal strips of thickness \( \Delta y = (b - a)/N \). If \( \Delta y \) is small, the \( j \)th strip is nearly rectangular of area \( f(y_j) \Delta y \). Since the strip lies at depth \( y_j = a + j \Delta y \), the force \( F_j \) on the \( j \)th strip can be approximated:

\[
F_j \approx \rho g f(y_j) \Delta y = \left( \frac{\rho g}{\text{Area}} \right) f(y_j) \Delta y
\]

The force \( F \) is approximated by a Riemann sum that converges to an integral:

\[
F = F_1 + \ldots + F_N \approx \rho g \sum_{j=1}^{N} y_j f(y_j) \Delta y \quad \Rightarrow \quad F = \rho g \int_{a}^{b} y f(y) \, dy
\]

**THEOREM 1** Fluid Force on a Flat Surface Submerged Vertically

The fluid force \( F \) on a flat side of an object submerged vertically in a fluid is

\[
F = \rho g \int_{a}^{b} y f(y) \, dy
\]

where \( f(y) \) is the horizontal width of the side at depth \( y \), and the object extends from depth \( y = a \) to depth \( y = b \).

**EXAMPLE 3** Calculate the fluid force \( F \) on one side of an equilateral triangular plate of side 2 m submerged vertically in a tank of oil of mass density \( \rho = 900 \text{ kg/m}^3 \) (Figure 5).

Solution To use Eq. (2), we need to find the horizontal width \( f(y) \) of the plate at depth \( y \). An equilateral triangle of side \( s = 2 \) has height \( \sqrt{3}s/2 = \sqrt{3} \). By similar triangles, \( y/f(y) = \sqrt{3}/2 \) and thus \( f(y) = 2y/\sqrt{3} \). By Eq. (2),

\[
F = \rho g \int_{0}^{\sqrt{3}} y f(y) \, dy = (900)(9.8) \int_{0}^{\sqrt{3}} \frac{2}{\sqrt{3}} y^2 \, dy
\]

\[
= \left( \frac{17,640}{\sqrt{3}} \right) \frac{y^3}{3} \bigg|_{0}^{\sqrt{3}} = 17,640 \text{ newtons}
\]

The next example shows how to modify the force calculation when the side of the submerged object is inclined at an angle.

**EXAMPLE 4** Force on an Inclined Surface

The side of a dam is inclined at an angle of 45°. The dam has height 700 ft and width 1500 ft as in Figure 6. Calculate the force \( F \) on the dam if the reservoir is filled to the top of the dam. Water has weight density \( w = 62.4 \text{ pounds per cubic foot} \).

Solution The vertical height of the dam is 700 ft, so we divide the vertical axis from 0 to 700 into \( N \) subintervals of length \( \Delta y = 700/N \). This divides the face of the dam into \( N \) strips as in Figure 6. By trigonometry, each strip has a width equal to \( \Delta y/\sin(45°) = \sqrt{2} \Delta y \). Therefore,

\[
\text{area of each strip} = \text{length} \times \text{width} = 1500(\sqrt{2} \Delta y)
\]
As usual, we approximate the force $F_j$ on the $j$th strip. The term $\rho g$ is equal to weight per unit volume, so we use $w = 62.4$ lb/ft$^3$ in place of $\rho g$:

$$F_j \approx \frac{\text{Pressure}}{\text{Area of strip}} = \frac{w \Delta y}{1500 \sqrt{2} \Delta y} \times 1500 \sqrt{2} \Delta y = w y_j \times 1500 \sqrt{2} \Delta y \text{ lb}$$

$$F = \sum_{j=1}^{N} F_j \approx \sum_{j=1}^{N} w y_j (1500 \sqrt{2} \Delta y) = 1500 \sqrt{2} \sum_{j=1}^{N} y_j \Delta y$$

This is a Riemann sum for the integral $1500 \sqrt{2} w \int_{0}^{700} y \, dy$. Therefore,

$$F = 1500 \sqrt{2} w \int_{0}^{700} y \, dy = 1500 \sqrt{2} (62.4) \frac{700^2}{2} \approx 3.24 \times 10^{10} \text{ lb}$$

### 9.3 SUMMARY

- If pressure is constant, then force $= \text{pressure} \times \text{area}$.
- The fluid pressure at depth $h$ is equal to $\rho gh$, where $\rho$ is the fluid density (mass per unit volume) and $g = 9.8$ m/s$^2$ is the acceleration due to gravity. Fluid pressure acts on a surface in the direction perpendicular to the surface. Water has mass density 1000 kg/m$^3$.
- If an object is submerged vertically in a fluid and extends from depth $y = a$ to $y = b$, then the total fluid force on a side of the object is

$$F = \rho g \int_{a}^{b} yf(y) \, dy$$

where $f(y)$ is the horizontal width of the side at depth $y$.
- If fluid density is given as weight per unit volume, we use $w$ in place of $\rho g$. Water has weight density 62.4 lb/ft$^3$. 
9.3 EXERCISES

Preliminary Questions

1. How is pressure defined?
2. Fluid pressure is proportional to depth. What is the factor of proportionality?
3. When fluid force acts on the side of a submerged object, in which direction does it act?

Exercises

1. A box of height 6 m and square base of side 3 m is submerged in a pool of water. The top of the box is 2 m below the surface of the water.
   (a) Calculate the fluid force on the top and bottom of the box.
   (b) Write a Riemann sum that approximates the fluid force on a side of the box by dividing the side into $N$ horizontal strips of thickness $\Delta y = 6/N$.
   (c) To which integral does the Riemann sum converge?
   (d) Compute the fluid force on a side of the box.

2. A square plate that is 2 by 2 m is submerged in water so that its top edge is level with the surface of the water. Calculate the fluid force on one side of it.

3. If a rectangular plate that is 1 by 2 m is dipped into a pool of water so that initially its top edge of length 1 is even with the surface of the water, and then it is lowered so that its top edge is at a depth of 1 m, calculate the increase in fluid force on one side of it.

4. A plate in the shape of an isosceles triangle with base 1 m and height 2 m is submerged vertically in a tank of water so that its vertex touches the surface of the water (Figure 7).
   (a) Show that the width of the triangle at depth $y$ is $f(y) = \frac{1}{2} y$.
   (b) Consider a thin strip of thickness $\Delta y$ at depth $y$. Explain why the fluid force on a side of this strip is approximately equal to $\rho g \frac{1}{2} y^2 \Delta y$.
   (c) Write an approximation for the total fluid force $F$ on a side of the plate as a Riemann sum and indicate the integral to which it converges.
   (d) Calculate $F$.

5. Repeat Exercise 4, but assume that the top of the triangle is located 3 m below the surface of the water.

6. The plate $R$ in Figure 8, bounded by the parabola $y = x^2$ and $y = 1$, is submerged vertically in water (distance in meters).
   (a) Show that the width of $R$ at height $y$ is $f(y) = \sqrt{y}$ and the fluid force on a side of a horizontal strip of thickness $\Delta y$ at height $y$ is approximately $\rho g \frac{1}{2} y^{1/2}(1 - y) \Delta y$.

7. Let $F$ be the fluid force on a side of a semicircular plate of radius $r$ meters, submerged vertically in water so that its diameter is level with the water’s surface (Figure 9).
   (a) Show that the width of the plate at depth $y$ is $2\sqrt{r^2 - y^2}$.
   (b) Calculate $F$ as a function of $r$ using Eq. (2).

8. Calculate the force on one side of a circular plate with radius 2 m, submerged vertically in a tank of water so that the top of the circle is tangent to the water surface.

9. A semicircular plate of radius $r$ meters, oriented as in Figure 9, is submerged in water so that its diameter is located at a depth of $m$ meters. Calculate the fluid force on one side of the plate in terms of $m$ and $r$.

10. A plate extending from depth $y = 2$ m to $y = 5$ m is submerged in a fluid of density $\rho = 850$ kg/m$^3$. The horizontal width of the plate at depth $y$ is $f(y) = 2(1 + y^2)^{-1}$. Calculate the fluid force on one side of the plate.

11. Figure 10 shows the wall of a dam on a water reservoir. Use the Trapezoidal Rule and the width and depth measurements in the figure to estimate the fluid force on the wall.
12. Assume in Figure 10 that the depth of water in the reservoir dropped 20 ft in a drought. Use the Trapezoidal Rule and the measurements in the figure to estimate the fluid force on the wall.

13. Calculate the fluid force on a side of the plate in Figure 11(A), submerged in water, assuming that the top of the plate is at a depth of $D = 2$ m.

14. Calculate the fluid force on a side of the plate in Figure 11(A), submerged in water, assuming that the top of the plate is at a depth of $D = 4$ m.

15. Calculate the fluid force on a side of the plate in Figure 11(B), submerged in a fluid of mass density $\rho = 800$ kg/m$^3$.

16. Find the fluid force on the side of the plate in Figure 12, submerged in a fluid of density $\rho = 1200$ kg/m$^3$. The top of the plate is level with the fluid surface. The edges of the plate are the curves $y = x^{1/3}$ and $y = -x^{1/3}$.

17. Let $R$ be the plate in the shape of the region under $y = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$ in Figure 13(A). If $R$ is rotated counterclockwise by $90^\circ$ and then submerged in a fluid of density $1100$ kg/m$^3$ with its top edge level with the surface of the fluid as in Figure 13(B), find the fluid force on a side of $R$.

18. In the notation of Exercise 17, calculate the fluid force on a side of the plate $R$ if it is oriented as in Figure 13(A).

19. Calculate the fluid force on one side of a plate in the shape of region $A$ shown in Figure 14. The water surface is at $y = 1$, and the fluid has density $\rho = 900$ kg/m$^3$.

20. Calculate the fluid force on one side of the “infinite” plate $B$ in Figure 14, assuming the fluid has density $\rho = 900$ kg/m$^3$.

21. Figure 15(A) shows a ramp inclined at $30^\circ$ leading into a swimming pool. Calculate the fluid force on the ramp.

22. Calculate the fluid force on one side of the plate (an isosceles triangle) shown in Figure 15(B).

23. The massive Three Gorges Dam on China’s Yangtze River has height 185 m (Figure 16). Calculate the force on the dam, assuming that the dam is a trapezoid of base 2000 m and upper edge 3000 m, inclined at an angle of $55^\circ$ to the horizontal (Figure 17).

24. A square plate of side 3 m is submerged in water at an incline of $30^\circ$ with the horizontal. Calculate the fluid force on one side of the plate if the top edge of the plate lies at a depth of 6 m.
25. The trough in Figure 18 is filled with corn syrup, whose weight density is 90 lb/ft³. Calculate the force on the front side of the trough.

26. Calculate the fluid pressure on one of the slanted sides of the trough in Figure 18 when it is filled with corn syrup as in Exercise 25.

Further Insights and Challenges

27. The end of the trough in Figure 19 is an equilateral triangle of side 3. Assume that the trough is filled with water to height $H$. Calculate the fluid force on each side of the trough as a function of $H$ and the length $l$ of the trough.

28. A rectangular plate of side $a$ is submerged vertically in a fluid of density $w$, with its top edge at depth $h$. Show that if the depth is increased by an amount $\Delta h$, then the force on a side of the plate increases by $wA\Delta h$, where $A$ is the area of the plate.

29. Prove that the force on the side of a rectangular plate of area $A$ submerged vertically in a fluid is equal to $pghA$, where $p$ is the fluid pressure at the center point of the rectangle.

30. If the density of a fluid varies with depth, then the pressure at depth $y$ is $p(y)$ (which need not equal $w$ as in the case of constant density). Use Riemann sums to argue that the total force $F$ on the flat side of a submerged object submerged vertically is $F = \int_0^h f(y)p(y)\,dy$, where $f(y)$ is the width of the side at depth $y$.

9.4 Center of Mass

Every object has a balance point called the center of mass (Figure 1). When a rigid object such as a hammer is tossed in the air, it may rotate in a complicated fashion, but its center of mass follows the same simple parabolic trajectory as a stone tossed in the air. In this section, we use integration to compute the center of mass of a thin plate (also called a lamina) of constant mass density $\rho$.

Consider a seesaw with a lineback for the Green Bay Packers on one end and a ballerina from the New York City Ballet on the other end, as in Figure 2. Clearly, the balance point $x$ must be closer to the linebacker than the ballerina.

As Archimedes realized over 2000 years ago, if two objects balance on opposite sides of a lever, then the mass of each object times its distance to the balance point must be equal. In this case, $m_1d_1 = m_2d_2$. If we use the coordinates given on the line, this becomes

$$m_1(x - x_1) = m_2(x_2 - x)$$
We solve this for $\bar{x}$:

\[
\begin{align*}
m_1 \bar{x} - m_1 x_1 &= m_2 x_2 - m_2 \bar{x} \\
m_1 \bar{x} + m_2 \bar{x} &= m_1 x_1 + m_2 x_2 \\
(m_1 + m_2) \bar{x} &= m_1 x_1 + m_2 x_2 \\
\bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}
\end{align*}
\]

In other words, the balance point, called the center of mass (COM), occurs at the coordinate given by taking the sum of the product of each mass times its position on the line and then dividing this sum by the total mass. These quantities $m_1 x_1$ and $m_2 x_2$ are called moments (with respect to the origin).

This idea generalizes to finitely many particles on a line. If we have particles of mass $m_1, m_2, \ldots, m_n$ at positions $x_1, x_2, \ldots, x_n$, respectively, then the COM is located at

\[
\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}
\]

**EXAMPLE 1** On the line, we have a mass of 3 at $-2$, a mass of 2 at 1, and a mass of 15 at 6 (Figure 3). What is the location of the center of mass?

**Solution** We have

\[
\bar{x} = \frac{(3)(-2) + (2)(1) + (15)(6)}{3 + 2 + 6} = 4.35
\]

It makes sense that the COM is closest to the mass of 15 at 6 because most of the mass in the system is concentrated at $x = 6$.

Now, how do we extend this idea to particles in the $xy$-plane (Figure 4)? Suppose we have a collection of particles with masses $m_1, m_2, \ldots, m_n$ at positions $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively. The COM is a point in the plane with coordinates $(\bar{x}, \bar{y})$. These coordinates are determined in the same manner as the COM of the particles on the line. That is,

\[
\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}, \quad \text{and} \quad \bar{y} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n}
\]

Like the particles on the line, the terms in the numerators in these COM equations are referred to as moments. For a particle of mass $m_1$ located at the point $(x, y)$, the moment with respect to the $x$-axis, $M_x$, and the moment with respect to the $y$-axis, $M_y$, are given by

\[
M_x = m_1 y \quad \text{(mass times directed distance to $x$-axis)} \\
M_y = m_1 x \quad \text{(mass times directed distance to $y$-axis)}
\]

Moments are additive: The moment of a system of $n$ particles with coordinates $(x_i, y_i)$ and mass $m_i$ (Figure 5) is the sum

\[
M_x = m_1 y_1 + m_2 y_2 + \cdots + m_n y_n = \sum_{i=1}^{n} m_i y_i \\
M_y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n = \sum_{i=1}^{n} m_i x_i
\]

With this notation, the center of mass $(\bar{x}, \bar{y})$ is given by

\[
\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}
\]
where \( M = m_1 + m_2 + \ldots + m_6 \) is the total mass of the system. That is, the \( x \)-coordinate of the COM of the system is the moment with respect to the \( y \)-axis divided by the total mass, and the \( y \)-coordinate is the moment with respect to the \( x \)-axis divided by the total mass.

**EXAMPLE 2** Find the COM of the system of three particles in Figure 6, having masses 2, 4, and 8 at locations (0, 2), (3, 1), and (6, 4), respectively.

**Solution** The total mass is \( M = 2 + 4 + 8 = 14 \) and the moments are

\[
\begin{align*}
M_x &= m_1y_1 + m_2y_2 + m_3y_3 = 2 \cdot 2 + 4 \cdot 1 + 8 \cdot 4 = 40 \\
M_y &= m_1x_1 + m_2x_2 + m_3x_3 = 2 \cdot 0 + 4 \cdot 3 + 8 \cdot 6 = 60
\end{align*}
\]

Therefore, \( \bar{x} = \frac{60}{14} \approx \frac{30}{7} \) and \( \bar{y} = \frac{40}{14} = \frac{20}{7} \). The COM is \( \left( \frac{30}{7}, \frac{20}{7} \right) \).

---

**Laminas (Thin Plates)**

Now consider a lamina (thin plate) of constant mass density \( \rho \) occupying the region under the graph of \( f \) over an interval \([a, b]\), where \( f \) is continuous and \( f(x) \geq 0 \) (Figure 7). In our calculations, we will use the principle of additivity of moments mentioned above for point masses:

If a region is decomposed into smaller, nonoverlapping regions, then the moment of the region is the sum of the moments of the smaller regions.

To compute the moment with respect to the \( y \)-axis, \( M_y \), we begin, as usual, by dividing \([a, b]\) into \( N \) subintervals of width \( \Delta x = (b - a)/N \) and endpoints \( x_j = a + j\Delta x \). This divides the lamina into \( N \) vertical strips (Figure 8). If \( \Delta x \) is small, the \( j \)-th strip is nearly rectangular of area \( f(x_j) \Delta x \) and mass \( \rho f(x_j) \Delta x \). Since all points in the strip lie at approximately the same distance \( x_j \) from the \( y \)-axis, the moment \( M_{y,j} \) of the \( j \)-th strip is approximately

\[
M_{y,j} \approx (\text{mass}) \times (\text{directed distance to } y\text{-axis}) = (\rho f(x_j) \Delta x)x_j
\]

By additivity of moments,

\[
M_y = \sum_{j=1}^{N} M_{y,j} \approx \rho \sum_{j=1}^{N} x_j f(x_j) \Delta x
\]

This is a Riemann sum whose value approaches \( \rho \int_{a}^{b} xf(x)\,dx \) as \( N \to \infty \), and thus,

\[
M_y = \rho \int_{a}^{b} xf(x)\,dx
\]
More generally, if the lamina occupies a vertically simple region between the graphs of two functions \( f_1 \) and \( f_2 \) over \([a, b]\), where \( f_1(x) \geq f_2(x) \), then

\[
M_y = \rho \int_a^b x(\text{length of vertical cut}) \, dx = \rho \int_a^b x (f_1(x) - f_2(x)) \, dx
\]

Think of the lamina as made up of vertical strips of length \( f_1(x) - f_2(x) \) at distance \( x \) from the y-axis (Figure 9).

In a similar manner, we can compute the \( x \)-moment by dividing the lamina into horizontal strips. This computation is straightforward to set up when the lamina occupies a horizontally simple region between two curves \( x = g_1(y) \) and \( x = g_2(y) \) with \( g_1(y) \geq g_2(y) \) over an interval \([c, d]\) along the y-axis (Figure 10):

\[
M_x = \rho \int_c^d y(\text{length of horizontal cut}) \, dy = \rho \int_c^d y (g_1(y) - g_2(y)) \, dy
\]

The total mass of the lamina is \( M = \rho A \), where \( A \) is the area of the lamina:

- For a vertically simple region: \( M = \rho A = \rho \int_a^b (f_1(x) - f_2(x)) \, dx \)
- For a horizontally simple region: \( M = \rho A = \rho \int_c^d (g_1(y) - g_2(y)) \, dy \)

The center-of-mass coordinates are the moments divided by the total mass:

\[
\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}
\]

The lamina will balance at the point \((\bar{x}, \bar{y})\) as in Figure 11.

**EXAMPLE 3** Find the moments and COM of the lamina of uniform density \( \rho \) occupying the region underneath the graph of \( f(x) = x^2 \) and above the \( x \)-axis for \( 0 \leq x \leq 2 \).

**Solution** The region is both vertically simple and horizontally simple. First, compute \( M_y \) using the fact that the region is vertically simple. By Eq. (1):

\[
M_y = \rho \int_0^2 x f(x) \, dx = \rho \int_0^2 x (x^2) \, dx = \rho \frac{x^4}{4} \bigg|_0^2 = 4 \rho
\]
Then compute \( M_x \) using the fact that the lamina occupies the vertically simple region between \( x = \sqrt{y} \) and \( x = 2 \) over the interval \([0, 4]\) along the y-axis (Figure 12). By Eq. (2),

\[
M_x = \rho \int_0^4 y (g_1(y) - g_2(y)) \, dy = \rho \int_0^4 y (2 - \sqrt{y}) \, dy
\]

\[
= \rho \left( \frac{y^2}{2} - \frac{2}{3} \sqrt{y^3} \right) \bigg|_0^4 = \rho \left( \frac{16}{2} - \frac{2}{3} \cdot 32 \right) = \frac{16}{5} \rho
\]

The plate has area \( A = \int_0^2 x^2 \, dx = \frac{x^3}{3} \) and total mass \( M = \frac{8}{3} \rho \). Therefore as shown in Figure 13,

\[
\overline{x} = \frac{M_x}{M} = \frac{\frac{16}{5} \rho}{\frac{8}{3} \rho} = \frac{6}{5} \quad \text{and} \quad \overline{y} = \frac{M_y}{M} = \frac{3}{2} \rho
\]

**CONCEPTUAL INSIGHT** The COM of a lamina of constant mass density \( \rho \) is also called the centroid. The centroid depends on the shape of the lamina, but not on its mass density because the factor \( \rho \) cancels in the ratios \( M_x/M \) and \( M_y/M \). In particular, in calculating the centroid, we can take \( \rho = 1 \). When mass density is not constant, the COM depends on both shape and mass density. We compute the COM for lamina with nonconstant density in Section 16.5.

So far, our process for computing the COM requires the lamina to occupy a region that is both vertically simple and horizontally simple (the former to compute \( M_y \), the latter to compute \( M_x \)). Fortunately, there are formulas that apply in the opposite circumstances: for computing \( M_x \) in the vertically simple case, \( M_y \) in the horizontally simple case.

Let us consider \( M_x \) for a lamina occupying a vertically simple region between the x-axis and the graph of \( y = f(x) \). As before, divide the region into \( N \) thin vertical strips of width \( \Delta x \) (see Figure 14). Let \( M_{x,j} \) be the moment with respect to the x-axis of the \( j \)th strip and let \( m_j \) be its mass. The strip is nearly rectangular with height \( f(x_j) \) and width \( \Delta x \), so \( m_j \approx \rho f(x_j) \Delta x \). Furthermore, \( M_{x,j} = \overline{y} m_j \), where \( \overline{y} \) is the y-coordinate of the COM of the strip. However, \( \overline{y} \approx \frac{1}{2} f(x_j) \) because the COM of a rectangle with uniform mass density is located at its center. Thus,

\[
M_{x,j} = m_j \overline{y} \approx \rho f(x_j) \Delta x \cdot \frac{1}{2} f(x_j) = \frac{1}{2} \rho f(x_j)^2 \Delta x
\]

\[
M_x = \sum_{j=1}^N M_{x,j} \approx \frac{1}{2} \rho \sum_{j=1}^N f(x_j)^2 \Delta x
\]

This is a Riemann sum whose value approaches \( \frac{1}{2} \rho \int_a^b f(x)^2 \, dx \) as \( N \to \infty \). The case of a region between the graphs of functions \( f_1 \) and \( f_2 \) where \( f_1(x) \geq f_2(x) \geq 0 \) is the difference of the moments corresponding to \( f_1(x) \) and \( f_2(x) \), by the principle of additivity of moments, so we obtain the following formulas for \( M_x \) for a lamina occupying a vertically simple region:

\[
M_x = \frac{1}{2} \rho \int_a^b f(x)^2 \, dx \quad \text{or} \quad M_x = \frac{1}{2} \rho \int_a^b (f_1(x)^2 - f_2(x)^2) \, dx
\]

The same idea holds for determining \( M_y \) in the horizontally simple case:

\[
M_y = \frac{1}{2} \rho \int_c^d g(y)^2 \, dy \quad \text{or} \quad M_y = \frac{1}{2} \rho \int_c^d (g_1(y)^2 - g_2(y)^2) \, dy
\]
EXAMPLE 4 Find the centroid of the shaded region in Figure 15.

Solution The centroid does not depend on \( \rho \), so we may set \( \rho = 1 \) and apply Eqs. (1) and (3) with \( f(x) = e^x \):

\[
M_x = \frac{1}{2} \int_1^3 f(x)^2 \, dx = \frac{1}{2} \int_1^3 e^{2x} \, dx = \left. \frac{1}{4} e^{2x} \right|_1^3 = \frac{e^6 - e^4}{4}
\]

Using Integration by Parts, we get

\[
M_y = \int_1^3 x f(x) \, dx = \left. \int_1^3 x e^x \, dx = (x-1)e^x \right|_1^3 = 2e^3
\]

The total mass is \( M = \int_1^3 e^x \, dx = (e^3 - e) \). The centroid has coordinates

\[
\bar{x} = \frac{M_y}{M} = \frac{2e^3}{e^3 - e} \approx 2.313, \quad \bar{y} = \frac{M_x}{M} = \frac{e^6 - e^4}{4(e^3 - e)} \approx 5.701
\]

The symmetry properties of an object give information about its centroid (Figure 16). For instance, the centroid of a square or circular plate is located at its center. Here is a precise formulation (see Exercise 49):

**THEOREM 1 Symmetry Principle** If a lamina is symmetric with respect to a line, then its centroid lies on that line.

EXAMPLE 5 Using Symmetry Find the centroid of the half-disk of radius 3, between the x-axis and the graph of \( f(x) = \sqrt{9 - x^2} \), as shown in Figure 17.

Solution Symmetry cuts our work in half. The half-disk is symmetric with respect to the y-axis, so the centroid lies on the y-axis, and hence \( \bar{x} = 0 \). It remains to calculate \( M_x \) and \( \bar{y} \). By Eq. (3) with \( \rho = 1 \),

\[
M_x = \frac{1}{2} \int_{-3}^{3} f(x)^2 \, dx = \frac{1}{2} \int_{-3}^{3} (9 - x^2) \, dx = \frac{1}{2} \left( 9x - \frac{3}{2} x^3 \right) \bigg|_{-3}^{3} = 9 - (-9) = 18
\]

The half-disk has area (and mass) equal to \( A = \frac{1}{2} \pi (3^2) = 9\pi/2 \), so

\[
\bar{y} = \frac{M_x}{M} = \frac{18}{9\pi/2} = \frac{4}{\pi} \approx 1.27
\]

EXAMPLE 6 Using Additivity and Symmetry Find the centroid of the region \( R \) in Figure 18.

Solution We set \( \rho = 1 \) because we are computing a centroid. The region \( R \) is symmetric with respect to the y-axis, and therefore, \( \bar{x} = 0 \). To find \( \bar{y} \), we compute the moment \( M_x \).

Step 1. Use additivity of moments.

Let \( M_x^{\triangle} \) and \( M_x^{\text{circle}} \) be the \( x \)-moments of the triangle and the circle. Then

\[
M_x = M_x^{\triangle} + M_x^{\text{circle}}
\]

Step 2. Moment of the circle.

To save work, we use the fact that the centroid of the circle is located at the center (0, 0) by symmetry. Thus, \( M_x^{\text{circle}} = 5 \) and we can solve for the moment:

\[
\bar{y} = \frac{M_x^{\text{circle}}}{M_{\text{circle}}} = \frac{5}{4\pi} = 5 \quad \Rightarrow \quad M_x^{\text{circle}} = 20\pi
\]

Here, the mass of the circle is its area \( M_{\text{circle}} = \pi (2^2) = 4\pi \) (since \( \rho = 1 \)).
Step 3. Moment of a triangle.

Let’s compute \( M^\triangle \) for an arbitrary triangle of height \( h \) and base \( b \) (Figure 19). Let \( \ell(y) \) be the width of the triangle at height \( y \). By similar triangles,

\[
\frac{\ell(y)}{h-y} = \frac{b}{h} \Rightarrow \ell(y) = b - \frac{b}{h} y
\]

By Eq. (2),

\[
M^\triangle_x = \int_0^h y \ell(y) \, dy = \left[ \frac{h^2}{2} - \frac{by^2}{2h} \right]_0^h = \frac{bh^2}{6}
\]

In our case, \( b = 4, h = 3 \), and \( M^\triangle_x = \frac{4 \cdot 3^2}{6} = 6 \).

Step 4. Computation of \( \bar{y} \).

\[
M_x = M^\triangle_x + M^\text{circle} = 6 + 20\pi
\]

The triangle has mass \( \frac{1}{2} \cdot 4 \cdot 3 = 6 \), and the circle has mass \( 4\pi \), so \( R \) has mass \( M = 6 + 4\pi \) and

\[
\bar{y} = \frac{M_x}{M} = \frac{6 + 20\pi}{6 + 4\pi} \approx 3.71
\]

We end this section with the Theorem of Pappus, attributed to Pappus of Alexandria, a mathematician of the fourth century BCE:

**Theorem 2. Theorem of Pappus** Let \( R \) be a region of area \( A \) in the plane. If we rotate \( R \) about an axis that is disjoint from \( R \), then the volume of the resulting solid is the product of \( A \) with the distance traveled by the centroid of \( R \).

**Proof** Since we assume a uniform density of 1, the area \( A \) of the region is equal to the mass \( M \). We will prove the theorem only in the special case that we are rotating about the \( x \)-axis and that we have a region bounded by \( y = f_1(x) \) and \( y = f_2(x) \) for \( a \leq x \leq b \) and \( f_1(x) \geq f_2(x) > 0 \). In this case, we know that the volume is given by

\[
V = \pi \int_a^b (f_1(x)^2 - f_2(x)^2) \, dx = 2\pi \left( \frac{1}{2} \int_a^b (f_1(x)^2 - f_2(x)^2) \, dx \right)
\]

\[
= 2\pi M_x = A \cdot 2\pi \frac{M_x}{A} = A \cdot 2\pi \bar{y}
\]

Thus \( V = A \cdot 2\pi \bar{y} \). This is the desired result because \( \bar{y} \) is the distance from the rotation axis to the centroid (since we are rotating about the \( x \)-axis), and therefore, \( 2\pi \bar{y} \) is the distance traveled by the centroid.

**Example 7** Find the formula for the volume of the solid torus obtained by rotating the disk of radius \( a \) centered at \((b, 0)\) about the \( y \)-axis, where \( a < b \), as in Figure 20.

**Solution** The centroid of the disk occurs at its center. So, \((\bar{x}, \bar{y}) = (b, 0)\). The Theorem of Pappus then says that

\[
V = A \cdot 2\pi \bar{x} = \pi a^2 2\pi b = 2\pi^2 a^2 b
\]
HISTORICAL PERSPECTIVE

We take it for granted that physical laws are best expressed as mathematical relationships. Think of \( F = ma \) or the universal law of gravitation. However, the fundamental insight that mathematics could be used to formulate laws of nature (and not just for counting or measuring) developed gradually, beginning with the philosophers of ancient Greece and culminating some 2000 years later in the discoveries of Galileo and Newton. Archimedes was one of the first scientists (perhaps the first) to formulate a precise physical law. Concerning the principle of the lever, Archimedes wrote, “Commensurable magnitudes balance at distances reciprocally proportional to their weight.” In other words, if weights of mass \( m_1 \) and \( m_2 \) are placed on a weightless lever at distances \( L_1 \) and \( L_2 \) from the fulcrum \( P \) (Figure 21), then the lever will balance if \( m_1 L_1 = m_2 L_2 \).

\[
\frac{m_1}{m_2} = \frac{L_2}{L_1}
\]

In our terminology, what Archimedes had discovered was the center of mass \( P \) of the system of weights (see Exercises 47 and 48).

9.4 SUMMARY

- The moments of a system of particles of mass \( m_j \) located at \((x_j, y_j)\) are

\[
\begin{align*}
M_x &= m_1 y_1 + \cdots + m_n y_n, \\
M_y &= m_1 x_1 + \cdots + m_n x_n
\end{align*}
\]

The center of mass (COM) has coordinates

\[
\bar{x} = \frac{M_y}{M} \quad \text{and} \quad \bar{y} = \frac{M_x}{M}
\]

where \( M = m_1 + \cdots + m_n \).

- Lamina (thin plate) of constant mass density \( \rho \) (Figure 22):

  - Vertically simple region. Mass: \( M = \rho \int_a^b (f_1(x) - f_2(x)) \, dx \)
  
  Moments: \( M_x = \frac{1}{2} \rho \int_a^b (f_1(x)^2 - f_2(x)^2) \, dx \), \( M_y = \rho \int_a^b x(f_1(x) - f_2(x)) \, dx \)

  - Horizontally simple region. Mass: \( M = \rho \int_c^d (g_1(y) - g_2(y)) \, dy \)
  
  Moments: \( M_x = \rho \int_c^d y(g_1(y) - g_2(y)) \, dy \), \( M_y = \frac{1}{2} \rho \int_c^d (g_1(y)^2 - g_2(y)^2) \, dy \)

- The coordinates of the center of mass (also called the centroid) are

\[
\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}
\]

- Additivity: If a region is decomposed into smaller nonoverlapping regions, then the moment of the region is the sum of the moments of the smaller regions.

- Symmetry Principle: If a lamina of constant mass density is symmetric with respect to a given line, then the center of mass (centroid) lies on that line.

- The Theorem of Pappus: If a lamina is rotated about a disjoint axis, then the volume of the resulting solid of revolution is the area of the lamina times the distance traveled by the centroid.
9.4 EXERCISES

Preliminary Questions

1. What are the x- and y-moments of a lamina whose center of mass is located at the origin?
2. A thin plate has mass 3. What is the x-moment of the plate if its center of mass has coordinates (2, 7)?
3. The center of mass of a lamina of total mass 5 has coordinates (2, 1). What are the lamina’s x- and y-moments?

Exercises

1. On a line, there are particles located at −3, −1, 1, 2, and 5. Their masses are 8, 2, 3, 2, and 1, respectively.
   (a) What is the center of mass of the system?
   (b) Keeping the other four masses the same, what would the mass at 5 need to be in order to have the center of mass be 0?
2. On a line, there are particles located at 1, 2, 3, 4, and 5. Their masses are 1, 2, 3, 4, and 5, respectively.
   (a) What is the center of mass of the system?
   (b) If we add a particle of mass 6 at 6, what is the center of mass?
   (c) If we add particles of mass 4 at j for j = 6 to n, what is the center of mass? Hint: Power sums that were introduced in Section 5.1 will be helpful here.
3. Four particles are located at points (1, 1), (1, 2), (4, 0), and (3, 1).
   (a) Find the moments 41 and 42 and the center of mass of the system,
       assuming that the particles have equal mass m.
   (b) Find the center of mass of the system, assuming the particles have masses 3, 2, 5, and 7, respectively.
4. Find the center of mass of the system of particles of masses 4, 2, 5, and 1 located at (1, 2), (−3, 2), (2, −1), and (4, 0).
5. Point masses of equal size are placed at the vertices of the triangle with coordinates (a, 0), (b, 0), and (0, c).
   Show that the center of mass of the system of masses has coordinates \((a+b+c)/3, c/3\).
6. Points of mass m1, m2, and m3 are placed at the points (−1, 0), (3, 0), and (0, 4).
   (a) Suppose that m1 = 6. Find m2 such that the center of mass lies on the y-axis.
   (b) Suppose that m1 = 6 and m2 = 4. Find the value of m3 such that \( y = 2 \).
7. Sketch the lamina \( S \) of constant density \( \rho = 3 \text{ g/cm}^2 \) occupying the region beneath the graph of \( y = \sqrt{x} \) for \( 0 \leq x \leq 3 \).
   (a) Use Eqs. (1) and (2) to compute \( M_x \) and \( M_y \).
   (b) Find the area and the center of mass of \( S \).
8. Use Eqs. (1) and (3) to find the moments and center of mass of the lamina \( S \) of constant density \( \rho = 2 \text{ g/cm}^2 \) occupying the region between \( y = x^2 \) and \( y = 9x \) over \([0, 3] \).
   Sketch \( S \), indicating the location of the center of mass.
9. Find the moments and center of mass of the lamina of uniform density \( \rho \) occupying the region \( R \) under the graph of \( y = 1 - x^2 \) for \( 0 \leq x \leq 1 \) in two ways, first using Eq. (2) and then using Eq. (3).
10. Calculate \( M_y \) (assuming \( \rho = 1 \)) for the region under the graph of \( y = 1 - x^2 \) for \( 0 \leq x \leq 1 \) in two ways, first using Eq. (2) and then using Eq. (3).
11. Let \( T \) be the triangular lamina in Figure 23 and assume \( P = 6 \).
   (a) Show that the horizontal cut at height \( y \) has length \( 4 - \frac{3}{2} y \) and use Eq. (2) to compute \( M_y \) (with \( \rho = 1 \)).
   (b) Use the Symmetry Principle to show that \( M_x = 0 \) and find the center of mass.

4. Explain how the Symmetry Principle is used to conclude that the centroid of a rectangle is the center of the rectangle.
5. Give an example of a plate such that its center of mass does not occur at any point on the plate.
6. Draw a plate such that its center of mass occurs on its boundary. (You do not need to verify this fact. It should just be believable from the drawing.)

FIGURE 23 Isosceles triangle.

12. Let \( T \) be the triangular lamina in Figure 23, and assume \( P = 8 \) and \( \rho = 1 \). Find the center of mass.

In Exercises 13–21, find the centroid of the region lying underneath the graph of the function over the given interval.

13. \( f(x) = 4x \), \([0, 1]\)
14. \( f(x) = 6 - 2x \), \([0, 3]\)
15. \( f(x) = \sqrt{x} \), \([1, 4]\)
16. \( f(x) = x^3 \), \([0, 1]\)
17. \( f(x) = 9 - x^2 \), \([0, 3]\)
18. \( f(x) = (1 + x^2)^{-1/2} \), \([0, 3]\)
19. \( f(x) = e^{-x} \), \([0, 4]\)
20. \( f(x) = \ln x \), \([1, 2]\)
21. \( f(x) = \sin x \), \([0, \pi]\)
22. Calculate the moments and center of mass of the lamina occupying the region between the curves \( y = x + 4 \) and \( y = 2 - x \) for \( 0 \leq x \leq 2 \).
   Using symmetry, explain why the centroid of the region lies on the line \( y = 3 \).
   Verify this by computing the moments and the centroid.

In Exercises 24–29, find the centroid of the region lying between the graphs of the functions over the given interval.

24. \( y = x \), \( y = \sqrt{x} \), \([0, 1]\)
25. \( y = x^3 \), \( y = \sqrt{x} \), \([0, 1]\)
26. \( y = x^{-1} \), \( y = 2 - x \), \([1, 2]\)
27. \( y = e^x \), \( y = 1 \), \([0, 1]\)
28. \( y = \ln x \), \( y = x - 1 \), \([1, 3]\)
29. \( y = \sin x \), \( y = \cos x \), \([0, \pi/4]\)
30. Sketch the region enclosed by \( y = x + 1 \) and \( y = (x - 1)^2 \) and find its centroid.
31. Sketch the region enclosed by \( y = 0 \), \( y = (x + 1)^2 \), and \( y = (1 - x)^2 \), and find its centroid.
In Exercises 32–36, find the centroid of the region.

32. Top half of the ellipse \( \frac{x^2}{4} + \frac{y^2}{2} = 1 \)
33. Top half of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) for arbitrary \( a, b > 0 \)
34. Semicircle of radius \( r \) with center at the origin
35. Quarter of the unit circle lying in the first quadrant
36. Region between \( y = x(a - x) \) and the \( x \)-axis for \( a > 0 \)

37. Find the centroid of the shaded region of the semicircle of radius \( r \) in Figure 24. What is the centroid when \( r = 1 \) and \( h = \frac{1}{2} \)? Hint: Use geometry rather than integration to show that the area of the region is \( r^2 \sin^{-1} \left( \frac{1}{2} \right) - \frac{1}{2} \sqrt{2} - \frac{1}{2} \).

\[ \text{FIGURE 24} \]

38. Sketch the region between \( y = x^m \) and \( y = x^n \) for \( 0 \leq x \leq 1 \), where \( m > n \geq 0 \), and find the COM of the region. Find a pair \((n, m)\) such that the COM lies outside the region.

39. Find the formula for the volume of a right circular cone of height \( H \) and radius \( R \) using the Theorem of Pappus as applied to the triangle bounded by the \( x \)-axis, the \( y \)-axis, and the line \( y = \frac{2H}{R} x + H \), rotated about the \( y \)-axis.

40. Use the Theorem of Pappus to find the centroid of the half-disk bounded by \( y = \sqrt{R^2 - x^2} \) and the \( x \)-axis.

In Exercises 41–43, use the additivity of moments to find the COM of the region.

41. Isosceles triangle of height 2 on top of a rectangle of base 4 and height 3 (Figure 25)

\[ \text{FIGURE 25} \]

42. An ice cream cone consisting of a semicircle on top of an equilateral triangle of side 6 (Figure 26)

\[ \text{FIGURE 26} \]

43. Three-quarters of the unit circle (remove the part in the fourth quadrant)

44. Let \( S \) be the lamina of mass density \( \rho = 1 \) obtained by removing a circle of radius \( r \) from the circle of radius \( 2r \) shown in Figure 27. Let \( M_x^S \) and \( M_y^S \) denote the moments of \( S \). Similarly, let \( M_x^{big} \) and \( M_y^{big} \) be the \( y \)-moments of the larger and smaller circles.

(a) Use the Symmetry Principle to show that \( M_y^S = 0 \).

(b) Show that \( M_y^S = M_y^{big} - M_y^{small} \) using the additivity of moments.

(c) Find \( M_y^{big} \) and \( M_y^{small} \) using the fact that the COM of a circle is its center. Then compute \( M_y^S \) using (b).

(d) Determine the COM of \( S \).

\[ \text{FIGURE 27} \]

45. Find the COM of the lamina in Figure 28 obtained by removing squares of side 2 from a square of side 8.

\[ \text{FIGURE 28} \]

Further Insights and Challenges

46. A median of a triangle is a segment joining a vertex to the midpoint of the opposite side. Show that the centroid of a triangle lies on each of its medians, at a distance two-thirds down from the vertex. Then use this fact to prove that the three medians intersect at a single point. Hint: Simplify the calculation by assuming that one vertex lies at the origin and another on the \( x \)-axis.

47. Let \( P \) be the COM of a system of two weights with masses \( m_1 \) and \( m_2 \) separated by a distance \( d \). Prove Archimedes's Law of the (weightless) Lever: \( P \) is the point on a line between the two weights such that \( m_1 L_1 = m_2 L_2 \), where \( L_j \) is the distance from mass \( j \) to \( P \).
48. Find the COM of a system of two weights of masses $m_1$ and $m_2$ connected by a lever of length $d$ whose mass density $\rho$ is uniform. Hint: The moment of the system is the sum of the moments of the weights and the lever.

49. **Symmetry Principle** Let $R$ be the region under the graph of $y = f(x)$ over the interval $[-a, a]$, where $f(x) \geq 0$. Assume that $R$ is symmetric with respect to the $y$-axis.

(a) Explain why $y = f(x)$ is even—that is, why $f(x) = f(-x)$.
(b) Show that $y = xf(x)$ is an odd function.
(c) Use (b) to prove that $M_y = 0$.
(d) Prove that the COM of $R$ lies on the $y$-axis (a similar argument applies to symmetry with respect to the $x$-axis).

50. Prove directly that Eqs. (2) and (3) are equivalent in the following situation. Let $f$ be a positive decreasing function on $[0, b]$ such that $f(b) = 0$. Set $d = f(0)$ and $g(y) = f^{-1}(y)$. Show that

$$\frac{1}{2} \int_0^b f(x)^2 \, dx = \int_0^d yg(y) \, dy$$

**CHAPTER REVIEW EXERCISES**

1. Compute $p(X \leq 1)$, where $X$ is a continuous random variable with probability density $p(x) = \frac{1}{\pi(x^2 + 1)}$.

2. Show that $p(x) = \frac{1}{2} e^{-x^2/2} + \frac{1}{6} e^{-x^2/3}$ is a probability density over the domain $[0, \infty)$ and find its mean.

3. Find a constant $C$ such that $p(x) = C x^2 e^{-x^2}$ is a probability density over the domain $[0, \infty)$ and compute $P(0 \leq X \leq 1)$.

4. The interval between patient arrivals in an emergency department is a random variable with exponential density function $p(t) = 0.125 e^{-0.125t}$ (in minutes). What is the average time between patient arrivals? What is the probability of two patients arriving within 3 min of each other?

5. Calculate the following probabilities, assuming that $X$ is normally distributed with mean $\mu = 40$ and $\sigma = 5$.
   (a) $P(X \geq 45)$
   (b) $P(0 \leq X \leq 40)$

6. According to kinetic theory, the molecules of ordinary matter are in constant random motion. The energy $E$ of a molecule is a random variable with density function $p(E) = \frac{1}{kT} e^{-E/(kT)}$, where $T$ is the temperature (in kelvins) and $k$ is Boltzmann’s constant. Compute the mean kinetic energy $\overline{E}$ in terms of $k$ and $T$.

In Exercises 7–10, calculate the arc length over the given interval.

7. $y = \frac{x^5}{10} + \frac{x^3}{6}$, $[1, 2]$
8. $y = e^{x^2} + e^{-x^2/2}$, $[0, 2]$
9. $y = 4x - 2$, $[-2, 2]$
10. $y = x^2/3$, $[1, 8]$

11. Show that the arc length of $y = 2\sqrt{x}$ over $[0, a]$ is equal to $\sqrt{a(a + 1) + \ln(ab + a + 1)}$. Hint: Apply the substitution $x = \tan^2 \theta$ to the arc length integral.

12. **CAS** Compute the trapezoidal approximation $T_3$ to the arc length $s$ of $y = \tan x$ over $[0, \frac{\pi}{4}]$.

In Exercises 13–16, calculate the surface area of the solid obtained by rotating the curve over the given interval about the $x$-axis.

13. $y = x + 1$, $[0, 4]$
14. $y = \frac{2}{3}x^{3/4} - \frac{2}{5}x^{5/4}$, $[0, 1]$
15. $y = \frac{2}{3}x^{3/2} - \frac{x}{2}^{1/2}$, $[1, 2]$
16. $y = \frac{1}{2}x^2$, $[0, 2]$

17. Compute the total surface area of the coin obtained by rotating the region in Figure 1 about the $x$-axis. The top and bottom parts of the region are semicircles with a radius of 1 mm.

18. Calculate the fluid force on the side of a right triangle of height 3 m and base 2 m submerged in water vertically, with its upper vertex at the surface of the water.

19. Calculate the fluid force on the side of a right triangle of height 3 m and base 2 m submerged in water vertically, with its upper vertex located at a depth of 4 m.
20. A plate in the shape of the shaded region in Figure 2 is submerged in water. Calculate the fluid force on a side of the plate if the water surface is \( y = 1 \).

![Figure 2](image)

21. Figure 3 shows an object whose face is an equilateral triangle with 5-m sides. The object is 2 m thick and is submerged in water with its vertex 3 m below the water surface. Calculate the fluid force on both a triangular face and a slanted rectangular edge of the object.

![Figure 3](image)

22. The end of a horizontal oil tank is an ellipse (Figure 4) with equation \((x/4)^2 + (y/3)^2 = 1\) (length in meters). Assume that the tank is filled with oil of density 900 kg/m³.

(a) Calculate the total force \( F \) on the end of the tank when the tank is full.

(b) Would you expect the total force on the lower half of the tank to be greater than, less than, or equal to \( \frac{1}{2} F \)? Explain. Then compute the force on the lower half exactly and confirm (or refute) your expectation.

23. Calculate the moments and COM of the lamina occupying the region under \( y = x(4-x) \) for \( 0 \leq x \leq 4 \), assuming a density of \( \rho = 1200 \) kg/m³.

24. Sketch the region between \( y = 4(x + 1)^{-1} \) and \( y = 1 \) for \( 0 \leq x \leq 3 \), and find its centroid.

25. Find the centroid of the region between the semicircle \( y = \sqrt{1-x^2} \) and the top half of the ellipse \( y = \frac{1}{2} \sqrt{1-x^2} \) (Figure 2).

26. Find the centroid of the shaded region in Figure 5 bounded on the left by \( x = 2y^2 - 2 \) and on the right by a semicircle of radius 1. *Hint:* Use symmetry and additivity of moments.

![Figure 5](image)

27. Use the Theorem of Pappus to find the volume of the solid of revolution obtained by rotating the region in the first quadrant bounded by \( y = x^2 \) and \( y = \sqrt{x} \) about the y-axis.

28. Use the Theorem of Pappus to find a formula for the volume of the solid obtained by rotating the triangle with vertices \((1,0), (3,0),\) and \((2,2)\) about the y-axis.
10 INTRODUCTION TO DIFFERENTIAL EQUATIONS

Differential equations are among the most powerful tools we have for analyzing the world with mathematics. They are used to formulate the fundamental laws of nature (from Newton’s laws to Maxwell’s equations and the laws of quantum mechanics) and to model the most diverse physical phenomena. This chapter provides a brief introduction to some elementary techniques and applications of this important subject.

10.1 Solving Differential Equations

A differential equation is an equation that involves an unknown function and its first or higher derivatives. The following differential equations are among the ones that we will consider at various points in this chapter:

\[
\frac{dy}{dx} = 1 - 6e^{2x}, \quad \frac{dP}{dt} = rP - N, \quad \frac{dy}{dt} + \frac{1}{t + 30} y = 4
\]

The order of a differential equation is the highest order derivative that appears in the equation. In this chapter, we restrict our attention to first-order differential equations. A solution to a differential equation is a function that satisfies the equation, and a primary goal when working with a differential equation is to find the solutions. The next example considers a simple differential equation like the ones we considered in Section 5.3.

**EXAMPLE 1** Consider the differential equation \( \frac{dy}{dx} = 1 - 6e^{2x} \).

(a) Find the solutions.

(b) Find the solution satisfying \( y(0) = 5 \).

**Solution**

(a) The form of the differential equation enables us to solve it directly by antidifferentiation:

\[
y(x) = \int \left(1 - 6e^{2x}\right) dx = x - 3e^{2x} + C
\]

(b) To satisfy \( y(0) = 5 \), we must have \( 5 = 0 - 3e^{0} + C \). Therefore, \( C = 8 \). So the function

\[
y(x) = x - 3e^{2x} + 8
\]

satisfies the differential equation and the condition \( y(0) = 5 \).

Generally, a differential equation has a family of solutions. For example, the solutions to the differential equation \( \frac{dy}{dx} = 1 - 6e^{2x} \) in Example 1 are the functions \( y(x) = x - 3e^{2x} + C \). The graphs of these solutions form a collection of curves in the \( xy \)-plane (Figure 1). An expression for the family of solutions to a differential equation, such as \( y(x) = x - 3e^{2x} + C \) in Example 1, is called a general solution. For each value of \( C \) in the general solution, we obtain a particular solution. The function \( y(x) = x - 3e^{2x} + 8 \) is the particular solution to \( \frac{dy}{dx} = 1 - 6e^{2x} \) satisfying \( y(0) = 5 \). A condition \( y(x_0) = y_0 \) for identifying a particular solution is called an initial condition. A problem consisting of a differential equation and an initial condition is called an Initial Value Problem.
Separation of Variables

A differential equation is called separable if it can be written in the form

\[
\frac{dy}{dx} = f(x)g(y)
\]

For example,

- \(\frac{dy}{dx} = y \sin x\) and \(\frac{dy}{dx} = x + xy\) are separable.

- \(\frac{dy}{dx} = x + y\) is not separable because \(x + y\) is not a product \(f(x)g(y)\).

Separable equations are solved using the method of Separation of Variables: Move the terms involving \(y\) to the left and those involving \(x\) to the right. Then set up integrals and evaluate.

\[
\frac{dy}{dx} = f(x)g(y)
\]

\[
\int \frac{1}{g(y)} \, dy = \int f(x) \, dx
\]

**EXAMPLE 2** Find the general solution to \(\frac{dy}{dx} = 6xy^2\) and the particular solution satisfying \(y(0) = -1\).

**Solution** First, we observe that \(y(x) = 0\) is a solution to the differential equation. Then, assuming \(y \neq 0\), we use Separation of Variables as follows:

\[
y^{-2} \, dy = 6x \, dx
\]

\[
\int y^{-2} \, dy = \int 6x \, dx
\]

\[
-y^{-1} = 3x^2 + C
\]

\[
y = \frac{-1}{3x^2 + C}
\]

Thus, the general solution to \(\frac{dy}{dx} = 6xy^2\) consists of the functions

\[
y(x) = 0 \quad \text{and} \quad y(x) = \frac{-1}{3x^2 + C} \quad \text{for all} \quad C
\]

This family of solutions is sketched in Figure 2, and the particular solution satisfying the initial condition \(y(0) = -1\) is highlighted. To find the particular solution satisfying \(y(0) = -1\), we set \(x = 0\) and \(y = -1\) in the general solution and solve for \(C\):

\[
-1 = \frac{-1}{3(0)^2 + C}
\]

\[
-1 = \frac{-1}{C}
\]

\[
C = 1
\]

Thus, the particular solution satisfying the given initial condition is

\[
y = \frac{-1}{3x^2 + 1}
\]
It is beneficial to check your Initial Value Problem solution: First, \( y(0) = \frac{-1}{3x^2+1} = -1 \), as required. Also,

\[
\frac{dy}{dx} = \frac{d}{dx} \left( \frac{-1}{3x^2+1} \right) = \frac{6x}{(3x^2+1)^2} = 6xy^2
\]

Therefore, \( y = \frac{-1}{3x^2+1} \) satisfies the Initial Value Problem.

In the field of glaciology, a simple separable differential equation is used to model a glacier's thickness, the vertical distance from its surface to its base (Figure 3). A glacier is a river of ice, and like a river of water, it flows (but does so relatively slowly). Mass is added to a glacier by precipitation and is lost by sublimation, evaporation, and meltwater runoff. We assume that the amount of mass lost equals the amount that is gained, and therefore, the glacier's shape does not change. We model the glacier's thickness in meters by \( T(x) \) where \( x \) is the distance in meters from the front of the glacier (the location where the glacier ends on land or water, also known as the terminus).

In a simple force-balance model, where fluid pressure forces in a glacier are balanced with a friction force at its base, the following differential equation for \( T(x) \) results:

\[
\frac{T}{dx} = \frac{\tau}{\rho g}
\]

In the equation, \( \tau \) is the friction at the base of the glacier, \( \rho \) is the ice density, and \( g \) is acceleration due to gravity.

**EXAMPLE 3** A Glacial Thickness Model Let \( \rho = 917 \text{ kg/m}^3 \), \( g = 9.8 \text{ m/s}^2 \), and \( \tau = 75,000 \text{ N/m}^2 \) in Eq. (2). Use \( T(0) = 0 \) for an initial condition, and solve for \( T(x) \). Then use \( T(x) \) to determine the thickness of the glacier 1 km from its terminus.

**Solution** The differential equation that we need to solve is

\[
T \frac{dT}{dx} = \frac{75,000}{(917)(9.8)}
\]

It is a separable differential equation. We use the approximate value of 8.35 for the right-hand side, and proceed as follows:

\[
\int T \, dT = \int 8.35 \, dx
\]

\[
\frac{1}{2} T^2 = 8.35x + C
\]

\[
T(x) = \sqrt{16.7x + C}
\]

Since \( T(0) = 0 \), we obtain \( T(x) = \sqrt{16.7x} \) (Figure 4).

At a distance of 1 km from the terminus, the thickness is \( T(1000) = \sqrt{16700} \approx 129 \text{ m} \).

Another model that results in a separable differential equation involves the water level in a container that has a leak through a hole in the bottom (Figure 5). We let \( y(t) \) be the height of water in the container as a function of time, \( A(y) \) be the cross-sectional area of the container at height \( y \), and \( B \) be the area of the hole in the bottom of the container.

In a model where we assume the volume lost from the container equals the volume that flows out, we apply an important fluid-flow law known as Torricelli's Law to obtain the following differential equation:

\[
\frac{dy}{dt} = -\frac{B \sqrt{2gy}}{A(y)}
\]

In the equation, \( g \) is acceleration due to gravity.
EXAMPLE 4 A cylindrical container of height 400 cm and radius 100 cm is filled with water. In the bottom of the container is a square hole of side length 2 cm through which water leaks. Determine the water level \( y(t) \), in cm, as a function of time \( t \), in seconds. How long does it take for the tank to go from full to empty?

**Solution** The horizontal cross section of the cylinder is a circle of radius \( r = 100 \) cm and area \( A(y) = \pi r^2 = 10,000\pi \) cm\(^2\) (Figure 6). The hole has area \( B = 4 \) cm\(^2\). With the units we are employing, we set \( g = 980 \) cm/s\(^2\). Now, Eq. (3) becomes

\[
\frac{dy}{dt} = -\frac{4\sqrt{1950y}}{10,000\pi} \approx -0.0056\sqrt{y}
\]

Since the tank is full at \( t = 0 \), we have the initial condition \( y(0) = 400 \).

Now, \( y(t) = 0 \) is a solution to the differential equation, but it does not satisfy the initial condition. We assume \( y \neq 0 \) and solve using Separation of Variables:

\[
\int \frac{dy}{\sqrt{y}} = -0.0056 \int dt
\]

\[
2y^{1/2} = -0.0056t + C
\]

At this point, rather than solving for general \( y(t) \), it is convenient to determine the value of \( C \) for which the initial condition is satisfied. Substituting \( y = 400 \) and \( t = 0 \) into Eq. (5), we obtain \( C = 40 \). Thus, we have

\[
2y^{1/2} = -0.0056t + 40
\]

\[
y(t) = (20 - 0.0028t)^2
\]

To determine the time \( t_e \) that it takes to empty the tank, we solve

\[
y(t_e) = (20 - 0.0028t_e)^2 = 0 \quad \Rightarrow \quad t_e \approx 7142 \text{ s}
\]

So, the tank is empty after 7142 s, or nearly 2 hours (Figure 7).

Note that the solution \( y(t) = (20 - 0.0028t)^2 \) in the previous example is valid only for \( 0 \leq t \leq t_e \). For \( t > t_e \) the function \( y(t) = (20 - 0.0028t)^2 \) does not satisfy the differential equation; it is increasing while the differential equation requires that \( \frac{dy}{dt} \leq 0 \) for all \( t \). This “extraneous solution” (for \( t > t_e \)) arose as a result of squaring both sides of Eq. (6). It is clear physically that after the tank empties, \( y(t) = 0 \) for all \( t \). Mathematically, it also works to extend the solution so that \( y(t) = 0 \) for \( t \geq t_e \) because \( y(t) = 0 \) also satisfies the differential equation. Thus, the solution to the Initial Value Problem, defined for all \( t \geq 0 \), is

\[
y(t) = \begin{cases} 
(20 - 0.0028t)^2 & 0 \leq t \leq t_e \\
0 & t \geq t_e
\end{cases}
\]

### Exponential Growth and Decay

Many phenomena are modeled by the differential equation \( \frac{dy}{dt} = ky \), which assumes that: 

**The rate of change of \( y \) is proportional to the amount of \( y \) present.**

Examples include the growth of a population, the spread of a computer virus, the decay of a radioactive substance, and the elimination of a drug from a patient's bloodstream. For “growth” and “spread,” the proportionality constant \( k \) in the differential equation is positive (\( y \) would be increasing); for “decay” and “elimination,” it is negative.
EXAMPLE 5 Find the general solution to $\frac{dy}{dt} = ky$.

Solution First, note that $y(t) = 0$ is a solution to the differential equation. Then, assuming $y \neq 0$, we employ Separation of Variables:

$\int \frac{1}{y} dy = \int k \, dt$

$\ln |y| = kt + C$

$|y| = e^{kt+C}$

$y = D e^{kt}$

Two simplifications occurred at the last step. First, we write $e^{kt+C} = e^k e^C = D e^k$, where $e$ to the power of an arbitrary constant is replaced by a positive constant term $D$. Second, if $|y| = D e^k$, then $y = \pm D e^k$, which can be expressed as $y = D e^k$, allowing $D$ to be positive or negative. Furthermore, since $y = 0$ solves the differential equation, the possibility that $D = 0$ is also allowed. Therefore, the functions $y(t) = D e^k$, for all $D$, satisfy the differential equation. Summarizing,

The general solution to $\frac{dy}{dt} = ky$ is $y(t) = D e^{kt}$.

The general solution with $k > 0$ is illustrated in Figure 8.

The function $y = D e^k$ models exponential growth when $k > 0$, and it models exponential decay when $k < 0$. For a simple population growth model, it is natural to assume that the rate of growth of the population is proportional to the population present because we expect some fixed percentage of all individuals will contribute to reproduction. Of course, we cannot expect population growth to occur at an increasing rate forever. However, at least for an initial growth period, as in the next example, exponential population growth is feasible.

EXAMPLE 6 In the laboratory, the *Escherichia coli* bacteria grows such that the rate of change of the population is proportional to the population present. Assume that 1000 bacteria are initially present, and 1500 are present after 1 hour.

(a) Determine $P(t)$, the population after $t$ hours.

(b) How large is the population after 5 h?

(c) How long does it take for the population to double in size?

Solution

(a) Since the rate of change $P'(t)$ is proportional to $P(t)$, we have the differential equation $P'(t) = k P(t)$. Furthermore, we are given the initial conditions $P(0) = 1000$ and $P(1) = 1500$. Together, the differential equation and initial conditions make up the Initial Value Problem that we must solve.

Note that we have two initial conditions. Both are necessary because we need to determine the constant term $D$ in the general solution to the differential equation and the term $k$ in the differential equation itself.

The general solution to the differential equation is $P(t) = D e^{kt}$, and the initial condition $P(0) = 1000$ implies that $D = 1000$. Thus, the solution is in the form

$P(t) = 1000e^{kt}$
To determine $k$, we substitute $P = 1500$ and $t = 1$ into $P(t) = 1000e^{kt}$ and solve for $k$:

\[
1500 = 1000e^k \\
1.5 = e^k \\
\ln 1.5 = k \\
k \approx 0.405
\]

Therefore, with an approximate value for $k$, we have that $P(t) = 1000e^{0.405t}$ (Figure 9).

(b) After 5 h the population is $P(5) = 1000e^{0.405(5)} \approx 7576$ bacteria.

(c) To determine when the population doubles, we find $t$ such that $P = 2000$:

\[
2000 = 1000e^{0.405t} \\
2 = e^{0.405t} \\
\ln 2 = 0.405t \\
t = \frac{\ln 2}{0.405} \approx 1.71 \text{ h}
\]

In the previous example, we computed the time it takes for the initial population to double. In exponential growth, the time to double in size is called the doubling time. Similarly, in exponential decay, the half-life is the time it takes for an initial amount to decrease to half its size. Doubling time and half-life do not depend on the initial amount, but depend only on the value of the term $k$, as follows:

**Half-life and Doubling Time for $P(t) = P_0e^{kt}$**

- **Exponential decay:** $k < 0$  
  half-life $= \frac{\ln 0.5}{k}$
- **Exponential growth:** $k > 0$  
  doubling time $= \frac{\ln 2}{k}$

It is straightforward to verify these formulas. For example, suppose that $t = \frac{\ln 2}{k}$; then

\[
P\left(\frac{\ln 2}{k}\right) = P_0e^{k\left(\frac{\ln 2}{k}\right)} = P_0e^{\ln 2} = 2P_0
\]

Thus, $\frac{\ln 2}{k}$ is the value of $t$ at which $P(t)$ is equal to twice the initial amount $P_0$.

**EXAMPLE 7** A patient is administered 400 mg of penicillin, and after 50 minutes 300 mg remains in her bloodstream. Let $A(t)$ represent the amount of penicillin (in mg) in her bloodstream at minutes after the drug was administered. Assume that the drug is leaving her bloodstream at a rate proportional to the amount in her bloodstream. Determine $A(t)$ and the half-life of the resulting exponential decay.

**Solution**

Since the rate of change is proportional to the amount present, $A'(t) = kA(t)$. Furthermore, we are given the initial conditions $A(0) = 400$ and $A(50) = 300$, and therefore, we have an Initial Value Problem that we must solve.

The general solution to the differential equation is $A(t) = De^{kt}$, and the initial condition $A(0) = 400$ implies that $D = 400$. The solution is therefore in the form $A(t) = 400e^{kt}$. 

---

**Exponential growth cannot continue over long periods of time.** In this example, the population would grow to over $10^{94}$ bacteria cells after 3 weeks—much more than the estimated number of atoms in the observable universe. In typical population growth, an initial rapid growth phase is followed by a period in which growth slows. In Section 10.4, we adjust the population growth model to take into account a limited resource base for the population. This results in solutions where the population initially grows rapidly but then levels off.
To determine $k$, we substitute $A = 300$ and $t = 50$ into $A(t) = 400e^{kt}$ and solve for $k$:

$$300 = 400e^{50k}$$

$$0.75 = e^{50k}$$

$$\ln 0.75 = 50k$$

$$k = \frac{\ln 0.75}{50} \approx -0.0058$$

Thus, with this approximate value for $k$, we have that $A(t) = 400e^{-0.0058t}$ (Figure 10). The half-life is $\frac{\ln 0.5}{k} \approx \frac{\ln 0.5}{-0.0058} \approx 120$ min. \hfill \blacksquare

If an Initial Value Problem models a physical relationship, we would want to know whether or not a solution exists and whether or not the solution is unique. Having no solution suggests that the model is not correct. Perhaps we would need to reevaluate the assumptions used to build the model. Having a unique solution would be desirable if we expect the phenomenon to change in only one way and we wish to make predictions.

Note that the Initial Value Problem, $\frac{dy}{dt} = ky$ and $y(0) = y_0$, has the solution $y = y_0e^{kt}$ [obtained from Eq. (8) using the initial condition $y(0) = y_0$]. It is the only solution that is provided by the general solution formula. We can prove that this solution is unique by showing that all solutions to the differential equation are provided by the general solution (see Exercise 61).

### 10.1 SUMMARY

- A **directly integrable** differential equation is in the form $\frac{dy}{dx} = f(x)$. It is solved by direct ant differentiation.
- A **separable first-order** differential equation is in the form $\frac{dy}{dx} = f(x)g(y)$

Differential equations of this type are solved by **Separation of Variables**: Move all terms involving $y$ to the left and all terms involving $x$ to the right and integrate

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

- A simple differential equation model for exponential growth and decay is $\frac{dy}{dt} = ky$

The general solution is $y(t) = De^{kt}$. Exponential growth occurs when $k > 0$, and the doubling time is $\frac{\ln 2}{k}$. Exponential decay occurs when $k < 0$, and the half-life is $\frac{\ln 0.5}{k}$.

### 10.1 EXERCISES

**Preliminary Questions**

1. Determine the order of the following differential equations:
   - (a) $x^2y' = 1$
   - (b) $(y')^3 + x = 1$
   - (c) $y'' + x^2y' = 2$
   - (d) $\sin(y') + x = y$
   - (e) $\frac{dx}{dt} = t^2e^{-3t}$
   - (f) $\frac{dy}{dx} = x - 1$

2. Which of the following differential equations are directly integrable?
3. Which of the following differential equations are first-order?
   (a) \( y' = x^2 \)  
   (b) \( y' = y^2 \)  
   (c) \( (y')^2 + y' = \sin x \)  
   (d) \( x^3 y' - e^y = \sin y \)  
   (e) \( y'' + 3y' = \frac{y}{x} \)  
   (f) \( yy' + x + y = 0 \)

4. Which of the following differential equations are separable?
   (a) \( \frac{dy}{dx} = x - 2y \)  
   (b) \( y' + 3ye^{x^2} = 0 \)  
   (c) \( y' = x^2 y^2 \)  
   (d) \( y' = 1 - y^2 \)  
   (e) \( \frac{dy}{dt} = 3\sqrt{1 + y} \)  
   (f) \( \frac{dP}{dt} = \frac{P + t}{t} \)

Exercises

In Exercises 1–6, verify that the given function is a solution of the differential equation.

1. \( y' - 8x = 0, \ y = 4x^2 \)
2. \( yy' + 4x = 0, \ y = \sqrt{12 - 4x^2} \)
3. \( y' + 4x = 0, \ y = 25e^{-2x} \)
4. \( (x^2 - 1)y' + xy = 0, \ y = 4(x^2 - 1)^{-1/2} \)
5. \( y'' - 2xy' + 5y = 0, \ y = e^x \sin 2x \)

7. The following differential equations appear similar but have very different solutions:
   \( \frac{dy}{dx} = 0, \quad \frac{dy}{dx} = 0.001 \)

Solve both subject to the initial condition \( y(1) = -1 \).

8. The following differential equations appear similar but have very different solutions:
   \( \frac{dy}{dx} = x, \quad \frac{dy}{dx} = y \)

Solve both subject to the initial condition \( y(1) = 2 \).

9. Verify that \( x^2 y'' + e^{-y} = 0 \) is separable.
   (a) Write it as \( \frac{d}{dx} y' = -e^{-y} dx \).
   (b) Integrate both sides to obtain \( y' = e^{x^2} - C \).
   (c) Verify that \( y = \ln(x^2 - C) \) is the general solution.
   (d) Find the particular solution satisfying \( y(2) = 4 \).

10. Consider the differential equation \( xy'' - 9x^2 = 0 \).
    (a) Write it as \( \frac{d}{dx} y' = 9x^2 dx \).
    (b) Integrate both sides to obtain \( \frac{1}{4}y' = 3x^3 + C \).
    (c) Verify that \( y' = (12x^3 + C)^{1/4} \) is the general solution.

11. In Exercises 11–28, use Separation of Variables to find the general solution.
    (a) \( y' - 6x^2 y' = 0 \)
    (b) \( y' + 4x^2 = 0 \)
    (c) \( y' + x^2 y = 0 \)
    (d) \( y' - e^{x+y} = 0 \)
    (e) \( 2y'' + 5y = 4 \)
    (f) \( \frac{dy}{dt} = 8 \sqrt{y} \)
    (g) \( \sqrt{1 - x^2} y' = xy \)
    (h) \( y' = y^2(1 - x^2) \)
    (i) \( (x+y) y' = xy \)
    (j) \( \frac{dx}{dt} = (t + 1)(x^2 + 1) \)
    (k) \( 3\sqrt{1 + y} \)

25. \( y' = x \sec y \)
26. \( \frac{dy}{dx} = \tan y \)
27. \( \frac{dy}{dt} = y \tan t \)
28. \( \frac{dy}{dt} = \tan x \)

In Exercises 29–42, solve the Initial Value Problem.

29. \( y' + 2y = 0, \ y(5) = 3 \)
30. \( y' - 3y + 12 = 0, \ y(2) = 1 \)
31. \( y' = xe^{-y^2}, \ y(0) = -2 \)
32. \( \frac{dy}{dx} = x^3, \ y(1) = 0 \)
33. \( y' = (x-1)(y-2), \ y(2) = 4 \)
34. \( y' = (x-1)(y-2), \ y(2) = 2 \)
35. \( y' = x(y^2 + 1), \ y(0) = 0 \)
36. \( (1-x) \frac{dy}{dt} - y = 0, \ y(2) = -4 \)
37. \( \frac{dy}{dt} = ye^{-x}, \ y(0) = 1 \)
38. \( \frac{dy}{dt} = 1e^{-y}, \ y(1) = 0 \)
39. \( t \frac{dy}{dt} = t + y + ty, \ y(1) = 0 \)
40. \( \sqrt{1 - x^2} y' = y^2 + 1, \ y(0) = 0 \)
41. \( y' = \tan y, \ y(\pi/2) = \frac{\pi}{2} \)
42. \( y' = y^2 \sin x, \ y(\pi) = 2 \)

In Example 3, we calculated the thickness of a glacier assuming the friction at the base of the glacier is constant. In Exercises 43–44, we consider cases where the friction varies along the length of the glacier.

43. (a) Solve the glacier thickness differential equation (Eq. 2) for \( T(x) \) with \( r(x) = 75x \) N/m and initial condition \( T(0) = 0 \).
    (b) Sketch the graph of \( T \) for \( 0 \leq x \leq 1000 \) m.

44. (a) Solve the glacier thickness differential equation (Eq. 2) for \( T(x) \) with \( r(x) = 0.3x(1000 - x) \) N/m and initial condition \( T(0) = 0 \).
    (b) Sketch the graph of \( T \) for \( 0 \leq x \leq 1000 \) m.

In Exercises 45–50, use the differential equation for a leaking container, Eq. 3.

45. Water leaks through a hole of area \( B = 0.002 \) m² at the bottom of a cylindrical tank that is filled with water and has height 3 m and a base of area 10 m². How long does it take (a) for half of the water to leak out and (b) for the tank to empty?
46. At \( t = 0 \), a conical tank of height 300 cm and top radius 100 cm (Figure 11(A)) is filled with water. Water leaks through a hole in the bottom of area \( B = 3 \text{ cm}^2 \). Let \( y(t) \) be the water level at time \( t \).
   (a) Show that the tank's cross-sectional area at height \( y \) is \( A(y) = \frac{\pi}{3} y^2 \).
   (b) Solve the differential equation satisfied by \( y(t) \).
   (c) How long does it take for the tank to empty?

![Conical tank](image)

**FIGURE 11**

47. The tank in Figure 11(B) is a cylinder of radius 4 m and height 15 m. Assume that the tank is half-filled with water and that water leaks through a hole in the bottom of area \( B = 0.001 \text{ m}^2 \). Determine the water level \( y(t) \) and the time \( t_e \) when the tank is empty.

48. A tank has the shape of the parabola \( y = x^2 \), revolved around the y-axis. Water leaks from a hole of area \( B = 0.0005 \text{ m}^2 \) at the bottom of the tank. Let \( y(t) \) be the water level at time \( t \). How long does it take for the tank to empty if it is initially filled to height \( y_0 = 1 \text{ m} \)?

49. A tank has the shape of the parabola \( y = ax^2 \) (where \( a \) is a constant) revolved around the y-axis. Water drains from a hole of area \( B \text{ m}^2 \) at the bottom of the tank.
   (a) Show that the water level at time \( t \) is
      \[
y(t) = \left( \frac{2B}{3} t \right)^{1/3}
      \]
      where \( y_0 \) is the water level at time \( t = 0 \).
   (b) Show that if the total volume of water in the tank has volume \( V \) at time \( t = 0 \), then \( y_0 = \sqrt{V/\pi} \). Hint: Compute the volume of the tank as a function of rotation.
   (c) Show that the tank is empty at time
      \[
t_e = \left( \frac{3B}{2a} \right)^{1/4}
      \]

We see that for fixed initial water volume \( V \), the time \( t_e \) is proportional to \( a^{-1/4} \). A large value of \( a \) corresponds to a tall thin tank. Such a tank drains more quickly than a short wide tank of the same initial volume.

50. A cylindrical tank filled with water has height \( h \) and a base of area \( A \). Water leaks through a hole in the bottom of area \( B \).
   (a) Show that the time required for the tank to empty is proportional to \( \sqrt{h/B} \).
   (b) Show that the emptying time is proportional to \( Vh^{-1/2} \), where \( V \) is the volume of the tank.
   (c) Two tanks have the same volume and a hole of the same size, but they have different heights and bases. Which tank empties first: the taller or the shorter tank?

51. When cane sugar is dissolved in water, it converts to invert sugar over a period of several hours. The amount \( A(t) \) of unconverted cane sugar at time \( t \) (in hours) satisfies \( A' = -0.2A \). If there is initially 500 g of unconverted cane sugar, how much unconverted cane sugar remains after 5 h? After 10 h?

52. A certain RNA molecule replicates every 3 minutes. Find the differential equation for the number \( N(t) \) of molecules present at time \( t \) in minutes. How many molecules will be present after 1 h if there is one molecule at \( t = 0 \)?

53. Assume that during periods of job growth, the rate of increase of the number employed is proportional to the number who are employed. At the outset of a job growth period, the number employed in a certain country grew from 11.3 million to 11.7 million in 10 weeks. Let \( N(t) \) represent the number employed in millions as a function of \( t \) in weeks since the start of the growth period.
   (a) Set up and solve an Initial Value Problem to determine \( N(t) \).
   (b) The growth period lasted for a year (52 weeks). What was the increase in the number of jobs?

54. Bismuth-210 decays at a rate proportional to the amount present. A sample of Bismuth-210 that initially had a mass of 1000 mg decayed 500 mg in 5 days. Let \( M(t) \) be the mass of Bismuth-210 in milligrams \( t \) days after the initial sample of 1000 mg began to decay. Set up and solve an Initial Value Problem for determining \( M(t) \).

55. (a) With \( y(t) = y_0 e^{kt} \), at what value of \( t \) (in terms of \( p \) and \( k \)) is \( y(t) = py_0 \)?
   (b) If \( y(t) = y_0 e^{kt} \), with \( t \) in hours, how long does it take for \( y \) to double? To triple? To increase 10-fold?

56. Drug Dosage Interval. Let \( y(t) \) be the drug concentration (in micrograms per milliliter) in a patient's body at time \( t \). The initial concentration is \( y(0) = L \). Additional doses that increase the concentration by an amount \( d \) are administered at regular time intervals of length \( T \). In between doses, \( y(t) \) decays exponentially—that is, \( y' = -ky \). Find the value of \( T \) in terms of \( k \) and \( d \) for which the concentration varies between \( L \) and \( L + d \) as in Figure 12.

![Drug concentration with periodic doses](image)

**FIGURE 12**

57. Figure 13 shows a circuit consisting of a resistor of \( R \) ohms, a capacitor of \( C \) farads, and a battery of voltage \( V \). When the circuit is completed, the amount of charge \( q(t) \) (in coulombs) on the plates of the capacitor varies according to the differential equation \( t \) (in seconds)

\[
R \frac{dq}{dt} + \frac{1}{C} q = V
\]

where \( R, C, \) and \( V \) are constants.
58. Assume in the circuit of Figure 13 that $R = 200$ ohms, $C = 0.02$ farad, and $V = 12$ volts. How many seconds does it take for the charge on the capacitor plates to reach half of its limiting value?

59. According to one hypothesis, the growth rate $dV/dt$ of a cell’s volume $V$ is proportional to its surface area $A$. Since $V$ has cubic units such as cm$^3$ and $A$ has square units such as cm$^2$, we may assume roughly that $A \propto V^{2/3}$, and hence $dV/dt = kV^{2/3}$ (for some constant $k$). If this hypothesis is correct, which dependence of volume on time would we expect to see (again, roughly speaking) in the laboratory?

(a) linear
(b) quadratic
(c) cubic

60. We might also guess that the volume $V$ of a melting snowball decreases at a rate proportional to its surface area. Argue as in Exercise 59 to find a differential equation satisfied by $V$. Suppose the snowball has volume 1000 cm$^3$ and that it loses half of its volume after 5 minutes. According to this model, when will the snowball disappear?

61. The general solution to $\frac{dy}{dt} = ky$ is $y = De^{kt}$. Here we prove that every solution to the differential equation, defined on an interval, is given by the general solution.

(a) Show that if $y(t)$ satisfies the given differential equation, then $\frac{d}{dt} (ye^{-kt}) = 0$.

(b) Assume that $y(t)$ satisfies the differential equation on an interval $I$. Use the result from (a) and the Corollary to the Mean Value Theorem in Section 4.3 to prove that on $I$, $y(t) = De^{kt}$ for some $D$.

62. Captain Qant is standing at point $B$ on a dock and is holding a rope of length $\ell$ attached to a boat at point $A$ [Figure 14(A)]. As the captain walks along the dock, holding the rope taut, the boat moves along a curve called a tractrix (from the Latin tractus, meaning "pulled"). The segment from a point $P$ on the curve to the $x$-axis along the tangent line has constant length $\ell$ [Figure 14(B)]. Let $y = f(x)$ be the equation of the tractrix.

(a) Show that $y^2 + (y/x)^2 = \ell^2$ and conclude $y' = -\frac{y}{\sqrt{x^2 - y^2}}$. Why must we choose the negative square root?

(b) Prove that the tractrix is the graph of

$$x = \ell \ln \left( \frac{\ell + \sqrt{\ell^2 - y^2}}{y} \right) - \sqrt{\ell^2 - y^2}$$

63. Show that the differential equations $y' = 3y/x$ and $y' = -x/3y$ define orthogonal families of curves; that is, the graphs of solutions to the first equation intersect the graphs of the solutions to the second equation in right angles (Figure 15). Find these curves explicitly.

64. Find the family of curves satisfying $y' = x/y$ and sketch several members of the family. Then find the differential equation for the orthogonal family (see Exercise 63), find its general solution, and add some members of this orthogonal family to your plot.

65. A 50-kg model rocket lifts off by expelling fuel downward at a rate of $k = 4.75$ kg/s for 10 s. The fuel leaves the end of the rocket with an exhaust velocity of $b = 100$ m/s. Let $m(t)$ be the mass of the rocket at time $t$. From the law of conservation of momentum, we find the following differential equation for the rocket’s velocity $v(t)$ (in meters per second):

$$m(t)v'(t) = -9.8m(t) + \frac{dm}{dt}$$

(a) Show that $m(t) = 50 - 4.75t$ kg.

(b) Solve for $v(t)$ and compute the rocket’s velocity at rocket burnout (after 10 s).

66. Let $v(t)$ be the velocity of an object of mass $m$ in free-fall near the earth’s surface. If we assume that air resistance is proportional to $v^2$, then $v$ satisfies the differential equation $m\frac{dv}{dt} = -g + kv^2$ for some constant $k > 0$.

(a) Set $a = (g/k)^{1/2}$ and rewrite the differential equation as

$$\frac{dx}{dt} = -\frac{k}{m}(a^2 - v^2)$$

Then solve using Separation of Variables with initial condition $v(0) = 0$.

(b) Show that the terminal velocity $\lim_{t \to \infty} v(t)$ is equal to $-a$. 
67. If a bucket of water spins about a vertical axis with constant angular velocity \( \omega \) (in radians per second), the water climbs up the side of the bucket until it reaches an equilibrium position (Figure 16). Two forces act on a particle located at a distance \( x \) from the vertical axis: the gravitational force \(-mg\) acting downward and the force of the bucket on the particle (transmitted indirectly through the liquid) in the direction perpendicular to the surface of the water. These two forces must combine to supply a centripetal force \( m \omega^2 x \), and this occurs if the diagonal of the rectangle in Figure 16 is normal to the water's surface (i.e., perpendicular to the tangent line). Prove that if \( y = f(x) \) is the equation of the curve obtained by taking a vertical cross section through the axis, then \(-1/y' = -g/(\omega^2 x)\). Show that \( y = f(x) \) is a parabola.

Further Insights and Challenges

68. In Section 6.2, we computed the volume \( V \) of a solid as the integral of cross-sectional area. Explain this formula in terms of differential equations. Let \( V(y) \) be the volume of the solid up to height \( y \), and let \( A(y) \) be the cross-sectional area at height \( y \) as in Figure 17.

(a) Explain the following approximation for small \( \Delta y \):

\[
V(y + \Delta y) - V(y) \approx A(y) \Delta y
\]

(b) Use Eq. (11) to justify the differential equation \( dV/dy = A(y) \). Then derive the formula

\[
V = \int_a^b A(y) \, dy
\]

69. A basic theorem states that a linear differential equation of order \( n \) has a general solution that depends on \( n \) arbitrary constants. The following examples show that, in general, the theorem does not hold for nonlinear differential equations. In each case the differential equation is not linear because of the \((y')^2\) term.

(a) Show that \((y')^2 + y^2 = 0\) is a first-order equation with only one solution \( y = 0 \).

(b) Show that \((y')^2 + y^2 + 1 = 0\) is a first-order equation with no solutions.

70. Show that \( y = C_1 e^{rx} \) is a solution of \( y'' + ay' + by = 0 \) if and only if \( r \) is a root of \( P(r) = r^2 + ar + b \). Verify directly that \( y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \) is a solution of \( y'' - 2y' + y = 0 \) for any constants \( C_1, C_2 \).

71. A spherical tank of radius \( R \) is filled with water. Suppose that water leaks through a hole in the bottom of area \( B \). Let \( y(r) \) be the water level at time \( t \) (seconds).

(a) Show that \( \frac{dy}{dt} = \frac{\sqrt{2g} B \sqrt{y}}{\pi(2R y^2 - y^2)} \).

(b) Show that for some constant \( C \),

\[
\frac{2\pi}{15B \sqrt{2g}} \left( 10R y\sqrt{y} - 3y^{3/2} \right) = C - t
\]

(c) Use the initial condition \( y(0) = R \) to compute \( C \), and show that \( C = t_0 \), the time at which the tank is empty.

(d) Show that \( t_0 \) is proportional to \( R^{5/2} \) and inversely proportional to \( B \).

10.2 Models Involving \( y' = k(y - b) \)

In this section we examine the differential equation

\[
\frac{dy}{dt} = k(y - b)
\]

where \( k \) and \( b \) are constants. This differential equation describes a quantity \( y \) whose rate of change is proportional to the difference \( y - b \). It arises in many different modelling situations. We will use it to model a cooling object in a fixed-temperature environment, vertical motion under the influence of gravity and air resistance, and the changing value of an annuity.

This differential equation can be written in the form \( \frac{dy}{dt} = ky + c \) and is called linear because the relationship between \( \frac{dy}{dt} \) and \( y \) is linear. Furthermore, because the
coefficient \( k \) and the term \( c \) are constant, the differential equation is said to have constant coefficients. Thus, the differential equations we examine in this section are first-order linear constant coefficient differential equations. In Section 10.5, we examine the general first-order linear differential equation where the terms \( k \) and \( c \) could be functions of \( t \).

We can use Separation of Variables to show that the general solution to Eq. (1) is

\[
y(t) = b + Ce^{kt}
\]

Alternatively, we may observe that \((y - b)' = y'\) since \( b \) is a constant, so Eq. (1) may be rewritten

\[
\frac{d}{dt}(y - b) = k(y - b)
\]

In other words, \( y - b \) satisfies the differential equation of an exponential function and thus \( y - b = Ce^{kt} \), or \( y = b + Ce^{kt} \), as claimed.

**GRAPHICAL INSIGHT** The behavior of the solution \( y(t) \) as \( t \to \infty \) depends on whether \( C \) and \( k \) are positive or negative:

- When \( k > 0 \), \( e^{kt} \) tends to \( \infty \) and, therefore, \( y(t) \) tends to \( \infty \) if \( C > 0 \) and \( y(t) \) tends to \( -\infty \) if \( C < 0 \).
- When \( k < 0 \), we usually rewrite the differential equation as \( y' = -k(y - b) \) with \( k > 0 \). In this case, \( y(t) = b + Ce^{-kt} \) and \( y(t) \) approaches the horizontal asymptote \( y = b \) since \( Ce^{-kt} \) tends to zero as \( t \to \infty \) (Figure 1). Note that \( y(t) \) approaches the asymptote from above or below, depending on whether \( C > 0 \) or \( C < 0 \).

We now consider some applications of Eq. (1), beginning with Newton’s Law of Cooling. Let \( y(t) \) be the temperature of a hot object that is cooling off in an environment where the ambient temperature is \( T_0 \). Newton assumed that the rate of cooling is proportional to the temperature difference \( y - T_0 \). We express this hypothesis in a precise way by the differential equation

\[
y' = -k(y - T_0) \quad (T_0 = \text{ambient temperature})
\]

The constant \( k \), in units of \((\text{time})^{-1}\), is called the cooling constant and depends on the physical properties of the object.

**EXAMPLE 1** Newton’s Law of Cooling A hot metal bar with cooling constant \( k = 2.1 \text{ min}^{-1} \) is submerged in a large tank of water held at temperature \( T_0 = 10^\circ \text{C} \). Let \( y(t) \) be the bar’s temperature at time \( t \) (in minutes).

(a) Find the differential equation satisfied by \( y(t) \) and find its general solution.
(b) What is the bar’s temperature after 1 min if its initial temperature was \( 180^\circ \text{C} \)?
(c) What was the bar’s initial temperature if it cooled to \( 80^\circ \text{C} \) in 30 seconds?

**Solution**

(a) Since \( k = 2.1 \text{ min}^{-1} \), \( y(t) \) (with \( t \) in minutes) satisfies

\[
y' = -2.1(y - 10)
\]

By Eq. (2), the general solution is \( y(t) = 10 + Ce^{-2.1t} \) for some constant \( C \).

(b) If the initial temperature was \( 180^\circ \text{C} \), then \( y(0) = 10 + C = 180 \). Thus, \( C = 170 \) and \( y(t) = 10 + 170e^{-2.1t} \) (Figure 2). After 1 min,

\[
y(1) = 10 + 170e^{-2.1(1)} \approx 30.8^\circ \text{C}
\]
The effect of air resistance depends on the physical situation. A high-speed bullet is affected differently than a skydiver. Our model is fairly realistic for a large object such as a skydiver falling from high altitudes.

In this model, \( k \) has units of mass per time, such as kilograms per second.

(c) If the temperature after 30 s is \( 80^\circ \text{C} \), then \( y(0.5) = 80 \), and we have

\[
10 + Ce^{-2.1(0.5)} = 80 \quad \Rightarrow \quad Ce^{-1.05} = 70 \quad \Rightarrow \quad C = 70e^{1.05} \approx 200
\]

It follows that \( y(t) = 10 + 200e^{-2.1t} \) and the initial temperature was

\[
y(0) = 10 + 200e^{-2.1(0)} = 10 + 200 = 210^\circ \text{C}
\]

The differential equation \( y' = k(y - b) \) is also used to model vertical motion near the surface of the earth when air resistance is taken into account. Assume that the force due to air resistance is proportional to the velocity \( v \) and acts opposite to the direction of motion. We write this force as \(-kv\), where \( k > 0 \). We take the upward direction to be positive, so \( v < 0 \) for a falling object and \(-kv\) is an upward acting force, while \( v > 0 \) for a rising object and \(-kv\) is a downward acting force.

The force due to gravity on an object of mass \( m \) is \(-mg\), where \( g \) is the acceleration due to gravity, so the total force is \( F = -mg - kv \). By Newton's Second Law of Motion,

\[
F = ma = mv' \quad (a = v' \text{ is the acceleration})
\]

Thus, \( mv' = -mg - kv \), which can be written as

\[
v' = -\frac{k}{m}(v + \frac{mg}{k})
\]

This equation has the form \( v' = -k(v - b) \) with \( k \) replaced by \( k/m \) and \( b = -mg/k \). By Eq. (2), the general solution is

\[
v(t) = -\frac{mg}{k} + Ce^{-(k/m)t}
\]

Since \( Ce^{-(k/m)t} \) tends to zero as \( t \to \infty \), \( v(t) \) tends to a limiting terminal velocity:

\[
\text{terminal velocity} = \lim_{t \to \infty} v(t) = -\frac{mg}{k}
\]

Without air resistance, the speed (absolute value of velocity) of a falling object would increase without bound until a sudden collision with the ground occurs. On the other hand, with air resistance the speed also increases, but levels off and approaches a limiting value of \( mg/k \).

**EXAMPLE 2**

An 80-kg skydiver steps out of an airplane.

(a) What is her terminal velocity if \( k = 8 \text{ kg/s} \)?

(b) What is her velocity after 30 s?

**Solution**

(a) By Eq. (5), with \( k = 8 \text{ kg/s} \) and \( g = 9.8 \text{ m/s}^2 \), the terminal velocity is

\[
-\frac{mg}{k} = -\frac{(80)(9.8)}{8} = -98 \text{ m/s}
\]

(b) With \( t \) in seconds, we have, by Eq. (4),

\[
v(t) = -98 + Ce^{-(k/m)t} = -98 + Ce^{-(8/90)t} = -98 + Ce^{-0.1t}
\]

We assume that the skydiver leaves the airplane with no initial vertical velocity, so \( v(0) = -98 + C = 0 \), and \( C = 98 \). Thus, we have \( v(t) = -98(1 - e^{-0.1t}) \) (Figure 3). The skydiver's velocity after 30 s is

\[
v(30) = -98(1 - e^{-0.1(30)}) \approx -93.1 \text{ m/s}
\]
In Example 8 in Section 7.4 and Example 5 in Section 7.5, we investigated limits involving the maximum height attained by a 1-kg ball that is launched upward at 30 m/s and acted on by gravity and air resistance. We observed that the maximum height (in meters) depends on the strength of the air resistance (i.e., on the proportionality constant \( k \)), a relationship given by

\[
H(k) = \frac{30k - 9.8 \ln \left( \frac{150k}{49} + 1 \right)}{k^2}
\]

Using Eq. (4), we can now derive this equation. In the next example, we find equations for the projectile's velocity \( v(t) \) and height \( y(t) \). The maximum height \( H(k) \) is then found by determining the height when the velocity is zero (see Exercise 17).

**Example 3** A 1-kg ball is launched from the ground at 30 m/s and is acted on by air resistance that is expressed in the form \(-kv\) and by gravity. Determine its velocity \( v(t) \) and height \( y(t) \).

**Solution** By Eq. (4), the projectile's velocity is

\[
v(t) = -\frac{9.8}{k} + Ce^{-kt}
\]

where \( C \) must be chosen so that \( v(0) = 30 \). That is, \( C \) must satisfy

\[
30 = -\frac{9.8}{k} + C
\]

Thus, \( C = 30 + \frac{9.8}{k} \), and therefore,

\[
v(t) = -\frac{9.8}{k} + \left( 30 + \frac{9.8}{k} \right) e^{-kt}
\]

Next, we take the antiderivative of \( v(t) \) to find \( y(t) \). We have

\[
y(t) = -\frac{9.8}{k} t - \frac{1}{k} \left( 30 + \frac{9.8}{k} \right) e^{-kt} + C
\]

To satisfy \( y(0) = 0 \), we must have \( C = \frac{1}{k} \left( 30 + \frac{9.8}{k} \right) \). Using this equation to substitute for \( C \) in \( y(t) \), and simplifying, we find

\[
y(t) = -\frac{9.8}{k} t + \frac{30k + 9.8}{k^2} (1 - e^{-kt})
\]

An **annuity** is an investment in which an amount of money \( P_0 \), called the principal, is placed in an account that earns interest. Let \( P(t) \) be the balance in the annuity (in dollars) after \( t \) years. When interest on the balance is compounded continuously at rate \( r \), the rate of growth of the balance is proportional to the balance, and the proportionality constant is \( r \). That is, \( P'(t) = r P(t) \). If we withdraw from the annuity at a constant rate of \( N \) dollars per year, we can model the balance in the annuity by the differential equation

\[
P'(t) = \frac{r P(t) - N}{r}
\]

This equation has the form \( y' = ky - b \) with \( k = r \) and \( b = N/r \), so by Eq. (2), the general solution is

\[
P(t) = \frac{N}{r} + Ce^{rt}
\]
Since \( P(0) = P_0 \), we know \( P_0 = \frac{N}{r} + C \) and, therefore, \( C \) is given by \( C = P_0 - \frac{N}{r} \). Because \( e^{rt} \) tends to infinity as \( t \to \infty \), the balance \( P(t) \) tends to \( \infty \) if \( C > 0 \). If \( C < 0 \), then \( P(t) \) tends to \( -\infty \) (i.e., the annuity eventually runs out of money). If \( C = 0 \), then \( P(t) \) remains constant with value \( N/r \).

**EXAMPLE 4 Does an Annuity Pay Out Forever?** An annuity earns continuously compounded interest at the rate \( r = 0.07 \), and withdrawals are made continuously at a rate of \( N = 55000 \) per year.

(a) When will the annuity run out of money if the initial deposit is \( P(0) = 55000 \)?

(b) Show that the balance increases indefinitely if \( P(0) = 59000 \).

**Solution** We have \( N/r = \frac{500}{0.07} \approx 7143 \), so \( P(t) = 7143 + Ce^{0.07t} \) by Eq. (7).

(a) If \( P(0) = 5000 \), then \( C = -2143 \) and

\[
P(t) = 7143 - 2143e^{0.07t}
\]

The account runs out of money when \( P(t) = 7143 - 2143e^{0.07t} = 0 \), or

\[
e^{0.07t} = \frac{7143}{2143} \Rightarrow 0.07t = \ln \left( \frac{7143}{2143} \right) \approx 1.2
\]

The annuity runs out at time \( t = \frac{12}{0.07} \approx 17 \) years.

(b) If \( P(0) = 9000 \), then \( C = 1857 \) and

\[
P(t) = 7143 + 1857e^{0.07t}
\]

Since the coefficient \( C = 1857 \) is positive, the account never runs out of money. In fact, \( P(t) \) increases indefinitely as \( t \to \infty \). Figure 4 illustrates the two cases.

**10.2 SUMMARY**

- The general solution of \( y' = k(y - b) \) is \( y = b + Ce^{kt} \), where \( C \) is a constant.
- The following table describes the solutions to \( y' = k(y - b) \) (see Figure 5):

<table>
<thead>
<tr>
<th>Equation ( (k &gt; 0) )</th>
<th>Solution</th>
<th>Behavior as ( t \to \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y' = k(y - b) )</td>
<td>( y(t) = b + Ce^{kt} )</td>
<td>( \lim_{t \to \infty} y(t) = \begin{cases} \infty &amp; \text{if } C &gt; 0 \ -\infty &amp; \text{if } C &lt; 0 \end{cases} )</td>
</tr>
<tr>
<td>( y' = -k(y - b) )</td>
<td>( y(t) = b + Ce^{-kt} )</td>
<td>( \lim_{t \to \infty} y(t) = b )</td>
</tr>
</tbody>
</table>

**Figure 5**

Solutions to \( y' = k(y - b) \) with \( k > 0 \)

Solutions to \( y' = -k(y - b) \) with \( k > 0 \)
10.2 EXERCISES

Preliminary Questions
1. Write a solution to \( y' = 4(y - 5) \) that tends to \(-\infty\) as \( t \to -\infty \).
2. Does \( y' = -4(y - 5) \) have a solution that tends to \(-\infty\) as \( t \to -\infty \)?
3. True or false? If \( k > 0 \), then all solutions of \( y' = -k(y - b) \) approach the same limit as \( t \to -\infty \).

Exercises
1. Find the general solution of \( y' = 2(y - 10) \). Then find the two solutions satisfying \( y(0) = 25 \) and \( y(0) = 5 \), and sketch their graphs.
2. Verify directly that \( y = 12 + Ce^{-3t} \) satisfies \( y' = -3(y - 12) \) for all \( C \). Then find the two solutions satisfying \( y(0) = 20 \) and \( y(0) = 0 \), and sketch their graphs.
3. Solve \( y' = 4y + 24 \) subject to \( y(0) = 5 \).
4. Solve \( y' + 6y = 12 \) subject to \( y(2) = 10 \).

In Exercises 5–12, use Newton’s Law of Cooling.
5. A hot anvil with cooling constant \( k = 0.02 \text{ s}^{-1} \) is submerged in a large pool of water whose temperature is \( 10^\circ C \). Let \( y(t) \) be the anvil’s temperature \( t \) seconds later.
   a. What is the differential equation satisfied by \( y(t) \)?
   b. Find a formula for \( y(t) \), assuming the object’s initial temperature is \( 100^\circ C \).
   c. How long does it take the object to cool down to \( 20^\circ C \)?
6. Frank’s automobile engine runs at \( 100^\circ C \). On a day when the outside temperature was \( 21^\circ C \), he turns off the ignition and notes that 5 minutes later, the engine has cooled to \( 70^\circ C \).
   a. Determine the engine’s cooling constant \( k \).
   b. What is the formula for \( y(t) \)?
   c. When will the engine cool to \( 40^\circ C \)?
7. At 10:30 AM, detectives discover a dead body in a room and measure its temperature at \( 20^\circ C \). One hour later, the body’s temperature had dropped to \( 24.8^\circ C \). Determine the time of death (when the body temperature was a normal \( 37^\circ C \)), assuming that the temperature in the room was held constant at \( 20^\circ C \).
8. A cup of coffee with cooling constant \( k = 0.09 \text{ min}^{-1} \) is placed in a room at temperature \( 20^\circ C \).
   a. How fast is the coffee cooling (in degrees per minute) when its temperature is \( T = 80^\circ C \)?
   b. Use the Linear Approximation to estimate the change in temperature over the next 6 s when \( T = 80^\circ C \).
   c. If the coffee is served at \( 90^\circ C \), how long will it take to reach an optimal drinking temperature of \( 65^\circ C \)?
9. A cold metal bar at \(-30^\circ C \) is submerged in a pool maintained at a temperature of \( 40^\circ C \). Half a minute later, the temperature of the bar is \( 20^\circ C \). How long will it take for the bar to attain a temperature of \( 30^\circ C \)?
10. When a hot object is placed in a water bath whose temperature is \( 25^\circ C \), it cools from \( 100^\circ C \) to \( 50^\circ C \) in 150 seconds. In another bath, the same cooling occurs in \( 120 \) s. Find the temperature of the second bath.
11. Objects \( A \) and \( B \) are placed in a warm bath at temperature \( T_0 = 40^\circ C \). Object \( A \) has initial temperature \( -20^\circ C \) and cooling constant \( k = 0.004 \text{ s}^{-1} \). Object \( B \) has initial temperature \( 0^\circ C \) and cooling constant \( k = 0.002 \text{ s}^{-1} \). Plot the temperatures of \( A \) and \( B \) for \( 0 \leq t \leq 1000 \). After how many seconds will the objects have the same temperature?
12. In Newton’s Law of Cooling, the constant \( r = 1/k \) is called the characteristic time. Show that \( r \) is the time required for the temperature difference \( (y - T_0) \) to decrease by the factor \( e^{-r} = 0.37 \). For example, if \( y(0) = 100^\circ C \) and \( T_0 = 0^\circ C \), then the object cools to \( 100/e \approx 37^\circ C \) in time \( r \), to \( 100/e^2 \approx 13.5^\circ C \) in time \( 2r \), and so on.

In Exercises 13–16, use Eq. (3) as a model for free-fall with air resistance.
13. A 60-kg skydiver jumps out of an airplane. What is her terminal velocity, in meters per second, assuming that \( k = 10 \text{ kg/s} \) for free-fall (no parachute)?
14. Find the terminal velocity of a skydiver of weight \( w = 192 \text{ pounds} \) if \( k = 1.2 \text{ lb-ft/s} \). How long does it take him to reach half of his terminal velocity if his initial velocity is zero? Mass and weight are related by \( w = mg \), and Eq. (3) becomes \( v' = -(kg/w)(v + w/k) \) with \( g = 32 \text{ ft/s}^2 \).
15. An 80-kg skydiver jumps out of an airplane (with zero initial velocity). Assume that \( k = 12 \text{ kg/s} \) with a closed parachute and \( k = 70 \text{ kg/s} \) with an open parachute. What is the skydiver’s velocity at \( t = 25 \text{ s} \) if the parachute opens after 20 s of free-fall?
16. Does a heavier or a lighter skydiver reach terminal velocity more quickly?
17. As in Example 3, a 1-kg ball is launched upward at 30 m/s and is acted on by gravity and air resistance.

(a) Show that the ball’s velocity is zero at time \( t^* = \frac{1}{k} \ln \left( \frac{30k}{9.8} + 1 \right) \).

(b) Show that \( y(t^*) = \frac{30k - 9.8 \ln \left( \frac{150k}{49} + 1 \right)}{k^2} \) (thereby establishing the formula for \( H(k) \) given prior to the example).

18. A 500 g ball is launched upward at 30 m/s and is acted on by gravity and air resistance that can be expressed in the form \(-kv\), where \( v \) is the ball’s velocity.

(a) Determine the ball’s velocity \( v(t) \), expressed in terms of \( k \).

(b) Determine the ball’s height \( y(t) \), expressed in terms of \( k \).

(c) Determine \( H(k) \), the maximum height reached by the ball, as a function of \( k \).

In Exercises 19(a)-(g), use Eqs. (5) and (7) that describe the balance in a continuous annuity.

19. (a) A continuous annuity with withdrawal rate \( N = $5000/\text{year} \) and interest rate \( r = 5\% \) is funded by an initial deposit of \( P_0 = $50,000 \).

(b) What is the balance in the annuity after 10 years?

(c) When will the annuity run out of funds?

(d) Show that a continuous annuity with withdrawal rate \( N = $5000/\text{year} \) and interest rate \( r = 8\% \), funded by an initial deposit of \( P_0 = $75,000 \), never runs out of money.

(e) Find the minimum initial deposit \( P_0 \) that will allow an annuity to pay out $5000/year indefinitely if it earns interest at a rate of 5%.

(f) Find the minimum initial deposit \( P_0 \) necessary to fund an annuity for 20 years if withdrawals are made at a rate of $10,000/year and interest is earned at a rate of 7%.

(g) An initial deposit of 100,000 euros is placed in an annuity with a French bank. What is the minimum interest rate the annuity must earn to allow withdrawals at a rate of 8000 euros/year to continue indefinitely?

20. Sam borrows $10,000 from a bank at an interest rate of 9% and pays back the loan continuously at a rate of \( N \) dollars per year. Let \( P(t) \) denote the amount still owed at time \( t \).

(a) Explain why \( P(t) \) satisfies the differential equation

\[ y' = 0.09 y - N \]

(b) How long will it take Sam to pay back the loan if \( N = $12000 \)?

(c) Will the loan ever be paid back if \( N = $8000 \)?

21. April borrows $18,000 at an interest rate of 5% to purchase a new automobile. At what rate (in dollars per year) must the pay back the loan, if the loan must be paid off in 5 years? Hint: Set up the differential equation as in Exercise 20.

22. Let \( N(t) \) be the fraction of the population who have heard a given piece of news \( h \) hours after its initial release. According to one model, the rate \( N'(t) \) of which the news spreads is equal to \( k \) times the fraction of the population that has not yet heard the news, for some constant \( k > 0 \).

(a) Determine the differential equation satisfied by \( N(t) \).

(b) Find the solution of this differential equation with the initial condition \( N(0) = 0 \) in terms of \( k \).

(c) Suppose that half of the population is aware of an earthquake 8 hours after it occurs. Use the model to calculate \( k \) and estimate the percentage that will know about the earthquake 12 hours after it occurs.

23. Current in a Circuit When the circuit in Figure 6 (which consists of a battery of \( V \) volts, a resistor of \( R \) ohms, and an inductor of \( L \) henries) is connected, the current \( I(t) \) flowing in the circuit satisfies

\[ \frac{dI}{dt} + RI = V \]

with the initial condition \( I(0) = 0 \).

(a) Find a formula for \( I(t) \) in terms of \( L \), \( V \), and \( R \).

(b) Show that \( \lim_{t \to \infty} I(t) = V/R \).

(c) Show that \( I(t) \) reaches approximately 63% of its maximum value at the characteristic time \( t = L/R \).

**FIGURE 6** Current flow approaches the level \( I_{\text{max}} = V/R \).
10.3 Graphical and Numerical Methods

In the previous two sections, we focused on finding solutions to differential equations. Differential equations cannot always be solved explicitly. Fortunately, there are techniques for analyzing the solutions that do not rely on explicit formulas. Even for differential equations that have exact solutions, these techniques are valuable because they provide us with extra insight into the behavior of the solutions. In this section, we discuss the method of slope fields, which provides us with a good visual understanding of first-order equations. We also discuss Euler's Method for finding numerical approximations to solutions.

We use $t$ as the independent variable. A first-order differential equation can then be written in the form

$$\frac{dy}{dt} = F(t, y)$$

where $F(t, y)$ is a function of $t$ and $y$. The differential equation indicates that $\frac{dy}{dt}$, the slope of the graph of a solution $y = y(t)$, at a point $(t, y)$, is given by $F(t, y)$.

It is useful to think of Eq. (1) as a set of instructions that "tells a solution" which direction to go in. Thus, a solution passing through a point $(t, y)$ is "instructed" to do so in the direction of the slope $F(t, y)$ because the differential equation requires $F(t, y)$ to be the slope of the graph of the solution at that point. To visualize this set of instructions, we draw a slope field, which is an array of small segments of slope $F(t, y)$ at points $(t, y)$ lying on a rectangular grid in the plane as in Figure 1. Solutions to the differential equation $\frac{dy}{dt} = F(t, y)$ must have graphs that are everywhere tangent to the slope field, and therefore, without knowing the solutions, we can picture how their graphs must appear within the slope field.

To illustrate, let's consider the differential equation

$$\frac{dy}{dt} = -ty$$

In this case, $F(t, y) = -ty$. To sketch a slope field, we first compute $F(t, y)$ for an array of points in the $ty$-plane:

<table>
<thead>
<tr>
<th>Values of $F(t, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(-1,1) = 1$</td>
</tr>
<tr>
<td>$F(0,1) = 0$</td>
</tr>
<tr>
<td>$F(1,1) = -1$</td>
</tr>
<tr>
<td>$F(2,1) = -2$</td>
</tr>
<tr>
<td>$F(-1,0) = 0$</td>
</tr>
<tr>
<td>$F(0,0) = 0$</td>
</tr>
<tr>
<td>$F(1,0) = 0$</td>
</tr>
<tr>
<td>$F(2,0) = 0$</td>
</tr>
<tr>
<td>$F(-1,-1) = -1$</td>
</tr>
<tr>
<td>$F(0,-1) = 0$</td>
</tr>
<tr>
<td>$F(1,-1) = 1$</td>
</tr>
<tr>
<td>$F(2,-1) = 2$</td>
</tr>
</tbody>
</table>

Then, at each point $(t, y)$ plot a segment of slope $F(t, y)$ as in Figure 2.

A slope field is much easier to generate using technology than by hand. Many graphing calculators and computer algebra systems can produce slope fields. There are online programs available as well. We used technology to generate the slope field for $\frac{dy}{dt} = -ty$ in Figure 3(A).

We can use the slope field to see how solutions appear. The slope field enables us to visualize many solutions at a glance. Starting at any point, and going both to the left and to the right from the point, we can sketch the graph of a solution by drawing a curve that runs tangent to the slope segments at each point (Figure 3(B)). The graph of a solution is also called an integral curve.

**Example 1 Using Isochrones** Draw the slope field for

$$\frac{dy}{dt} = y - t$$
and sketch the integral curves satisfying the initial conditions

(a) \( y(0) = 1 \) and (b) \( y(1) = -2 \).

**Solution** A good way to sketch the slope field of \( \frac{dy}{dt} = F(t, y) \) is to choose several values \( c \) and identify the curve \( F(t, y) = c \), called the isocline of slope \( c \). The isocline is the curve consisting of all points where the slope field has slope \( c \).

In our case, \( F(t, y) = y - t \), so the isocline of fixed slope \( c \) has equation \( y - t = c \), or \( y = t + c \), which is a line. Consider the following values:

- \( c = 0 \): This isocline is \( y - t = 0 \), or \( y = t \). We draw segments of slope \( c = 0 \) at points along the line \( y = t \), as in Figure 4(A).
- \( c = 1 \): This isocline is \( y - t = 1 \), or \( y = t + 1 \). We draw segments of slope 1 at points along \( y = t + 1 \), as in Figure 4(B).
- \( c = 2 \): This isocline is \( y - t = 2 \), or \( y = t + 2 \). We draw segments of slope 2 at points along \( y = t + 2 \), as in Figure 4(C).
- \( c = -1 \): This isocline is \( y - t = -1 \), or \( y = t - 1 \) [Figure 4(C)]

A more detailed slope field is shown in Figure 4(D). To sketch the solution satisfying \( y(0) = 1 \), begin at the point \((t_0, y_0) = (0, 1)\) and draw the integral curve passing through \((0, 1)\) that is everywhere tangent to the slope field. The solution satisfying \( y(1) = -2 \) is sketched similarly. Figure 4(E) shows several other solutions (integral curves).

**Graphical Insight** Slope fields often let us see the asymptotic behavior of solutions (as \( t \to \infty \)) at a glance. Figure 4(E) suggests that the asymptotic behavior depends on the initial value. For example, if \( y(0) > 1 \), then \( y(t) \) tends to \( \infty \), and if \( y(0) < 1 \), then \( y(t) \) tends to \( -\infty \). We can check this using the general solution \( y(t) = 1 + Ce^t \), where \( y(0) = 1 + C \). If \( y(0) > 1 \), then \( C > 0 \) and \( y(t) \) tends to \( \infty \), but if \( y(0) < 1 \), then \( C < 0 \) and \( y(t) \) tends to \( -\infty \). The solution \( y = 1 + t \) with initial condition \( y(0) = 1 \) is the straight line shown in Figure 4(D).

**Example 2** Newton’s Law of Cooling Revisited The temperature \( y(t) \) (in degrees Celsius) of an object placed in a refrigerator that is kept at 4°C satisfies \( \frac{dy}{dt} = -0.5(y - 4) \) (\( t \) in minutes). Draw the slope field and describe the behavior of the solutions.
Chapter 10 — Introduction to Differential Equations

**Figure 4** Drawing the slope field for \( \frac{dy}{dt} = y - t \) using isoclines.

**Solution** The function \( F(t, y) = -0.5(y - 4) \) depends only on \( y \), so slopes of the segments in the slope field do not vary in the \( t \)-direction. The slope \( F(t, y) \) is positive for \( y < 4 \) and negative for \( y > 4 \). More precisely, the slope at height \( y \) is \( -0.5(y - 4) = -0.5y + 2 \), so the segments grow steeper with positive slope as \( y \to -\infty \), and they grow steeper with negative slope as \( y \to \infty \) (Figure 5).

The slope field shows that if the initial temperature satisfies \( y_0 > 4 \), then \( y(t) \) decreases to \( y = 4 \) as \( t \to \infty \). In other words, the object cools down to 4°C when placed in the refrigerator. If \( y_0 < 4 \), then \( y(t) \) increases to \( y = 4 \) as \( t \to \infty \). The object warms up when placed in the refrigerator. If \( y_0 = 4 \), then \( y \) remains at 4°C for all time \( t \).

**Conceptual Insight** Most first-order equations arising in applications have a uniqueness property: There is precisely one solution \( y(t) \) satisfying a given initial condition \( y(t_0) = y_0 \). Graphically, this means that precisely one integral curve (solution) passes through the point \( (t_0, y_0) \). Thus, when uniqueness holds, distinct integral curves never cross or overlap. Figure 6 shows the slope field of \( \frac{dy}{dt} = -\sqrt{|y|} \), where uniqueness fails. We can prove that once an integral curve touches the \( t \)-axis, it either remains on the \( t \)-axis or continues along the \( t \)-axis for a period of time before moving below the \( t \)-axis. Therefore, infinitely many integral curves pass through each point on the \( t \)-axis. However, the slope field does not show this clearly. Thus, when possible, it is important to obtain and analyze solutions rather than just rely on visual impressions alone.

**Euler's Method**

Euler's Method produces numerical approximations to the solution \( y(t) \) of a first-order Initial Value Problem:

\[
\frac{dy}{dt} = F(t, y), \quad y(t_0) = y_0
\]
Euler's Method is the simplest method for solving Initial Value Problems numerically, but it is not very efficient. Computer systems use more sophisticated schemes, making it possible to plot and analyze solutions to the complex systems of differential equations arising in areas such as weather prediction, aerodynamic modeling, and economic forecasting.

We begin by choosing a small number $h$, called the time step, and consider the sequence of times starting at the initial value $t_0$ and spaced at intervals of size $h$:

$$t_0, \quad t_1 = t_0 + h, \quad t_2 = t_0 + 2h, \quad t_3 = t_0 + 3h, \quad \ldots$$

In general, $t_k = t_0 + kh$ for $k = 0, 1, 2, \ldots$. Euler's Method consists of computing a sequence of values $y_1, y_2, y_3, \ldots, y_n$ successively using the formula

$$y_k = y_{k-1} + hF(t_{k-1}, y_{k-1})$$

Each $y_k$ is an approximation to the value of the solution to the Initial Value Problem at $t_k$; that is, $y_k \approx y(t_k)$. Starting with the initial value $y_0 = y(t_0)$, we compute

$$y_1 = y_0 + hF(t_0, y_0), \quad y_2 = y_1 + hF(t_1, y_1), \quad \text{ etc.}$$

We connect the points $(t_k, y_k)$ by segments to obtain an approximation to the graph of $y(t)$ (Figure 7).

**GRAPHICAL INSIGHT** The values $y_k$ are defined so that the line segment joining $(t_{k-1}, y_{k-1})$ to $(t_k, y_k)$ has slope

$$\frac{y_k - y_{k-1}}{t_k - t_{k-1}} = \frac{(y_{k-1} + hF(t_{k-1}, y_{k-1})) - y_{k-1}}{h} = F(t_{k-1}, y_{k-1})$$

Thus, in Euler's Method, we move from $(t_{k-1}, y_{k-1})$ to $(t_k, y_k)$ by traveling on a line segment in the direction specified by the slope field at $(t_{k-1}, y_{k-1})$ for a time interval of length $h$ (Figure 7).

**EXAMPLE 3** Use Euler's Method with time step $h = 0.2$ and $n = 4$ steps to approximate the solution of $\frac{dy}{dt} = y - t^2$, $y(0) = 3$ at $t = 0.2, 0.4, 0.6$, and 0.8.

**Solution** Our initial value at $t_0 = 0$ is $y_0 = 3$. The time values are $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$, and $t_4 = 0.8$. We use Eq. (3) with $F(t, y) = y - t^2$ to calculate

$$y(0.2) \approx y_1 = y_0 + hF(t_0, y_0) = 3 + 0.2(3 - (0)^2) = 3.6$$

$$y(0.4) \approx y_2 = y_1 + hF(t_1, y_1) = 3.6 + 0.2(3.6 - (0.2)^2) \approx 4.3$$

$$y(0.6) \approx y_3 = y_2 + hF(t_2, y_2) = 4.3 + 0.2(4.3 - (0.4)^2) \approx 5.14$$

$$y(0.8) \approx y_4 = y_3 + hF(t_3, y_3) = 5.14 + 0.2(5.14 - (0.6)^2) \approx 6.1$$

For comparison, Figure 8(A) shows the exact solution, $y(t) = 2 + 2t + t^2 + e^t$, to the Initial Value Problem in the previous example, together with a plot of the points $(t_k, y_k)$ for $k = 0, 1, 2, 3, 4$ connected by line segments.
• Euler's Method: To approximate a solution to \( \frac{dy}{dt} = F(t, y) \) with initial condition \( y(t_0) = y_0 \), fix a time step \( h \) and set \( t_k = t_0 + kh \). Define \( y_1, y_2, \ldots \) successively by the formula

\[
y_k = y_{k-1} + hF(t_{k-1}, y_{k-1})
\]

The values \( y_0, y_1, y_2, \ldots \) are approximations to the values \( y(t_0), y(t_1), y(t_2), \ldots \).

### 10.3 Exercises

#### Preliminary Questions

1. What is the slope of the segment in the slope field for \( \frac{dy}{dt} = ty + 1 \) at the point (2, 3)?

2. What is the equation of the isocline of slope \( c = 1 \) for \( \frac{dy}{dt} = y^2 - t \)?

3. For which of the following differential equations are the slopes at points on a vertical line \( t = C \) all equal?
   (a) \( \frac{dy}{dt} = \ln y \)
   (b) \( \frac{dy}{dt} = \ln t \)

4. Let \( y(t) \) be the solution to \( \frac{dy}{dt} = F(t, y) \) with \( y(1) = 3 \). How many iterations of Euler’s Method are required to approximate \( y(3) \) if the time step is \( h = 0.17 \)?

#### Exercises

1. Figure 9 shows the slope field for \( \frac{dy}{dt} = \sin y \sin t \). Sketch the graphs of the solutions with initial conditions \( y(0) = 1 \) and \( y(0) = -1 \). Show that \( y(t) = 0 \) is a solution and add its graph to the plot.

![Figure 9](image)

**FIGURE 9** Slope field for \( \frac{dy}{dt} = \sin y \sin t \).

2. Figure 10 shows the slope field for \( \frac{dy}{dt} = y^2 - t^2 \). Sketch the integral curve passing through the point \((0, -1)\), the curve through \((0, 0)\), and the curve through \((0, 2)\). Is \( y(t) = 0 \) a solution?

![Figure 10](image)

**FIGURE 10** Slope field for \( \frac{dy}{dt} = y^2 - t^2 \).

3. Show that \( f(t) = \frac{1}{2}(t - \frac{1}{2}) \) is a solution to \( \frac{dy}{dt} = t - 2y \). Sketch the four solutions with \( y(0) = \pm 0.5, \pm 1 \) on the slope field in Figure 11. The slope field suggests that every solution approaches \( f(t) \) as \( t \to \infty \). Confirm this by showing that \( y = f(t) + Ce^{-t} \) is the general solution.

![Figure 11](image)

**FIGURE 11** Slope field for \( \frac{dy}{dt} = t - 2y \).

4. One of the slope fields in Figures 12(A) and (B) is the slope field for \( \frac{dy}{dt} = t^2 \). The other is for \( \frac{dy}{dt} = y^2 \). Identify which is which. In each case, sketch the solutions with initial conditions \( y(0) = 1, y(0) = 0, \) and \( y(0) = -1 \).

![Figure 12](image)

**FIGURE 12**
5. Consider the differential equation \( \frac{dy}{dt} = t - y \).

(a) Sketch the slope field of the differential equation \( \frac{dy}{dt} = t - y \) in the range \(-1 \leq t \leq 3, -1 \leq y \leq 3\). As an aid, observe that the isolines of slope \( c \) are the lines \( t - y = c \), so the segments have slope \( c \) at points on the line \( y = t - c \).

(b) Show that \( y = t - 1 + Ce^{-t} \) is a solution for all \( C \). Since \( \lim_{t \to \infty} e^{-t} = 0 \), these solutions approach the particular solution \( y = t - 1 \) as \( t \to \infty \). Explain how this behavior is reflected in your slope field.

6. Show that the isolines of \( \frac{dy}{dt} = 1/y \) are horizontal lines. Sketch the slope field for \(-2 \leq t \leq 2, -2 \leq y \leq 2\) and plot the solutions with initial conditions \( y(0) = 0 \) and \( y(0) = 1 \).

7. Sketch the slope field for \( \frac{dy}{dt} = y + t \) for \(-2 \leq t \leq 2, -2 \leq y \leq 2\).

8. Sketch the slope field for \( \frac{dy}{dt} = \frac{t}{y} \) for \(-2 \leq t \leq 2, -2 \leq y \leq 2\).

9. Show that the isolines of \( \frac{dy}{dt} = t \) are vertical lines. Sketch the slope field for \(-2 \leq t \leq 2, -2 \leq y \leq 2\) and plot the integral curves passing through \((0, -1)\) and \((0, 1)\).

10. Sketch the slope field of \( \frac{dy}{dt} = ty \) for \(-2 \leq t \leq 2, -2 \leq y \leq 2\). Based on the sketch, determine \( \lim_{t \to \infty} y(t) \), where \( y(t) \) is a solution with \( y(0) > 0 \). What is \( \lim_{t \to \infty} y(t) \) if \( y(0) < 0 \)?

11. Match each differential equation with its slope field in Figures 13(A)-(F).

(i) \( \frac{dy}{dt} = 1 \)

(ii) \( \frac{dy}{dt} = \frac{y}{t} \)

(iii) \( \frac{dy}{dt} = t y \)

(iv) \( \frac{dy}{dt} = t y^2 \)

(v) \( \frac{dy}{dt} = t^2 + y^2 \)

(vi) \( \frac{dy}{dt} = t \)

12. Sketch the solution of \( \frac{dy}{dt} = ty^2 \) satisfying \( y(0) = 1 \) in the appropriate slope field of Figure 13(A)-(F). Then show, using Separation of Variables, that if \( y(t) \) is a solution such that \( y(0) > 0 \), then \( y(t) \) tends to infinity as \( t \to \infty \).

13. (a) Sketch the slope field of \( \frac{dy}{dt} = y/t \) in the region \(-2 \leq t \leq 2, -2 \leq y \leq 2\).

(b) Check that \( y = \frac{t}{t^2 + C} \) is the general solution.

(c) Sketch the solutions on the slope field with initial conditions \( y(0) = 1 \) and \( y(0) = -1 \).

14. Sketch the slope field of \( \frac{dy}{dt} = t^2 - y \) in the region \(-3 \leq t \leq 3, -3 \leq y \leq 3\) and sketch the solutions satisfying \( y(1) = 0 \), \( y(1) = 1 \), and \( y(1) = -1 \).

15. Let \( F(t, y) = t^2 - y \) and let \( y(t) \) be the solution of \( \frac{dy}{dt} = F(t, y) \) satisfying \( y(0) = 3 \). Let \( h = 0.1 \) be the time step in Euler's Method, and set \( y_0 = y(2) = 3 \).

(a) Calculate \( y_1 = y_0 + hF(2, 3) \).

(b) Calculate \( y_2 = y_1 + hF(2, 1, y_1) \).

(c) Calculate \( y_3 = y_2 + hF(2, 2, y_2) \) and continue computing \( y_4, y_5, \) and \( y_6 \).

(d) Find approximations to \( y(2.2) \) and \( y(2.5) \).

16. Let \( y(t) \) be the solution to \( \frac{dy}{dt} = t e^{-y} \) satisfying \( y(0) = 0 \).

(a) Use Euler's Method with time step \( h = 0.1 \) to approximate \( y(0.1), y(0.2), \ldots, y(0.5) \).

(b) Use Separation of Variables to find \( y(t) \) exactly.

(c) Compute the errors in the approximations to \( y(0.1) \) and \( y(0.5) \).

In Exercises 17–22, use Euler's Method to approximate the given value of \( y(t) \) with the time step \( h \) indicated.

17. \( y(0.5); \quad \frac{dy}{dt} = y + t, \quad y(0) = 1, \quad h = 0.1 \)

18. \( y(0.7); \quad \frac{dy}{dt} = 2y, \quad y(0) = 3, \quad h = 0.1 \)

19. \( y(3.3); \quad \frac{dy}{dt} = t^2 - y, \quad y(3) = 1, \quad h = 0.05 \)

20. \( y(3); \quad \frac{dy}{dt} = \sqrt{t + y}, \quad y(2.7) = 5, \quad h = 0.05 \)

21. \( y(2); \quad \frac{dy}{dt} = t \sin y, \quad y(1) = 2, \quad h = 0.2 \)

22. \( y(5.2); \quad \frac{dy}{dt} = t - \sec y, \quad y(4) = -2, \quad h = 0.2 \)
Further Insights and Challenges

23. If \( f \) is continuous on \([a, b]\), then the solution to \( \frac{dy}{dt} = f(t) \) with initial condition \( y(a) = 0 \) is \( y(t) = \int_a^t f(u) \, du \). Show that Euler's Method with time step \( h = (b - a)/N \) for \( N \) steps yields the \( N \)th left-endpoint approximation to \( y(b) = \int_a^b f(u) \, du \).

Exercises 24–29: Euler's Midpoint Method is a variation on Euler's Method that is significantly more accurate in general. For time step \( h \) and initial value \( y_0 = y(a) \), the values \( y_k \) are defined successively by

\[
y_k = y_{k-1} + h m_{k-1}
\]

where \( m_{k-1} = f \left( x_{k-1} + \frac{h}{2}, y_{k-1} + \frac{h}{2} f(x_{k-1}, y_{k-1}) \right) \).

24. Apply both Euler's Method and the Euler Midpoint Method with \( h = 0.1 \) to estimate \( y(1.5) \), where \( y(t) \) satisfies \( \frac{dy}{dt} = y + t \) with \( y(0) = 1 \). Find \( y(t) \) exactly and compute the errors in these two approximations.

In Exercises 25–28, use Euler's Midpoint Method with the time step indicated to approximate the given value of \( y(t) \).

25. \( y(0.5); \quad \frac{dy}{dt} = y + t, \quad y(0) = 1, \quad h = 0.1 \)

26. \( y(2); \quad \frac{dy}{dt} = t^2 - y, \quad y(1) = 3, \quad h = 0.2 \)

27. \( y(0.25); \quad \frac{dy}{dt} = \cos(y + t), \quad y(0) = 1, \quad h = 0.05 \)

28. \( y(2.3); \quad \frac{dy}{dt} = y + t^2, \quad y(2) = 1, \quad h = 0.05 \)

29. Assume that \( f \) is continuous on \([a, b]\). Show that Euler's Midpoint Method applied to \( \frac{dy}{dt} = f(t) \) with initial condition \( y(a) = 0 \) and time step \( h = (b - a)/N \) for \( N \) steps yields the \( N \)th midpoint approximation to

\[
y(b) = \int_a^b f(u) \, du
\]

10.4 The Logistic Equation

In Section 10.1, we introduced the population growth model \( \frac{dy}{dt} = ky \). The solutions to this differential equation (with \( k > 0 \)) imply that the population grows exponentially. Although such a growth model might be valid over an initial short time period, no population can increase without limit because needed resources such as food or land are finite. To model a population subject to finite resources, we adjust the assumptions underlying the differential equation. If \( y(t) \) represents the population at time \( t \) and \( A \) denotes the maximum population that the environment can support, we let \( A - y(t) \) represent the room available for growth. Now we assume:

The rate of change of \( y \) is proportional to the amount \( y(t) \) present and the amount \( A - y(t) \) of room for growth.

Translating this relationship into a differential equation, we obtain what is known as the logistic differential equation:

\[
\frac{dy}{dt} = Ky(A - y)
\]

where \( K \) is a proportionality constant. For convenience, we rewrite the right side slightly:

\[
K y(A - y) = KyA \left( 1 - \frac{y}{A} \right) = ky \left( 1 - \frac{y}{A} \right)
\]

In the last equality, we replaced the product of constant terms \( KA \) with constant \( k \). Therefore, we have

\[
\frac{dy}{dt} = ky \left( 1 - \frac{y}{A} \right)
\]
Here $k > 0$ and is called the growth constant, while $A > 0$ and is called the carrying capacity. Figure 1 shows a typical S-shaped solution of Eq. (1).

**CONCEPTUAL INSIGHT** The logistic equation $\frac{dy}{dt} = ky(1 - y/A)$ differs from the exponential differential equation $\frac{dy}{dt} = ky$ only by the additional factor $(1 - y/A)$. As long as $y$ is small relative to $A$, this factor is close to 1 and can be ignored, yielding $\frac{dy}{dt} \approx ky$.

Thus, $y(t)$ grows nearly exponentially when the population is small (Figure 1). As $y(t)$ approaches $A$, the factor $(1 - y/A)$ tends to zero. This causes $\frac{dy}{dt}$ to decrease to zero and prevents $y(t)$ from exceeding the carrying capacity $A$.

The slope field in Figure 2 shows clearly that there are three families of solutions, depending on the initial value $y_0 = y(0)$.

- If $y_0 > A$, then $y(t)$ is decreasing and approaches $A$ as $t \to \infty$.
- If $0 < y_0 < A$, then $y(t)$ is increasing and approaches $A$ as $t \to \infty$.
- If $y_0 < 0$, then $y(t)$ is decreasing and there is a time $t_0$ such that $\lim_{t \to t_0} y(t) = -\infty$.

Equation (1) also has two constant solutions: $y = 0$ and $y = A$. They correspond to the roots of $ky(1 - y/A) = 0$, and they satisfy Eq. (1) because $\frac{dy}{dt} = 0$ when $y$ is a constant. Constant solutions are called equilibrium or steady-state solutions. The equilibrium solution $y = A$ is a stable equilibrium because every solution with initial value $y_0$ close to $A$ approaches the equilibrium $y = A$ as $t \to \infty$. By contrast, $y = 0$ is an unstable equilibrium because every nonequilibrium solution with initial value $y_0$ near $y = 0$ either increases to $A$ or decreases to $-\infty$. These nonequilibrium solutions deviate away from the unstable equilibrium solution as time moves forward.

Having described the solutions qualitatively, let us now find the nonequilibrium solutions explicitly using Separation of Variables. Assuming that $y \neq 0$ and $y \neq A$, we have

\[
\frac{dy}{dt} = ky \left(1 - \frac{y}{A}\right)
\]

\[
\frac{dy}{y \left(1 - \frac{y}{A}\right)} = k \, dt
\]

\[
\int \left(\frac{1}{y} - \frac{1}{y - A}\right) \, dy = \int k \, dt
\]
\[
\ln |y| - \ln |y - A| = kt + C
\]
\[
\left| \frac{y}{y - A} \right| = e^{kt + C} = \frac{y}{y - A} = \pm e^C e^{kt}
\]
Since \( \pm e^C \) takes on arbitrary nonzero values, we replace \( \pm e^C \) with \( B \) (nonzero):
\[
\frac{y}{y - A} = Be^{kt}
\]
For \( t = 0 \), this gives a useful relation between \( B \) and the initial value \( y_0 = y(0) \):
\[
\frac{y_0}{y_0 - A} = B
\]
To solve for \( y \), multiply each side of Eq. (3) by \( (y - A) \):
\[
y = (y - A)Be^{kt}
\]
\[
y(1 - Be^{kt}) = -ABe^{kt}
\]
\[
y = \frac{ABe^{kt}}{Be^{kt} - 1}
\]
As \( B \neq 0 \), we may divide by \( Be^{kt} \) to obtain the general nonequilibrium solution:
\[
\frac{dy}{dt} = ky \left( 1 - \frac{y}{A} \right), \quad y = \frac{A}{1 - e^{-kt}/B}
\]
We use this formula in the next two examples. In each case, the differential equation may instead be solved by Separation of Variables, the method used to derive this solution.

EXAMPLE 1 Solve \( \frac{dy}{dt} = 0.3y(4 - y) \) with initial condition \( y(0) = 1 \).

Solution To apply Eq. (5), we must rewrite the equation in the form
\[
\frac{dy}{dt} = 1.2y \left( 1 - \frac{y}{4} \right)
\]
Thus, \( k = 1.2 \) and \( A = 4 \), and the general solution is
\[
y = \frac{4}{1 - e^{-1.2t}/B}
\]
There are two ways to find \( B \). One way is to solve \( y(0) = 1 \) for \( B \) directly. An easier way is to use Eq. (4):
\[
B = \frac{y_0}{y_0 - A} = \frac{1}{1 - \frac{4}{1 - 4}} = \frac{1}{3}
\]
We find that the particular solution is \( y = \frac{4}{1 + 3e^{-1.2t}} \) (Figure 3).

EXAMPLE 2 Deer Population A deer population grows logistically with growth constant \( k = 0.4 \) year\(^{-1} \) in a forest with a carrying capacity of 1000 deer.

(a) Find the deer population \( P(t) \) if the initial population is \( P_0 = 100 \).
(b) How long does it take for the deer population to reach 500?
Solution The time unit is the year because the unit of \( k \) is year\(^{-1} \).

(a) Since \( k = 0.4 \) and \( A = 1000 \), \( P(t) \) satisfies the differential equation

\[
\frac{dP}{dt} = 0.4P \left(1 - \frac{P}{1000}\right)
\]

The general solution is given by Eq. (5):

\[
P(t) = \frac{1000}{1 - e^{-0.4t}/B}
\]

Using Eq. (4) to compute \( B \), we find (Figure 4)

\[
B = \frac{P_0}{P_0 - A} = \frac{100}{100 - 1000} = \frac{1}{9} \quad \Rightarrow \quad P(t) = \frac{1000}{1 + 9e^{-0.4t}}
\]

(b) To find the time \( t \) when \( P(t) = 500 \), we could solve the equation

\[
P(t) = \frac{1000}{1 + 9e^{-0.4t}} = 500
\]

But it is easier to use Eq. (3):

\[
\frac{P}{P - A} = Be^{kt}
\]

\[
\frac{P}{P - 1000} = -\frac{1}{9}e^{0.4t}
\]

Set \( P = 500 \) and solve for \( t \):

\[
-\frac{500}{9} = \frac{500}{500 - 1000} = -1 \quad \Rightarrow \quad e^{0.4t} = 9 \quad \Rightarrow \quad 0.4t = \ln 9
\]

This gives \( t = (\ln 9)/0.4 \approx 5.5 \) years.

\section*{10.4 SUMMARY}

- The logistic equation and its general nonequilibrium solution \((k > 0 \text{ and } A > 0)\):

\[
\frac{dy}{dt} = ky \left(1 - \frac{y}{A}\right), \quad y = \frac{A}{1 - e^{-kt}/B}, \quad \text{or equivalently,} \quad \frac{y}{y - A} = Be^{kt}
\]

- Two equilibrium (constant) solutions:
  - \( y = 0 \) is an unstable equilibrium.
  - \( y = A \) is a stable equilibrium.

- If the initial value \( y_0 = y(0) \) satisfies \( y_0 > 0 \), then \( y(t) \) approaches the stable equilibrium \( y = A \); that is, \( \lim_{t \to \infty} y(t) = A \).

\section*{10.4 EXERCISES}

\underline{Preliminary Questions}

1. Which of the following differential equations is a logistic differential equation?
   (a) \( \frac{dy}{dt} = 2y(1 - y^2) \) \hspace{1cm} (b) \( \frac{dy}{dt} = 2y \left(1 - \frac{y}{3}\right)\)
   (c) \( \frac{dy}{dt} = 2y \left(1 - \frac{t}{4}\right) \) \hspace{1cm} (d) \( \frac{dy}{dt} = 2y(1 - 3y)\)

2. What are the constant solutions to \( \frac{dy}{dt} = ky \left(1 - \frac{y}{A}\right) \)?

3. Is the logistic equation separable?
Exercises

1. Find the general solution of the logistic equation

\[ \frac{dy}{dt} = 3y \left(1 - \frac{y}{5}\right) \]

Then find the particular solution satisfying \( y(0) = 2 \).

2. Find the solution of \( \frac{dy}{dt} = 2y(3 - y) \), \( y(0) = 10 \).

In Exercises 3–4, for each of (a)–(c), give the solution \( y(t) \) satisfying the initial condition. (Note: the general solution formula for the logistic equation, Eq. (5), applies when a solution is not constant.)

3. \( \frac{dy}{dt} = 3y(6 - y) \)
   (a) \( y(0) = 6 \)  
   (b) \( y(0) = 4 \)  
   (c) \( y(4) = 0 \)

4. \( \frac{dy}{dt} = y \left(2 - \frac{y}{4}\right) \)
   (a) \( y(0) = 6 \)  
   (b) \( y(0) = 8 \)  
   (c) \( y(0) = -2 \)

5. A population of squirrels lives in a forest with a carrying capacity of 2000. Assume logistic growth with growth constant \( k = 0.6 \) yr\(^{-1}\).
   (a) Find a formula for the squirrel population \( P(t) \), assuming an initial population of 500 squirrels.
   (b) How long will it take for the squirrel population to double?

6. The population \( P(t) \) of mosquito larvae growing in a tree hole increases according to the logistic equation with growth constant \( k = 0.3 \) day\(^{-1}\) and carrying capacity \( M = 500 \).
   (a) Find a formula for the larva population \( P(t) \), assuming an initial population of \( P_0 = 50 \) larvae.
   (b) After how many days will the larva population reach 200?

7. Sunset Lake is stocked with 2000 rainbow trout, and after 1 year the population has grown to 4500. Assuming logistic growth with a carrying capacity of 20,000, find the growth constant \( k \) (specify the units) and determine when the population will increase to 10,000.

8. Spread of a Rumor
   A rumor spreads through a small town. Let \( y(t) \) be the fraction of the population that has heard the rumor at time \( t \) and assume that the rate at which the rumor spreads is proportional to the product of the fraction \( y \) of the population that has heard the rumor and the fraction \( 1 - y \) that has not yet heard the rumor.
   (a) Write the differential equation satisfied by \( y \) in terms of a proportionality factor \( k \).
   (b) Find \( k \) (in units of \( \text{day}^{-1} \)), assuming that 10% of the population knows the rumor at \( t = 0 \) and 40% knows it at \( t = 2 \) days.
   (c) Using the assumptions of part (b), determine when 75% of the population will know the rumor.

9. A rumor spreads through a school with 1000 students. At 8 AM, 80 students have heard the rumor, and by noon, half the school has heard it. Using the logistic model of Exercise 8, determine when 90% of the students will have heard the rumor.

10. \( \text{GU} \)
   A simpler model for the spread of a rumor assumes that the rate at which the rumor spreads is proportional (with factor \( k \)) to the fraction of the population that has not yet heard the rumor.
   (a) Compute the solutions to this model and the model of Exercise 8 with the values \( k = 0.9 \) and \( y_0 = 0.1 \).
   (b) Graph the two solutions on the same axis.
   (c) Which model seems more realistic? Why?

11. Let \( k = 1 \) and \( A = 1 \) in the logistic equation.
   (a) Find the solutions satisfying \( y(0) = 10 \) and \( y(0) = -1 \).
   (b) Find the time \( t \) when \( y(t) = 5 \).
   (c) When does \( y(t) \) become infinite?

12. A tissue culture grows until it has a maximum area of \( M \) square centimeters. The area \( A(t) \) of the culture at time \( t \) may be modeled by the differential equation

\[ \frac{dA}{dt} = k \sqrt{A} \left(1 - \frac{A}{M}\right) \]

where \( k \) is a growth constant.
   (a) Show that if we set \( A = a^2 \), then

\[ \frac{du}{dt} = \frac{1}{2} k \left(1 - \frac{a^2}{M}\right) \]

Then find the general solution using Separation of Variables.
   (b) Show that the general solution to Eq. (7) is

\[ A(t) = M \left(\frac{C e^{k \sqrt{t}} - 1}{C e^{k \sqrt{t}} + 1}\right)^2 \]

13. \( \text{GU} \)
   In the model of Exercise 12, let \( A(t) \) be the area at time \( t \) (hours) of a growing tissue culture with initial size \( A(0) = 1 \) cm\(^2\), assuming that the maximum area is \( M = 16 \) cm\(^2\) and the growth constant is \( k = 0.1 \).
   (a) Find a formula for \( A(t) \). Note: The initial condition is satisfied for two values of the constant \( C \). Choose the value of \( C \) for which \( A(t) \) is increasing.
   (b) Determine the area of the culture at \( t = 10 \) hours.

14. Graph the solution using a graphing utility.

15. In 1751, Benjamin Franklin predicted that the U.S. population \( P(t) \) would increase with growth constant \( k = 0.028 \) year\(^{-1}\). According to the census, the U.S. population was 5 million in 1800 and 76 million in 1900. Assuming logistic growth with \( k = 0.028 \), find the predicted carrying capacity for the U.S. population. Hint: Use Eqs. (3) and (4) to show that

\[ \frac{P(t)}{P_0} = 1 + e^{-kt} \]

16. Reverse Logistic Equation
   Consider the following logistic equation (with \( k, B > 0 \)):

\[ \frac{dP}{dt} = -kP \left(1 - \frac{P}{B}\right) \]

(a) Sketch the slope field of this equation.
   (b) The general solution is \( P(t) = B / (1 - e^{kt}/C) \), where \( C \) is a nonzero constant. Show that \( P(0) > B \) if \( C > 1 \) and \( 0 < P(0) < B \) if \( C < 0 \).
   (c) Show that Eq. (8) models an extinction–explosion population. That is, \( P(t) \) tends to zero if the initial population satisfies \( 0 < P(0) < B \), and it tends to infinity after a finite amount of time if \( P(0) > B \).
   (d) Show that \( P = 0 \) is a stable equilibrium and \( P = B \) is an unstable equilibrium.
Further Insights and Challenges

In Exercises 17 and 18, let \( y(t) \) be a solution of the logistic equation

\[
\frac{dy}{dt} = ky \left( 1 - \frac{y}{A} \right)
\]

where \( A > 0 \) and \( k > 0 \).

17. (a) Differentiate Eq. (9) with respect to \( t \) and use the Chain Rule to show that

\[
\frac{d^2y}{dt^2} = k^2y \left( 1 - \frac{y}{A} \right) \left( 1 - \frac{2y}{A} \right)
\]

(b) Show that the graph of the function \( y \) is concave up if \( 0 < y < A/2 \) and concave down if \( A/2 < y < A \).

(c) Show that if \( 0 < y(0) < A/2 \), then \( y \) has a point of inflection at \( y = A/2 \) (Figure 5).

(d) Assume that \( 0 < y(0) < A/2 \). Find the time \( t \) when \( y(t) \) reaches the inflection point.

18. Let \( y = \frac{A}{1 - e^{-kt}} \) be the general nonequilibrium solution to Eq. (9). If \( y(t) \) has a vertical asymptote at \( t = t_0 \), that is, if

\[
\lim_{t \to t_0^-} y(t) = \pm \infty,
\]

we say that the solution "blows up" at \( t = t_0 \).

(a) Show that if \( 0 < y(0) < A \), then \( y \) does not blow up at any time \( t_0 \).

(b) Show that if \( y(0) > A \), then \( y \) blows up at a time \( t_0 \), which is negative (and hence does not correspond to a real time).

(c) Show that \( y \) blows up at some positive time \( t_0 \) if and only if \( y(0) < 0 \) (and hence does not correspond to a real population).

10.5 First-Order Linear Equations

In Section 10.2, we investigated the first-order linear constant coefficient differential equation \( \frac{dy}{dt} = ky + c \). In this section, we work with a more general version of this differential equation, where the terms \( k \) and \( c \) are functions of the independent variable (which here is \( x \)). The method of "integrating factors" is used to solve these differential equations. Although some of these equations are separable and can be solved by Separation of Variables, the method that we present here applies to all first-order linear differential equations, whether separable or not (Figure 1).

A first-order linear differential equation is one that can be put in the following form:

\[
y' + P(x)y = Q(x)
\]

To solve Eq. (1), we shall multiply through by a function \( \alpha(x) \), called an integrating factor, that turns the left-hand side into the derivative of \( \alpha(x)y \):

\[
\alpha(x)(y' + P(x)y) = \left( \alpha(x)y \right)'
\]

Suppose we can find \( \alpha(x) \) satisfying Eq. (2) that is nonzero. Then Eq. (1) yields

\[
\alpha(x)(y' + P(x)y) = \alpha(x)Q(x)
\]

\[
\left( \alpha(x)y \right)' = \alpha(x)Q(x)
\]

We can solve this equation by integration:

\[
\alpha(x)y = \int \alpha(x)Q(x)\,dx \quad \text{or} \quad y = \frac{1}{\alpha(x)} \left( \int \alpha(x)Q(x)\,dx \right)
\]

To find \( \alpha(x) \), expand Eq. (2), using the Product Rule on the right-hand side:

\[
\alpha(x)y' + \alpha(x)P(x)y = \alpha(x)y' + \alpha(x)y \quad \Rightarrow \quad \alpha(x)P(x)y = \alpha(x)y
\]
Dividing by \( y \), we obtain

\[
\frac{da}{dx} = \alpha(x)P(x)
\]

We solve this equation using Separation of Variables:

\[
\frac{da}{\alpha} = P(x)\,dx \quad \Rightarrow \quad \int \frac{da}{\alpha} = \int P(x)\,dx
\]

Therefore, \( \ln |\alpha(x)| = \int P(x)\,dx \), and by exponentiation, \( \alpha(x) = \pm e^{\int P(x)\,dx} \). Since we need just one solution of Eq. (3), we choose the positive sign in the expression for \( \alpha(x) \).

**THEOREM 1** The general solution of \( y' + P(x)y = Q(x) \) is

\[
y = \frac{1}{\alpha(x)} \left( \int \alpha(x)Q(x)\,dx \right)
\]

where \( \alpha(x) \) is an integrating factor:

\[
\alpha(x) = e^{\int P(x)\,dx}
\]

To solve differential equations \( y' + P(x)y = Q(x) \), we can either find an integrating factor and use Eq. (4) in Theorem 1, or we can use the method that yielded the solution in Eq. (4). That method is

- Find an integrating factor \( \alpha(x) = e^{\int P(x)\,dx} \).
- Multiply both sides of the equation \( y' + P(x)y = Q(x) \) by the integrating factor \( \alpha(x) \). As a result, the left side of the equation is then the derivative of a product.
- Integrate both sides and solve for \( y \) to obtain the solution.

We follow this method to obtain the solution in the next example.

**EXAMPLE 1** Solve \( xy' - 3y = x^2 \), \( y(1) = 2 \).

**Solution** First, divide by \( x \) to put the equation in the form \( y' + P(x)y = Q(x) \):

\[
y' - \frac{3}{x}y = x
\]

Thus, \( P(x) = -3x^{-1} \) and \( Q(x) = x \).

**Step 1. Find an integrating factor.**

In our case, \( P(x) = -3x^{-1} \), and by Eq. (5),

\[
\alpha(x) = e^{\int P(x)\,dx} = e^{\int (-3/x)\,dx} = e^{-3\ln x} = e^{\ln(x^{-3})} = x^{-3}
\]

**Step 2. Multiply the equation by the integrating factor.**

\[
x^{-3}(y' - \frac{3}{x}y) = x^{-3}(x)
\]

\[
(x^{-3}y)' = x^{-2}
\]

**Step 3. Integrate both sides.**

\[
x^{-3}y = -x^{-1} + C
\]
Step 4. Solve for y.

\[
y = x^3(-x^{-1} + C) = -x^2 + Cx^3
\]

Step 5. Solve the Initial Value Problem.

Now solve for C using the initial condition \(y(1) = 2\):

\[
y(1) = -1^2 + C \cdot 1^3 = 2 \quad \text{or} \quad C = 3
\]

Therefore, the solution of the Initial Value Problem is \(y = -x^2 + Cx^3\).

Finally, let’s check that \(y = -x^2 + 3x^3\) satisfies our equation \(xy' - 3y = x^2\):

\[
x y' - 3y = x(-2x^2 + 9x^3) - 3(-x^2 + 3x^3) = (-2x^2 + 9x^3) + (3x^2 - 9x^3) = x^2
\]

**Example 2** Solve the Initial Value Problem: \(y' + (1 - x^{-1})y = x^2\), \(y(1) = 2\).

Solution This equation has the form \(y' + P(x)y = Q(x)\) with \(P(x) = 1 - x^{-1}\). By Eq. (5), an integrating factor is

\[
\alpha(x) = e^{\int (1 - x^{-1}) dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} e^x
\]

By either multiplying by the integration factor and then integrating both sides of the resulting equation or by applying Eq. (4) with \(Q(x) = x^2\), we obtain the general solution:

\[
y = \alpha(x)^{-1} \left( \int \alpha(x)Q(x) \, dx \right) = x e^{-x} \left( \int (x^{-1} e^x) x^2 \, dx \right)
\]

Integration by Parts shows that \(\int x e^x \, dx = (x - 1) e^x + C\), so we obtain

\[
y = x e^{-x}((x - 1) e^x + C) = x(x - 1) + Cxe^{-x}
\]

The initial condition \(y(1) = 2\) gives

\[
y(1) = 1(1 - 1) + C e^{-1} = C e^{-1} = 2 \quad \Rightarrow \quad C = 2e
\]

The desired particular solution is

\[
y = x(x - 1) + (2e)xe^{-x} = x(x - 1) + 2xe^{1-x}
\]

**Conceptual Insight** We have expressed the general solution of a first-order linear differential equation in terms of the integrals in Eqs. (4) and (5). Keep in mind, however, that this is not always possible to evaluate these integrals explicitly. For example, the general solution of \(y' + xy = 1\) is

\[
y = e^{-x^2/2} \left( \int e^{x^2/2} \, dx + C \right)
\]

The integral \(\int e^{x^2/2} \, dx\) cannot be evaluated in elementary terms. However, we can approximate the integral numerically and plot the solutions by computer (Figure 2).

In the next example, we use a differential equation to model a mixing problem, which has applications in biology, chemistry, and medicine.
**EXAMPLE 3  A Mixing Problem**  A tank contains 600 liters of water with a sucrose concentration of 0.2 kg/L. We begin adding water with a sucrose concentration of 0.1 kg/L at a rate of $R_{in} = 40$ L/min (Figure 3). The water mixes instantaneously and exits the bottom of the tank at a rate of $R_{out} = 20$ L/min. Let $y(t)$ be the quantity (in kilograms) of sucrose in the tank at time $t$ (in minutes). Set up a differential equation for $y(t)$ and solve for $y(t)$.

**Solution**

**Step 1. Set up the differential equation.**

The derivative $dy/dt$ is the difference of two rates of change, namely the rate at which sucrose enters the tank and the rate at which it leaves:

$$\frac{dy}{dt} = \text{sucrose rate in} - \text{sucrose rate out}$$

The rate at which sucrose enters the tank is:

$$\text{sucrose rate in} = (0.1 \text{ kg/L})(40 \text{ L/min}) = 4 \text{ kg/min}$$

Concentration times water rate in

Next, we compute the sucrose concentration in the tank at time $t$. Water flows in at 40 L/min and out at 20 L/min, so there is a net inflow of 20 L/min. The tank has 600 L at time $t = 0$, so it has $600 + 20t$ L at time $t$, and

$$\text{concentration at time } t = \frac{\text{kilograms of sucrose in tank}}{\text{liters of water in tank}} = \frac{y(t)}{600 + 20t} \text{ kg/L}$$

The rate at which sucrose leaves the tank is the product of the concentration and the rate at which water flows out:

$$\text{sucrose rate out} = \left(\frac{y}{600 + 20t}\right) \left(\frac{20}{\text{min}}\right) = \frac{20y}{600 + 20t} = \frac{y}{t + 30} \text{ kg/min}$$

Concentration times water rate out

Now Eq. (6) gives us the differential equation:

$$\frac{dy}{dt} = 4 - \frac{y}{t + 30}$$

**Step 2. Find the general solution.**

We write Eq. (7) in standard form:

$$\frac{dy}{dt} + \frac{1}{t + 30} \frac{y}{P(t)} = \frac{4}{Q(t)}$$

An integrating factor is:

$$\alpha(t) = e^{\int P(t)dt} = e^{\int \frac{1}{t + 30} dt} = e^{\ln(t + 30)} = t + 30$$

The general solution is:

$$y(t) = \alpha(t)^{-1} \left( \int \alpha(t)Q(t) \, dt + C \right)$$

$$= \frac{1}{t + 30} \left( \int (t + 30)(4) \, dt + C \right)$$

$$= \frac{1}{t + 30} \left( 2(t + 30)^2 + C \right) = 2t + 60 + \frac{C}{t + 30}$$
Step 3. Solve the Initial Value Problem.
At \( t = 0 \), the tank contains 600 L of water with a sucrose concentration of 0.2 kg/L. Thus, the total sucrose at \( t = 0 \) is \( y(0) = (600)(0.2) = 120 \) kg. Also, from the general solution formula, we have

\[
y(0) = 2(0) + 60 + \frac{C}{0 + 30} = 60 + \frac{C}{30}
\]

Therefore,

\[
60 + \frac{C}{30} = 120 \quad \Rightarrow \quad C = 1800
\]

We obtain the following formula (\( t \) in minutes), which is valid until the tank overflows:

\[
y(t) = 2t + 60 + \frac{1800}{t + 30} \text{ kg sucrose}
\]

10.5 SUMMARY

- A first-order linear differential equation is a differential equation in the form

\[
y' + P(x)y = Q(x)
\]

- The general solution is

\[
y = \alpha(x)^{-1} \left( \int \alpha(x)Q(x) \, dx + C \right)
\]

where \( \alpha(x) \) is an integrating factor: \( \alpha(x) = e^{\int P(x) \, dx} \).

10.5 EXERCISES

Preliminary Questions

1. Which of the following are first-order linear equations?
   (a) \( y' + x^2y = 1 \)  
   (b) \( y' + xy^2 = 1 \)  
   (c) \( x^2y' + y = x^4 \)  
   (d) \( x^2y' + y = e^y \)

2. If \( \alpha(x) \) is an integrating factor for \( y' + A(x)y = B(x) \), then \( \alpha'(x) \) is equal to (choose the correct answer):
   (a) \( B(x) \)  
   (b) \( \alpha(x)A(x) \)  
   (c) \( \alpha(x)A'(x) \)  
   (d) \( \alpha(x)B(x) \)

3. For what function \( P \) is the integrating factor \( \alpha(x) \) equal to \( x \)?
4. For what function \( P \) is the integrating factor \( \alpha(x) \) equal to \( e^y \)?

Exercises

1. Consider \( y' + x^{-1}y = x^3 \).
   (a) Verify that \( \alpha(x) = x \) is an integrating factor.
   (b) Show that when multiplied by \( \alpha(x) \), the differential equation can be written \( (xy)' = x^4 \).
   (c) Conclude that \( xy \) is an antiderivative of \( x^4 \) and use this information to find the general solution.
   (d) Find the particular solution satisfying \( y(1) = 0 \).

2. Consider \( \frac{dy}{dx} + 2y = e^{-3x} \).
   (a) Verify that \( \alpha(x) = e^{2x} \) is an integrating factor.
   (b) Use Eq. (4) to find the general solution.
   (c) Find the particular solution with initial condition \( y(0) = 1 \).

3. Let \( \alpha(x) = e^{2x} \). Verify the identity
   \[
   (\alpha(x)y)' = \alpha(x)(y' + 2xy)
   \]
   and explain how it is used to find the general solution of
   \[
y' + 2xy = x
   \]

4. Find the solution of \( y' - y = e^{-2x}, y(0) = 1 \).

5. In Exercises 5–20, find the general solution of the first-order linear differential equation.
   (a) \( xy' + y = x \)
   (b) \( xy' - y = x^2 - x \)
   (c) \( 3xy' - y = x^2 \)
   (d) \( y' + xy = x \)
   (e) \( y' + 3x^{-1}y = x + x^{-1} \)
   (f) \( y' + x^{-1}y = \cos(x^3) \)
   (g) \( xy' = y - x \)
   (h) \( xy' = x^2 - \frac{3y}{x} \)
13. \[ y' + y = e^x \]
14. \[ \gamma' - y = e^x \]
15. \[ y' + (\tan x)y = \cos x \]
16. \[ \gamma' + (\sec x)y = \cos x \]
17. \[ e^x y' = 1 + 2e^x y \]
18. \[ e^x y' = 1 - e^x y \]
19. \[ \gamma' - (\ln x)y = x^2 \]
20. \[ \gamma' + y = \cos x \]

In Exercises 21–28, solve the Initial Value Problem.

21. \[ y' + 3y = e^{2x}, \quad y(0) = -1 \]
22. \[ xy' + y = x e^x, \quad y(1) = 3 \]
23. \[ y' + \frac{1}{x+1} y = x^{-2}, \quad y(1) = 2 \]
24. \[ y' + y = \sin x, \quad y(0) = 1 \]
25. \[ (\sin x)y' = (\cos x)y + 1, \quad y\left(\frac{\pi}{4}\right) = 0 \]
26. \[ y' + (\sec x)y = \sec x, \quad y\left(\frac{\pi}{4}\right) = 1 \]
27. \[ y' + (\tan x)y = 1, \quad y(0) = 3 \]
28. \[ y' + \frac{x}{1 + x^2} y = \frac{1}{(1 + x^2)/x}, \quad y(1) = 0 \]

29. The differential equation \( \frac{dy}{dx} = x \) is directly integrable and also first-order linear. Show that solving the differential equation using Theorem 1 leads to solving it by direct integration.

30. The differential equation \( \frac{dy}{dx} = 1 - y \) can be solved using Eq. (2) in Section 10.2 and using Theorem 1 in this section. Show that both approaches lead to the same general solution.

31. Find the general solution of \( y' + ny = e^{mx} \) for all \( m, n \). Note: The case \( m = -n \) must be treated separately.

32. Find the general solution of \( y' + ny = \cos x \) for all \( n \).

In Exercises 33–36, a 1000-liter tank contains 500 L of water with a salt concentration of 10 g/L. Water with a salt concentration of 50 g/L flows into the tank at a rate of \( R_{\text{in}} = 80 \text{ L/min} \). The fluid mixes instantaneously and is pumped out at a specified rate \( R_{\text{out}} \). Let \( y(t) \) denote the quantity of salt in the tank at time \( t \).

33. Assume that \( R_{\text{out}} = 40 \text{ L/min} \).
(a) Set up and solve the differential equation for \( y(t) \).
(b) What is the salt concentration when the tank overflows?

34. Find the salt concentration when the tank overflows, assuming that \( R_{\text{out}} = 60 \text{ L/min} \).

35. Find the limiting salt concentration as \( t \to \infty \), assuming that \( R_{\text{out}} = 80 \text{ L/min} \).

36. Assuming that \( R_{\text{out}} = 120 \text{ L/min} \), find \( y(t) \). Then calculate the tank volume and the salt concentration at \( t = 10 \text{ min} \).

37. Water flows into a tank at the variable rate of \( R_{\text{in}} = 20(1 + t) \text{ gal/min} \) and out at the constant rate \( R_{\text{out}} = 5 \text{ gal/min} \). Let \( V(t) \) be the volume of water in the tank at time \( t \).
(a) Set up a differential equation for \( V(t) \) and solve it with the initial condition \( V(0) = 100 \).
(b) Find the maximum value of \( V \).
(c) CAS Plot \( V(t) \) and estimate the time \( t \) when the tank is empty.

38. A stream feeds into a lake at a rate of 1000 m³/day. The stream is polluted with a toxin whose concentration is 5 g/m³. Assume that the lake has volume 10⁶ m³ and that water flows out of the lake at the same rate of 1000 m³/day.
(a) Set up a differential equation for the concentration \( c(t) \) of the toxin in the lake and solve for \( c(t) \), assuming that \( c(0) = 0 \). Hint: Find the differential equation for the quantity of toxin \( y(t) \), and observe that \( c(t) = y(t)/10^6 \).
(b) What is the limiting concentration for large \( t \) ?

In Exercises 39–42, consider a series circuit (Figure 4) consisting of a resistor of \( R \) ohms, an inductor of \( L \) henries, and a variable voltage source of \( V(t) \) volts (time \( t \) in seconds). The current through the circuit \( i(t) \) (in amperes) satisfies the differential equation

\[ \frac{di}{dt} + \frac{R}{L} i = \frac{1}{L} V(t) \]

39. Solve Eq. (9) with initial condition \( i(0) = 0 \), assuming that \( R = 100 \text{ ohms} \), \( L = 5 \text{ henries} \), and \( V(t) \) is constant with \( V(t) = 10 \text{ volts} \).

40. Assume that \( R = 110 \text{ ohms} \), \( L = 10 \text{ henries} \), and \( V(t) = e^{-t} \text{ volts} \).
(a) Solve Eq. (9) with initial condition \( i(0) = 0 \).
(b) Calculate \( t_{\text{fin}} \) and \( (t_{\text{fin}}) \), where \( t_{\text{fin}} \) is the time at which \( i(t) \) has a maximum value.
(c) (GU) Use a computer algebra system to sketch the graph of the solution for \( 0 \leq t \leq 3 \).

41. Assume that \( V(t) = V \) is constant and \( i(0) = 0 \).
(a) Solve for \( i(t) \).
(b) Show that \( \lim_{t \to \infty} i(t) = \frac{V}{R} \) and that \( i(t) \) reaches approximately 63% of its limiting value after \( L/R \) seconds.
(c) How long does it take for \( i(t) \) to reach 90% of its limiting value if \( R = 500 \text{ ohms} \), \( L = 4 \text{ henries} \), and \( V = 20 \text{ volts} \)?

42. Solve for \( i(t) \), assuming that \( R = 500 \text{ ohms} \), \( L = 4 \text{ henries} \), and \( V = 20 \text{ volts} \).

**Figure 4** RL circuit.

43. ✅ Tank 1 in Figure 5 is filled with \( V_1 \) liters of water containing blue dye at an initial concentration of \( c_0 \text{ g/L} \). Water flows into the tank at a rate of \( R \text{ L/min} \), is mixed instantaneously with the dye solution, and flows out through the bottom at the same rate \( R \). Let \( c_1(t) \) be the dye concentration in the tank at time \( t \).
(a) Explain why \( c_1(t) \) satisfies the differential equation

\[ \frac{dc_1}{dt} = -\frac{R}{V_1} c_1 \]

(b) Solve for \( c_1(t) \) with \( V_1 = 300 \text{ L} \), \( R = 50 \), and \( c_0 = 10 \text{ g/L} \).
41. Continuing with the previous exercise, let tank 2 be another tank filled with \( V_2 \) gallons of water. Assume that the dye solution from tank 1 empties into tank 2 as in Figure 5, mixes instantaneously, and leaves tank 2 at the same rate \( R \). Let \( c_2(t) \) be the dye concentration in tank 2 at time \( t \).

(a) Explain why \( c_2 \) satisfies the differential equation

\[
\frac{dc_2}{dt} = \frac{R}{V_2} (c_1 - c_2)
\]

(b) Use the solution to Exercise 43 to solve for \( c_2(t) \) if \( V_1 = 300 \), \( V_2 = 200 \), \( R = 50 \), and \( c_0 = 10 \).

(c) Find the maximum concentration in tank 2.

(d) Plot the solution.

45. Let \( a, b, r \) be constants. Show that

\[
y = Ce^{-rt} + a + b \left( \frac{k \sin rt - r \cos rt}{k^2 + r^2} \right)
\]

is a general solution of

\[
\frac{dy}{dt} = -k \left( y - a - b \sin rt \right)
\]

46. Assume that the outside temperature varies as

\[
T(t) = 15 + 5 \sin(\pi t / 12)
\]

where \( t \) is 0 at 12 noon. A house is heated to 25°C at \( t = 0 \) and after that, its temperature \( y(t) \) varies according to Newton’s Law of Cooling (Figure 6):

\[
\frac{dy}{dt} = -0.1 \left( y(t) - T(t) \right)
\]

Use Exercise 45 to solve for \( y(t) \).

Further Insights and Challenges

47. Let \( a(x) \) be an integrating factor for \( y' + P(x)y = Q(x) \). The differential equation \( y' + P(x)y = 0 \) is called the associated homogeneous equation.

(a) Show that \( y = 1/a(x) \) is a solution of the associated homogeneous equation.

(b) Show that if \( y = f(x) \) is a particular solution of \( y' + P(x)y = Q(x) \), then \( f(x) + C/a(x) \) is also a solution for any constant \( C \).

48. Use the Fundamental Theorem of Calculus and the Product Rule to verify directly that for any \( x_0 \), the function

\[
f(x) = a(x)^{-1} \int_{x_0}^{x} a(t)Q(t) \, dt
\]

is a solution of the Initial Value Problem

\[
y' + P(x)y = Q(x), \quad y(x_0) = 0
\]

where \( a(x) \) is an integrating factor [a solution to Eq. (3)].

CHAPTER REVIEW EXERCISES

1. Which of the following differential equations are first-order linear?

   (a) \( y' = y^2 - 3x^4y \)
   (b) \( y' = x^3 - 3x^4y \)
   (c) \( y = y' - 3x \sqrt{y} \)
   (d) \( \sin x \cdot y' = y - 1 \)

2. Find a value of \( c \) such that \( y = xc - 2 + e^{cx} \) is a solution of \( 2y' + y = x \).

   In Exercises 3-6, solve using Separation of Variables.

3. \( \frac{dy}{dt} = t^2 y^{-3} \)

4. \( xyy' = 1 - x^2 \)
5. \( x \frac{dy}{dx} - 2y = 3 \)

6. \( y' = \frac{xy^2}{x^2 + 1} \)

In Exercises 7–10, solve the Initial Value Problem using Separation of Variables.

7. \( y' = \cos^2 x, \quad y(0) = \frac{\pi}{4} \)

8. \( y' = \cos^2 y, \quad y(0) = \frac{\pi}{4} \)

9. \( y' = 6xy^2, \quad y(1) = 4 \)

10. \( xyy' = 1, \quad y(3) = 2 \)

11. Figure 1 shows the slope field for \( \frac{dy}{dt} = \sin y + ty. \) Sketch the graphs of the solutions with the initial conditions \( y(0) = 1, \quad y(0) = 0, \quad \) and \( y(0) = -1. \)

![Figure 1](image)

12. Sketch the slope field for \( \frac{dy}{dt} = t^2y \) for \(-2 \leq t \leq 2, -2 \leq y \leq 2. \)

13. Sketch the slope field for \( \frac{dy}{dt} = y \sin t \) for \(-2\pi \leq t \leq 2\pi, -2 \leq y \leq 2. \)

14. Which of the equations (i)–(iii) corresponds to the slope field in Figure 2?

   (i) \( \frac{dy}{dt} = 1 - y^2 \)
   (ii) \( \frac{dy}{dt} = 1 + y^2 \)
   (iii) \( \frac{dy}{dt} = y^2 \)

![Figure 2](image)

15. Let \( y(t) \) be the solution to the differential equation with the slope field as shown in Figure 2, satisfying \( y(0) = 0. \) Sketch the graph of \( y(t) \). Then use your answer to Exercise 14 to solve for \( y(t) \).

16. Let \( y(t) \) be the solution of \( 4 \frac{dy}{dt} = y^2 + t \) satisfying \( y(2) = 1. \) Carry out Euler’s Method with time step \( h = 0.05 \) for \( n = 6 \) steps.

17. Let \( y(t) \) be the solution of \( (x^2 + 1) \frac{dy}{dt} = y \) satisfying \( y(0) = 1. \) Compute approximations to \( y(0.1), y(0.2), \) and \( y(0.3) \) using Euler’s Method with time step \( h = 0.1. \)

In Exercises 18–21, solve using the method of integrating factors.

18. \( \frac{dy}{dt} = y + t^2, \quad y(0) = 0 \)

19. \( \frac{dy}{dt} = \frac{y}{2x} - x, \quad y(1) = 1 \)

20. \( \frac{dy}{dt} = y - 3x, \quad y(-1) = 2 \)

21. \( y' + 2y = 1 + e^{-x}, \quad y(0) = -4 \)

In Exercises 22–29, solve using an appropriate method.

22. \( x^2y' = x^2 + 1, \quad y(1) = 10 \)

23. \( y' + (\tan x)y = \cos^2 x, \quad y(x) = 2 \)

24. \( xyz' = 2y + x - 1, \quad y(1) = 9 \)

25. \( (y - 1)z' = t, \quad y(1) = -3 \)

26. \( (\sqrt{x} + 1)y' = yx^2, \quad y(0) = 1 \)

27. \( \frac{dw}{dx} = k \frac{1 + u^2}{x}, \quad u(1) = 1 \)

28. \( y' + \frac{3y - 1}{t} = t + 2 \)

29. \( y' + \frac{y}{x} = \sin x \)

30. Find the solutions to \( y' = 4(y - 12) \) satisfying \( y(0) = 20 \) and \( y(0) = 0, \) and sketch their graphs.

31. Find the solutions to \( y' = -2y + 8 \) satisfying \( y(0) = 3 \) and \( y(0) = 4, \) and sketch their graphs.

32. Show that \( y = \sin^{-1} x \) satisfies the differential equation \( y' = \tan^{-1} x \) with initial condition \( y(0) = 0. \)

33. What is the limit \( \lim_{t \to \infty} y(t) \) if \( y(t) \) is a solution of each of the following?

   (a) \( \frac{dy}{dt} = -4(y - 12) \)
   (b) \( \frac{dy}{dt} = 4(y - 12) \)
   (c) \( \frac{dy}{dt} = -4y - 12 \)

In Exercises 34–37, let \( P(t) \) denote the balance at time \( t \) (years) of an annuity that earns \( 5\% \) interest continuously compounded and pays out \$20,000/year continuously.

34. Find the differential equation satisfied by \( P(t). \)

35. Determine \( P(5) \) if \( P(0) = 200,000. \)

36. When does the annuity run out of money if \( P(0) = 300,000? \)

37. What is the minimum initial balance that will allow the annuity to make payments indefinitely?

38. State whether the differential equation can be solved using Separation of Variables, the method of integrating factors, both, or neither.

   (a) \( y' = y + x^2 \)
   (b) \( xy' = y + 1 \)
   (c) \( y' = y^2 + x^2 \)
   (d) \( xy' = y^2 \)
39. In the laboratory, the *Escherichia coli* bacteria grows such that the rate of change of the population is proportional to the population present. Assume that 500 bacteria are initially present, and 650 are present after 1 hour.
   (a) Determine $P(t)$, the population after $t$ hours.
   (b) How long does it take for the population to double in size?
40. Uranium-238 is a radioactive material with a half-life of 4,468 billion years. With $t$ in billions of years, let $M(t)$ be the mass in grams of a sample of uranium-238 that initially consisted of 100 g. Set up and solve an Initial Value Problem for determining $M(t)$.
41. Let $A$ and $B$ be constants. Prove that if $A > 0$, then all solutions of $\frac{dy}{dt} + Ay = B$ approach the same limit as $t \to \infty$.
42. At time $t = 0$, a tank of height 5 m in the shape of an inverted pyramid whose cross section at the top is a square of side 2 m is filled with water. Water flows through a hole at the bottom of area 0.002 m$^2$. Determine the time required for the tank to empty.
43. The trough in Figure 3 (dimensions in centimeters) is filled with water. At time $t = 0$ (in seconds), water begins leaking through a hole at the bottom of area 4 cm$^2$. Find $y(t)$ be the water height at time $t$. Find a differential equation for $y(t)$ and solve it to determine when the water level decreases to 60 cm.

**Figure 3**

44. Find the solutions of the logistic equation $\frac{dy}{dt} = y(4 - y)$ satisfying the initial conditions:
   (a) $y(0) = 1$  
   (b) $y(0) = 4$  
   (c) $y(0) = 6$
45. Let $y(t)$ be the solution of $\frac{dy}{dt} = 0.3y(2 - y)$ with $y(0) = 1$. Determine $\lim_{t \to \infty} y(t)$ without solving for $y$ explicitly.
46. Suppose that $y = ky(1 - y/8)$ has a solution satisfying $y(0) = 12$ and $y(10) = 24$. Find $k$.
47. A lake has a carrying capacity of 1000 fish. Assume that the fish population grows logistically with growth constant $k = 0.2$ day$^{-1}$. How many days will it take for the population to reach 900 fish if the initial population is 20 fish?
48. A rabbit population on an island increases exponentially with growth rate $k = 0.12$ months$^{-1}$. When the population reaches 300 rabbits (say, at time $t = 0$), wolves begin eating the rabbits at a rate of $r$ rabbits per month.
   (a) Find a differential equation satisfied by the rabbit population $P(t)$.
   (b) How large can $r$ be without the rabbit population becoming extinct?
49. Show that $y = \sin(\tan^{-1} x + C)$ is the general solution of $y = \sqrt{1 - y^2}/(1 + x^2)$. Then use the addition formula for the sine function to show that the general solution may be written

$$y = \frac{(\cos C)x + \sin C}{\sqrt{1 + x^2}}$$

50. A tank is filled with 300 liters of contaminated water containing 3 kg of toxin. Pure water is pumped in at a rate of 40 L/min, mixes instantaneously, and is then pumped out at the same rate. Let $y(t)$ be the quantity of toxin present in the tank at time $t$.
   (a) Find a differential equation satisfied by $y(t)$.
   (b) Solve for $y(t)$.
   (c) Find the time at which there is 0.01 kg of toxin present.
51. At $t = 0$, a tank of volume 300 liters is filled with 100 L of water containing salt at a concentration of 8 g/L. Fresh water flows in at a rate of 40 L/min, mixes instantaneously, and exits at the same rate. Let $c_1(t)$ be the salt concentration at time $t$.
   (a) Find a differential equation satisfied by $c_1(t)$. HINT: Find the differential equation for the quantity of salt $y(t)$, and observe that $c_1(t) = y(t)/100$.
   (b) Find the salt concentration $c_1(t)$ in the tank as a function of time.
52. The outflow of the tank in Exercise 51 is directed into a second tank containing $V$ liters of fresh water where it mixes instantaneously and exits at the same rate of 40 L/min. Determine the salt concentration $c_2(t)$ in the second tank as a function of time in the following two cases:
   (a) $V = 200$  
   (b) $V = 300$
In each case, determine the maximum concentration.
11 INFINITE SERIES

The theory of infinite series is a third branch of calculus, in addition to differential and integral calculus. Infinite series provide us with convenient and useful ways of expressing functions as infinite sums of simple functions. For example, we will see that we can express the exponential function as

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \]

The idea behind infinite series is that we add infinitely many numbers. We will see that although this is more complicated than adding finitely many numbers, sometimes adding infinitely many numbers yields a sum, but other times it does not. For example, we will learn that \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \) adds up to 1, but \( 1 - 1 + 1 - 1 + \cdots \) does not add up to any value (and is said to diverge). To make the idea of infinite series precise, we employ limits to determine what happens to the sum as we add more and more terms in a series.

We start the chapter with a section about sequences and their limits, important concepts behind infinite series. We then introduce infinite series in Section 11.2. After further developing infinite series in Sections 11.3 through 11.5, we close the chapter with three sections (Power Series, Taylor Polynomials, and Taylor Series) where we examine the idea of representing functions as infinite series.

11.1 Sequences

Limits and convergence played a fundamental role in the definitions of the derivative and definite integral. The limit concept will be significant throughout this chapter. We start by developing the basic ideas of a sequence of numbers and the limit of such a sequence.

A simple sequence of numbers arises if you eat half of a cake, and eat half of the remaining half, and continue eating half of what's left indefinitely (Figure 1). The fraction of the whole cake that remains after each step forms the sequence

\[ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots \]

This is the sequence of values of the function \( f(n) = \frac{1}{2^n} \) for \( n = 1, 2, \ldots \).

**Definition**: A sequence \( \{a_n\} \) is an ordered collection of numbers defined by a function \( f \) on a set of sequential integers. The values \( a_n = f(n) \) are called the terms of the sequence, and \( n \) is called the index. Informally, we think of a sequence \( \{a_n\} \) as a list of terms:

\[ a_1, a_2, a_3, a_4, \ldots \]

The sequence does not have to start at \( n = 1 \). It can start at \( n = 0, n = 2, \) or any other integer.

When \( a_n \) is given by a formula, we refer to \( a_n \) as the general term, and we refer to the set of the values \( n \) on which the sequence is defined as the domain of the sequence.

For example, in the cake sequence, \( a_n = \frac{1}{2^n} \) is the general term and the domain is \( n \geq 1 \).

Not all sequences are generated by a formula. For instance, the sequence of digits in the decimal expansion of \( \pi \) is

\[ 3, 1, 4, 1, 5, 9, 2, 6, \ldots \]

There is no specific formula for the \( n \)th digit of \( \pi \) and therefore, there is no formula for the general term in this sequence.
The following are examples of some sequences and their general terms.

<table>
<thead>
<tr>
<th>General term</th>
<th>Domain</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n = 1 - \frac{1}{n} )</td>
<td>( n \geq 1 )</td>
<td>0, ( \frac{1}{2} ), ( \frac{2}{3} ), ( \frac{3}{4} ), ( \frac{4}{5} ), ...</td>
</tr>
<tr>
<td>( b_n = \frac{364.5n^2}{n^2 - 4} )</td>
<td>( n \geq 3 )</td>
<td>656.1, 486, 433.9, 410.1, 396.9, ...</td>
</tr>
<tr>
<td>( c_n = \cos\left(\frac{n\pi}{2}\right) )</td>
<td>( n \geq 0 )</td>
<td>1, 0, -1, 0, 1, 0, ...</td>
</tr>
<tr>
<td>( d_n = (-1)^n n )</td>
<td>( n \geq 0 )</td>
<td>0, -1, 2, -3, 4, ...</td>
</tr>
</tbody>
</table>

The sequence in the next example is defined recursively. For such a sequence, the first one or more terms may be given, and then the \( n \)th term is computed in terms of the preceding terms using some formula.

**EXAMPLE 1 The Fibonacci Sequence** We define the sequence by taking \( F_1 = 1 \), \( F_2 = 1 \), and \( F_n = F_{n-1} + F_{n-2} \) for all integers \( n > 2 \). In other words, each subsequent term is obtained by adding together the two preceding terms. Determine the first 10 terms in the sequence.

**Solution** Given the first two terms, we can easily find each subsequent term by adding the previous two. The sequence is

\[
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots
\]

The Fibonacci sequence appears in a surprisingly wide variety of situations, particularly in nature. For instance, the number of spiral arms in a sunflower almost always turns out to be a number from the Fibonacci sequence, as in Figure 2.

**EXAMPLE 2 Recursive Sequence** Compute the three terms \( a_2, a_3, a_4 \) for the sequence defined recursively by

\[
a_1 = 1, \quad a_n = \frac{1}{2} \left( a_{n-1} + \frac{2}{a_{n-1}} \right)
\]

**Solution**

\[
a_2 = \frac{1}{2} \left( a_1 + \frac{2}{a_1} \right) = \frac{1}{2} \left( 1 + \frac{2}{1} \right) = \frac{3}{2} = 1.5
\]

\[
a_3 = \frac{1}{2} \left( a_2 + \frac{2}{a_2} \right) = \frac{1}{2} \left( \frac{3}{2} + \frac{2}{\frac{3}{2}} \right) = \frac{17}{12} \approx 1.4167
\]

\[
a_4 = \frac{1}{2} \left( a_3 + \frac{2}{a_3} \right) = \frac{1}{2} \left( \frac{17}{12} + \frac{2}{\frac{17}{12}} \right) = \frac{577}{408} \approx 1.414216
\]
Our main goal is to study convergence of sequences. A sequence \( \{a_n\} \) converges to a limit \( L \) if \( |a_n - L| \) becomes arbitrarily small when \( n \) is sufficiently large. Here is the formal definition.

**Definition** Limit of a Sequence  We say \( \{a_n\} \) converges to a limit \( L \) and write
\[
\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L
\]
if, for every \( \epsilon > 0 \), there is a number \( M \) such that \( |a_n - L| < \epsilon \) for all \( n > M \).

- If no limit exists, we say that \( \{a_n\} \) diverges.
- If the terms increase without bound, we say that \( \{a_n\} \) diverges to infinity.

If \( \{a_n\} \) converges, then its limit \( L \) is unique. To visualize the limit plot the points \((1, a_1), (2, a_2), (3, a_3), \ldots, \) as in Figure 3. The sequence converges to \( L \) if, for every \( \epsilon > 0 \), the plotted points eventually remain within an \( \epsilon \)-band around the horizontal line \( y = L \). Figure 4 shows the plot of a sequence converging to \( L = 1 \). On the other hand, since it continually cycles through 1, 0, −1, 0, \( c_n = \cos \left( \frac{2\pi}{n} \right) \) in Figure 5 has no limit.

![Figure 3](image1.png)  Plot of a sequence with limit \( L \). For any \( \epsilon \), the dots eventually remain within an \( \epsilon \)-band around \( L \).

![Figure 4](image2.png)  The sequence \( a_n = \frac{n + 4}{n + 3} \).

![Figure 5](image3.png)  The sequence \( c_n = \cos \left( \frac{2\pi}{n} \right) \) has no limit.

**Example 3** Proving Convergence  Let \( a_n = \frac{n + 4}{n + 3} \). Prove that \( \lim_{n \to \infty} a_n = 1 \).

**Solution** The definition requires us to find, for every \( \epsilon > 0 \), a number \( M \) such that
\[
|a_n - 1| < \epsilon \quad \text{for all } n > M
\]
We have
\[
|a_n - 1| = \left| \frac{n + 4}{n + 3} - 1 \right| = \frac{1}{n + 3}
\]
Therefore, \( |a_n - 1| < \epsilon \) if
\[
\frac{1}{n + 3} < \epsilon \quad \text{or} \quad n > \frac{1}{\epsilon} - 3
\]
In other words, \( |a_n - 1| < \epsilon \) for all \( n > \frac{1}{\epsilon} - 3 \). This proves that \( \lim_{n \to \infty} a_n = 1 \).

Note the following two facts about sequences:
- The limit does not change if we change or drop finitely many terms of the sequence.
- If \( C \) is a constant and \( a_n = C \) for all \( n \) greater than some fixed value \( N \), then \( \lim_{n \to \infty} a_n = C \).

Many of the sequences we consider are defined by functions; that is, \( a_n = f(n) \) for some function \( f \). For example,
\[
a_n = \frac{n - 1}{n} \quad \text{is defined by} \quad f(x) = \frac{x - 1}{x}
\]
We will often use the fact that if \( f(x) \) approaches a limit \( L \) as \( x \to \infty \), then the sequence \( a_n = f(n) \) approaches the same limit \( L \) (Figure 6). Indeed, if for all \( \epsilon > 0 \) we can find a positive real number \( M \) so that \( |f(x) - L| < \epsilon \) for all \( x > M \), then it follows automatically that \( |f(n) - L| < \epsilon \) for all integers \( n > M \).

**THEOREM 1** Sequence Defined by a Function If \( \lim_{x \to \infty} f(x) \) exists, then the sequence \( a_n = f(n) \) converges to the same limit:

\[
\lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x)
\]

**EXAMPLE 4** Find the limit of the sequence

\[
\frac{2^2 - 2}{2^2}, \quad \frac{3^2 - 2}{3^2}, \quad \frac{4^2 - 2}{4^2}, \quad \frac{5^2 - 2}{5^2}, \quad \ldots
\]

Solution This is the sequence with general term

\[
a_n = \frac{n^2 - 2}{n^2} = 1 - \frac{2}{n^2}
\]

Therefore, we apply Theorem 1 with \( f(x) = 1 - \frac{2}{x^2} \):

\[
\lim_{n \to \infty} a_n = \lim_{x \to \infty} \left( 1 - \frac{2}{x^2} \right) = 1 - \lim_{x \to \infty} \frac{2}{x^2} = 1 - 0 = 1
\]

**EXAMPLE 5** Calculate \( \lim_{n \to \infty} \frac{n + \ln n}{n^2} \).

Solution Apply Theorem 1, using L'Hôpital's Rule in the second step:

\[
\lim_{n \to \infty} \frac{n + \ln n}{n^2} = \lim_{x \to \infty} \frac{x + \ln x}{x^2} = \lim_{x \to \infty} \frac{1 + (1/x)}{2x} = 0
\]

The limit of the Balmer wavelengths \( b_n \) in the next example plays a role in physics and chemistry because it determines the ionization energy of the hydrogen atom. Figure 7 plots the sequence and the graph of a function \( f \) that defines the sequence. In Figure 8, the wavelengths are shown "crowding in" toward their limiting value.

**EXAMPLE 6** Balmer Wavelengths Calculate the limit of the Balmer wavelengths \( b_n = \frac{364.5n^2}{n^2 - 4} \) in nanometers, where \( n \geq 3 \).
Solution: Apply Theorem 1 with \( f(x) = \frac{364.5x^2}{x^2 - 4} \):

\[
\lim_{x \to \infty} b_n = \lim_{x \to \infty} \frac{364.5x^2}{x^2 - 4} = \lim_{x \to \infty} \frac{364.5x^2 \frac{1}{x^2}}{x^2 - 4} = \lim_{x \to \infty} \frac{364.5}{1 - \frac{4}{x^2}} = \frac{364.5}{1} = 364.5 \text{ nm}
\]

A geometric sequence is a sequence \( a_n = cr^n \), where \( c \) and \( r \) are nonzero constants. Each term is \( r \) times the previous term; that is, \( a_n/a_{n-1} = r \). The number \( r \) is called the common ratio. For instance, if \( r = 3 \) and \( c = 2 \), we obtain the sequence (starting at \( n = 0 \))

\[
2, \ 2 \cdot 3, \ 2 \cdot 3^2, \ 2 \cdot 3^3, \ 2 \cdot 3^4, \ 2 \cdot 3^5, \ldots
\]

In the next example, we determine when a geometric series converges. Recall that \( \{a_n\} \) converges to \( \infty \) if the terms \( a_n \) increase beyond all bounds (Figure 9); that is,

\[
\lim_{n \to \infty} a_n = \infty \text{ if, for every number } N, \ a_n > N \text{ for all sufficiently large } n
\]

We define \( \lim_{n \to \infty} a_n = -\infty \) similarly.

**EXAMPLE 7 Geometric Sequences with \( r \geq 0 \)** Prove that for \( r \geq 0 \) and \( c > 0 \),

\[
\lim_{n \to \infty} cr^n = \begin{cases} 
0 & \text{if } 0 \leq r < 1 \\
c & \text{if } r = 1 \\
\infty & \text{if } r > 1
\end{cases}
\]

Solution: Set \( f(r) = cr^r \). If \( 0 \leq r < 1 \), then (Figure 10)

\[
\lim_{r \to \infty} cr^n = \lim_{x \to \infty} f(x) = c \lim_{x \to \infty} r^x = 0
\]

If \( r > 1 \), then since \( c > 0 \), both \( f(x) \) and the sequence \( \{cr^n\} \) diverge to \( \infty \) (Figure 9). If \( r = 1 \), then \( cr^n = c \) for all \( n \), and the limit is \( c \).

This last example will prove extremely useful when we consider geometric series in Section 11.2.

The limit laws we have used for functions also apply to sequences and are proved in a similar fashion.

**THEOREM 2 Limit Laws for Sequences** Assume that \( \{a_n\} \) and \( \{b_n\} \) are convergent sequences with

\[
\lim_{n \to \infty} a_n = L, \quad \lim_{n \to \infty} b_n = M
\]

Then:

(i) \( \lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n = L \pm M \)

(ii) \( \lim_{n \to \infty} a_n b_n = \left( \lim_{n \to \infty} a_n \right) \left( \lim_{n \to \infty} b_n \right) = LM \)

(iii) \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} = \frac{L}{M} \text{ if } M \neq 0 \)

(iv) \( \lim_{n \to \infty} ca_n = c \lim_{n \to \infty} a_n = cL \text{ for any constant } c \)
THEOREM 3 Squeeze Theorem for Sequences  Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be sequences such that for some number \( M \),
\[
b_n \leq a_n \leq c_n \quad \text{for} \quad n > M \quad \text{and} \quad \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = L
\]
Then \( \lim_{n \to \infty} a_n = L \).

EXAMPLE 8 Show that if \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

Solution  We have
\[
-|a_n| \leq a_n \leq |a_n|
\]
By hypothesis, \( \lim_{n \to \infty} |a_n| = 0 \), and thus also \( \lim_{n \to \infty} -|a_n| = -\lim_{n \to \infty} |a_n| = 0 \). Therefore, we can apply the Squeeze Theorem to conclude that \( \lim_{n \to \infty} a_n = 0 \).

EXAMPLE 9 Geometric Sequences with \( r < 0 \) Prove that for \( c \neq 0 \),
\[
\lim_{n \to \infty} cr^n = \begin{cases} 
0 & \text{if } -1 < r < 0 \\
\text{diverges} & \text{if } r \leq -1
\end{cases}
\]
Solution If \(-1 < r < 0\), then \( 0 < |r| < 1 \) and \( \lim_{n \to \infty} |cr^n| = 0 \) by Example 7. Thus, \( \lim_{n \to \infty} cr^n = 0 \) by Example 8. If \( r = -1 \), then the sequence \( cr^n = (-1)^n c \) alternates between \( c \) and \(-c\) and therefore does not approach a limit. The sequence also diverges if \( r \leq -1 \) because \(|cr^n|\) grows arbitrarily large.

As another application of the Squeeze Theorem, consider the sequence
\[
a_n = \frac{5^n}{n!}
\]
Both the numerator and the denominator grow without bound, so it is not clear in advance whether \( \{a_n\} \) converges. Figure 11 and Table I suggest that \( a_n \) increases initially and then tends to zero. In the next example, we verify that \( a_n = R^n/n! \) converges to zero for all \( R \). This fact is used in the discussion of Taylor series in Section 11.8.

EXAMPLE 10 Prove that \( \lim_{n \to \infty} \frac{R^n}{n!} = 0 \) for all \( R \).

Solution Assume first that \( R > 0 \) and let \( M \) be the nonnegative integer such that
\[
M \leq R < M + 1
\]
For \( n > M \), we write \( R^n/n! \) as a product of \( n \) factors:
\[
\frac{R^n}{n!} = \left( \frac{R}{1} \right) \left( \frac{R}{2} \right) \cdots \left( \frac{R}{M} \right) \left( \frac{R}{M+1} \right) \left( \frac{R}{M+2} \right) \cdots \left( \frac{R}{n} \right) \leq C \left( \frac{R}{n} \right)
\]
where \( C \) is a constant. Each factor is less than 1.

The first \( M \) factors are greater than or equal to 1 and the last \( n - M \) factors are less than 1. If we lump together the first \( M \) factors and call the product \( C \), and replace all the remaining factors except \( R/n \) with 1, we see that
\[
0 \leq \frac{R^n}{n!} \leq \frac{CR}{n}
\]
Since \( CR/n \to 0 \), the Squeeze Theorem gives us \( \lim_{n \to \infty} R^n/n! = 0 \) as claimed. If \( R < 0 \), the limit is also zero by Example 8 because \( |R^n/n!| \) tends to zero.
Given a sequence \( \{a_n\} \) and a function \( f \), we can form the new sequence \( \{f(a_n)\} \). It is useful to know that if \( f \) is continuous and \( a_n \to L \), then \( f(a_n) \to f(L) \). A proof is given in Appendix D.

**Theorem 4**  
If \( f \) is continuous and \( \lim_{n \to \infty} a_n = L \), then  
\[
\lim_{n \to \infty} f(a_n) = f \left( \lim_{n \to \infty} a_n \right) = f(L)
\]

In other words, we may pass a limit of a sequence inside a continuous function.

**Example 11**  
Determine the limit of the sequence \( a_n = \frac{3n}{n + 1} \), and then apply Theorem 4 to determine the limits of the sequences \( \{f(a_n)\} \) and \( \{g(a_n)\} \), where \( f(x) = e^x \) and \( g(x) = x^2 \).

**Solution**  
First,  
\[
L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{3n}{n + 1} = \lim_{n \to \infty} \frac{3}{1 + \frac{1}{n}} = 3
\]

Now, with \( f(x) = e^x \), we have \( f(a_n) = e^{a_n} = e^{3/2} \). According to Theorem 4,  
\[
\lim_{n \to \infty} f(a_n) = f \left( \lim_{n \to \infty} a_n \right) = e^{\lim_{n \to \infty} \frac{3}{2}} = e^3
\]

Finally, with \( g(x) = x^2 \), we have \( g(a_n) = a_n^2 \). According to Theorem 4,  
\[
\lim_{n \to \infty} g(a_n) = g \left( \lim_{n \to \infty} a_n \right) = \left( \lim_{n \to \infty} \frac{3n}{n + 1} \right)^2 = 3^2 = 9
\]

Next, we define the concepts of a bounded sequence and a monotonic sequence, concepts of great importance for understanding convergence.

**Definition** Bounded Sequences  
A sequence \( \{a_n\} \) is
- Bounded from above if there is a number \( M \) such that \( a_n \leq M \) for all \( n \). The number \( M \) is called an upper bound.
- Bounded from below if there is a number \( m \) such that \( a_n \geq m \) for all \( n \). The number \( m \) is called a lower bound.

The sequence \( \{a_n\} \) is called bounded if it is bounded from above and below. A sequence that is not bounded is called an unbounded sequence.

Thus, for instance, the sequence given by \( a_n = 3 - \frac{1}{n} \) is clearly bounded above by 3. It is also bounded below by 0, since all the terms are positive. Hence, this sequence is bounded.

Upper and lower bounds are not unique. If \( M \) is an upper bound, then any number greater than \( M \) is also an upper bound, and if \( m \) is a lower bound, then any number less than \( m \) is also a lower bound (Figure 12).

As we might expect, a convergent sequence \( \{a_n\} \) is necessarily bounded because the terms \( a_n \) get closer and closer to the limit. This fact is stated in the next theorem.

**Theorem 5** Convergent Sequences Are Bounded  
If \( \{a_n\} \) converges, then \( \{a_n\} \) is bounded.
The Fibonacci sequence \( \{F_n\} \) diverges since it is unbounded \((F_n \geq n\) for all \(n)\), but the sequence defined by the ratios \(a_n = \frac{F_{n+1}}{F_n}\) converges. The limit is an important number known as the golden ratio (see Exercises 33 and 34).

![Diagram of an increasing sequence with upper bound M approaching a limit L.](image)

**Theorem 6** Bounded Monotonic Sequences Converge

- If \(\{a_n\}\) is increasing and \(a_n \leq M\), then \(\{a_n\}\) converges and \(\lim_{n \to \infty} a_n = M\).
- If \(\{a_n\}\) is decreasing and \(a_n \geq m\), then \(\{a_n\}\) converges and \(\lim_{n \to \infty} a_n = m\).

**Example 12** Verify that \(a_n = \sqrt{n+1} - \sqrt{n}\) is decreasing and bounded below. Does \(\lim_{n \to \infty} a_n\) exist?

**Solution** The function \(f(x) = \sqrt{x+1} - \sqrt{x}\) is decreasing because its derivative is negative:

\[
 f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0 \quad \text{for } x > 0
\]

It follows that \(a_n = f(n)\) is decreasing (see Table 2). Furthermore, \(a_n > 0\) for all \(n\), so the sequence has lower bound \(m = 0\). Theorem 6 guarantees that \(L = \lim_{n \to \infty} a_n\) exists and \(L \geq 0\). In fact, we can show that \(L = 0\) by noting that \(f(x)\) can be rewritten as

\[
 f(x) = \frac{1}{\sqrt{x+1} + \sqrt{x}}.
\]

Hence, \(\lim_{x \to \infty} f(x) = 0\).

**Example 13** Show that the following sequence is bounded and increasing:

\(a_1 = \sqrt{2}, \ a_2 = \sqrt{2\sqrt{2}}, \ a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\)

Then prove that \(L = \lim_{n \to \infty} a_n\) exists and compute its value.
Solution

**Step 1. Show that \( \{a_n\} \) is bounded above.**
We claim that \( M = 2 \) is an upper bound. We certainly have \( a_1 < 2 \) because \( a_1 = \sqrt{2} \approx 1.414 \). On the other hand,

\[
\text{if } a_n < 2, \quad \text{then } a_{n+1} < 2
\]

is true because \( a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2 \). Now, since \( a_1 < 2 \), we can apply (2) to conclude that \( a_2 < 2 \). Similarly, \( a_2 < 2 \) implies \( a_3 < 2 \), and so on. It follows that \( a_n < 2 \) for all \( n \). (Formally speaking, this is a proof by induction.)

**Step 2. Show that \( \{a_n\} \) is increasing.**
Since \( a_n \) is positive and \( a_n < 2 \), we have

\[
a_{n+1} = \sqrt{2a_n} > \sqrt{a_n \cdot a_n} = a_n
\]

This shows that \( \{a_n\} \) is increasing. Since the sequence is bounded above and increasing, we conclude that the limit \( L \) exists.

Now that we know the limit \( L \) exists, we can find its value as follows. The idea is that \( L \) "contains a copy" of itself under the square root sign:

\[
L = \sqrt{2 \sqrt{2 \sqrt{2 \cdots}}} = \sqrt{2L}
\]

Thus, \( L^2 = 2L \), which implies that \( L = 0 \) or \( L = 2 \). We eliminate \( L = 0 \) because the terms \( a_n \) are positive and increasing, so we must have \( L = 2 \) (see Table 3).

In the previous example, the argument that \( L = \sqrt{2L} \) is more formally expressed by noting that the sequence is defined recursively by

\[
a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n}
\]

If \( a_n \) converges to \( L \), then the sequence \( b_n = a_{n+1} \) also converges to \( L \) (because it is the same sequence, with terms shifted one to the left). Then, applying Theorem 4 to \( f(x) = \sqrt{x} \), we have

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{2a_n} = \sqrt{2 \lim_{n \to \infty} a_n} = \sqrt{2L}
\]

### 11.1 SUMMARY

- A sequence \( \{a_n\} \) converges to a limit \( L \) if, for every \( \epsilon > 0 \), there is a number \( M \) such that

\[
|a_n - L| < \epsilon \quad \text{for all } n > M
\]

We write \( \lim_{n \to \infty} a_n = L \) or \( a_n \to L \).
- If no limit exists, we say that \( \{a_n\} \) diverges.
- In particular, if the terms increase without bound, we say that \( \{a_n\} \) diverges to infinity.
- If \( a_n = f(n) \) and \( \lim_{n \to \infty} f(x) = L \), then \( \lim_{n \to \infty} a_n = L \).
- A geometric sequence is a sequence \( a_n = cr^n \), where \( c \) and \( r \) are nonzero. It converges to 0 for \(-1 < r < 1\), converges to \( c \) for \( r = 1 \), and diverges otherwise.
- The Basic Limit Laws and the Squeeze Theorem apply to sequences.
- If \( f \) is continuous and \( \lim_{n \to \infty} a_n = L \), then \( \lim_{n \to \infty} f(a_n) = f(L) \).
- A sequence \( \{a_n\} \) is
  - bounded above by \( M \) if \( a_n \leq M \) for all \( n \).
  - bounded below by \( m \) if \( a_n \geq m \) for all \( n \).
If \( \{a_n\} \) is bounded above and below, \( \{a_n\} \) is called bounded.

- A sequence \( \{a_n\} \) is monotonic if it is increasing \( (a_n < a_{n+1}) \) or decreasing \( (a_{n+1} < a_n) \).
- Bounded monotonic sequences converge (Theorem 6).

### 11.1 EXERCISES

#### Preliminary Questions

1. What is \( a_n \) for the sequence \( a_n = n^2 - n \)?

2. Which of the following sequences converge to zero?
   (a) \( \frac{n^2}{n^2+1} \)
   (b) \( 2^n \)
   (c) \( \left( -\frac{1}{2} \right)^n \)

3. Let \( a_n \) be the \( n \)th decimal approximation to \( \sqrt{2} \). That is, \( a_1 = 1, a_2 = 1.4, a_3 = 1.41 \), and so on. What is \( \lim_{n \to \infty} a_n \)?

4. Which of the following sequences is defined recursively?  
   (a) \( a_n = \sqrt{4 + \frac{1}{n}} \)
   (b) \( b_n = \sqrt{4 + \frac{1}{n-1}} \)

5. Theorem 5 says that every convergent sequence is bounded. Determine if the following statements are true or false, and if false, give a counterexample:
   (a) If \( \{a_n\} \) is bounded, then it converges.
   (b) If \( \{a_n\} \) is not bounded, then it diverges.
   (c) \( \{a_n\} \) converges, then it is not bounded.

#### Exercises

1. Match each sequence with its general term:

   \[
   \begin{array}{c|c}
   a_1, a_2, a_3, a_4, \ldots & \text{General term} \\
   \hline
   (a) \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \ldots & (i) \cos \pi n \\
   (b) -1, 1, -1, 1, \ldots & (ii) \frac{n!}{2^n} \\
   (c) 1, -1, 1, -1, \ldots & (iii) (-1)^{n+1} \\
   (d) \frac{n}{2}, \frac{3}{4}, \frac{5}{8}, \ldots & (iv) \frac{n}{n+1} \\
   \end{array}
   \]

2. Let \( a_n = \frac{1}{2n-1} \) for \( n = 1, 2, 3, \ldots \). Write out the first three terms of the following sequences.
   (a) \( b_1 = a_1 + 1 
   (b) c_n = a_{n+3} 
   (c) d_n = \frac{1}{a_n} 
   (d) e_n = 2a_n - a_{n+1} 

3. In Exercises 3–12, calculate the first four terms of the sequence, starting with \( n = 1 \).
   \[
   \begin{align*}
   &3. c_n = \frac{3^n}{n!} \\
   &4. b_n = \frac{(2n-1)!}{n!} \\
   &5. a_1 = 2, a_{n+1} = 2a_n^2 - 3 \\
   &6. b_1 = 1, b_n = b_{n-1} + \frac{1}{b_{n-1}} \\
   &7. b_n = 5 + \cos \pi n \\
   &8. c_n = (-1)^{2n+1} \\
   &9. c_1 = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \\
   &10. w_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \\
   &11. b_1 = 2, b_2 = 3, b_n = 2b_{n-1} - b_{n-2} \\
   &12. b_n = \frac{F_{n+1}}{F_n} \text{ where } F_n \text{ is the } n \text{th Fibonacci number.} 
   \end{align*}
   \]

4. Find a formula for the \( n \)th term of each sequence.
   (a) \( \frac{1}{1}, \frac{-1}{8}, \frac{1}{27}, \ldots \)
   (b) \( \frac{2}{6}, \frac{3}{7}, \frac{4}{8}, \ldots \)

5. Suppose that \( \lim_{n \to \infty} a_n = 4 \) and \( \lim_{n \to \infty} b_n = 7 \). Determine:
   (a) \( \lim_{n \to \infty} (a_n + b_n) \)
   (b) \( \lim_{n \to \infty} a_n^3 \)
   (c) \( \lim_{n \to \infty} \cos(n \pi) \)
   (d) \( \lim_{n \to \infty} (a_n^2 - 2a_nb_n) \)

6. In Exercises 15–28, use Theorem 1 to determine the limit of the sequence or state that the sequence diverges.
   \[
   \begin{align*}
   &15. a_n = 5 - 2n \\
   &16. a_n = 20 - \frac{4}{n^2} \\
   &17. b_n = \frac{5n - 1}{12n + 9} \\
   &18. a_n = \frac{4 + n - 3n^2}{4n^2 + 1} \\
   &19. a_n = \left( \frac{1}{2} \right)^n \\
   &20. a_n = \left( \frac{1}{3} \right)^n \\
   &21. c_n = 9^n \\
   &22. a_n = 10^{-4/n} \\
   &23. a_n = \frac{n}{\sqrt{n^2 + 1}} \\
   &24. a_n = \sqrt{n^2 + 1} \\
   &25. a_n = \ln \left( \frac{12n + 2}{-9 + 4n} \right) \\
   &26. r_n = \ln n - \ln (n^2 + 1) \\
   &27. z_n = \frac{n + 1}{\ln n} \\
   &28. y_n = n e^{1/n} 
   \end{align*}
   \]

7. In Exercises 29–32, use Theorem 4 to determine the limit of the sequence.
   \[
   \begin{align*}
   &29. a_n = \sqrt{4 + \frac{1}{n}} \\
   &30. a_n = e^{3n/2n+9} \\
   &31. a_n = \cos^{-1} \left( \frac{n^3}{2n^3 + 1} \right) \\
   &32. a_n = \tan^{-1} (e^{-n}) 
   \end{align*}
   \]

8. In Exercises 33–34 let \( a_n = \frac{F_{n+1}}{F_n} \), where \( \{F_n\} \) is the Fibonacci sequence. The sequence \( \{a_n\} \) has a limit. We do not prove this fact, but investigate the value of the limit in these exercises.

9. \textbf{(CAS)} Estimate \( \lim_{n \to \infty} a_n \) to five decimal places by computing \( a_n \) for sufficiently large \( n \).

10. Denote the limit of \( \{a_n\} \) by \( L \). Given that the limit exists, we can determine \( L \) as follows:
    (a) Show that \( a_{n+1} = 1 + \frac{1}{a_n} \).
(b) Given that \( \{a_n\} \) converges to \( L \), it follows that \( \{a_{n+1}\} \) also converges to \( L \) (see Exercise 65). Show that \( L^2 - L - 1 = 0 \) and solve this equation to determine \( L \). (The value of \( L \) is known as the golden ratio. It arises in many different situations in mathematics.)

35. Let \( a_n = \frac{n}{n+1} \). Find a number \( M \) such that:
(a) \( |a_n - 1| < 0.001 \) for \( n \geq M \).
(b) \( |a_n - 1| < 0.00001 \) for \( n \geq M \).

Then use the limit definition to prove that \( \lim_{n \to \infty} a_n = 1 \).

36. Let \( b_n = \left( \frac{1}{2} \right)^n \).
(a) Find a value of \( M \) such that \( |b_n| \leq 10^{-3} \) for \( n \geq M \).
(b) Use the limit definition to prove that \( \lim_{n \to \infty} b_n = 0 \).

37. Use the limit definition to prove that \( \lim_{n \to \infty} n^{-2} = 0 \).

38. Use the limit definition to prove that \( \lim_{n \to \infty} \frac{n}{n^2 + n + 1} = 1 \).

In Exercises 39–66, use the appropriate limit laws and theorems to determine the limit of the sequence or show that it diverges.

39. \( a_n = 10 + \left( -\frac{1}{9} \right)^n \)
40. \( a_n = \sqrt{n} + 3 - \sqrt{n} \)
41. \( c_n = 1.01^n \)
42. \( b_n = e - n^2 \)
43. \( a_n = 2/n \)
44. \( b_n = n^2/n \)
45. \( c_n = \frac{9^n}{n!} \)
46. \( a_n = \frac{2^n}{n!} \)
47. \( a_n = \frac{3n^2 + n + 2}{\sqrt{n} - 3} \)
48. \( a_n = \frac{\sqrt{n}}{\sqrt{n} + 4} \)
49. \( a_n = \cos \frac{n\pi}{n} \)
50. \( a_n = \frac{(-1)^n}{\sqrt{n}} \)
51. \( a_n = \ln 5^n - \ln n! \)
52. \( d_n = \ln(\sqrt{n^2 + 4}) - \ln(n^2 - 1) \)
53. \( a_n = \left( 1 + \frac{1}{n^2} \right)^{1/2} \)
54. \( b_n = \tan^{-1}(1 - \frac{2}{n}) \)
55. \( c_n = \ln \left( \frac{2n + 1}{3n + 4} \right) \)
56. \( a_n = \frac{n}{n + 1/2} \)
57. \( y_n = \frac{e^n}{2^n} \)
58. \( a_n = \frac{n}{n^2 + n + 1} \)
59. \( y_n = \frac{e^n + (-3)^n}{5^n} \)
60. \( b_n = \frac{(-1)^n n^2 + 2^n}{3^n + 4^n} \)
61. \( a_n = n \sin \frac{\pi}{n} \)
62. \( b_n = \frac{n}{n^2 + n + 1} \)
63. \( b_n = \frac{3 - n^2}{2 + 7 \cdot 4^n} \)
64. \( a_n = \frac{3 - 4^n}{2 + 7 \cdot 3^n} \)
65. \( a_n = \left( 1 + \frac{1}{n} \right)^n \)
66. \( a_n = \left( 1 + \frac{1}{n} \right)^n \)

In Exercises 67–70, find the limit of the sequence using L'Hôpital's Rule.

67. \( a_n = \frac{\ln(n)^2}{n} \)
68. \( b_n = \sqrt{n} \ln \left( 1 + \frac{1}{n} \right) \)
69. \( c_n = n/(\sqrt{n^2 + 1} - n) \)
70. \( a_n = n^2/(\sqrt{n^2 + 1} - n) \)

In Exercises 71–74, use the Squeeze Theorem to evaluate \( \lim_{n \to \infty} a_n \) by verifying the given inequality.

71. \( a_n = \frac{1}{\sqrt{n^2 + n^2}}, \quad \frac{1}{\sqrt{2n^2}} \leq a_n \leq \frac{1}{\sqrt{2n^2}} \)
72. \( c_n = \frac{1}{\sqrt{n^2 + 1}} + \frac{1}{\sqrt{n^2 + 2}} + \cdots + \frac{1}{\sqrt{n^2 + n}}, \quad \frac{1}{\sqrt{n^2 + n}} \leq c_n \leq \frac{1}{\sqrt{n^2 + 1}} \)
73. \( a_n = (2^n + 3^n)/n, \quad 3 \leq a_n \leq (2 \cdot 3^n)/n = 2^{n+1}/3 \)
74. \( a_n = (n + 10^n)/n, \quad 10 \leq a_n \leq (2 \cdot 10^n)/n \)
75. Which of the following statements is equivalent to the assertion \( \lim_{n \to \infty} a_n = L \)? Explain.
   (a) For every \( \epsilon > 0 \), the interval \( (L - \epsilon, L + \epsilon) \) contains at least one element of the sequence \( \{a_n\} \).
   (b) For every \( \epsilon > 0 \), the interval \( (L - \epsilon, L + \epsilon) \) contains at most one element of the sequence \( \{a_n\} \).
76. Show that \( a_n = \frac{1}{2n + 1} \) is decreasing.
77. Show that \( a_n = \frac{3n^2}{n^2 + 2} \) is increasing. Find an upper bound.
78. Show that \( a_n = \frac{\sqrt{n + 1} - n}{2n - 1} \) is decreasing.
79. Give an example of a divergent sequence \( \{a_n\} \) such that \( \lim_{n \to \infty} |a_n| \) converges.
80. Give an example of divergent sequences \( \{a_n\} \) and \( \{b_n\} \) such that \( \{a_n + b_n\} \) converges.
81. Using the limit definition, prove that if \( \{a_n\} \) converges and \( \{b_n\} \) diverges, then \( \{a_n + b_n\} \) diverges.
82. Use the limit definition to prove that if \( \{a_n\} \) is a convergent sequence of integrals with limit \( L \), then there exists a number \( M \) such that \( a_n = L \) for all \( n \geq M \).
83. Theorem 1 states that if \( \lim_{n \to \infty} f(x) = L \), then the sequence \( a_n = f(n) \) converges and \( \lim_{n \to \infty} a_n = L \). Show that the converse is false. In other words, find a function \( f \) such that \( a_n = f(n) \) converges but \( \lim_{n \to \infty} f(x) \) does not exist.
84. Use the limit definition to prove that the limit does not change if a finite number of terms are added or removed from a convergent sequence.
85. Let \( b_n = a_{n+1} \). Use the limit definition to prove that if \( \{a_n\} \) converges, then \( \{b_n\} \) also converges and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n \).
86. Let \( \{a_n\} \) be a sequence such that \( \lim_{n \to \infty} a_n \) exists and is nonzero. Show that \( \lim_{n \to \infty} a_n \) exists if and only if there exists an integer \( M \) such that the sign of \( a_n \) does not change for \( n > M \).
87. Proceed as in Example 13 to show that the sequence \( \sqrt[3]{3}, \sqrt[3]{3^2}, \sqrt[3]{3^3}, \ldots \) is increasing and bounded above by \( M = 3 \). Then prove that the limit exists and find its value.
88. Let \( \{a_n\} \) be the sequence defined recursively by \( a_0 = 0, \quad a_{n+1} = \sqrt{2 + a_n} \).
   (a) Show that if \( a_n < 2 \), then \( a_{n+1} < 2 \). Conclude by induction that \( a_n < 2 \) for all \( n \).
   (b) Show that if \( a_n < 2 \), then \( a_n \leq a_{n+1} \). Conclude by induction that \( \{a_n\} \) is increasing.
   (c) Use (a) and (b) to conclude that \( L = \lim_{n \to \infty} a_n \) exists. Then compute \( L \) by showing that \( L = \sqrt{2 + L} \).
Further Insights and Challenges

89. Show that \( \lim_{n \to \infty} \sqrt[n]{n} = \infty \). \textbf{Hint:} Verify that \( n! \geq (n/2)^{n/2} \) by observing that half of the factors of \( n! \) are greater than or equal to \( n/2 \).

90. Let \( b_n = \frac{\sqrt[n]{n}}{n} \).
(a) Show that \( \ln b_n = \frac{1}{n} \sum_{k=1}^{n} \ln \frac{k}{n} \).
(b) Show that \( \ln b_n \) converges to \( \int_0^1 \ln x \, dx \), and conclude that \( b_n \to e^{-1} \).

91. Given positive numbers \( a_1 < b_1 \), define two sequences recursively by \( a_{n+1} = \sqrt[n]{a_n b_n} \), \( b_{n+1} = \frac{a_n + b_n}{2} \).
(a) Show that \( a_n \leq b_n \) for all \( n \) (Figure 14).
(b) Show that \( \{a_n\} \) is increasing and \( \{b_n\} \) is decreasing.
(c) Show that \( b_{n+1} - a_{n+1} \leq \frac{b_n - a_n}{2} \).
(d) Prove that both \( \{a_n\} \) and \( \{b_n\} \) converge and have the same limit. This limit, denoted \( \text{AGM}(a_1, b_1) \), is called the arithmetic-geometric mean of \( a_1 \) and \( b_1 \).
(e) Estimate \( \text{AGM}(1, \sqrt{2}) \) to three decimal places.

\[ \begin{array}{ccc}
\text{Geometric} & \text{Arithmetic} & \text{Mean} \\
 a_n & a_{n+1} & b_{n+1} & b_n \\
 \text{AGM}(a_1, b_1) & & & \int_0^x \\
\end{array} \]

FIGURE 14

92. Let \( c_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \).
(a) Calculate \( c_1, c_2, c_3, c_4 \).
(b) Use a comparison of rectangles with the area under \( y = x^{-1} \) over the interval \([n, 2n]\) to prove that
\[ \int_n^{2n} \frac{dx}{x} + \frac{1}{2n} \leq c_n \leq \int_n^{2n} \frac{dx}{x} + \frac{1}{n} \]
(c) Use the Squeeze Theorem to determine \( \lim_{n \to \infty} c_n \).

93. \( \square \) Let \( a_n = H_n - \ln n \), where \( H_n \) is the \( n \)th harmonic number:
\[ H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \]
(a) Show that \( a_n \geq 0 \) for \( n \geq 1 \). \textbf{Hint:} Show that \( H_n \geq \int_1^{n+1} \frac{dx}{x} \).
(b) Show that \( \{a_n\} \) is decreasing by interpreting \( a_n - a_{n+1} \) as an area.
(c) Prove that \( \lim_{n \to \infty} a_n \) exists.
This limit, denoted \( \gamma \), is known as Euler's Constant. It appears in many areas of mathematics, including analysis and number theory, and has been calculated to more than 100 million decimal places, but it is still not known whether \( \gamma \) is an irrational number. The first 10 digits are \( \gamma \approx 0.5772156649 \).

11.2 Summing an Infinite Series

Many quantities that arise in mathematics and its applications cannot be computed exactly. We cannot write down an exact decimal expression for the number \( \pi \) or for values of the sine function such as \( \sin 1 \). However, sometimes these quantities can be represented as infinite sums. For example, using Taylor series (Section 11.8), we can show that
\[ \sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!} + \cdots \]  

\[ \boxed{1} \]

Infinite sums of this type are called \textbf{infinite series}. We think of them as having been obtained by adding up all of the terms in a sequence of numbers.

But what precisely does Eq. (1) mean? How do we make sense of a sum of infinitely many terms? The idea is to examine finite sums of terms at the start of the series and see how they behave. We add progressively more terms and determine whether or not the sums approach a limiting value. More specifically, for the infinite series
\[ a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots \]
define the partial sums:
\[ S_1 = a_1 \]
\[ S_2 = a_1 + a_2 \]
\[ S_3 = a_1 + a_2 + a_3 \]
\[ \cdots \]
\[ S_N = a_1 + a_2 + a_3 + \cdots + a_N \]
The idea then is to consider the sequence of values, \( S_1, S_2, S_3, \ldots, S_N, \ldots \), and whether the limit of this sequence exists.

For example, here are the first five partial sums of the infinite series for \( \sin 1 \):

\[
S_1 = 1 \\
S_2 = 1 - \frac{1}{3!} = 1 - \frac{1}{6} \approx 0.833 \\
S_3 = 1 - \frac{1}{3!} + \frac{1}{5!} = 1 - \frac{1}{6} + \frac{1}{120} \approx 0.841667 \\
S_4 = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} \approx 0.841468 \\
S_5 = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \frac{1}{362,880} \approx 0.8414709846
\]

Compare these values with the value obtained from a calculator:

\[
\sin 1 \approx 0.8414709848079
\]

We see that \( S_5 \) differs from \( \sin 1 \) by less than \( 10^{-9} \). This suggests that the partial sums converge to \( \sin 1 \), and in fact, in Section 11.8 we will prove that

\[
\sin 1 = \lim_{N \to \infty} S_N
\]

(see Example 2). It makes sense then to define the sum of an infinite series as a limit of partial sums.

In general, an infinite series is an expression of the form

\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots
\]

where \( \{a_n\} \) is any sequence. For example,

<table>
<thead>
<tr>
<th>Sequence</th>
<th>General term</th>
<th>Infinite series</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{3} ) ( \frac{1}{9} ) ( \frac{1}{27} ) \ldots</td>
<td>( a_n = \frac{1}{3^n} )</td>
<td>( \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots )</td>
</tr>
<tr>
<td>( \frac{1}{4} ) ( \frac{1}{9} ) ( \frac{1}{16} ) \ldots</td>
<td>( a_n = \frac{1}{n^2} )</td>
<td>( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots )</td>
</tr>
</tbody>
</table>

The \( N \)th partial sum \( S_N \) is the finite sum of the terms up to and including \( a_N \):

\[
S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N
\]

If the series begins at \( k \), then \( S_N = a_k + a_{k+1} + \cdots + a_N \).

**DEFINITION Convergence of an Infinite Series** An infinite series \( \sum_{n=k}^{\infty} a_n \) converges to the sum \( S \) if the sequence of its partial sums \( \{S_N\} \) converges to \( S \):

\[
\lim_{N \to \infty} S_N = S
\]

In this case, we write \( S = \sum_{n=k}^{\infty} a_n \).

- If the limit does not exist, we say that the infinite series diverges.
- If the limit is infinite, we say that the infinite series diverges to infinity.
We can investigate series numerically by computing several partial sums \( S_N \). If the sequence of partial sums shows a trend of convergence to some number \( S \), then we have evidence (but not proof) that the series converges to \( S \). The next example treats a telescoping series, where the partial sums are particularly easy to evaluate.

**EXAMPLE 1  Telescoping Series** Investigate numerically:

\[
\sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \cdots
\]

Then compute the sum of the series using the identity:

\[
\frac{1}{n(n + 1)} = \frac{1}{n} - \frac{1}{n + 1}
\]

**Solution** The values of the partial sums listed in Table 1 suggest convergence to \( S = 1 \).

To prove this, we observe that because of the identity, each partial sum collapses down to just two terms:

\[
S_1 = \frac{1}{1(2)} = \frac{1}{1} - \frac{1}{2}
\]

\[
S_2 = \frac{1}{1(2)} + \frac{1}{2(3)} = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) = 1 - \frac{1}{3}
\]

\[
S_3 = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) = 1 - \frac{1}{4}
\]

In general,

\[
S_N = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots + \left( \frac{1}{N - 1} - \frac{1}{N} \right) + \left( \frac{1}{N} - \frac{1}{N + 1} \right) = 1 - \frac{1}{N + 1}
\]

The sum \( S \) is the limit of the sequence of partial sums:

\[
S = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \left( 1 - \frac{1}{N + 1} \right) = 1
\]

It is important to keep in mind the difference between a sequence \( \{a_n\} \) and an infinite series \( \sum_{n=1}^{\infty} a_n \).

**EXAMPLE 2  Sequences Versus Series** Discuss the difference between \( \{a_n\} \) and \( \sum_{n=1}^{\infty} a_n \), where \( a_n = \frac{1}{n(n + 1)} \).

**Solution** The sequence is the list of numbers \( \frac{1}{1(2)}, \frac{1}{2(3)}, \frac{1}{3(4)}, \ldots \). This sequence converges to zero:

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n(n + 1)} = 0
\]

The infinite series is the sum of the numbers \( a_n \), defined as the limit of the sequence of partial sums. This sum is not zero. In fact, the sum is equal to 1 by Example 1:

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots = 1
\]
The next theorem shows that infinite series may be added or subtracted like ordinary sums, provided that the series converge.

**Theorem 1** Linearity of Infinite Series

If \( \sum a_n \) and \( \sum b_n \) converge, then \( \sum (a_n + b_n) \), \( \sum (a_n - b_n) \), and \( \sum c a_n \) also converge, the latter for any constant \( c \).

Furthermore,

\[
\sum (a_n + b_n) = \sum a_n + \sum b_n
\]

\[
\sum (a_n - b_n) = \sum a_n - \sum b_n
\]

\[
\sum c a_n = c \sum a_n \quad (c \text{ any constant})
\]

**Proof** These rules follow from the corresponding linearity rules for limits. For example,

\[
\sum_{n=1}^{\infty} (a_n + b_n) = \lim_{N \to \infty} \sum_{n=1}^{N} (a_n + b_n) = \lim_{N \to \infty} \left( \sum_{n=1}^{N} a_n + \sum_{n=1}^{N} b_n \right)
\]

\[
= \lim_{N \to \infty} \sum_{n=1}^{N} a_n + \lim_{N \to \infty} \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n
\]

A main goal in this chapter is to develop techniques for determining whether a series converges or diverges. It is easy to give examples of series that diverge:

- \( \sum_{n=1}^{\infty} 1 \) diverges to infinity (the partial sums increase without bound):
  
  \[S_1 = 1, \quad S_2 = 1 + 1 = 2, \quad S_3 = 1 + 1 + 1 = 3, \quad S_4 = 1 + 1 + 1 + 1 = 4, \ldots\]

- \( \sum_{n=1}^{\infty} (-1)^{n-1} \) diverges (the partial sums jump between 1 and 0):
  
  \[S_1 = 1, \quad S_2 = 1 - 1 = 0, \quad S_3 = 1 - 1 + 1 = 1, \quad S_4 = 1 - 1 + 1 - 1 = 0, \ldots\]

Next, we study geometric series, which converge or diverge depending on the common ratio \( r \).

A **geometric series** with common ratio \( r \neq 0 \) is a series defined by a geometric sequence \( cr^n \), where \( c \neq 0 \). If the series begins at \( n = 0 \), then

\[\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + cr^4 + cr^5 + \cdots\]

For \( r = \frac{1}{2} \) and \( c = 1 \), we have the following series:

\[\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\]

Figure 1 demonstrates that adding successive terms in the series corresponds to moving stepwise from 0 to 1, where each step is a move to the right by half of the remaining distance. Thus it appears that the series converges to 1.
There is a simple formula for computing the partial sums of a geometric series:

**Theorem 2** Partial Sums of a Geometric Series  
For the geometric series $\sum_{n=0}^{\infty} cr^n$ with $r \neq 1$,

$$S_N = c + cr + cr^2 + \cdots + cr^N = \frac{c(1 - r^{N+1})}{1 - r}$$  

**Proof**  
In the steps below, we start with the expression for $S_N$, multiply each side by $r$, take the difference between the first two lines, and then simplify:

$$S_N = c + cr + cr^2 + \cdots + cr^N$$

$$rS_N = cr + cr^2 + cr^3 + \cdots + cr^N + cr^{N+1}$$

$$S_N - rS_N = c - cr^{N+1}$$

$$S_N(1 - r) = c(1 - r^{N+1})$$

Since $r \neq 1$, we may divide by $(1 - r)$ to obtain

$$S_N = \frac{c(1 - r^{N+1})}{1 - r}$$

Now, the partial sum formula enables us to compute the sum of the geometric series when $|r| < 1$.

**Theorem 3** Sum of a Geometric Series  
Let $c \neq 0$. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + \cdots = \frac{c}{1 - r}$$

If $|r| \geq 1$, then the geometric series diverges.

**Proof**  
If $r = 1$, then the series certainly diverges because the partial sums $S_N = Nc$ grow arbitrarily large. If $r \neq 1$, then Eq. (3) yields

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{c(1 - r^{N+1})}{1 - r} = \frac{c}{1 - r} - \frac{c}{1 - r} \lim_{N \to \infty} r^{N+1}$$

If $|r| < 1$, then $\lim_{N \to \infty} r^{N+1} = 0$ and we obtain Eq. (4). If $|r| \geq 1$ and $r \neq 1$, then $\lim_{N \to \infty} r^{N+1}$ does not exist and the geometric series diverges.
EXAMPLE 3 Evaluate $\sum_{n=0}^{\infty} 5^{-n}$.

Solution This is a geometric series with common ratio $r = 5^{-1}$ and first term $c = 1$. By Eq. (4),

$$\sum_{n=0}^{\infty} 5^{-n} = 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots = \frac{1}{1 - 5^{-1}} = \frac{5}{4}$$

EXAMPLE 4 Evaluate $\sum_{n=3}^{\infty} 7 \left(-\frac{3}{4}\right)^n = 7 \left(-\frac{3}{4}\right)^3 + 7 \left(-\frac{3}{4}\right)^4 + 7 \left(-\frac{3}{4}\right)^5 + \cdots$.

Solution This is a geometric series with common ratio $r = -\frac{3}{4}$ and first term $c = 7 \left(-\frac{3}{4}\right)^3$. Therefore, it converges to

$$\frac{c}{1 - r} = \frac{7 \left(-\frac{3}{4}\right)^3}{1 - (-\frac{3}{4})} = \frac{27}{16}$$

EXAMPLE 5 Find a fraction that has repeated decimal expansion 0.212121....

Solution We can write this decimal as the series $\frac{21}{100} + \frac{21}{100^2} + \frac{21}{100^3} + \cdots$. This is a geometric series with $c = \frac{21}{100}$ and $r = \frac{1}{100}$. Thus, it converges to

$$\frac{c}{1 - r} = \frac{\frac{21}{100}}{1 - \frac{1}{100}} = \frac{21}{99} = \frac{7}{33}$$

EXAMPLE 6 A Probability Computation Nina and Brook are participating in an archery competition where they take turns shooting at a target. The first one to hit the bullseye wins. Nina's success rate hitting the bullseye is 45%, while Brook's is 52%. Nina pointed out this difference, arguing that she should go first. Brook agreed to give the first turn to Nina. Should he have?

Solution We can answer this question by determining the probability that Nina wins the competition. It is done via a geometric series.

Nina wins in each of the following cases.

- By hitting the bullseye on her first turn (which happens with probability 0.45), or
- By having both players miss on their first turn and Nina hit on her second turn [which happens with probability $(0.55)(0.48)(0.45)$], or
- By having both players miss on their first two turns and Nina hit on her third [which happens with probability $(0.55)(0.48)(0.55)(0.48)(0.45)$], and so on...

There are infinitely many different cases that result in a win for Nina, and because they are distinct from each other (that is, no two of them can occur at the same time) the probability that some one of them occurs is the sum of each of the individual probabilities. That is, the probability that Nina hits the bullseye first is:

$$0.45 + (0.55)(0.48)(0.45) + (0.55)^2(0.48)^2(0.45) + \cdots$$

This is a geometric series with $c = 0.45$ and $r = (0.55)(0.48) = 0.264$. It follows that the probability that Nina wins is $\frac{0.45}{1 - 0.264} = 0.61$. Thus, Brook would have been wise not to let Nina go first.
EXAMPLE 7 Evaluate \( \sum_{n=0}^{\infty} \frac{2 + 3^n}{5^n} \).

Solution Write the series as a sum of two geometric series. This is valid by Theorem 1 because both geometric series converge:

\[
\sum_{n=0}^{\infty} \frac{2 + 3^n}{5^n} = \sum_{n=0}^{\infty} \frac{2}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n} = 2 \cdot \frac{1}{1 - \frac{1}{5}} + \frac{3}{1 - \frac{1}{5}} = 5
\]

Both geometric series converge.

CONCEPTUAL INSIGHT Assumptions Matter Knowing that a series converges, sometimes we can determine its sum through simple algebraic manipulation. For example, suppose we know that the geometric series with \( r = 1/2 \) and \( c = 1/2 \) converges. Let us say that the sum of the series is \( S \), and we write

\[
S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots
\]

\[
2S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1 + S
\]

Thus, \( 2S = 1 + S \), or \( S = 1 \). Therefore, the sum of the series is 1.

Observe what happens when this approach is applied to a divergent series:

\[
S = 1 + 2 + 4 + 8 + 16 + \ldots
\]

\[
2S = 2 + 4 + 8 + 16 + \ldots = S - 1
\]

This would yield \( 2S = S - 1 \), or \( S = -1 \), which is absurd because the series diverges. Thus, without the assumption that a series converges, we cannot employ such algebraic techniques to determine its sum.

The infinite series \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges because the \( N \)th partial sum \( S_N = N \) diverges to infinity. It is less clear whether the following series converges or diverges:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \ldots
\]

We now introduce a useful test that allows us to conclude that this series diverges. The idea is that if the terms are not shrinking to 0 in size, then the series will not converge. This is typically the first test one applies when attempting to determine whether a series diverges.

THEOREM 4 \( n \)th Term Divergence Test If \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges.

Proof First, note that \( a_n = S_n - S_{n-1} \) because

\[
S_n = (a_1 + a_2 + \ldots + a_{n-1}) + a_n = S_{n-1} + a_n
\]

If \( \sum_{n=1}^{\infty} a_n \) converges with sum \( S \), then

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = S - S = 0
\]

Therefore, if \( a_n \) does not converge to zero, \( \sum_{n=1}^{\infty} a_n \) cannot converge.
EXAMPLE 8 Prove the divergence of \( \sum_{n=1}^{\infty} \frac{n}{4n+1} \).

Solution We have

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{4n+1} = \lim_{n \to \infty} \frac{1}{4 + 1/n} = \frac{1}{4}
\]

The \( n \)-th term \( a_n \) does not converge to zero, so the series diverges by the \( n \)-th Term Divergence Test (Theorem 4).

EXAMPLE 9 Determine the convergence or divergence of

\[
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots
\]

Solution The general term \( a_n = (-1)^{n-1} \frac{n}{n+1} \) does not approach a limit. Indeed, \( \frac{n}{n+1} \) tends to 1, so the odd terms \( a_{2n+1} \) tend to 1, and the even terms \( a_{2n} \) tend to -1. Because \( \lim_{n \to \infty} a_n \) does not exist, the series diverges by the \( n \)-th Term Divergence Test.

The \( n \)-th Term Divergence Test tells only part of the story. If \( a_n \) does not tend to zero, then \( \sum a_n \) certainly diverges. But what if \( a_n \) does tend to zero? In this case, the series may converge or it may diverge. In other words, \( \lim_{n \to \infty} a_n = 0 \) is a necessary condition of convergence, but it is not sufficient. As we show in the next example, it is possible for a series to diverge even though its terms tend to zero.

EXAMPLE 10 Sequence Tends to Zero, Yet the Series Diverges Prove the divergence of

\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots
\]

Solution The general term \( 1/\sqrt{n} \) tends to zero. However, because each term in the partial sum \( S_N \) is greater than or equal to \( 1/\sqrt{N} \), we have

\[
S_N = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{N}}
\]

\[
\geq \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \cdots + \frac{1}{\sqrt{N}}
\]

\[
= N \left( \frac{1}{\sqrt{N}} \right) = \sqrt{N}
\]

This shows that \( S_N \geq \sqrt{N} \). But \( \sqrt{N} \) increases without bound (Figure 2). Therefore, \( S_N \) also increases without bound. This proves that the series diverges.

11.2 SUMMARY

- An infinite series is an expression

\[
\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots
\]

We call \( a_n \) the general term of the series. An infinite series can begin at \( n = k \) for any integer \( k \).
- The *Nth partial sum* is the finite sum of the terms up to and including the *N*th term:

\[ S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N \]

- By definition, the sum of an infinite series is the limit \( \lim_{N \to \infty} S_N \). If the limit exists, we say that the infinite series is *convergent* or converges to the sum \( S \). If the limit does not exist, we say that the infinite series *diverges*.

- If the sequence of partial sums of a series increases without bound, we say that the series diverges to infinity.

- *nth Term Divergence Test*: If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges. However, a series may diverge even if its general term \( a_n \) tends to zero.

- Partial sum of a geometric series:

\[ c + cr + cr^2 + cr^3 + \cdots + cr^N = \frac{c(1-r^{N+1})}{1-r} \]

- Geometric series: Assume \( c \neq 0 \). If \( |r| < 1 \), then

\[ \sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \cdots = \frac{c}{1-r} \]

The geometric series converges if \( |r| \geq 1 \).

---

**HISTORICAL PERSPECTIVE**

Archimedes (287–212 BCE), who discovered the law of the lever, said, "Give me a place to stand on, and I can move the Earth" (quoted by Pappus of Alexandria c. 340 CE).

Archimedes showed that the area \( S \) of the parabolic segment is \( \frac{1}{3} T \), where \( T \) is the area of \( \triangle ABC \).

**Figure 3**

Geometric series were used as early as the third century BCE by Archimedes in a brilliant argument for determining the area \( S \) of a "parabolic segment" (shaded region in Figure 3). Given two points \( A \) and \( C \) on a parabola, there is a point \( B \) between \( A \) and \( C \) where the tangent line is parallel to \( AC \) (apparently, Archimedes was aware of the Mean Value Theorem more than 2000 years before the invention of calculus). Let \( T \) be the area of triangle \( \triangle ABC \). Archimedes proved that if \( D \) is chosen in a similar fashion relative to \( AB \) and \( E \) is chosen relative to \( BC \), then

\[ \frac{1}{3} T = \text{area}(\triangle ADB) + \text{area}(\triangle BEC) \]

This construction of triangles can be continued. The next step would be to construct the four triangles on the segments \( AD, DB, BE, EC \), of total area \( \left(\frac{1}{3}\right)^2 T \). Then construct eight triangles of total area \( \left(\frac{1}{3}\right)^3 T \), and so on. In this way, we obtain infinitely many triangles that completely fill up the parabolic segment. By the formula for the sum of a geometric series, we get

\[ S = T + \frac{1}{4} T + \frac{1}{16} T + \cdots = T \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3} T \]

For this and many other achievements, Archimedes is ranked together with Newton and Gauss as one of the greatest scientists of all time.

The modern study of infinite series began in the seventeenth century with Newton, Leibniz, and their contemporaries. The divergence of \( \sum_{n=1}^{\infty} 1/n \) (called the *harmonic series*) was known to the medieval scholar Nicole d'Oresme (1323–1382), but his proof was lost for centuries, and the result was rediscovered on more than one occasion. It was also known that the sum of the reciprocal squares \( \sum_{n=1}^{\infty} 1/n^2 \) converges, and in the 1640s, the Italian Pietro Mengoli put forward the challenge of finding its sum. Despite the efforts of the best
11.2 EXERCISES

Preliminary Questions

1. What role do partial sums play in defining the sum of an infinite series?
2. What is the sum of the following infinite series?
   \[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \]
3. What happens if you apply the formula for the sum of a geometric series to the following series? Is the formula valid?
   \[ 1 + 3 + 9 + 27 + 81 + \cdots \]
4. Indicate whether or not the reasoning in the following statement is correct: \( \sum_{n=1}^{\infty} \frac{1}{n^2} = 0 \) because \( \frac{1}{n^2} \) tends to zero.

Exercises

1. Find a formula for the general term \( a_n \) (not the partial sum) of the infinite series.
   (a) \( \frac{1}{3} + \frac{1}{4} + \frac{1}{27} + \frac{1}{81} + \cdots \)  
   (b) \( \frac{1}{1} + \frac{5}{2} + \frac{25}{4} + \frac{125}{8} + \cdots \)
   (c) \( \frac{1}{1} + \frac{2}{2} + \frac{3}{3} + \frac{4}{4} + \cdots \)
   (d) \( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \)

2. Write in summation notation:
   (a) \( 1 + \frac{1}{9} + \frac{1}{16} + \cdots \)  
   (b) \( \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots \)
   (c) \( \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots \)
   (d) \( \frac{125}{9} + \frac{625}{16} + \frac{3125}{25} + \frac{15,625}{36} + \cdots \)

In Exercises 3–6, compute the partial sums \( S_1, S_2, \) and \( S_6. \)

3. \( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \)
4. \( \sum_{k=1}^{\infty} (-1)^k k^{-1} \)
5. \( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \)
6. \( \sum_{j=1}^{\infty} \frac{1}{j!} \)

7. The series \( 1 + \left( \frac{1}{2} \right) + \left( \frac{1}{2} \right)^2 + \left( \frac{1}{2} \right)^3 + \cdots \) converges to \( \frac{1}{2} \). Calculate \( S_N \) for \( N = 1, 2, \ldots \) until you find an \( S_N \) that approximates \( \frac{1}{2} \) with an error less than 0.0001.

8. The series \( \frac{1}{11} + \frac{1}{21} + \frac{1}{31} + \cdots \) is known to converge to \( e^{-1} \) (recall that \( e^1 = 1 \)). Calculate \( S_N \) for \( N = 1, 2, \ldots \) until you find an \( S_N \) that approximates \( e^{-1} \) with an error less than 0.0001.

5. Indicate whether or not the reasoning in the following statement is correct:
   \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) converges because
   \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \)

6. Find an \( N \) such that \( S_N > 25 \) for the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

7. Does there exist an \( N \) such that \( S_N > 25 \) for the series \( \sum_{n=1}^{\infty} \frac{1}{2^{-n}} \)? Explain.

8. Give an example of a divergent infinite series whose general term tends to zero.

9. \[ \frac{\pi - 3}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots \]
10. \[ \frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \]

11. Calculate \( S_1, S_2, \) and \( S_3 \) and then find the sum of the telescoping series
    \[ \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) \]

12. Write \( \sum_{n=3}^{\infty} \frac{1}{n(n-1)} \) as a telescoping series and find its sum.

13. Calculate \( S_1, S_2, \) and \( S_3 \) and then find the sum \( \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \) using the identity
    \[ \frac{1}{4n^2 - 1} = \frac{1}{2} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right) \]

14. Use partial fractions to rewrite \( \sum_{n=1}^{\infty} \frac{1}{n(n+3)} \) as a telescoping series and find its sum.

15. Find the sum of \( \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \).

16. Find a formula for the partial sum \( S_N \) of \( \sum_{n=1}^{\infty} (-1)^{n+1} \) and show that the series diverges.

In Exercises 9 and 10, use a computer algebra system to compute \( S_{10}, S_{100}, S_{1000}, \) and \( S_{10000} \) for the series. Do these values suggest convergence to the given value?
In Exercises 17–22, use the nth Term Divergence Test (Theorem 4) to prove that the following series diverge.

17. \( \sum_{n=1}^{\infty} \frac{n}{10n^2 + 12} \)

18. \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}} \)

19. \( \frac{1}{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots} \)

20. \( \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2} \)

21. \( \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right) \sum_{n=1}^{\infty} \frac{\sqrt{4n^2 + 1} - n}{n} \)

In Exercises 23–28, either use the formula for the sum of a geometric series to find the sum, or state that the series diverges.

23. \( \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \cdots \)

24. \( \frac{4^3}{3^3} + \frac{4^4}{3^4} + \frac{4^5}{3^5} + \cdots \)

25. \( \frac{7}{3} + \frac{7}{33} + \frac{7}{333} + \cdots \)

26. \( \frac{7}{3} + \left( \frac{7}{3} \right)^2 + \left( \frac{7}{3} \right)^3 + \left( \frac{7}{5} \right)^4 + \cdots \)

27. \( \sum_{n=3}^{\infty} \left( \frac{3}{11} \right)^n \)

28. \( \sum_{n=2}^{\infty} \frac{7 \cdot (-3)^n}{5^n} \)

29. \( \sum_{n=4}^{\infty} \left( -\frac{4}{9} \right)^n \)

30. \( \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n \)

31. \( \sum_{n=1}^{\infty} e^{-n} \)

32. \( \sum_{n=1}^{\infty} 3^{2n} \)

33. \( \sum_{n=0}^{\infty} \frac{8 + 2^n}{5^n} \)

34. \( \sum_{n=0}^{\infty} \frac{3(-2)^n - 5^n}{8^n} \)

35. \( \frac{3}{4} + \frac{5}{4^2} + \frac{5}{4^3} + \cdots \)

36. \( \frac{2^3}{7} + \frac{2^4}{7^2} + \frac{2^5}{7^3} + \frac{2^6}{7^4} + \cdots \)

37. \( \frac{21}{8} = 64 + \frac{343}{12} + \frac{2401}{4} + \cdots \)

38. \( \frac{25}{9} + \frac{5}{3} + \frac{1}{5} + \frac{3}{5} + \frac{9}{5} + \frac{27}{5} + \frac{125}{3} + \cdots \)

In Exercises 39–44, determine a reduced fraction that has this decimal expansion.

39. 0.222…

40. 0.454545…

41. 0.31313131…

42. 0.217217217…

43. 0.1233333333…

44. 0.8088888888…

45. Verify that 0.9999999… = 1 by expressing the left side as a geometric series and determining the sum of the series.

46. The repeating decimal

0.012345678901234567890123456789…

can be expressed as a fraction with denominator 1,111,111,111. What is the numerator?

47. Which of the following are not geometric series?

(a) \( \sum_{n=0}^{\infty} \frac{n^2}{29^n} \)

(b) \( \sum_{n=3}^{\infty} \frac{1}{n^4} \)

48. Use the method of Example 10 to show that \( \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} \) diverges.

49. Prove that if \( \sum_{n=1}^{\infty} a_n \) converges and \( \sum_{n=1}^{\infty} b_n \) diverges, then

\( \sum_{n=1}^{\infty} (a_n + b_n) \) diverges. Hint: If not, derive a contradiction by writing

\( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n) - \sum_{n=1}^{\infty} a_n \)

50. Prove the divergence of \( \sum_{n=0}^{\infty} \frac{2^n + 2^n}{5^n} \).

51. Give a counterexample to show that each of the following statements is false.

(a) If the general term \( a_n \) tends to zero, then \( \sum_{n=1}^{\infty} a_n \) converges.

(b) The nth partial sum of the infinite series defined by \( \{a_n\} \) is \( a_n \).

(c) If \( a_n \) tends to zero, then \( \sum_{n=1}^{\infty} a_n \) converges.

(d) If \( a_n \) tends to \( L \), then \( \sum_{n=1}^{\infty} a_n = L \).

52. Suppose that \( \sum_{n=1}^{\infty} a_n \) is an infinite series with partial sum

\( S_N = 5 - \frac{2}{N^2} \).

(a) What are the values of \( \sum_{n=1}^{10} a_n \) and \( \sum_{n=5}^{16} a_n \)?

(b) What is the value of \( a_N \)?

(c) Find a general formula for \( a_n \).

(d) Find the sum \( \sum_{n=1}^{\infty} a_n \).

53. Consider the archery competition in Example 6.

(a) Assume that Nina goes first. Let \( p_N \) represent the probability that Brooke wins on his nth turn. Give an expression for \( p_N \).

(b) Use the result from (a) and a geometric series to determine the probability that Brooke wins when Nina goes first.

(c) Now assume that Brooke goes first. Use a geometric series to compute the probability that Brooke wins the competition.

54. Consider the archery competition in Example 6. Assume that Nina's probability of hitting the bullseye on a turn is 0.45 and that Brooke's probability is \( p \). Assume that Nina goes first. For what value of \( p \) do both players have a probability of 1/2 of winning the competition?

55. Compute the total area of the (infinitely many) triangles in Figure 4.
56. The winner of a lottery receives $m$ dollars at the end of each year for $N$ years. The present value (PV) of this prize in today’s dollars is $PV = \sum_{i=1}^{N} m(1 + r)^{-i}$, where $r$ is the interest rate. Calculate PV if $m = 50,000, r = 0.06$ (corresponding to 6%), and $N = 20$. What is PV if $N = \infty$?

57. If a patient takes a dose of $D$ units of a particular drug, the amount of the dosage that remains in the patient’s bloodstream after $t$ days is $De^{-kt}$, where $k$ is a positive constant depending on the particular drug.

(a) Show that if the patient takes a dose $D$ every day for an extended period, the amount of drug in the bloodstream approaches $R = \frac{De^{-k}}{1-e^{-k}}$.

(b) Show that if the patient takes a dose $D$ once every $i$ days for an extended period, the amount of drug in the bloodstream approaches $R = \frac{De^{-ki}}{1-e^{-ki}}$.

(c) Suppose that it is considered dangerous to have more than $S$ units of the drug in the bloodstream. What is the minimal time between doses that is safe? Hint: $D + R \leq S$.

58. In economics, the multiplier effect refers to the fact that when there is an injection of money to consumers, the consumers spend a certain percentage of it. That amount recirculates through the economy and adds additional income, which comes back to the consumers and of which they spend the same percentage. This process repeats indefinitely, circulating additional money through the economy. Suppose that in order to stimulate the economy, the government institutes a tax cut of $10$ billion. If taxpayers are known to save $10\%$ of any additional money they receive, and to spend $90\%$, how much total money will be circulated through the economy by that single $10$ billion tax cut?

59. Find the total length of the infinite zigzag path in Figure 5 (each zag occurs at an angle of $\frac{\pi}{4}$).

![Figure 5](image)

60. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$. Hint: Find constants $A$, $B$, and $C$ such that $\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$ and use the result to evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

61. Show that if $a$ is a positive integer, then $\sum_{n=1}^{\infty} \frac{1}{n(n+a)} = \frac{1}{a} \left(1 + \frac{1}{2} + \cdots + \frac{1}{a}\right)$.

62. A ball dropped from a height of $10$ ft begins to bounce vertically. Each time it strikes the ground, it returns to two-thirds of its previous height. What is the total vertical distance traveled by the ball if it bounces infinitely many times?

63. In this exercise, we resolve the paradox of Gabriel’s Horn (Example 3 in Section 8.7 and Example 7 in Section 9.2). Recall that the horn is the surface formed by rotating $y = \frac{1}{x}$ for $x \geq 1$ around the $x$-axis. The surface encloses a finite volume and has an infinite surface area. Thus, apparently we can fill the surface with a finite volume of paint, but an infinite volume of paint is required to paint the surface.

(a) Explain that if we can fill the horn with paint, then the paint must be Magic Paint that can be spread arbitrarily thin, thinner than the thickness of the molecules in normal paint.

(b) Explain that if we use Magic Paint, then we can paint the surface of the horn with a finite volume of paint, in fact with just a milliliter of it. Hint: A geometric series helps here. Use half of a milliliter to paint that part of the surface between $x = 1$ and $x = 2$.

64. A unit sphere is cut into nine equal regions as in Figure 6(A). The central subsphere is painted red. Each of the unpainted squares is then cut into nine equal subsquares and the central square of each is painted red as in Figure 6(B). This procedure is repeated for each of the resulting unpainted squares. After continuing this process an infinite number of times, what fraction of the total area of the original square is painted?

![Figure 6](image)

65. Let $\{b_n\}$ be a sequence and let $a_n = b_n - b_{n-1}$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n \to \infty} b_n$ exists.

66. Assumptions Matter Show, by giving counterexamples, that the assertions of Theorem 1 are not valid if the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are not convergent.

Further Insights and Challenges

In Exercises 67–69, use the formula

$$1 + r + r^2 + \cdots + r^{N-1} = \frac{1 - r^N}{1 - r}$$

67. Professor George Andrews of Pennsylvania State University observed that we can use Eq. (6) to calculate the derivative of $f(x) = x^N$ (for $N \geq 0$). Assume that $a \neq 0$ and let $x = ra$. Show that

$$f'(a) = \lim_{x \to a} \frac{x^N - a^N}{x - a} = a^{N-1} \lim_{x \to a} \frac{r^N - 1}{r - 1}$$

and evaluate the limit.

68. Pierre de Fermat used geometric series to compute the area under the graph of $f(x) = \frac{x}{a}$ over $[0, A]$. For $0 < r < 1$, let $F(r)$ be the sum of the
areas of the infinitely many right-endpoint rectangles with endpoints \( A \cdot r^n \), as in Figure 7. As \( r \) tends to 1, the rectangles become narrower and \( F(r) \) tends to the area under the graph.

(a) Show that \( F(r) = A \cdot \frac{1 - r^{N+1}}{1 - r} \).

(b) Use Eq. (6) to evaluate \( \int_0^A x^N \, dx = \lim_{r \to 1^-} F(r) \).

![Figure 7](image.png)

**FIGURE 7**

69. Verify the Gregory–Leibniz formula in part (d) as follows.

(a) Set \( r = -x^2 \) in Eq. (6) and rearrange to show that

\[
\frac{1}{1 + x^2} = 1 - x^2 + x^4 - \cdots + (-1)^{N-1}x^{2N-2} + \frac{(-1)^N x^{2N}}{1 + x^2}
\]

(b) Show, by integrating over \([0, 1]\), that

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{N-1}}{2N-1} + \frac{(-1)^N}{2N+1}
\]

(c) Use the Comparison Theorem for integrals to prove that

\[
0 \leq \int_0^1 x^{2N} \, dx \leq \frac{1}{2N+1}
\]

Hint: Observe that the integrand is \( \leq x^{2N} \).

(d) Prove that

\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots
\]

Hint: Use (b) and (c) to show that the partial sums \( S_N \) satisfy \( |S_N - \frac{\pi}{4}| \leq \frac{1}{2N+1} \), and therefor conclude that \( \lim_{N \to \infty} S_N = \frac{\pi}{4} \).

70. Cantor's Disappearing Table (following Larry Knop of Hamilton College)

Take a table of length \( L \) (Figure 8). At Stage 1, remove the section of length \( L/4 \) centered at the midpoint. Two sections remain, each with length less than \( L/2 \). At Stage 2, remove sections of length \( L/4^2 \) from each of these two sections (this stage removes \( L/8 \) of the table). Now four sections remain, each of length less than \( L/4 \). At Stage 3, remove the four central sections of length \( L/4^2 \), and so on.

(a) Show that at the \( N \)th stage, each remaining section has length less than \( L/2^N \) and that the total amount of table removed is

\[
L \left( \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{N+1}} \right)
\]

(b) Show that in the limit as \( N \to \infty \), precisely one-half of the table remains.

This result is intriguing, because there are no nonzero intervals of table left (at each stage, the remaining sections have a length less than \( L/2^N \)). So, the table has "disappeared." However, we can place any object longer than \( L/4 \) on the table. The object will not fall through because it will not fit through any of the removed sections.

![Figure 8](image.png)

**FIGURE 8**

71. The Koch Snowflake (described in 1904 by Swedish mathematician Helge von Koch) is an infinitely jagged "fractal" curve obtained as a limit of polygonal curves (it is continuous but has no tangent line at any point). Begin with an equilateral triangle (Stage 0) and produce Stage 1 by replacing each edge with four edges of one-third the length, arranged as in Figure 9. Continue the process: At the \( n \)th stage, replace each edge with four edges of one-third the length of the edge from the \((n-1)\)st stage.

(a) Show that the perimeter \( P_n \) of the polygon at the \( n \)th stage satisfies \( P_n = \frac{4}{3} P_{n-1} \). Prove that \( \lim_{n \to \infty} P_n = \infty \). The snowflake has infinite length.

(b) Let \( A_0 \) be the area of the original equilateral triangle. Show that \( (3/4)^{n-1} \) new triangles are added at the \( n \)th stage, each with area \( A_0/9^n \) (for \( n \geq 1 \)). Show that the total area of the Koch snowflake is \( \frac{8}{5} A_0 \).

![Figure 9](image.png)

**FIGURE 9**

### 11.3 Convergence of Series with Positive Terms

The next three sections develop techniques for determining whether an infinite series converges or diverges. This is easier than finding the sum of an infinite series, which is possible only in special cases.

In this section, we consider positive series \( \sum a_n \), where \( a_n > 0 \) for all \( n \). We can visualize the terms of a positive series as rectangles of width 1 and height \( a_n \) (Figure 1).

The partial sum

\[
S_N = a_1 + a_2 + \cdots + a_N
\]

is equal to the area of the first \( N \) rectangles.
The key feature of positive series is that their partial sums form an increasing sequence

\[ S_N < S_{N+1} \]

for all \( N \). This is because \( S_{N+1} \) is obtained from \( S_N \) by adding a positive number:

\[ S_{N+1} = (a_1 + a_2 + \cdots + a_N) + a_{N+1} = S_N + a_{N+1} \]

Recall that an increasing sequence converges if it is bounded above. Otherwise, it diverges (Theorem 6, Section 11.1). It follows that a positive series behaves in one of two ways.

**Theorem 1** Partial Sum Theorem for Positive Series

If \( \sum_{n=1}^{\infty} a_n \) is a positive series, then either

(i) The partial sums \( S_N \) are bounded above. In this case, \( \sum_{n=1}^{\infty} a_n \) converges. Or,

(ii) The partial sums \( S_N \) are not bounded above. In this case, \( \sum_{n=1}^{\infty} a_n \) diverges.

- Theorem 1 remains true if \( a_n \geq 0 \). It is not necessary to assume that \( a_n > 0 \).
- It also remains true if \( a_n > 0 \) for all \( n \geq M \) for some \( M \), because the convergence or divergence of a series is not affected by the first \( M \) terms.

**Assumptions Matter**

The theorem does not hold for nonpositive series. Consider

\[ \sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots \]

The partial sums are bounded (because \( S_N = 1 \) or 0), but the series diverges.

Our first application of Theorem 1 is the following Integral Test. It is extremely useful because in many cases, integrals are easier to evaluate than series.

**Theorem 2** Integral Test

Let \( a_n = f(n) \), where \( f \) is a positive, decreasing, and continuous function of \( x \) for \( x \geq 1 \).

(i) If \( \int_1^{\infty} f(x) \, dx \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

(ii) If \( \int_1^{\infty} f(x) \, dx \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

**Proof** Because \( f \) is decreasing, the shaded rectangles in Figure 2 lie below the graph of \( f \), and therefore for all \( N \),

\[ a_1 + a_2 + \cdots + a_N \leq \int_1^{N} f(x) \, dx \leq \int_1^{\infty} f(x) \, dx \]

If the improper integral on the right converges, then the sums \( a_1 + \cdots + a_N \) are bounded above. That is, the partial sums \( S_N \) are bounded above, and therefore the infinite series converges by the Partial Sum Theorem for Positive Series (Theorem 1). This proves (i).

On the other hand, the rectangles in Figure 3 lie above the graph of \( f \), so

\[ \int_1^{N} f(x) \, dx \geq a_1 + a_2 + \cdots + a_{N-1} \]

Area of shaded rectangles in Figure 3
The integral test is valid for any series \( \sum_{n=1}^{\infty} a_n \), provided that for some \( M > 0 \), \( f \) is a positive, decreasing, and continuous function of \( x \) for \( x \geq M \). The convergence of the series is determined by the convergence of
\[
\int_{M}^{\infty} f(x) \, dx
\]
The infinite series
\[
\sum_{n=1}^{\infty} \frac{1}{n}
\]
is called the harmonic series.

If \( \int_{1}^{\infty} f(x) \, dx \) diverges, then \( \int_{1}^{N} f(x) \, dx \) increases without bound as \( N \) increases. The inequality in (1) shows that \( S_N \) also increases without bound, and therefore, the series diverges. This proves (ii).

**EXAMPLE 1** The Harmonic Series Diverges
Show that \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

Solution Let \( f(x) = \frac{1}{x} \). Then \( f(n) = \frac{1}{n} \), and the integral test applies because \( f \) is positive, decreasing, and continuous for \( x \geq 1 \). The integral diverges:
\[
\int_{1}^{\infty} \frac{dx}{x} = \lim_{R \to \infty} \int_{1}^{R} \frac{dx}{x} = \lim_{R \to \infty} \ln R = \infty
\]

Therefore, the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

**EXAMPLE 2** Does \( \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2} \) converge?

Solution The function \( f(x) = \frac{x}{(x^2 + 1)^2} \) is positive and continuous for \( x \geq 1 \). It is decreasing because \( f'(x) \) is negative:
\[
f'(x) = \frac{1 - 3x^2}{(x^2 + 1)^3} < 0 \quad \text{for} \quad x \geq 1
\]
Therefore, the integral test applies. Using the substitution \( u = x^2 + 1, \, du = 2x \, dx \), we have
\[
\int_{1}^{\infty} \frac{x}{(x^2 + 1)^2} \, dx = \lim_{R \to \infty} \int_{1}^{R} \frac{x}{(x^2 + 1)^2} \, dx = \lim_{R \to \infty} \frac{1}{2} \int_{2}^{R^2 + 1} \frac{du}{u^2}
\]
\[
= \lim_{R \to \infty} \frac{-1}{2u} \bigg|_{2}^{R^2 + 1} = \lim_{R \to \infty} \left( \frac{1}{4} - \frac{1}{2(R^2 + 1)} \right) = \frac{1}{4}
\]
Thus, the integral converges, and therefore, \( \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2} \) also converges by the integral test.

The sum of the reciprocal powers \( n^{-p} \) is called a \( p \)-series. As the next theorem shows, the convergence or divergence of these series is determined by the value of \( p \).

**THEOREM 3** Convergence of \( p \)-Series
The infinite series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges if \( p > 1 \) and diverges otherwise.

**Proof** If \( p \leq 0 \), then the general term \( n^{-p} \) does not tend to zero, so the series diverges by the nth Term Divergence Test. If \( p > 0 \), then \( f(x) = x^{-p} \) is positive and decreasing for \( x \geq 1 \), so the integral test applies. According to Theorem 1 in Section 8.7,
\[
\int_{1}^{\infty} \frac{1}{x^p} \, dx = \begin{cases} 
\frac{1}{p - 1} & \text{if} \quad p > 1 \\
\infty & \text{if} \quad p \leq 1
\end{cases}
\]

Therefore, \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges for \( p > 1 \) and diverges for \( p \leq 1 \).
Here are two examples of $p$-series:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \quad \text{diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{converges}$$

Another powerful method for determining convergence of positive series occurs via comparison with other series. Suppose that $0 \leq a_n \leq b_n$. Figure 4 suggests that if the larger sum $\sum b_n$ converges, then the smaller sum $\sum a_n$ also converges. Similarly, if the smaller sum diverges, then the larger sum also diverges.

**Theorem 4 Direct Comparison Test**

Assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq M$.

(i) If $\sum b_n$ converges, then $\sum a_n$ also converges.

(ii) If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

**Proof** We can assume, without loss of generality, that $M = 1$. If $\sum b_n$ converges to $S$, then the partial sums of $\sum_{n=1}^{\infty} a_n$ are bounded above by $S$ because

$$a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N \leq \sum_{n=1}^{\infty} b_n = S$$

Note that the first inequality in (2) holds since $a_n \leq b_n$ for all $n$, and the second holds since $b_n \geq 0$ for all $n$.

Under the assumption that $\sum b_n$ converges, it now follows that $\sum a_n$ converges by the Partial Sum Theorem for Positive Series (Theorem 1). This proves (i). On the other hand, if $\sum a_n$ diverges, then $\sum b_n$ must also diverge. Otherwise, we would have a contradiction to (i).

**Example 3** Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \, 3^n}$ converge?

**Solution** For $n \geq 1$, we have

$$\frac{1}{\sqrt{n} \, 3^n} \leq \frac{1}{3^n}$$

The larger series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges because it is a geometric series with $r = \frac{1}{3} < 1$. By the Direct Comparison Test, the smaller series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \, 3^n}$ also converges.
EXAMPLE 4  Does \( \sum_{n=2}^{\infty} \frac{1}{(n^2 + 3)^{1/3}} \) converge?

Solution  Let us show that

\[
\frac{1}{n^{1/3}} \leq \frac{1}{(n^2 + 3)^{1/3}} \quad \text{for } n \geq 2
\]

This inequality is equivalent to \( (n^2 + 3) \leq n^3 \), so we must show that

\[
f(x) = x^3 - (x^2 + 3) \geq 0 \quad \text{for } x \geq 2
\]

The function \( f \) is increasing because its derivative \( f'(x) = 3x^2 - 2x = 3x(x - \frac{2}{3}) \) is positive for \( x \geq 2 \). Since \( f(2) = 1 \), it follows that \( f(x) \geq 1 \) for \( x \geq 2 \), and our original inequality follows. We know that the smaller harmonic series \( \sum_{n=2}^{\infty} \frac{1}{n} \) diverges. Therefore, the larger series \( \sum_{n=2}^{\infty} \frac{1}{(n^2 + 1)^{1/3}} \) also diverges.

EXAMPLE 5  Determine the convergence of

\[
\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}
\]

Solution  We might be tempted to compare \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) to the harmonic series \( \sum_{n=3}^{\infty} \frac{1}{n} \) using the inequality (valid for \( n \geq 3 \) since \( \ln 3 > 1 \))

\[
\frac{1}{n(\ln n)^2} \leq \frac{1}{n}
\]

However, \( \sum_{n=2}^{\infty} \frac{1}{n} \) diverges, and this says nothing about the smaller series \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \).

Fortunately, the Integral Test can be used. The substitution \( u = \ln x \) yields

\[
\int_{2}^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2} = \lim_{R \to \infty} \left( \frac{1}{\ln 2} - \frac{1}{R} \right) = \frac{1}{\ln 2} < \infty
\]

The Integral Test shows that \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) converges.

The next test for convergence involves a comparison between two series \( \sum a_n \) and \( \sum b_n \) via a limit of the ratios, \( \frac{a_n}{b_n} \), of the terms in the series.

CAUTION The Limit Comparison Test is not valid if the series are not positive. See Exercise 44 in Section 11.4.

THEOREM 5  Limit Comparison Test  Let \( \{a_n\} \) and \( \{b_n\} \) be positive sequences. Assume that the following limit exists:

\[
L = \lim_{n \to \infty} \frac{a_n}{b_n}
\]

- If \( L > 0 \), then \( \sum a_n \) converges if and only if \( \sum b_n \) converges.
- If \( L = \infty \) and \( \sum a_n \) converges, then \( \sum b_n \) converges.
- If \( L = 0 \) and \( \sum b_n \) converges, then \( \sum a_n \) converges.
Proof Assume first that $L$ is finite (possibly zero) and that \( \sum b_n \) converges. Choose a positive number $R > L$. Then $0 \leq a_n/b_n \leq R$ for all $n$ sufficiently large because $a_n/b_n$ approaches $L$. Therefore, $a_n \leq R b_n$. The series \( \sum R b_n \) converges because it is a constant multiple of the convergent series \( \sum b_n \). Thus, \( \sum a_n \) converges by the Direct Comparison Test.

Next, suppose that $L$ is nonzero (positive or infinite) and that \( \sum a_n \) converges. Let $K = \lim_{n \to \infty} b_n/a_n$. Then either $K = L^{-1}$ (if $L$ is finite) or $K = 0$ (if $L$ is infinite). In either case, $K$ is finite and we can apply the result of the previous paragraph with the roles of \( \{a_n\} \) and \( \{b_n\} \) reversed to conclude that \( \sum b_n \) converges.

**CONCEPTUAL INSIGHT** To remember the different cases of the Limit Comparison Test, you can think of it this way: If $L > 0$, then \( a_n \approx L b_n \) for large $n$. In other words, the series \( \sum a_n \) and \( \sum b_n \) are roughly multiples of each other, so one converges if and only if the other converges. If $L = \infty$, then \( a_n \) is much larger than \( b_n \) (for large $n$), so if \( \sum a_n \) converges, \( \sum b_n \) certainly converges. Finally, if $L = 0$, then \( b_n \) is much larger than \( a_n \) and the convergence of \( \sum b_n \) yields the convergence of \( \sum a_n \).

**EXAMPLE 6** Show that \( \sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1} \) converges.

**Solution** If we divide the numerator and denominator by $n$, we can conclude that for large $n$,

\[
\frac{n^2}{n^4 - n - 1} \approx \frac{1}{n^2}
\]

To apply the Limit Comparison Test, we set

\[
a_n = \frac{n^2}{n^4 - n - 1} \quad \text{and} \quad b_n = \frac{1}{n^2}
\]

We observe that \( \lim_{n \to \infty} \frac{a_n}{b_n} \) exists and is positive:

\[
L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{n^4 - n - 1} \frac{1}{1} = \lim_{n \to \infty} \frac{1}{1 - n^{-3} - n^{-4}} = 1
\]

Since \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) converges, our series \( \sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1} \) also converges by Theorem 5.

**EXAMPLE 7** Determine whether \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 4}} \) converges.

**Solution** Apply the Limit Comparison Test with \( a_n = \frac{1}{\sqrt{n^2 + 4}} \) and \( b_n = \frac{1}{n} \). Then

\[
L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + 4}} = \lim_{n \to \infty} \frac{1}{\sqrt{1 + 4/n^2}} = 1
\]

Since \( \sum_{n=3}^{\infty} \frac{1}{n} \) diverges and $L > 0$, the series \( \sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2 + 4}} \) also diverges.

In the Limit Comparison Test, when attempting to find an appropriate $b_n$ to compare with $a_n$, we typically keep only the largest power of $n$ in the numerator and denominator of $a_n$, as we did in each of the previous examples.
11.3 SUMMARY

- The partial sums $S_N$ of a positive series $\sum a_n$ form an increasing sequence.
- Partial Sum Theorem for Positive Series: A positive series converges if its partial sums $S_N$ are bounded. Otherwise, it diverges.
- Integral Test: Assume that $f$ is positive, decreasing, and continuous for $x > M$. Set $a_n = f(n)$. If $\int_M^\infty f(x) \, dx$ converges, then $\sum a_n$ converges, and if $\int_M^\infty f(x) \, dx$ diverges, then $\sum a_n$ diverges.
- $p$-Series: The series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
- Direct Comparison Test: Assume there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for all $n \geq M$. If $\sum b_n$ converges, then $\sum a_n$ converges, and if $\sum a_n$ diverges, then $\sum b_n$ diverges.
- Limit Comparison Test: Assume that $(a_n)$ and $(b_n)$ are positive and that the following limit exists:
  \[
  L = \lim_{n \to \infty} \frac{a_n}{b_n}
  \]
  - If $L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.
  - If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.
  - If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

11.3 EXERCISES

Preliminary Questions

1. For the series $\sum a_n$, if the partial sums $S_N$ are increasing, then (choose the correct conclusion)
   (a) $(a_n)$ is an increasing sequence.
   (b) $(a_n)$ is a positive sequence.
2. What are the hypotheses of the Integral Test?
3. Which test would you use to determine whether $\sum_{n=1}^{\infty} n^{-3.2}$ converges?
4. Which test would you use to determine whether $\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$ converges?
5. Ralph hopes to investigate the convergence of $\sum_{n=1}^{\infty} \frac{e^{-n}}{n}$ by comparing it with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Is Ralph on the right track?

Exercises

In Exercises 1–12, use the Integral Test to determine whether the infinite series is convergent.

1. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$
2. $\sum_{n=1}^{\infty} \frac{1}{n + 3}$
3. $\sum_{n=1}^{\infty} n^{-1/3}$
4. $\sum_{n=5}^{\infty} \frac{1}{\sqrt{n-4}}$
5. $\sum_{n=23}^{\infty} \frac{n^2}{(n^2 + 9)^{3/2}}$
6. $\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^{3/2}}$
7. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$
8. $\sum_{n=4}^{\infty} \frac{1}{n^2 - 1}$
9. $\sum_{n=1}^{\infty} \frac{1}{n(n+5)}$
10. $\sum_{n=1}^{\infty} \frac{1}{n^n}$
11. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$
12. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$
13. Show that $\sum_{n=1}^{\infty} \frac{1}{n^3 + 8n}$ converges by using the Direct Comparison Test with $\sum_{n=1}^{\infty} \frac{n^{-3}}{n}$.
14. Show that $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2 - 3}}$ diverges by comparing with $\sum_{n=2}^{\infty} \frac{1}{n}$. 
15. For \( \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}} \), verify that for \( n \geq 1 \),
\[
\frac{1}{n + \sqrt{n}} \leq \frac{1}{n}, \quad \frac{1}{n + \sqrt{n}} \leq \frac{1}{\sqrt{n}}
\]
Can either inequality be used to show that the series diverges? Show that
\[
\frac{1}{n + \sqrt{n}} \geq \frac{1}{2n}
\]
for \( n \geq 1 \) and conclude that the series diverges.
16. Which of the following inequalities can be used to study the convergence of \( \sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n}} \)? Explain.
\[
\frac{1}{n^2 + \sqrt{n}} \leq \frac{1}{n}, \quad \frac{1}{n^2 + \sqrt{n}} \leq \frac{1}{n^2}
\]

In Exercises 17–28, use the Direct Comparison Test to determine whether the infinite series is convergent.

17. \( \sum_{n=1}^{\infty} \frac{1}{n^{2n}} \)
18. \( \sum_{n=1}^{\infty} \frac{n^3}{n^5 + 4n + 1} \)
19. \( \sum_{n=1}^{\infty} \frac{1}{n^{1/3} + 2n} \)
20. \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 2n - 1}} \)
21. \( \sum_{n=1}^{\infty} \frac{4}{n! + 4^n} \)
22. \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n - 3} \)
23. \( \sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2} \)
24. \( \sum_{k=1}^{\infty} \frac{k^2/9}{k/10 + 9k - 9} \)
25. \( \sum_{n=1}^{\infty} \frac{2}{3^n + 3^n} \)
26. \( \sum_{n=1}^{\infty} \frac{2^{-k^2}}{k!} \)
27. \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)!} \)
28. \( \sum_{n=1}^{\infty} \frac{n!}{n^2} \)

Exercise 29–34: For all \( a > 0 \) and \( b > 1 \), the inequalities \( \ln n \leq n^a \), \( n^a < b \) are true for \( n \) sufficiently large (this can be proved using L'Hopital's Rule). Use this, together with the Direct Comparison Test, to determine whether the series converges or diverges.

29. \( \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \)
30. \( \sum_{n=1}^{\infty} \frac{\ln n}{n^4} \)
31. \( \sum_{n=1}^{\infty} \frac{\ln n^{100}}{n^{10}} \)
32. \( \sum_{n=1}^{\infty} \frac{1}{(\ln n)^{10}} \)
33. \( \sum_{n=1}^{\infty} \frac{n}{3^n} \)
34. \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} \)

35. Show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges. Hint: Use \( \sin x \leq x \) for \( x \geq 0 \).

36. Does \( \sum_{n=1}^{\infty} \frac{\sin(1/n)}{\ln n} \) converge? Hint: By Theorem 3 in Section 2.6, \( \sin(1/n) > (\cos(1/n))/n \). Thus, \( \sin(1/n) > 1/(2n) \) for \( n > 2 \) [because \( \cos(1/n) > \frac{1}{2} \)].

In Exercises 37–46, use the Limit Comparison Test to prove convergence or divergence of the infinite series.

37. \( \sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1} \)
38. \( \sum_{n=2}^{\infty} \frac{1}{m(n-1)} \)
39. \( \sum_{n=1}^{\infty} \frac{n}{n^3 + 1} \)
40. \( \sum_{n=1}^{\infty} \frac{1}{n^3 + 2n^2 + 1} \)
41. \( \sum_{n=1}^{\infty} \frac{3n + 5}{n(n-1)(n-2)} \)
42. \( \sum_{n=1}^{\infty} \frac{e^{-n}}{n^2} \)
43. \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n + \ln n}} \)
44. \( \sum_{n=1}^{\infty} \frac{\ln(n + 4)}{n^{3/2}} \)
45. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) Hint: Compare with \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).
46. \( \sum_{n=1}^{\infty} \frac{1}{2^{n/2}} \) Hint: Compare with the harmonic series.

In Exercises 47–76, determine convergence or divergence using any method covered so far.

47. \( \sum_{n=1}^{\infty} \frac{1}{n^2 - 9} \)
48. \( \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2} \)
49. \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n - 5} \)
50. \( \sum_{n=1}^{\infty} \frac{n - \cos n}{n^3} \)
51. \( \sum_{n=1}^{\infty} \frac{n^3 - 1}{n^3 + 1} \)
52. \( \sum_{n=1}^{\infty} \frac{1}{n + \sin n} \)
53. \( \sum_{n=1}^{\infty} \frac{(4/5)^n}{n} \)
54. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)
55. \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2} \ln n} \)
56. \( \sum_{n=1}^{\infty} \frac{\ln(n)^{12}}{n^{5/8}} \)
57. \( \sum_{k=1}^{\infty} \frac{4^{1/k}}{5^k} \)
58. \( \sum_{n=1}^{\infty} \frac{4^n}{n^2 - 2n} \)
59. \( \sum_{n=1}^{\infty} \frac{1}{\ln n} \)
60. \( \sum_{n=1}^{\infty} \frac{2^n}{n^2 - n} \)
61. \( \sum_{n=1}^{\infty} \frac{1}{n \ln n - n} \)
62. \( \sum_{n=1}^{\infty} \frac{1}{n \ln n^2 - n} \)
63. \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)
64. \( \sum_{n=1}^{\infty} \frac{1}{n^3 - 4n + 1} \)
65. \( \sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n} \)
66. \( \sum_{n=1}^{\infty} \frac{2 + (-1)^n}{n^{3/2}} \)
67. \( \sum_{n=1}^{\infty} \frac{1}{n} \)
68. \( \sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}} \)
71. \[ \sum_{n=1}^{\infty} \frac{\ln n}{n^2 - 3n} \]

72. \[ \sum_{n=1}^{\infty} \ln \left( 1 + \frac{1}{n} \right) \]

73. \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/2} \ln n} \]

74. \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2} - (\ln n)^{}} \]

75. \[ \sum_{n=2}^{\infty} \frac{4n^2 + 15n}{3n^3 - 5n^2 - 17} \]

76. \[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 5n - 6} \]

77. For which \( a \) does \( \sum_{n=2}^{\infty} \frac{1}{n(n \ln n)^a} \) converge?

78. For which \( a \) does \( \sum_{n=2}^{\infty} \frac{1}{n^a \ln n} \) converge?

79. For which values of \( p \) does \( \sum_{n=1}^{\infty} \frac{n^2}{(n + 1)p} \) converge?

80. For which values of \( p \) does \( \sum_{n=1}^{\infty} \frac{e^n}{(1 + e^n)^p} \) converge?

### Approximating Infinite Sums

**In Exercises 81-83, let \( a_n = f(n) \), where \( f \) is a continuous, decreasing function such that \( f(x) \geq 0 \) and \( \int_1^\infty f(x) \, dx \) converges.**

**81.** Show that
\[
\int_1^\infty f(x) \, dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^\infty f(x) \, dx
\]

**CAS** Using the inequality in (3), show that
\[ 5 \leq \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \leq 6 \]

This series converges slowly. Use a computer algebra system to verify that \( S_n < 5 \) for \( N \leq 43,128 \) and \( S_{43,128} \approx 5.00000021 \).

**83.** Assume \( \sum_{n=1}^{\infty} a_n \) converges to \( S \). Arguing as in Exercise 81, show that
\[
\sum_{n=1}^{M} a_n + \int_{M+1}^{\infty} f(x) \, dx \leq S \leq \sum_{n=1}^{M} a_n + \int_{M+1}^{\infty} f(x) \, dx
\]

Conclude that
\[
0 \leq S - \left( \sum_{n=1}^{M} a_n + \int_{M+1}^{\infty} f(x) \, dx \right) \leq a_{M+1}
\]

This provides a method for approximating \( S \) with an error of at most \( a_{M+1} \).

**84. CAS** Use the inequalities in (4) from Exercise 83 with \( M = 43,129 \) to prove that
\[ 5.5915810 \leq \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \leq 5.5915839 \]

**85. CAS** Use the inequalities in (4) from Exercise 83 with \( M = 40,000 \) to show that
\[ 1.644934066 \leq \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \leq 1.644934068 \]

Is this consistent with Euler's result, according to which this infinite series has sum \( \pi^2/6 \)?

**86. CAS** Use a CAS and the inequalities in (5) from Exercise 83 to determine the value of \( \sum_{n=1}^{\infty} n^{-6} \) to within an error less than \( 10^{-4} \). Check that your result is consistent with that of Euler, who proved that the sum is equal to \( \pi^2/945 \).

**87. CAS** Use a CAS and the inequalities in (5) from Exercise 83 to determine the value of \( \sum_{n=1}^{\infty} n^{-5} \) to within an error less than \( 10^{-4} \).

88. How far can a stack of identical books (of mass \( m \) and unit length) extend without tipping over? The stack will not tip over if the \( (n + 1) \)st book is placed at the bottom of the stack with its right edge located at or before the center of mass of the first \( n \) books (Figure 6). Let \( c_n \) be the center of mass of the first \( n \) books, measured along the \( x \)-axis, where we take the positive \( x \)-axis to the left of the origin as in Figure 7. Recall that if an object of mass \( m_1 \) has center of mass at \( x_1 \) and a second object of mass \( m_2 \) has center of mass \( x_2 \), then the center of mass of the system has \( x \)-coordinate
\[
\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}
\]

(a) Show that if the \( (n + 1) \)st book is placed with its right edge at \( c_n \), then its center of mass is located at \( c_n + \frac{1}{2} \).

(b) Consider the first \( n \) books as a single object of mass \( nm \) with center of mass at \( c_n \) and the \( (n + 1) \)st book as a second object of mass \( m \). Show that if the \( (n + 1) \)st book is placed with its right edge at \( c_n \), then
\[ c_{n+1} = c_n + \frac{1}{2(n + 1)} \]

(c) Prove that \( \lim_{n \to \infty} c_n = \infty \). Thus, by using enough books, the stack can be extended as far as desired without tipping over.
89. The following argument proves the divergence of the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) without using the Integral Test. To begin, assume that the harmonic series converges to a value \( S \).
(a) Prove that the following two series must also converge:
\[
1 + \frac{1}{3} + \frac{1}{5} + \cdots \quad \text{and} \quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots
\]
(b) Prove that if \( S_1 \) and \( S_2 \) are the sums of the series on the left and right, respectively, then \( S = S_1 + S_2 \).
(c) Prove that \( S_1 \geq S_2 + \frac{1}{2} \), and \( S_2 = \frac{1}{2} S \). Explain how this leads to a contradiction and the conclusion that the harmonic series diverges.

Further Insights and Challenges

90. Consider the series \( \sum_{n=2}^{\infty} a_n \), where \( a_n = (\ln(n))^{-\ln n} \).
(a) Show, by taking logarithms, that \( a_n = e^{-\ln(n)^{\ln n}} \).
(b) Show that \( \ln(n)^{\ln n} \geq 2 \) if \( n > e^2 \), where \( e = \text{e}^2 \).
(c) Show that the series converges.

91. Kummer's Acceleration Method Suppose we wish to approximate \( S = \sum_{n=1}^{\infty} \frac{1}{n^2} \). There is a similar telescoping series whose value can be computed exactly (Example 2 in Section 11.2):
\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1
\]
(a) Verify that
\[
S = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{n+1} \right)
\]
Thus for \( M \) large,
\[
S \approx 1 + \sum_{n=1}^{M} \frac{1}{n^2(n+1)}
\]
(b) Explain what has been gained. Why is (6) a better approximation to \( S \) than \( \sum_{n=1}^{M} \frac{1}{n^2} \)?
(c) \text{CAS} Compute
\[
\sum_{n=1}^{100} \frac{1}{n^2} = 1 + \sum_{n=1}^{100} \frac{1}{n^2(n+1)}
\]
Which is a better approximation to \( S \), whose exact value is \( \pi^2/6 \)?

92. \text{CAS} The sum \( S = \sum_{n=1}^{\infty} \frac{1}{n^3} \) has been computed to more than 100 million digits. The first 30 digits are
\[
S = 1.2020569031595942853979738161511
\]
Approximate \( S \) using Kummer's Acceleration Method of Exercise 91 with the similar series \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} \) and \( M = 500 \). According to Exercise 60 in Section 11.2, the similar series is a telescoping series with a sum of \( \frac{1}{2} \).

11.4 Absolute and Conditional Convergence

In the previous section, we studied positive series, but we still lack the tools to analyze series with both positive and negative terms. One of the keys to understanding such series is the concept of absolute convergence.

**Definition** Absolute Convergence The series \( \sum a_n \) converges absolutely if \( \sum |a_n| \) converges.

**Example 1** Verify that the series
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots
\]
converges absolutely.
Solution This series converges absolutely because taking the absolute value of each term, we obtain a $p$-series with $p = 2 > 1$:

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \text{(convergent $p$-series)}
$$

The next theorem tells us that if the series of absolute values converges, then the original series also converges.

**THEOREM 1** Absolute Convergence Implies Convergence  
If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

**Proof** We have $-|a_n| \leq a_n \leq |a_n|$. By adding $|a_n|$ to all parts of the inequality, we get $0 \leq |a_n| + a_n \leq 2|a_n|$. If $\sum |a_n|$ converges, then $\sum 2|a_n|$ also converges, and therefore $\sum (a_n + |a_n|)$ converges by the Direct Comparison Test. Our original series converges because it is the difference of two convergent series:

$$
\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|
$$

**EXAMPLE 2** Verify that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges.

Solution We showed that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges absolutely in Example 1. By Theorem 1, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ itself converges.

**EXAMPLE 3** Does $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converge absolutely?

Solution The series of absolute values is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a $p$-series with $p = \frac{1}{2}$. It diverges because $p < 1$. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ does not converge absolutely.

The series in the previous example does not converge absolutely, but we still do not know whether or not it converges. A series $\sum a_n$ may converge without converging absolutely. In this case, we say that $\sum a_n$ is conditionally convergent.

**DEFINITION** Conditional Convergence  
An infinite series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

If a series is not absolutely convergent, how can we determine whether it is conditionally convergent? This is often a difficult question, because we cannot use the Integral Test or the Direct Comparison Test since they apply only to positive series. However, convergence is guaranteed in the particular case of an alternating series

$$
\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots
$$

where the terms $b_n$ are positive and decrease to zero (Figure 1).
Assumptions Matter The Alternating Series Test is not valid if we drop the assumption that \( b_n \) is decreasing (see Exercise 25).

The Alternating Series Test is the only test for conditional convergence developed in this text. Other tests, such as Abel's Criterion and the Dirichlet Test, are discussed in textbooks on analysis.

**Theorem 2 Alternating Series Test** Assume that \( \{b_n\} \) is a positive sequence that is decreasing and converges to 0:

\[
\begin{align*}
  b_1 &> b_2 > b_3 > b_4 > \cdots > 0, \\
  \lim_{n \to \infty} b_n & = 0
\end{align*}
\]

Then the following alternating series converges:

\[
\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots
\]

Furthermore, if \( S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n \), then

\[
0 < S < b_1 \quad \text{and} \quad S_p < S < S_q \quad \text{for} \quad p \text{ even and} \quad q \text{ odd}
\]

As we will see, this last fact allows the estimation of such a series to any level of accuracy needed.

Notice that under the same conditions, the series

\[
\sum_{n=1}^{\infty} (-1)^{n} b_n = -b_1 + b_2 - b_3 + b_4 - \cdots
\]

also converges since it is just \(-1\) times the series appearing in the theorem.

**Proof** We will prove that the partial sums zigzag above and below the sum \( S \) as in Figure 2. Note first that the even partial sums are increasing. Indeed, the odd-numbered terms occur with a plus sign and thus, for example,

\[
S_7 + b_8 - b_6 = S_5
\]

But \( b_5 - b_6 > 0 \) because \( b_n \) is decreasing, and therefore, \( S_8 < S_6 \). In general,

\[
S_{2N} + (b_{2N+1} - b_{2N+2}) = S_{2N+2}
\]

where \( b_{2N+1} - b_{2N+2} > 0 \). Thus, \( S_{2N} < S_{2N+2} \) and

\[
0 < S_2 < S_4 < S_6 < \cdots
\]

Similarly,

\[
S_{2N+1} - (b_{2N} - b_{2N+1}) = S_{2N+1}
\]

Therefore, \( S_{2N+1} < S_{2N-1} \), and the sequence of odd partial sums is decreasing:

\[
\cdots < S_7 < S_5 < S_3 < S_1
\]

Finally, \( S_{2N} < S_{2N} + b_{2N+1} = S_{2N+1} \). The partial sums compare as follows:

\[
0 < S_2 < S_4 < S_6 < \cdots < S_7 < S_5 < S_3 < S_1
\]

Now, because bounded monotonic sequences converge (Theorem 6 of Section 11.1), the even and odd partial sums approach limits that are sandwiched in the middle:

\[
0 < S_2 < S_4 < \cdots < \lim_{N \to \infty} S_{2N} \leq \lim_{N \to \infty} S_{2N+1} < \cdots < S_7 < S_5 < S_3 < S_1
\]

These two limits must have a common value \( S \) because

\[
\lim_{N \to \infty} S_{2N+1} - \lim_{N \to \infty} S_{2N} = \lim_{N \to \infty} (S_{2N+1} - S_{2N}) = \lim_{N \to \infty} b_{2N+1} = 0
\]
Therefore, \( \lim_{N \to \infty} S_N = S \) and the infinite series converges to \( S \). From the inequalities in (1) we also see that \( 0 < S < S_1 = b_1 \) and \( S_p < S < S_q \) for all \( p \) even and \( q \) odd as claimed.

**Example 4**

Show that \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots \) converges conditionally. Furthermore show that if \( S \) is the sum of the series, then \( 0 \leq S \leq 1 \).

**Solution**

The terms \( b_n = \frac{1}{\sqrt{n}} \) are positive and decreasing, and \( \lim_{n \to \infty} b_n = 0 \). Therefore, the series converges by the Alternating Series Test. Furthermore, if \( S \) is the sum of the series, then \( 0 \leq S \leq 1 \) because \( b_1 = 1 \). However, the positive series \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges because it is a \( p \)-series with \( p = \frac{1}{2} < 1 \). Thus, \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \) is conditionally convergent (Figure 3).

The next corollary, which is based on the inequality \( S_p < S < S_q \) in Theorem 2, gives us important information about the error involved in using a partial sum to approximate the sum of a convergent alternating series.

**Corollary**

Let \( S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n \), where \( \{b_n\} \) is a positive decreasing sequence that converges to 0. Then

\[
|S - S_N| < b_{N+1}
\]

In other words, when we approximate \( S \) by \( S_N \), the error is less than the size of the first omitted term \( b_{N+1} \).

**Proof**

If \( N \) is even, then \( N + 1 \) is odd and Theorem 2 implies that \( S_N < S < S_{N+1} \). Also, if \( N \) is odd, then \( N + 1 \) is even and Theorem 2 implies that \( S_{N+1} < S < S_N \). In either case,

\[
|S - S_N| < |S_{N+1} - S_N| = b_{N+1}
\]

**Example 5**

**Alternating Harmonic Series**

Show that \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \) converges conditionally. If \( S \) represents the sum, then

(a) Show that \( |S - S_0| < \frac{1}{2} \).

(b) Find an \( N \) such that \( S_N \) approximates \( S \) with an error less than \( 10^{-3} \).

**Solution**

The terms \( b_n = \frac{1}{n} \) are positive and decreasing, and \( \lim_{n \to \infty} b_n = 0 \). Therefore, the series converges by the Alternating Series Test. The harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, so the series converges conditionally. Now, applying the inequality in (2), we have

\[
|S - S_N| < b_{N+1} = \frac{1}{N+1}
\]

For \( N = 6 \), we obtain \( |S - S_6| < \frac{1}{7} \). We can make the error less than \( 10^{-3} \) by choosing \( N \) so that

\[
\frac{1}{N+1} \leq 10^{-3} \quad \Rightarrow \quad N + 1 \geq 10^3 \quad \Rightarrow \quad N \geq 999
\]

Therefore, with \( N > 999 \), \( S_N \) approximates \( S \) with error less than \( 10^{-3} \).
For the series in the previous example, a computer algebra system gives $S_{999} \approx 0.6937$, and therefore, $S$ is within $10^{-3}$ of this value. In fact, it can be shown that $S = \ln 2$ (see Exercise 92 of Section 11.8). Thus, $|S - \ln 2| \approx 0.6931$, which verifies the result in the example:

$$|S - S_{999}| \approx |\ln 2 - 0.6937| \approx 0.0006 < 10^{-3}$$

**CONCEPTUAL INSIGHT** The convergence of an infinite series $\sum a_n$ depends on two factors: (1) how quickly $a_n$ tends to zero, and (2) how much cancellation takes place among the terms. Consider:

- **Harmonic series (diverges):**
  $$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

- **$p$-Series with $p = 2$ (converges):**
  $$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

- **Alternating harmonic series (converges):**
  $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

The harmonic series diverges because reciprocals $1/n$ do not tend to zero quickly enough. By contrast, the reciprocal squares $1/n^2$ tend to zero quickly enough for the $p$-series with $p = 2$ to converge. The alternating harmonic series converges, but only due to the cancellation among the terms.

### 11.4 SUMMARY

- $\sum a_n$ converges absolutely if the positive series $\sum |a_n|$ converges.
- Absolute convergence implies convergence: If $\sum |a_n|$ converges, then $\sum a_n$ also converges.
- $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.
- **Alternating Series Test:** If $\{b_n\}$ is positive and decreasing and $\lim_{n \to \infty} b_n = 0$, then the alternating series
  $$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots$$

  converges. Furthermore, if $S$ is the sum of the series, then $|S - S_n| < b_{n+1}$.

- We have developed two ways to handle nonpositive series: show absolute convergence if possible, or use the Alternating Series Test if applicable.

### 11.4 EXERCISES

#### Preliminary Questions

1. Give an example of a series such that $\sum a_n$ converges but $\sum |a_n|$ diverges.
2. Which of the following statements is equivalent to Theorem 1?
   (a) If $\sum |a_n|$ diverges, then $\sum a_n$ also diverges.
   (b) If $\sum a_n$ diverges, then $\sum |a_n|$ also diverges.
   (c) If $\sum a_n$ converges, then $\sum |a_n|$ also converges.

3. Indicate whether or not the reasoning in the following statement is correct: Since $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ is an alternating series, it must converge.

4. Suppose that $b_n$ is positive, decreasing, and tends to 0, and let $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$. What can we say about $|S - S_{100}|$ if $a_{101} = 10^{-3}$? Is $S$ larger or smaller than $S_{100}$?
Exercises

1. Show that
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \]
converges absolutely.

2. Show that the following series converges conditionally:
\[ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^{1/3}} = \frac{1}{1^{1/3}} - \frac{1}{2^{1/3}} + \frac{1}{3^{1/3}} - \frac{1}{4^{1/3}} + \cdots \]

In Exercises 3–10, determine whether the series converges absolutely, conditionally, or not at all.

3. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}} \]

4. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \]

5. \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n^{0.9999}} \]

6. \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} \]

7. \[ \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n}}{n^2} \]

8. \[ \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{4}\right)}{n^2} \]

9. \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n \ln n} \]

10. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3} + 1} \]

11. Let \( S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \).
   (a) Calculate \( S_n \) for 1 \( \leq n \leq 10 \).
   (b) Use the inequality in (2) to show that 0.8 \( \leq S \leq 0.902 \).

12. Use the inequality in (2) to approximate
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \]
to four decimal places.

13. Approximate \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \) to three decimal places.

14. CAS Let
\[ S = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \]
Use a computer algebra system to calculate and plot the partial sums \( S_n \) for 1 \( \leq n \leq 100 \). Observe that the partial sums zigzag above and below the limit.

In Exercises 15–16, find a value of \( N \) such that \( S_N \) approximates the series with an error of at most 10\(^{-3}\). Using technology, compute this value of \( S_N \).

15. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/3} + 1} \]

16. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n} \]

In Exercises 17–32, determine convergence or divergence by any method.

17. \[ \sum_{n=0}^{\infty} \frac{7^{-n}}{n!} \]

18. \[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \]

19. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 - 3n} \]

20. \[ \sum_{n=1}^{\infty} \frac{1}{n + \frac{1}{n}} \]

21. \[ \sum_{n=1}^{\infty} \frac{1}{n^3 + 12n} \]

22. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2} + 1} \]

23. \[ \sum_{n=1}^{\infty} \frac{1}{9n^2 + 1} \]

24. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2 + 1} \]

25. \[ \sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{5^n} \]

26. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)!} \]

27. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - 3n} \]

28. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \]

29. \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/3} + 1} \]

30. \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/2} + (\ln n)^2} \]

31. \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/3}} \]

32. \[ \sum_{n=1}^{\infty} \frac{1}{(\ln n)^2} \]

33. Show that
\[ \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots \]
converges by computing the partial sums. Does it converge absolutely?

34. The Alternating Series Test cannot be applied to
\[ \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots \]
Why not? Show that it converges by another method.

35. Assumptions Matter Show that the following series diverges:
\[ \frac{1}{2} - \frac{1}{3} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{n - \frac{1}{2^n}} + \cdots \]
(Note: This demonstrates that in the Alternating Series Test, we need the assumption that the sequence \( a_n \) is decreasing. It is not enough to assume only that \( a_n \) tends to zero.)

36. Determine whether the following series converges conditionally:
\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots \]

37. Prove that if \( \sum a_n \) converges absolutely, then \( \sum a_n^2 \) also converges. Give an example where \( \sum a_n \) is only conditionally convergent and \( \sum a_n^2 \) diverges.

38. Use Exercise 38 to show that the following series converges:
\[ \frac{1}{1 + \ln 2} + \frac{1}{1 + \ln 3} - \frac{1}{1 + \ln 4} + \frac{1}{1 + \ln 5} + \frac{1}{1 + \ln 6} - \frac{1}{1 + \ln 7} + \cdots \]

39. Prove the conditional convergence of
\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \]
41. Show that the following series diverges:

\[ 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \]

*Hint:* Use the result of Exercise 40 to write the series as the sum of a convergent series and a divergent series.

42. Prove that

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\ln n)^a}{n} \]

converges for all exponents \( a \). *Hint:* Show that \( f(x) = (\ln x)^a/x \) is decreasing for \( x \) sufficiently large.

43. We say that \( (b_n) \) is a rearrangement of \( (a_n) \) if \( (b_n) \) has the same terms as \( (a_n) \) but occurring in a different order. Show that if \( \sum b_n \) is a rearrangement of \( \sum a_n \) and \( \sum a_n \) converges absolutely, then \( \sum b_n \) also converges absolutely. *Hint:* Prove that the partial sums \( \sum_{n=1}^{N} b_n \) are bounded. (It can be shown further that the two series converge to the same value. This result does not hold if \( \sum a_n \) is only conditionally convergent.)

44. Assumptions Matter In 1829, Lejeune Dirichlet pointed out that the great French mathematician Augustin Louis Cauchy made a mistake in a published paper by improperly assuming the Limit Comparison Test to be valid for nonpositive series. Here are Dirichlet's two series:

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} (1 + (-1)^n)} \]

Explain how they provide a counterexample to the Limit Comparison Test when the series are not assumed to be positive.

---

### 11.5 The Ratio and Root Tests and Strategies for Choosing Tests

In the previous sections, we developed a number of theorems and tests that are used to investigate whether a series converges or diverges. In this section, we present two more tests, the Ratio Test and the Root Test. Then we outline a strategy for choosing which test to apply to determine if a specific series converges. We begin with the Ratio Test.

**Theorem I: Ratio Test**  Assume that the following limit exists:

\[ \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \]

(i) If \( \rho < 1 \), then \( \sum a_n \) converges absolutely.

(ii) If \( \rho > 1 \), then \( \sum a_n \) diverges.

(iii) If \( \rho = 1 \), the test is inconclusive.

**Proof**  The idea is to compare with a geometric series. If \( \rho < 1 \), we may choose a number \( r \) such that \( 0 < r < 1 \). Since \( |a_{n+1}/a_n| \) converges to \( \rho \), there exists a number \( M \) such that \( |a_{n+1}/a_n| < r \) for all \( n \geq M \). Therefore,

\[
|a_{M+1}| < r |a_M| \\
|a_{M+2}| < r |a_{M+1}| < r^2 |a_M| \\
|a_{M+3}| < r |a_{M+2}| < r^3 |a_M|
\]

In general, \( |a_{M+n}| < r^n |a_M| \), and thus,

\[
\sum_{n=M}^{\infty} |a_n| = \sum_{n=0}^{\infty} |a_{M+n}| \leq \sum_{n=0}^{\infty} |a_M| r^n = |a_M| \sum_{n=0}^{\infty} r^n
\]

The geometric series on the right converges because \( 0 < r < 1 \), so \( \sum_{n=M}^{\infty} |a_n| \) converges by the Direct Comparison Test. Thus \( \sum a_n \) converges absolutely.

If \( \rho > 1 \), choose \( r \) such that \( 1 < r < \rho \). Then there exists a number \( M \) such that \( |a_{n+1}/a_n| > r \) for all \( n \geq M \). Arguing as before with the inequalities reversed, we find that \( |a_{M+n}| \geq r^n |a_M| \). Since \( r^n \) tends to \( 0 \), the terms \( a_{M+n} \) do not tend to zero, and consequently, \( \sum a_n \) diverges. Finally, Example 4 in this section shows that both convergence and divergence are possible when \( \rho = 1 \), so the test is inconclusive in this case.
EXAMPLE 1 Prove that $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

Solution Compute the ratio and its limit with $a_n = \frac{2^n}{n!}$. Note that $(n+1)! = (n+1)n!$.
Thus,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n(n+1)} = \frac{2}{n+1}$$

We obtain

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2}{n+1} = 0$$

Since $\rho < 1$, the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges by the Ratio Test.

EXAMPLE 2 Does $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge?

Solution Apply the Ratio Test with $a_n = \frac{n^2}{2^n}$:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \left( \frac{n^2 + 2n + 1}{n^2} \right) = \frac{1}{2} \left( 1 + \frac{2}{n} + \frac{1}{n^2} \right)$$

We obtain

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2} \lim_{n \to \infty} \left( 1 + \frac{2}{n} + \frac{1}{n^2} \right) = \frac{1}{2}$$

Since $\rho < 1$, the series converges by the Ratio Test.

EXAMPLE 3 Does $\sum_{n=0}^{\infty} (-1)^n \frac{n!}{1000^n}$ converge?

Solution This series diverges by the Ratio Test because $\rho > 1$:

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{n!} = \lim_{n \to \infty} \frac{n + 1}{1000} = \infty$$

In the next example, we demonstrate why the Ratio Test is inconclusive in the case where $\rho = 1$.

EXAMPLE 4 Ratio Test Inconclusive Show that both convergence and divergence are possible when $\rho = 1$ by considering $\sum_{n=1}^{\infty} n^2$ and $\sum_{n=1}^{\infty} n^{-2}$.

Solution For $a_n = n^2$, we have

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} = \lim_{n \to \infty} \frac{n^2 + 2n + 1}{n^2} = \lim_{n \to \infty} \left( 1 + \frac{2}{n} + \frac{1}{n^2} \right) = 1$$

Furthermore, for $b_n = n^{-2}$,

$$\rho = \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{\lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right)} = 1$$
Thus, $p = 1$ in both cases, but, in fact, $\sum_{n=1}^{\infty} n^2$ diverges by the nth Term Divergence Test since $\lim_{n \to \infty} n^2 = \infty$, and $\sum_{n=1}^{\infty} n^{-2}$ converges since it is a $p$-series with $p = 2 > 1$. This shows that both convergence and divergence are possible when $p = 1$.

Our next test is based on the limit of the $n$th roots $\sqrt[n]{|a_n|}$ rather than the ratios $a_{n+1}/a_n$. Its proof, like that of the Ratio Test, is based on a comparison with a geometric series (see Exercise 63).

**Theorem 2** Root Test  Assume that the following limit exists:

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

(i) If $L < 1$, then $\sum a_n$ converges absolutely.

(ii) If $L > 1$, then $\sum a_n$ diverges.

(iii) If $L = 1$, the test is inconclusive.

**Example 5** Does $\sum_{n=1}^{\infty} \left( \frac{n}{2n+3} \right)^n$ converge?

**Solution** We have $L = \lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{n}{2n+3} = \frac{1}{2}$. Since $L < 1$, the series converges by the Root Test.

---

**Determining Which Test to Apply**

We end this section with a brief review of all of the tests we have introduced for determining convergence so far and how one decides which test to apply.

Let $\sum a_n$ be given. Keep in mind that the series for which convergence or divergence is known include the geometric series $\sum_{n=0}^{\infty} ar^n$, which converge for $|r| < 1$, and the $p$-series $\sum_{n=0}^{\infty} \frac{1}{n^p}$, which converge for $p > 1$.

1. **The nth Term Divergence Test** Always check this test first. If $\lim_{n \to \infty} a_n \neq 0$, then the series diverges. But if $\lim_{n \to \infty} a_n = 0$, we do not know whether the series converges or diverges, and hence we move on to the next step.

2. **Positive Series** If all terms in the series are positive, try one of the following tests:

   (a) **The Direct Comparison Test** Consider whether dropping terms in the numerator or denominator gives a series that we know either converges or diverges. If a larger series converges or a smaller series diverges, then the original series does the same. For example, $\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n}}$ converges because $\frac{1}{n^2 + \sqrt{n}} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (since it is a $p$-series with $p = 2 > 1$). On the other hand, this does not work for $\sum_{n=1}^{\infty} \frac{1}{n^2 - \sqrt{n}}$ since then the comparison series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, while still converging, is
smaller than the original series, so we cannot compare the series with \( \sum \frac{1}{n^2} \) and apply the Direct Comparison Test. In this case, we can often apply the Limit Comparison Test as follows.

(b) The Limit Comparison Test Consider the dominant term in the numerator and denominator, and compare the original series to the ratio of those terms. For example, for \( \sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}} \), \( n^2 \) is dominant over \( \sqrt{n} \) as it grows faster as \( n \) increases. So, we let \( b_n = \frac{1}{n^2} \). Then

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{1}{n^2 - \sqrt{n}}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2}{n^2 - \sqrt{n}} = 1
\]

The limit is a positive number, so the Limit Comparison Test applies. Since \( \sum \frac{1}{n^2} \) converges, so does the original series.

(c) The Ratio Test The Ratio Test is often effective in the presence of a factorial such as \( n! \) since in the ratio, the factorial disappears after cancellation. It is also effective when there are constants to the power \( n \), such as \( 2^n \), since in the ratio, the power \( n \) disappears after cancellation. For example, if the series is \( \sum_{n=1}^{\infty} \frac{3^n}{n!} \), then applying the Ratio Test yields

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{3}{n+1} = 0 < 1
\]

Therefore, the series converges.

(d) The Root Test The Root Test is often effective when there is a term of the form \( f(n)^{g(n)} \). For example, \( \sum_{n=1}^{\infty} \frac{2^n}{n^{2n}} \) is a good example since applying the Root Test yields

\[
\lim_{n \to \infty} \left| a_n \right|^{1/n} = \lim_{n \to \infty} \left( \frac{2^n}{n^{2n}} \right)^{1/n} = \lim_{n \to \infty} \frac{2}{n^2} = 0 < 1
\]

Thus, the series converges.

(e) The Integral Test When the other tests fail on a positive series, consider the Integral Test. If \( a_n = f(n) \) is a decreasing function, then the series converges if and only if the improper integral \( \int_{1}^{\infty} f(x) \, dx \) converges. For example, the other tests do not easily apply to \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \). However, \( f(x) = \frac{1}{x \ln x} \) is a decreasing function and

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \ln(\ln x) \bigg|_{2}^{\infty} = \infty.
\]

Thus, the integral diverges, implying that the series does as well.

3. Series That Are Not Positive Series

(a) Alternating Series Test If the series is alternating of the form \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n \), show that \( 0 < b_{n+1} < b_n \) and \( \lim_{n \to \infty} b_n = 0 \). Then the Alternating Series Test shows the series converges.

(b) Absolute Convergence If the series \( \sum a_n \) is not alternating, then see if \( \sum |a_n| \), which is a positive series, converges using the tests for positive series. If so, the original series is absolutely convergent and therefore convergent.
11.5 SUMMARY

- Ratio Test: Assume that \( \rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \) exists.
  - Then \( \sum a_n \) converges absolutely if \( \rho < 1 \).
  - Then \( \sum a_n \) diverges if \( \rho > 1 \).
  - The test is inconclusive if \( \rho = 1 \).

- Root Test: Assume that \( L = \lim_{n \to \infty} \sqrt[n]{|a_n|} \) exists.
  - Then \( \sum a_n \) converges absolutely if \( L < 1 \).
  - Then \( \sum a_n \) diverges if \( L > 1 \).
  - The test is inconclusive if \( L = 1 \).

11.5 EXERCISES

Preliminary Questions
1. Consider the geometric series \( \sum_{n=0}^{\infty} c r^n \).
   (a) In the Ratio Test, what do the terms \( |\frac{a_{n+1}}{a_n}| \) equal?
   (b) In the Root Test, what do the terms \( \sqrt[n]{|a_n|} \) equal?

2. Consider the p-series \( \sum_{n=1}^{\infty} n^{-p} \).
   (a) In the Ratio Test, what do the terms \( \frac{a_{n+1}}{a_n} \) equal?
   (b) What can be concluded from the Ratio Test?

Exercises

In Exercises 1–20, apply the Ratio Test to determine convergence or divergence, or state that the Ratio Test is inconclusive.

1. \( \sum_{n=1}^{\infty} \frac{1}{5^n} \)
2. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{5^n} \)
3. \( \sum_{n=1}^{\infty} \frac{3n + 2}{5n^2 + 1} \)
4. \( \sum_{n=1}^{\infty} \frac{n^{3/2}}{n^2 + 1} \)
5. \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \)
6. \( \sum_{n=1}^{\infty} \frac{3n^2 + 1}{n^3} \)
7. \( \sum_{n=1}^{\infty} \frac{2n^3 + 1}{100} \)
8. \( \sum_{n=1}^{\infty} \frac{n}{3^n} \)
9. \( \sum_{n=1}^{\infty} \frac{10^n}{2^n} \)
10. \( \sum_{n=1}^{\infty} \frac{e^n}{n!} \)
11. \( \sum_{n=1}^{\infty} \frac{n^{1/2}}{10^n} \)
12. \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \)
13. \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \)
14. \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \)
15. \( \sum_{n=1}^{\infty} \frac{1}{n^{1/3} \ln n} \)
16. \( \sum_{n=1}^{\infty} \frac{1}{(2n)!} \)
17. \( \sum_{n=1}^{\infty} \frac{n^2}{(2n + 1)!} \)
18. \( \sum_{n=1}^{\infty} \frac{(n+1)^2}{(3n)!} \)
19. \( \sum_{n=1}^{\infty} \frac{1}{2^n + 1} \)
20. \( \sum_{n=1}^{\infty} \frac{1}{n!} \)
21. Show that \( \sum_{n=1}^{\infty} n^k 3^{-n} \) converges for all exponents \( k \).
22. Show that \( \sum_{n=1}^{\infty} n^k x^n \) converges if \( |x| < 1 \).
23. Show that \( \sum_{n=1}^{\infty} 2^n n^k \) converges if \( |x| < \frac{1}{2} \).
24. Show that \( \sum_{n=1}^{\infty} \frac{n}{n!} \) converges for all \( r \).
25. Show that \( \sum_{n=1}^{\infty} \frac{n}{n!} \) converges if \( |r| < 1 \).
26. Is there any value of \( k \) such that \( \sum_{n=1}^{\infty} \frac{n^k}{n!} \) converges?
In Exercises 27–28, the following limit could be helpful:
\[ \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e. \]

27. Does \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \) converge or diverge?

28. Does \( \sum_{n=1}^{\infty} \frac{(2n)!}{n^n} \) converge or diverge?

In Exercises 29–33, assume that \( |a_{n+1}/a_n| \) converges to \( \rho = \frac{1}{2} \). What can you say about the convergence of the given series?

29. \( \sum_{n=1}^{\infty} \frac{n^3}{3^n} \)

30. \( \sum_{n=1}^{\infty} \frac{2^n}{n^n} \)

31. \( \sum_{n=1}^{\infty} \frac{3^n}{n^n} \)

32. \( \sum_{n=1}^{\infty} \frac{4^n}{n^n} \)

33. \( \sum_{n=1}^{\infty} a_n^2 \)

34. Assume that \( |a_{n+1}/a_n| \) converges to \( \rho = 4 \). Does \( \sum_{n=1}^{\infty} a_n^{-1} \) converge (assume that \( a_n \neq 0 \) for all \( n \))?

35. Show that the Root Test is inconclusive for the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \).

In Exercises 36–41, use the Root Test to determine convergence or divergence (or state that the test is inconclusive).

36. \( \sum_{k=1}^{\infty} \frac{1}{k+10} \)

37. \( \sum_{n=1}^{\infty} \frac{1}{n^n} \)

38. \( \sum_{k=0}^{\infty} \left( \frac{k}{3k+1} \right)^k \)

39. \( \sum_{k=0}^{\infty} \left( \frac{k}{3k+1} \right)^k \)

40. \( \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^{-n} \)

41. \( \sum_{n=1}^{\infty} \frac{1}{n^n} \)

42. Prove that \( \sum_{n=1}^{\infty} \frac{n^2}{n!} \) diverges. Hint: Use \( 2n^3 = (2n)^3n! = n^n \).

In Exercises 43–62, determine convergence or divergence using any method covered in the text so far.

43. \( \sum_{n=1}^{\infty} \frac{2^n + 4^n}{7^n} \)

44. \( \sum_{n=1}^{\infty} \frac{n^3}{n!} \)

45. \( \sum_{n=1}^{\infty} \frac{n}{2^n + 1} \)

46. \( \sum_{n=1}^{\infty} \frac{n+1}{2n!} \)

47. \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \)

48. \( \sum_{n=1}^{\infty} \frac{n^2}{(2n)^n} \)

49. \( \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}} \)

50. \( \sum_{n=1}^{\infty} \frac{1}{n(n\ln n)^3} \)

51. \( \sum_{n=1}^{\infty} \frac{n^3}{n^2} \)

52. \( \sum_{n=1}^{\infty} \frac{n^2}{(n\ln n)^3} \)

53. \( \sum_{n=1}^{\infty} \frac{n^2}{n^2 + n} \)

54. \( \sum_{n=1}^{\infty} \frac{n^2 + 4n}{3n^3 + 9} \)

55. \( \sum_{n=1}^{\infty} \frac{n^{0.8}}{n^{0.8}} \)

56. \( \sum_{n=1}^{\infty} \frac{0.8^{0.3} - n^{-0.8}}{n} \)

57. \( \sum_{n=1}^{\infty} \frac{4 - 2n + 1}{n+1} \)

58. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \)

59. \( \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^n \)

60. \( \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^n \cos \frac{1}{n} \)

61. \( \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} \)

62. \( \sum_{n=1}^{\infty} \left( \frac{n}{n+12} \right)^n \)

Further Insights and Challenges

63. \( \square \) Proof of the Root Test Let \( \sum a_n \) be a positive series, and assume that \( L = \lim_{n \to \infty} \sqrt[n]{a_n} \) exists.

(a) Show that the series converges if \( L < 1 \). Hint: Choose \( R \) with \( L < R < 1 \) and show that \( a_n \leq R^n \) for \( n \) sufficiently large. Then compare with the geometric series \( \sum R^n \).

(b) Show that the series diverges if \( L > 1 \).

(c) Show that the Ratio Test does not apply, but verify convergence using the Direct Comparison Test for the series

\[ \frac{1}{2} + \frac{1}{32} + \frac{1}{256} + \frac{1}{16384} + \cdots \]

64. \( \square \) Proof of the Limit Comparison Test Let \( \sum \frac{a_n}{n} \) be a positive series, and assume that \( \lim_{n \to \infty} \frac{a_n}{b_n} = \ell \). Then \( \sum a_n \) converges absolutely if \( \sum b_n \) converges and diverges if \( \sum b_n \) diverges.

65. Let \( \sum \frac{c^n n!}{n^n} \), where \( c \) is a constant.

(a) Show that the series converges absolutely if \( |c| < e \) and diverges if \( |c| > e \).

(b) Show that the series converges if \( |c| = e \).

(c) Use the Limit Comparison Test to prove the series diverges for \( c = e \).

11.6 Power Series

With series we can make sense of the idea of a polynomial of infinite degree:

\[ F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]

Specifically, a power series with center \( c \) is an infinite series

\[ F(x) = \sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + a_3 (x - c)^3 + \cdots \]
where $x$ is a variable. For example,

$$F(x) = 1 - x + x^2 - x^3 + \cdots$$
$$G(x) = 1 + (x - 2) + 2(x - 2)^2 + 3(x - 2)^3 + \cdots$$

are power series where $F(x)$ has center $c = 0$ and $G(x)$ has center $c = 2$.

A power series $F(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$ converges for some values of $x$ and may diverge for others. For example, if we set $x = \frac{9}{4}$ in the power series of Eq. (1), we obtain the infinite series

$$G \left( \frac{9}{4} \right) = 1 + \left( \frac{9}{4} - 2 \right) + 2 \left( \frac{9}{4} - 2 \right)^2 + 3 \left( \frac{9}{4} - 2 \right)^3 + \cdots$$
$$= 1 + \left( \frac{1}{4} \right) + 2 \left( \frac{1}{4} \right)^2 + 3 \left( \frac{1}{4} \right)^3 + \cdots + n \left( \frac{1}{4} \right)^n + \cdots$$

This converges by the Ratio Test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{4} \left( \frac{n+1}{n} \right)^{1/n} = \lim_{n \to \infty} \frac{1}{4} \left( 1 + \frac{1}{n} \right) = \frac{1}{4}$$

On the other hand, the power series in Eq. (1) diverges for $x = 3$ by the $n$th Term Divergence Test:

$$G(3) = 1 + (3 - 2) + 2(3 - 2)^2 + 3(3 - 2)^3 + \cdots$$
$$= 1 + 1 + 2 + 3 + \cdots$$

There is a surprisingly simple way to describe the set of values $x$ at which a power series $F(x)$ converges. According to our next theorem, either $F(x)$ converges absolutely for all values of $x$ or there is a radius of convergence $R$ such that

$$F(x) \text{ converges absolutely when } |x - c| < R \text{ and diverges when } |x - c| > R.$$  

This means that $F(x)$ converges for $x$ in an interval of convergence consisting of the open interval $(c - R, c + R)$ and possibly one or both of the endpoints $c - R$ and $c + R$ (Figure 1). Note that $F(x)$ automatically converges at $x = c$ because

$$F(c) = a_0 + a_1(c - c) + a_2(c - c)^2 + a_3(c - c)^3 + \cdots = a_0$$

We set $R = 0$ if $F(x)$ converges only for $x = c$, and we set $R = \infty$ if $F(x)$ converges for all values of $x$.

**THEOREM 1 Radius of Convergence**

Every power series

$$F(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

has a radius of convergence $R$, which is either a nonnegative number $(R \geq 0)$ or infinity $(R = \infty)$. If $R$ is finite, $F(x)$ converges absolutely when $|x - c| < R$ and diverges when $|x - c| > R$. If $R = \infty$, then $F(x)$ converges absolutely for all $x$. 

**FIGURE 1** Interval of convergence of a power series.
Proof. We assume that \( c = 0 \) to simplify the notation. If \( F(x) \) converges, then \( R = 0 \). Otherwise, \( F(x) \) converges for some nonzero value \( x = B \). We claim that \( F(x) \) must then converge absolutely for all \(|x| < |B|\). To prove this, note that because \( F(B) = \sum_{n=0}^{\infty} a_n B^n \) converges, the general term \( a_n B^n \) tends to zero. In particular, there exists \( M > 0 \) such that \( |a_n B^n| < M \) for all \( n \). Therefore,

\[
\sum_{n=0}^{\infty} |a_n x^n| = \sum_{n=0}^{\infty} |a_n B^n| \left| \frac{x}{B} \right|^n < M \sum_{n=0}^{\infty} \left| \frac{x}{B} \right|^n
\]

If \(|x| < |B|\), then \(|x/B| < 1\) and the series on the right is a convergent geometric series. By the Direct Comparison Test, the series on the left also converges. This proves that \( F(x) \) converges absolutely if \(|x| < |B|\).

Now let \( S \) be the set of numbers \( x \) such that \( F(x) \) converges. Then \( S \) contains 0, and we have shown that if \( S \) contains a number \( B \neq 0 \), then \( S \) contains the open interval \((-B, B))\). If \( S \) is bounded, then \( S \) has a least upper bound \( L > 0 \) (see marginal note). In this case, there exist numbers \( B \in S \) smaller than but arbitrarily close to \( L \), and thus, \( S \) contains \((-B, B)\) for all \( 0 < B < L \). It follows that \( S \) contains the open interval \((-L, L)\). The set \( S \) cannot contain any number \( x \) with \(|x| > L \), but \( S \) may contain one or both of the endpoints \( x = \pm L \). So in this case, \( F \) has radius of convergence \( R = L \). If \( S \) is not bounded, then \( S \) contains intervals \((-B, B)\) for \( B \) arbitrarily large. In this case, \( S \) is the entire real line \( R \), and the radius of convergence is \( R = \infty \).

From Theorem 1, we see that there are two steps in determining the interval of convergence of \( F \):

1. Find the radius of convergence \( R \) (using the Ratio Test, in most cases).
2. Check convergence at the endpoints (if \( R \neq 0 \) or \( \infty \)).

**Example 1. Using the Ratio Test** Where does \( F(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n} \) converge?

**Solution.**

**Step 1.** Find the radius of convergence.

Let \( a_n = \frac{x^n}{2^n} \) and compute \( \rho \) from the Ratio Test:

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n \to \infty} \frac{1}{2} |x| = \frac{1}{2} |x|
\]

We find that

\[
\rho < 1 \quad \text{if} \quad \frac{1}{2} |x| < 1, \quad \text{that is, if} \quad |x| < 2
\]

Thus, \( F(x) \) converges if \(|x| < 2 \). Similarly, \( \rho > 1 \) if \( \frac{1}{2} |x| > 1 \), or \(|x| > 2 \). So, \( F(x) \) diverges if \(|x| > 2 \). Therefore, the radius of convergence is \( R = 2 \).

**Step 2. Check the endpoints.**

The Ratio Test is inconclusive for \( x = \pm 2 \), so we must check these cases directly:

\[
F(2) = \sum_{n=0}^{\infty} 2^n = 1 + 1 + 1 + 1 + 1 + \cdots
\]

\[
F(-2) = \sum_{n=0}^{\infty} (-2)^n = 1 - 1 + 1 - 1 + 1 - \cdots
\]

Both series diverge. We conclude that \( F(x) \) converges only for \(|x| < 2 \) (Figure 2).
EXAMPLE 2  Where does \( F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} (x - 5)^n \) converge?

Solution  We compute \( \rho \) with \( a_n = \frac{(-1)^n}{4^n} (x - 5)^n \):

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x - 5|^{n+1}}{4^{n+1}(n + 1)} \frac{4^n}{|x - 5|^n} = |x - 5| \lim_{n \to \infty} \frac{n}{4(n + 1)} = \frac{1}{4} |x - 5|
\]

We find that \( \rho < 1 \) if \( \frac{1}{4} |x - 5| < 1 \), that is, if \( |x - 5| < 4 \)

Thus, \( F(x) \) converges absolutely on the open interval \((1, 9)\) of radius 4 with center \( c = 5 \).

In other words, the radius of convergence is \( R = 4 \). Next, we check the endpoints:

\[
x = 9: \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} (9 - 5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges (Alternating Series Test)}
\]

\[
x = 1: \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} (4)^n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (harmonic series)}
\]

We conclude that \( F(x) \) converges for \( x \) in the half-open interval \((1, 9)\) shown in Figure 3.

Some power series contain only even powers or only odd powers of \( x \). The Ratio Test can still be used to find the radius of convergence.

EXAMPLE 3  An Even Power Series  Where does \( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \) converge?

Solution  Although this power series has only even powers of \( x \), we can still apply the Ratio Test with \( a_n = x^{2n}/(2n)! \). We have

\[
a_{n+1} = \frac{x^{2(n+1)}}{(2(n+1))!} = \frac{x^{2n+2}}{(2n+2)!}
\]

Furthermore, \((2n + 2)! = (2n + 2)(2n + 1)(2n)!\), so

\[
\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{x^{2n+2}}{(2n + 2)!} \frac{(2n)!}{x^{2n}} = |x|^2 \lim_{n \to \infty} \frac{1}{(2n + 2)(2n + 1)} = 0
\]

Thus, \( \rho = 0 \) for all \( x \), and \( F(x) \) converges for all \( x \). The radius of convergence is \( R = \infty \).

Geometric series are important examples of power series. Recall the formula \( \sum_{n=0}^{\infty} r^n = 1/(1 - r) \), valid for \( |r| < 1 \). Writing \( x \) in place of \( r \), we obtain a power series expansion with radius of convergence \( R = 1 \):

\[
\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \ |x| < 1
\]

The next two examples show that we can modify this formula to find the power series expansions of other functions.
EXAMPLE 4 Geometric Series  Prove that
\[ \frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n \quad \text{for } |x| < \frac{1}{2} \]

Solution  Substitute 2x for x in Eq. (2):
\[ \frac{1}{1 - 2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n \]

Expansion (2) is valid for |x| < 1, so Eq. (3) is valid for |2x| < 1, or |x| < \frac{1}{2}.

EXAMPLE 5  Find a power series expansion with center c = 0 for
\[ f(x) = \frac{1}{2 + x^2} \]
and find the interval of convergence.

Solution  We need to rewrite f(x) so we can use Eq. (2). We have
\[ \frac{1}{2 + x^2} = \frac{1}{2} \left( \frac{1}{1 + \frac{1}{2}x^2} \right) = \frac{1}{2} \left( \frac{1}{1 - \left(-\frac{1}{2}x^2\right)} \right) = \frac{1}{2} \left( \frac{1}{1 - u} \right) \]
where \( u = -\frac{1}{2}x^2 \). Now substitute \( u = -\frac{1}{2}x^2 \) for x in Eq. (2) to obtain
\[ f(x) = \frac{1}{2 + x^2} = \frac{1}{2} \sum_{n=0}^{\infty} \left( -\frac{x^2}{2} \right)^n \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{n+1}} \]

This expansion is valid if \(|-x^2/2| < 1\), or \(|x| < \sqrt{2}\). The interval of convergence is \((-\sqrt{2}, \sqrt{2})\).

Our next theorem tells us that within the interval of convergence, we can treat a power series as though it were a polynomial; that is, we can differentiate and integrate term by term.

THEOREM 2 Term-by-Term Differentiation and Integration  Assume that
\[ F(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \]
has radius of convergence \( R > 0 \). Then \( F \) is differentiable on \((c - R, c + R)\). Furthermore, we can integrate and differentiate term by term. For \( x \in (c - R, c + R)\),
\[ F'(x) = \sum_{n=1}^{\infty} na_n (x - c)^{n-1} \]
\[ \int F(x) \, dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - c)^{n+1} \quad (A \text{ any constant}) \]

For both the derivative series and the integral series the radius of convergence is also \( R \).

Theorem 2 is a powerful tool for working with power series. The next two examples show how to use differentiation or antiderivative of power series representations of functions to obtain power series for other functions.
EXAMPLE 6  Differentiating a Power Series  Prove that for \(-1 < x < 1\),
\[
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots
\]

Solution  First, note that
\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \left( \frac{1}{1-x} \right)
\]

For \( \frac{1}{1-x} \), we have the following geometric series with radius of convergence \( R = 1 \):
\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots
\]

By Theorem 2, we can differentiate term by term for \( |x| < 1 \) to obtain
\[
\frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} (1 + x + x^2 + x^3 + x^4 + \cdots)
\]
\[
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots
\]

EXAMPLE 7  Power Series for Arctangent  Prove that for \(-1 < x < 1\),
\[
\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]

Solution  Recall that \( \tan^{-1} x \) is an antiderivative of \((1 + x^2)^{-1}\). We obtain a power series expansion of \((1 + x^2)^{-1}\) by substituting \(-x^2\) for \(x\) in the geometric series of Eq. (2):
\[
\frac{1}{1-x^2} = 1 - x^2 + x^4 - x^6 + \cdots
\]

This expansion is valid for \( |x^2| < 1 \)—that is, for \( |x| < 1 \). By Theorem 2, we can integrate this series term by term. The resulting expansion is also valid for \( |x| < 1 \):
\[
\tan^{-1} x = \int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + \cdots) \, dx
\]
\[
= A + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]

Setting \( x = 0 \), we obtain \( A = \tan^{-1} 0 = 0 \). Thus, Eq. (4) is valid for \(-1 < x < 1\).

**GRAPHICAL INSIGHT**  Let's examine the expansion of the previous example graphically. The partial sums of the power series for \( f(x) = \tan^{-1} x \) are
\[
S_N(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^N \frac{x^{2N-1}}{2N-1}
\]

For large \( N \), we can expect \( S_N(x) \) to provide a good approximation to \( f(x) = \tan^{-1} x \) on the interval \((-1, 1)\), where the power series expansion is valid. Figure 4 confirms this expectation: The graphs of \( y = S_0(x) \) and \( y = S_1(x) \) are nearly indistinguishable from the graph of \( y = \tan^{-1} x \) on \((-1, 1)\). Thus, we may use the partial sums to approximate the arctangent. For example, using \( S_3(x) \) to approximate \( \tan^{-1} x \), we obtain \( \tan^{-1}(0.3) \) is approximated by
\[
S_3(0.3) = 0.3 - \frac{(0.3)^3}{3} + \frac{(0.3)^5}{5} - \frac{(0.3)^7}{7} + \frac{(0.3)^9}{9} \approx 0.2914569
\]
Since the power series is an alternating series, the error in this approximation is less than the first omitted term, the term with \((0.3)^{11}\). Therefore,
\[
\text{error} = |\tan^{-1}(0.3) - S_4(0.3)| < \frac{(0.3)^{11}}{11} \approx 1.61 \times 10^{-7}
\]
Approximating \(\tan^{-1}x\) with a partial sum \(S_N(x)\) works well in the region \(|x| < 1\). For \(|x| > 1\), the situation changes drastically since the power series diverges and the partial sums deviate sharply from \(\tan^{-1}x\).

### Power Series Solutions of Differential Equations

Power series are a basic tool in the study of differential equations. To illustrate, consider the differential equation with initial condition
\[
y' = y, \quad y(0) = 1
\]
From Example 5 in Section 10.1, it follows that \(f(x) = e^x\) is a solution to this Initial Value Problem. Here, we take a different approach and find a solution in the form of a power series, \(F(x) = \sum_{n=0}^{\infty} a_n x^n\). Ultimately, this approach will provide us with a power series representation of \(F(x) = e^x\). We have
\[
F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots
\]
\[
F'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots
\]
To satisfy the differential equation, we must have \(F'(x) = F(x)\), and therefore,
\[
a_0 = a_1, \quad a_1 = 2a_2, \quad a_2 = 3a_3, \quad a_3 = 4a_4, \quad \ldots
\]
In other words, \(F'(x) = F(x)\) if \(a_{n-1} = \frac{n}{n} a_n\), or
\[
a_n = \frac{a_{n-1}}{n}
\]
An equation of this type is called a recursion relation. It enables us to determine all of the coefficients \(a_n\) successively from the first coefficient \(a_0\), which may be chosen arbitrarily. For example,
\[
n = 1: \quad a_1 = \frac{a_0}{1}
\]
\[
n = 2: \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2 \cdot 1} = \frac{a_0}{2!}
\]
\[
n = 3: \quad a_3 = \frac{a_2}{3} = \frac{a_1}{3 \cdot 2} = \frac{a_0}{3 \cdot 2 \cdot 1} = \frac{a_0}{3!}
\]
To obtain a general formula for \(a_n\), apply the recursion relation \(n\) times:
\[
a_n = \frac{a_{n-1}}{n} = \frac{a_{n-2}}{n(n-1)} = \frac{a_{n-3}}{n(n-1)(n-2)} = \cdots = \frac{a_0}{n!}
\]
We conclude that
\[
F(x) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]
In Example 3, we showed that this power series has radius of convergence \( R = \infty \), so \( y = F(x) \) satisfies \( y = y \) for all \( x \). Moreover, \( F(0) = a_0 \), so the initial condition \( y(0) = 1 \) is satisfied with \( a_0 = 1 \). Therefore,

\[
F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

is the solution to the Initial Value Problem.

In Section 10.1, we showed that \( f(x) = e^x \) is not just a solution to the Initial Value Problem in (5), but the only solution. The uniqueness of the solution implies that \( e^x \) and the power series solution we obtained must be equal. Thus, we have found a power series representation for \( e^x \) that is valid for all \( x \):

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\]

In Section 11.8, we will see how to arrive at this power series representation of \( e^x \) via what are known as Taylor series.

In contrast to \( y' = y \), the differential equation in the next example cannot be solved using any method that is simpler than the process of finding a power series solution. As with the solution of \( y' = y \), the process involves solving a recursion relation that determines the coefficients \( a_n \) of a power series for the solution.

**EXAMPLE 8** Find a power series solution to the Initial Value Problem:

\[
x^2 y'' + xy' + (x^2 - 1)y = 0, \quad y'(0) = 1
\]

**Solution** Assume that Eq. (6) has a power series solution \( F(x) = \sum_{n=0}^{\infty} a_n x^n \). Then

\[
y' = F'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots
\]

\[
y'' = F''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots
\]

Now, substitute the series for \( y, y', \) and \( y'' \) into the differential equation (6) to determine the recursion relation satisfied by the coefficients \( a_n \):

\[
x^2 y'' + xy' + (x^2 - 1)y = x^2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + (x^2 - 1) \sum_{n=0}^{\infty} a_n x^n
\]

\[
= \sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2}
\]

\[
= \sum_{n=0}^{\infty} (n^2 - 1) a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0
\]

The differential equation is satisfied if

\[
\sum_{n=0}^{\infty} (n^2 - 1) a_n x^n = -\sum_{n=2}^{\infty} a_{n-2} x^n
\]
The first few terms on each side of this equation are
\[-a_0 + 0 \cdot x + 3a_2x^2 + 8a_3x^3 + 15a_4x^4 + \cdots = 0 + 0 \cdot x - a_0x^2 - a_1x^3 - a_2x^4 - \cdots\]
Matching up the coefficients of \(x^n\), we find that
\[-a_0 = 0, \quad 3a_2 = -a_0, \quad 8a_3 = -a_1, \quad 15a_4 = -a_2\]
In general, \((n^2 - 1)a_n = -a_{n-2}\), and this yields the recursion relation
\[a_n = -\frac{a_{n-2}}{n^2 - 1} \quad \text{for } n \geq 2\]
Note that \(a_0 = 0\) by Eq. (8). The recursion relation forces all of the even coefficients \(a_2, a_4, a_6, \ldots\) to be zero:
\[a_2 = \frac{a_0}{2^2 - 1} = 0\, \text{so} \, a_2 = 0,\]
and then
\[a_4 = \frac{a_2}{4^2 - 1} = 0\, \text{so} \, a_4 = 0,\]
and so on
As for the odd coefficients, \(a_1\) may be chosen arbitrarily. Because \(F'(0) = a_1\), we set \(a_1 = 1\) to obtain a solution \(y = F(x)\) satisfying \(F'(0) = 1\). Now, apply Eq. (9):
\[n = 3:\quad a_3 = -\frac{a_1}{3^2 - 1} = -\frac{1}{3^2 - 1}\]
\[n = 5:\quad a_5 = -\frac{a_3}{5^2 - 1} = \frac{1}{(5^2 - 1)(3^2 - 1)}\]
\[n = 7:\quad a_7 = -\frac{a_5}{7^2 - 1} = -\frac{1}{(7^2 - 1)(5^2 - 1)(3^2 - 1)}\]
This shows the general pattern of coefficients. To express the coefficients in a compact form, let \(n = 2k + 1\). Then the denominator in the recursion relation (9) can be written
\[n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 4k(k + 1)\]
and
\[a_{2k+1} = -\frac{a_{2k-1}}{4k(k + 1)}\]
Applying this recursion relation \(k\) times, we obtain the closed formula
\[a_{2k+1} = (-1)^k \left( \frac{1}{4k(k + 1)} \right) \left( \frac{1}{4(k - 1)(k + 1)} \right) \cdots \left( \frac{1}{4(1)(2)} \right) = \frac{(-1)^k}{4^k \cdot k! (k + 1)!}\]
Thus, we obtain a power series representation of our solution:
\[F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k \cdot k! (k + 1)!} x^{2k+1}\]
A straightforward application of the Ratio Test shows that \(F\) has an infinite radius of convergence. Therefore, \(F(x)\) is a solution of the Initial Value Problem for all \(x\).

11.6 SUMMARY

- A power series is an infinite series of the form
\[F(x) = \sum_{n=0}^{\infty} a_n (x - c)^n\]
The constant \(c\) is called the center of \(F(x)\).
Every power series \( F(x) \) has a radius of convergence \( R \) (Figure 5) such that

- \( F(x) \) converges absolutely for \( |x - c| < R \) and diverges for \( |x - c| > R \).
- \( F(x) \) may converge or diverge at the endpoints \( c - R \) and \( c + R \).

We set \( R = 0 \) if \( F(x) \) converges only for \( x = c \) and \( R = \infty \) if \( F(x) \) converges for all \( x \).

- The interval of convergence of \( F \) consists of the open interval \((c - R, c + R)\) and possibly one or both endpoints \( c - R \) and \( c + R \).
- In many cases, the Ratio Test can be used to find the radius of convergence \( R \). It is necessary to check convergence at the endpoints separately.
- If \( R > 0 \), then \( F \) is differentiable and has antiderivatives on \((c - R, c + R)\). The derivative and antiderivatives can be obtained by directly differentiating and antiderivating, respectively, the power series for \( F' \):

\[
F'(x) = \sum_{n=1}^{\infty} an(x - c)^{n-1}, \quad \int F(x) \, dx = A + \sum_{n=0}^{\infty} \frac{an}{n+1}(x - c)^{n+1}
\]

\( (A \) is any constant.) These two power series have the same radius of convergence \( R \).

- The expansion \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) is valid for \( |x| < 1 \). It can be used to derive expansions of other related functions by substitution, integration, or differentiation.

11.6 EXERCISES

**Preliminary Questions**

1. Suppose that \( \sum a_n x^n \) converges for \( x = 5 \). Must it also converge for \( x = -3 \)?
2. Suppose that \( \sum a_n(x - 6)^n \) converges for \( x = 10 \). At which of the points (a)–(d) must it also converge?
   - (a) \( x = 8 \)
   - (b) \( x = 11 \)
   - (c) \( x = 3 \)
   - (d) \( x = 0 \)
3. What is the radius of convergence of \( F(3x) \) if \( F(x) \) is a power series with radius of convergence \( R = 12 \)?
4. The power series \( F(x) = \sum_{n=1}^{\infty} nx^n \) has radius of convergence \( R = 1 \).
   What is the power series expansion of \( F'(x) \) and what is its radius of convergence?
5. Show that \( \sum_{n=0}^{\infty} x^n \) diverges for all \( x \neq 0 \).

**Exercises**

1. Use the Ratio Test to determine the radius of convergence \( R \) of \( \sum \frac{x^n}{n!} \). Does it converge at the endpoints \( x = \pm R \)?
2. Use the Ratio Test to show that \( \sum \frac{x^n}{n! \sqrt{n}} \) has radius of convergence \( R = 2 \). Then determine whether it converges at the endpoints \( R = \pm 2 \).
3. Show that the power series (a)–(c) have the same radius of convergence. Then show that (a) diverges at both endpoints, (b) converges at one endpoint but diverges at the other, and (c) converges at both endpoints.
   - (a) \( \sum_{n=1}^{\infty} \frac{x^n}{2^n} \)
   - (b) \( \sum_{n=1}^{\infty} \frac{x^n}{n!} \)
   - (c) \( \sum_{n=1}^{\infty} \frac{x^n}{n^n} \)
4. Repeat Exercise 3 for the following series:
   - (a) \( \sum_{n=1}^{\infty} \frac{(x - 5)^n}{9^n} \)
   - (b) \( \sum_{n=1}^{\infty} \frac{(x - 5)^n}{n9^n} \)
   - (c) \( \sum_{n=1}^{\infty} \frac{(x - 5)^n}{n^29^n} \)
5. Show that \( \sum_{n=0}^{\infty} x^n \) diverges for all \( x \neq 0 \).
6. For which values of \( x \) does \( \sum_{n=0}^{\infty} nx^n \) converge?
7. Use the Ratio Test to show that \( \sum_{n=0}^{\infty} \frac{x^{2n}}{3^n} \) has radius of convergence \( R = \sqrt{3} \).
8. Show that \( \sum_{n=0}^{\infty} \frac{x^{3n+1}}{64^n} \) has radius of convergence \( R = 4 \).

**In Exercises 9–34, find the interval of convergence.**

9. \( \sum_{n=0}^{\infty} nx^n \)
10. \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \)
11. \( \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \)
12. \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} \)
13. \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \)
14. \( \sum_{n=0}^{\infty} \frac{n^2}{n!} x^n \)
15. \( \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \)
16. \( \sum_{n=0}^{\infty} \frac{8^n}{n!} x^n \)
17. \[ \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n \]
18. \[ \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)!} x^{2n+1} \]
19. \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}} \]
20. \[ \sum_{n=0}^{\infty} \frac{x^n}{n^n + 2} \]
21. \[ \sum_{n=15}^{\infty} \frac{x^{2n+1}}{3n+1} \]
22. \[ \sum_{n=9}^{\infty} \frac{x^n}{n - 4 \ln n} \]
23. \[ \sum_{n=2}^{\infty} \frac{x^n}{n \ln n} \]
24. \[ \sum_{n=0}^{\infty} \frac{x^{3n+2}}{n^2} \]
25. \[ \sum_{n=1}^{\infty} n(x - 3)^n \]
26. \[ \sum_{n=1}^{\infty} \frac{(-5)^n (x - 3)^n}{n^2} \]
27. \[ \sum_{n=1}^{\infty} (-1)^n n^5 (x - 7)^n \]
28. \[ \sum_{n=0}^{\infty} \frac{27^n (x - 1)^{3n+2}}{n^2} \]
29. \[ \sum_{n=0}^{\infty} \frac{2^n (x + 3)^n}{3n+1} \]
30. \[ \sum_{n=0}^{\infty} \frac{(x - 4)^n}{n!} \]
31. \[ \sum_{n=0}^{\infty} \frac{(-5)^n}{n!} (x + 10)^n \]
32. \[ \sum_{n=0}^{\infty} n! (x + 5)^n \]
33. \[ \sum_{n=2}^{\infty} \frac{e^n (x - 2)^n}{(1 - x)^n} \]
34. \[ \sum_{n=2}^{\infty} \frac{(x + 4)^n}{(n\ln n)^2} \]

In Exercises 35-40, use Eq. (2) to expand the function in a power series with center \( c = 0 \) and determine the interval of convergence.

35. \( f(x) = \frac{1}{1 - 3x} \)
36. \( f(x) = \frac{1}{1 + 3x} \)
37. \( f(x) = \frac{1}{3 - x} \)
38. \( f(x) = \frac{1}{4 + 3x} \)
39. \( f(x) = \frac{1}{1 - x^2} \)
40. \( f(x) = \frac{1}{1 - x^4} \)

41. Differentiate the power series in Exercise 39 to obtain a power series for \( g(x) = \frac{3x^2}{(1 - x)^3} \).

42. Differentiate the power series in Exercise 40 to obtain a power series for \( g(x) = \frac{4x^3}{(1 - x^4)^2} \).

43. (a) Divide the power series in Exercise 41 by \( 3x^2 \) to obtain a power series for \( h(x) = \frac{1}{(1 - x)^3} \) and use the Ratio Test to show that the radius of convergence is 1.

(b) Another way to obtain a power series for \( h(x) \) is to square the power series for \( f(x) \) in Exercise 39. By multiplying term by term, determine the terms up to degree 9 in the resulting power series for \( f(x)^2 \) and show that they match the terms in the power series for \( h(x) \) found in part (a).

44. (a) Divide the power series in Exercise 42 by \( 4x^3 \) to obtain a power series for \( h(x) = \frac{1}{(1 - x^4)^2} \) and use the Ratio Test to show that the radius of convergence is 1.

(b) Another way to obtain a power series for \( h(x) \) is to square the power series for \( f(x) \) in Exercise 40. By multiplying term by term, determine the terms up to degree 12 in the resulting power series for \( f(x)^2 \) and show that they match the terms in the power series for \( h(x) \) found in part (a).

45. Use the equalities
\[ \frac{1}{1-x} = \frac{1}{1 - x - 4} = \frac{1}{1 + (x - 4)} \]
to show that for \( |x| < 3, \)
\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n (x - 4)^n \]

46. Use the method of Exercise 45 to expand \( 1/(1 - x) \) in power series with centers \( c = 2 \) and \( c = -2 \). Determine the interval of convergence for each.

47. Use the method of Exercise 45 to expand \( 1/(4 - x) \) in a power series with center \( c = 5 \). Determine the interval of convergence.

48. Find a power series that converges only for \( x \) in \([2, 6)\).

49. Apply integration to the expansion
\[ \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x^2 - x^3 + \cdots \]
to prove that for \( -1 < x < 1, \)
\[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \]

50. Use the result of Exercise 49 to prove that
\[ \frac{1}{2} = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{2} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \cdots \]

Use the fact that this is an alternating series to find an \( N \) such that the partial sum \( S_N \) approximates \( \ln \frac{1}{2} \) to within an error of at most \( 10^{-3} \). Confirm by using a calculator to compute both \( S_N \) and \( \ln \frac{1}{2} \).

51. Let \( F(x) = (x + 1) \ln(1 + x) - x \).

(a) Apply integration to the result of Exercise 49 to prove that the following power series holds for \( F(x) \) for \(-1 < x < 1, \)
\[ F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n + 1)} \]

(b) Evaluate at \( x = \frac{1}{2} \) to prove
\[ 3 \ln 2 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \cdots \]

(c) Use a calculator to verify that the partial sum \( S_N \) approximates the left-hand side with an error no greater than the term \( S_N \) of the series.

52. Prove that for \( |x| < 1, \)
\[ \int_0^1 \frac{dx}{x^4 + 1} = A + x^2 + \frac{x^3}{3} + \frac{x^4}{5} + \cdots \]

Use the first two terms to approximate \( \int_0^1 dx/(x^4 + 1) \) numerically. Use the fact that you have an alternating series to show that the error in this approximation is at most \( 0.000022 \).

53. Use the result of Example 7 to show that
\[ F(x) = \frac{x^2}{1 - 2} + \frac{x^4}{3 - 4} + \frac{x^6}{5 - 6} + \frac{x^8}{7 - 8} + \cdots \]
is an antiderivative of \( f(x) = \tan^{-1} x \) satisfying \( F(0) = 0 \). What is the radius of convergence of this power series?
54. Verify that function \( F(x) = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) \) is an antiderivative of \( f(x) = \tan^{-1} x \) satisfying \( F(0) = 0 \). Then use the result of Exercise 53 with \( x = \frac{1}{\sqrt{3}} \) to show that
\[
\frac{x}{\sqrt{3}} = \frac{1}{2} \ln \frac{4}{3} = \frac{1}{1 - 2(\sqrt{3})} - \frac{1}{3.4(3^2)} + \frac{1}{5.6(3^2)} - \frac{1}{7.8(3^3)} + \cdots
\]
Use a calculator to compare the value of the left-hand side with the partial sum \( S_N \) of the series on the right.

55. Evaluate \( \sum_{n=1}^{\infty} \frac{n}{n^2} \). Hint: Use differentiation to show that
\[
(1-x)^{-2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{for} \quad |x| < 1
\]

56. Use the power series for \((1 + x^2)^2\) and differentiation to prove that for \( |x| < 1 \),
\[
\frac{2x}{(x^2 + 1)^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n) x^{2n-1}}{n!}
\]

57. Show that the following series converges absolutely for \( |x| < 1 \) and compute its sum:
\[
F(x) = 1 - x - x^2 - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 + \cdots
\]
Hint: Write \( F(x) \) as a sum of three geometric series with common ratio \( x^3 \).

58. Show that for \( |x| < 1 \),
\[
\frac{1 + 2x}{1 + x + x^2} = 1 + x - 2x^2 + x^3 + x^4 - 2x^4 + x^5 + x^6 - 2x^6 + \cdots
\]
Hint: Use the hint from Exercise 57.

59. Find all values of \( x \) such that \( \sum_{n=1}^{\infty} \frac{n^2}{n^2} \) converges.

60. Find all values of \( x \) such that the following series converges:
\[
F(x) = 1 + 3x + x^2 + 27x^3 + x^4 + 243x^5 + \cdots
\]

61. Find a power series \( P(x) = \sum_{n=0}^{\infty} a_n x^n \) satisfying the differential equation \( y'' = -y \) with initial conditions \( y(0) = 1 \). Then use Eq. (8) in Section 10.1 to conclude that \( P(x) = e^{-x} \).

62. Let \( C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \).
   (a) Show that \( C(x) \) has an infinite radius of convergence.
   (b) Prove that \( C(x) \) and \( f(x) = \cos x \) are both solutions of \( y'' = -y \) with initial conditions \( y(0) = 1, y'(0) = 0 \). [This Initial Value Problem has a unique solution, so it follows that \( C(x) = \cos x \) for all \( x \).]

63. Use the power series for \( y = e^x \) to show that
\[
\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots
\]
Use the fact that this is an alternating series to find an \( N \) such that the partial sum \( S_N \) approximates \( e^{-1} \) to within an error of at most \( 10^{-3} \). Confirm this using a calculator to compute both \( S_N \) and \( e^{-1} \).

64. Let \( P(x) = \sum_{n=0}^{\infty} a_n x^n \) be a power series solution to \( y'' = 2xy \) with initial condition \( y(0) = 1 \).
   (a) Show that the odd coefficients \( a_{2k+1} \) are all zero.
   (b) Prove that \( a_{2k} = 2a_{2k-2}/k \) and use this result to determine the coefficients \( a_n \).

65. Find a power series \( P(x) \) satisfying the differential equation
\[
y'' = -xy' + y = 0
\]
with initial condition \( y(0) = 1, y'(0) = 0 \). What is the radius of convergence of the power series?

66. Find a power series satisfying Eq. (10) with initial condition \( y(0) = 0, y'(0) = 1 \).

67. Prove that
\[
J_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+2} k!(k+3)!} x^{2k+2}
\]
is a solution of the Bessel differential equation of order 2:
\[
x^2 y'' + xy' + (x^2 - 4)y = 0
\]

68. Why is it impossible to expand \( f(x) = |x| \) as a power series that converges in an interval around \( x = 0 \)? Explain using Theorem 2.

---

**Further Insights and Challenges**

69. Suppose that the coefficients of \( F(x) = \sum_{n=0}^{\infty} a_n x^n \) are periodic; that is, for some whole number \( M > 0 \), we have \( a_{M+n} = a_n \). Prove that \( F(x) \) converges absolutely for \( |x| < 1 \) and that
\[
F(x) = a_0 + a_1 x + \cdots + a_{M-1} x^{M-1} + \frac{a_M}{1-x^M}
\]
Hint: Use the hint for Exercise 57.

70. **Continuity of Power Series** Let \( F(x) = \sum_{n=0}^{\infty} a_n x^n \) be a power series with radius of convergence \( R > 0 \).
   (a) Prove the inequality
\[
|x^n - y^n| \leq n|x - y|(|x|^{n-1} + |y|^{n-1})
\]
Hint: \( x^n - y^n = (x - y)(x^{n-1} + x^{n-2} y + \cdots + y^{n-1}) \).

(b) Choose \( R_1 \) with \( 0 < R_1 < R \). Show that the infinite series
\[
M = \sum_{n=0}^{\infty} 2n! |a_n R_1^n|\]
converges. Hint: Show that \( n! |a_n| R_1^n < |a_n| x^n \) for all \( n \) sufficiently large if \( R_1 < x < R \).

(c) Use the inequality in (11) to show that if \( |x| < R_1 \) and \( |y| < R_1 \), then \( |F(x) - F(y)| \leq M|x - y| \).

(d) Prove that if \( |x| < R_1 \), then \( F \) is continuous at \( x \). Hint: Choose \( R_1 \) such that \( |x| < R_1 < R \). Show that if \( \epsilon > 0 \) is given, then \( |F(x) - F(y)| \leq \epsilon \) for all \( y \) such that \( |x - y| < \delta \), where \( \delta \) is any positive number that is less than \( \epsilon/M \) and \( R_1 - |x| \) (see Figure 6).

---

**Figure 6** If \( x > 0 \), choose \( \delta > 0 \) less than \( \epsilon/M \) and \( R_1 - x \).
11.7 Taylor Polynomials

Using power series, we have seen how we can express some functions as polynomials of infinite degree. We saw that we can take power series for specific functions and manipulate them by substitution, differentiation, integration, and algebraic operations to obtain power series for other functions.

Next, we consider how we can obtain a power series for a specific given function. To do so, first we introduce Taylor polynomials, special polynomial functions that turn out to be partial sums of the power series of a function. The Taylor polynomials are important in their own right since they are useful tools for approximating functions. In the next section, we extend these Taylor polynomials to Taylor series representations of functions.

Many functions are difficult to work with. For instance, $f(x) = \sin (x^2)$ cannot be integrated using elementary functions. Nor can $f(x) = e^{-x^2}$. In fact, even simple functions like $f(x) = \sin x$, $f(x) = \cos x$, $f(x) = e^x$, and $f(x) = \ln x$ can only be evaluated exactly at relatively few values of $x$ and otherwise they must be numerically approximated. On the other hand, polynomials such as $f(x) = 3x^4 - 7x^2 + 2x - 4$ can be easily differentiated and integrated. They can be evaluated at any value of $x$ using just multiplication and addition. Thus, given a function, it is natural to ask if there is a way to accurately approximate the function using a polynomial function.

We have worked with a simple polynomial approximation of a function before. In Section 4.1, we used the linearization $L(x) = f(a) + f'(a)(x-a)$ to approximate $f(x)$ near a point $x = a$:

$$f(x) \approx f(a) + f'(a)(x-a)$$

We refer to $L(x)$ as a “first-order” approximation to $f(x)$ at $x = a$ because $f(x)$ and $L(x)$ have the same value and the same first derivative at $x = a$ (Figure 1):

$$L(a) = f(a), \quad L'(a) = f'(a)$$

A first-order approximation is useful only in a small interval around $x = a$. In this section, we achieve greater accuracy over larger intervals using higher-order approximations (Figure 2). These higher-order approximations will simply be polynomials with higher powers, the Taylor polynomials. Along with using Taylor polynomials to approximate functions, we will develop tools for estimating the error in the approximation.

![Figure 1](image1.png) The linear approximation $L(x)$ is a first-order approximation to $f$.

![Figure 2](image2.png) A second-order approximation is more accurate over a larger interval.

In what follows, assume that $f$ is defined on an open interval $I$ and that all derivatives $f^{(k)}$ exist on $I$. Let $a \in I$. We say that two functions $f$ and $g$ agree to order $n$ at $x = a$ if their derivatives up to order $n$ at $x = a$ are equal:

$$f(a) = g(a), \quad f'(a) = g'(a), \quad f''(a) = g''(a), \quad \ldots, \quad f^{(n)}(a) = g^{(n)}(a)$$

We also say that $g$ approximates $f$ to order $n$ at $x = a$. 

Scanned with CamScanner
Define the $n$th Taylor polynomial $T_n$ of $f$ centered at $x = a$ as follows:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The first few Taylor polynomials are

- $T_0(x) = f(a)$
- $T_1(x) = f(a) + f'(a)(x-a)$
- $T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$
- $T_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{6} f'''(a)(x-a)^3$

Note that $T_0$ is a constant function equal to the value of $f$ at $a$, and that $T_1$ is the linearization of $f$ at $a$. Note also that $T_n$ is obtained from $T_{n-1}$ by adding on a term of degree $n$:

$$T_n(x) = T_{n-1}(x) + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The next theorem justifies our definition of $T_n$.

**Theorem 1** The polynomial $T_n$ centered at $a$ agrees with $f$ to order $n$ at $x = a$, and it is the only polynomial of degree at most $n$ with this property.

The verification of Theorem 1 is left to the exercises (Exercises 76–77), but we'll illustrate the idea by checking that $T_2$ agrees with $f$ to order $n = 2$:

- $T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$, \quad $T_2(a) = f(a)$
- $T'_2(x) = f'(a) + f''(a)(x-a)$, \quad $T'_2(a) = f'(a)$
- $T''_2(x) = f''(a)$, \quad $T''_2(a) = f''(a)$

This shows that the value and the derivatives of order up to $n = 2$ at $x = a$ are equal.

Before proceeding to the examples, we write $T_n$ in summation notation:

$$T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!} (x-a)^j$$

By convention, we regard $f$ as the zeroth derivative, and thus $f^{(0)}$ is $f$ itself. When $a = 0$, $T_n$ is also called the nth Maclaurin polynomial.

**Example 1** Maclaurin Polynomials for $f(x) = e^x$ Plot the third and fourth Maclaurin polynomials for $f(x) = e^x$. Compare with the linear approximation.

**Solution** All higher derivatives coincide with $f$ itself: $f^{(k)}(x) = e^x$. Therefore,

$$f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = e^0 = 1$$

The third Maclaurin polynomial (the case $a = 0$) is

$$T_3(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3$$
We obtain $T_4(x)$ by adding the term of degree 4 to $T_3(x)$:

$$T_4(x) = T_3(x) + \frac{1}{4!} f^{(4)}(0)x^4 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

Figure 3 shows that $T_3$ and $T_4$ approximate $f(x) = e^x$ much more closely than the linear approximation $T_1$ on an interval around $a = 0$. Higher-degree Maclaurin polynomials would provide even better approximations on larger intervals.

**EXAMPLE 2** For objects near the surface of the earth, to two decimal places the acceleration due to gravity is $g = 9.81$ m/s$^2$. For objects at higher altitudes, Newton's Law of Gravitation says that the acceleration due to gravity is

$$G(h) = \frac{g}{(1 + \frac{h}{6370})^2}$$

where $G(h)$ is in m/s$^2$, $h$ is the altitude above the surface of the earth in km, and 6370 is the radius of the earth in km.

(a) Find a power series representation of $G$ as a function of $h$. For what values of $h$ is the power series valid?

(b) Use the third Maclaurin polynomial to approximate the acceleration due to gravity on an object at an altitude of 1000 km, and estimate the error in the approximation.

**Solution**

(a) We use the power series representation for $\frac{1}{(1-x)^2}$ from Example 6 in the previous section. Substituting $\frac{h}{6370}$ for $x$, we obtain

$$G(h) = g \sum_{n=0}^{\infty} \frac{(n+1)(-h)^n}{6370^n} = 9.81 - \frac{19.6h}{6370} + \frac{29.4h^2}{6370^2} - \frac{39.2h^3}{6370^3} + \ldots$$

The power series for $\frac{1}{(1-x)^2}$ is valid for $-1 < x < 1$, and therefore, the series for $G(h)$ holds for $-1 < -\frac{h}{6370} < 1$. Since altitude is nonnegative, it follows that the power series for $G(h)$ is valid for $0 \leq h < 6370$.

(b) Using the third Maclaurin polynomial,

$$G(1000) \approx 9.81 - \frac{(19.6)(1000)}{6370} - \frac{29.4(1000)^2}{6370^2} - \frac{39.2(1000)^3}{6370^3} \approx 7.30 \text{ m/s}^2$$

Since we have an alternating series, we can apply the corollary to Theorem 2 in Section 11.4 and use the fourth power term in the series to estimate the error in our approximation. Thus,

$$\text{error} \leq \frac{49.0(1000)^4}{6370^4} \approx 0.03$$

**EXAMPLE 3** Computing Taylor Polynomials

Compute the Taylor polynomial $T_4$ centered at $a = 3$ for $f(x) = \sqrt{x+1}$.

**Solution** First, evaluate the derivatives up to degree 4 at $a = 3$:

$$f(x) = (x + 1)^{1/2}, \quad f(3) = 2$$

$$f'(x) = \frac{1}{2}(x + 1)^{-1/2}, \quad f'(3) = \frac{1}{4}$$

$$f''(x) = -\frac{1}{4}(x + 1)^{-3/2}, \quad f''(3) = -\frac{1}{32}$$

$$f'''(x) = \frac{3}{8}(x + 1)^{-5/2}, \quad f'''(3) = \frac{3}{256}$$

$$f^{(4)}(x) = -\frac{15}{16}(x + 1)^{-7/2}, \quad f^{(4)}(3) = -\frac{15}{2048}$$
The first term \( f(a) \) in the Taylor polynomial \( T_n \) is called the constant term.

**Figure 4** Graphs of \( f(x) = \sqrt{x+1} \) and \( T_4 \) centered at \( x = 3 \).

After computing several derivatives of \( f(x) = \ln x \), we begin to discern the pattern. For many functions of interest, however, the derivatives follow no simple pattern and there is no convenient formula for the general Taylor polynomial.

Taylor polynomials for \( \ln x \) at \( a = 1 \):

\[
\begin{align*}
T_0(x) &= (x - 1) \\
T_1(x) &= (x - 1) - \frac{1}{2} (x - 1)^2 \\
T_2(x) &= (x - 1) - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3
\end{align*}
\]

Then compute the coefficients \( \frac{f^{(j)}(3)}{j!} \):

- Constant term \( f(3) = 2 \)
- Coefficient of \( (x - 3) \) \( f'(3) = \frac{1}{4} \)
- Coefficient of \( (x - 3)^2 \) \( f''(3) = -\frac{1}{32} \)
- Coefficient of \( (x - 3)^3 \) \( f'''(3) = \frac{1}{256} \)
- Coefficient of \( (x - 3)^4 \) \( f^{(4)}(3) = -\frac{1}{16384} \)

The Taylor polynomial \( T_4 \) centered at \( a = 3 \) is (see Figure 4)

\[
T_4(x) = 2 + \frac{1}{4} (x - 3) - \frac{1}{64} (x - 3)^2 + \frac{1}{512} (x - 3)^3 - \frac{5}{16384} (x - 3)^4
\]

**Example 4** Finding a General Formula for \( T_n \)

Find the Taylor polynomials \( T_n \) of \( f(x) = \ln x \) centered at \( a = 1 \).

Solution For \( f(x) = \ln x \), the constant term of \( T_n \) at \( a = 1 \) is zero because \( f(1) = \ln 1 = 0 \). Next, we compute the derivatives:

\[
\begin{align*}
f'(x) &= x^{-1}, & f''(x) &= -x^{-2}, & f'''(x) &= 2x^{-3}, & f^{(4)}(x) &= -3 \cdot 2x^{-4}
\end{align*}
\]

Similarly, \( f^{(5)}(x) = 4 \cdot 3 \cdot 2x^{-3} \). The general pattern is that \( f^{(k)}(x) \) is a multiple of \( x^{-k} \), with a coefficient \( \pm(k - 1)! \) that alternates in sign:

\[
f^{(k)}(x) = (-1)^{k-1} (k - 1)! x^{-k}
\]

The coefficient of \( (x - 1)^k \) in \( T_n \) is

\[
\frac{f^{(k)}(1)}{k!} = \frac{(-1)^{k-1} (k - 1)!}{k!} = \frac{(-1)^{k-1}}{k} \quad \text{(for } k \geq 1\text{)}
\]

Thus, the coefficients for \( k \geq 1 \) form a sequence \( 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots \), and

\[
T_n(x) = (x - 1) - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 - \ldots + (-1)^n \cdot \frac{1}{n} (x - 1)^n
\]

**Example 5** Cosine

Find the Maclaurin polynomials of \( f(x) = \cos x \).

Solution The derivatives form a repeating pattern of period 4:

\[
\begin{align*}
f(x) &= \cos x, & f'(x) &= -\sin x, & f''(x) &= -\cos x, & f'''(x) &= \sin x, \\
f^{(4)}(x) &= \cos x, & f^{(5)}(x) &= -\sin x, & \ldots
\end{align*}
\]

In general, \( f^{(4j+1)}(x) = f^{(4j)}(x) \). The derivatives at \( x = 0 \) also form a pattern:

<table>
<thead>
<tr>
<th>( f(0) )</th>
<th>( f'(0) )</th>
<th>( f''(0) )</th>
<th>( f'''(0) )</th>
<th>( f^{(4)}(0) )</th>
<th>( f^{(5)}(0) )</th>
<th>( f^{(6)}(0) )</th>
<th>( f^{(7)}(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>
Therefore, the coefficients of the odd powers \( x^{2k+1} \) are zero, and the coefficients of the even powers \( x^{2k} \) alternate in sign with value \((-1)^k/(2k)!\):

\[
T_0(x) = T_1(x) = 1, \quad T_2(x) = T_3(x) = 1 - \frac{1}{2!}x^2
\]

\[
T_4(x) = T_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}
\]

\[
T_{2n}(x) = T_{2n+1}(x) = 1 - \frac{1}{2}x^2 + \frac{x^4}{4!} - \frac{1}{6!}x^6 + \cdots + (-1)^n\frac{1}{(2n)!}x^{2n}
\]

Figure 5 shows that as \( n \) increases, \( T_n \) approximates \( f(x) = \cos x \) well over larger and larger intervals, but outside this interval, the approximation fails.

**The Error Bound**

To use Taylor polynomials effectively to approximate a function, we need a way to estimate the size of the error in the approximation. This is provided by the next theorem, which shows that when approximating \( f \) with \( T_n \), the size of this error depends on the size of the \((n+1)\)st derivative.

**THEOREM 2** Error Bound Assume that \( f^{(n+1)}(x) \) exists and is continuous. Let \( K \) be a number such that \( |f^{(n+1)}(u)| \leq K \) for all \( u \) between \( a \) and \( x \). Then

\[
|f(x) - T_n(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}
\]

where \( T_n \) is the \( n \)th Taylor polynomial centered at \( x = a \).

**EXAMPLE 6 Using the Error Bound** Apply the Error Bound to

\[
|\ln 1.2 - T_3(1.2)|
\]

where \( T_3(x) \) is the third Taylor polynomial for \( f(x) = \ln x \) at \( a = 1 \). Check your result with a calculator.

**Solution**

**Step 1. Find a value of \( K \).**

To use the Error Bound with \( n = 3 \), we must find a value of \( K \) such that \( |f^{(4)}(u)| \leq K \) for all \( u \) between \( a = 1 \) and \( x = 1.2 \). As we computed in Example 4, \( f^{(4)}(x) = -6x^{-6} \). The absolute value \( |f^{(4)}(x)| \) is decreasing for \( x > 0 \), so its maximum value on \([1, 1.2]\) is \( |f^{(4)}(1)| = 6 \). Therefore, we may take \( K = 6 \).
Step 2. Apply the Error Bound.

\[ |\ln 1.2 - T_3(1.2)| \leq K \frac{|x - a|^{n+1}}{(n+1)!} = \frac{|1.2 - 1|^{4}}{4!} = 0.0004 \]

Step 3. Check the result.
Recall from Example 4 that

\[ T_3(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 \]

The following values from a calculator confirm that the error is at most 0.0004:

\[ |\ln 1.2 - T_3(1.2)| = \approx 0.00035 < 0.0004 \]

Observe in Figure 6 that \( y = \ln x \) and \( y = T_3(x) \) are indistinguishable near \( x = 1.2 \).

Example 7: Approximating with a Given Accuracy
Let \( T_n \) be the \( n \)th Maclaurin polynomial for \( f(x) = \cos x \). Find a value of \( n \) such that

\[ |\cos 0.2 - T_n(0.2)| < 10^{-5} \]

Solution

Step 1. Find a value of \( K \).
Since \( |f^{(n)}(x)| \) is \( |\cos x| \) or \( |\sin x| \), depending on whether \( n \) is even or odd, we have \( |f^{(n)}(u)| \leq 1 \) for all \( u \). Thus, we may apply the Error Bound with \( K = 1 \).

Step 2. Find a value of \( n \).
The Error Bound gives us

\[ |\cos 0.2 - T_n(0.2)| \leq \frac{|0.2|^{n+1}}{(n+1)!} \]

To make the error less than \( 10^{-5} \), we must choose \( n \) so that

\[ \frac{|0.2|^{n+1}}{(n+1)!} < 10^{-5} \]

It's not possible to solve this inequality for \( n \), but we can find a suitable \( n \) by checking several values:

<table>
<thead>
<tr>
<th>( n )</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.2^3/3! ≈ 0.0013</td>
<td>0.2^4/4! ≈ 6.67 \times 10^{-5}</td>
<td>0.2^5/5! ≈ 2.67 \times 10^{-6} &lt; 10^{-5}</td>
</tr>
</tbody>
</table>

We see that the error is less than \( 10^{-5} \) for \( n = 4 \).

Conceptual Insight

The term \( K \) in the Error Bound usually depends on \( n \), the number of terms in the Taylor polynomial. However, in some instances, \( K \) can be chosen independent of \( n \). For example, if \( f(x) = \sin x \) or \( f(x) = \cos x \), then we can let \( K = 1 \) for all \( n \) (since the absolute value of all derivatives of these functions is no larger than 1). Because the \( (n+1)! \) term in the denominator of the Error Bound grows very rapidly and dominates the fraction, the error goes to 0 as \( n \) increases. Thus, for these functions, the more terms in the Taylor polynomial, the better the approximation. Therefore, if we include infinitely many terms, we can ask if the resulting series and \( f \) are equal. This naturally leads to the subject of the next section, Taylor Series.

The rest of this section is devoted to a proof of the Error Bound (Theorem 2). Define the \( n \)th remainder:

\[ R_n(x) = f(x) - T_n(x) \]

The error in \( T_n(x) \) is the absolute value \( |R_n(x)| \). As a first step in proving the Error Bound, we show that \( R_n(x) \) can be represented as an integral.
THEOREM 3  Taylor's Theorem  Assume that $f^{(n+1)}$ exists and is continuous. Then

$$R_n(x) = \frac{1}{n!} \int_a^x (x-u)^n f^{(n+1)}(u) \, du$$

Proof  Set

$$I_n(x) = \frac{1}{n!} \int_a^x (x-u)^n f^{(n+1)}(u) \, du$$

Our goal is to show that $R_n(x) = I_n(x)$. For $n = 0$, $R_0(x) = f(x) - f(a)$ and the desired result is just a restatement of the Fundamental Theorem of Calculus:

$$I_0(x) = \int_a^x f'(u) \, du = f(x) - f(a) = R_0(x)$$

To prove the formula for $n > 0$, we apply Integration by Parts to $I_n(x)$ with

$$h(u) = \frac{1}{n!} (x-u)^n, \quad g(u) = f^{(n)}(u)$$

Then $g'(u) = f^{(n+1)}(u)$, and so

$$I_n(x) = \int_a^x h(u) g'(u) \, du = h(u)g(u) \bigg|_a^x - \int_a^x h'(u)g(u) \, du$$

$$= \frac{1}{n!} (x-u)^n f^{(n)}(u) \bigg|_a^x - \frac{1}{n!} \int_a^x (-n)(x-u)^{n-1} f^{(n)}(u) \, du$$

$$= -\frac{1}{n!} (x-a)^n f^{(n)}(a) + I_{n-1}(x)$$

This can be rewritten as

$$I_{n-1}(x) = \frac{f^{(n)}(a)}{n!} (x-a)^n + I_n(x)$$

Now, apply this relation $n$ times, noting that $I_0(x) = f(x) - f(a)$:

$$f(x) = f(a) + I_0(x)$$

$$= f(a) + \frac{f'(a)}{1!} (x-a) + I_1(x)$$

$$= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + I_2(x)$$

$$\vdots$$

$$= f(a) + \frac{f'(a)}{1!} (x-a) + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n + I_n(x)$$

This shows that $f(x) = T_n(x) + I_n(x)$ and hence $I_n(x) = R_n(x)$, as desired.

Proof  Now, we can prove Theorem 2. Assume first that $x \geq a$. Then

$$|f(x) - T_n(x)| = |R_n(x)| = \left| \frac{1}{n!} \int_a^x (x-u)^n f^{(n+1)}(u) \, du \right|$$

$$\leq \frac{1}{n!} \int_a^x |(x-u)^n f^{(n+1)}(u)| \, du$$

$$\leq \frac{K}{n!} \int_a^x |x-u|^n \, du$$

$$= \frac{K}{n!} \int_a^x |x-u|^{n+1} \, du$$

$$= \frac{K}{n!} \frac{(x-a)^{n+1}}{n+1} \bigg|_{x=a} = \frac{K}{n+1} \frac{(x-a)^{n+1}}{(n+1)!}$$

$$= \frac{K}{n+1} (x-a)^{n+1}$$

Scanned with CamScanner
Note that the absolute value is not needed in the inequality in (4) because \( x - u \geq 0 \) for \( a \leq u \leq x \). If \( x \leq a \), we must interchange the upper and lower limits of the integrals in (3) and (4).

### 11.7 SUMMARY

- The \( n \)th Taylor polynomial centered at \( x = a \) for the function \( f \) is
  \[
  T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n
  \]

  When \( a = 0 \), \( T_n \) is also called the \( n \)th Maclaurin polynomial.

- If \( f^{(n+1)} \) exists and is continuous, then we have the Error Bound
  \[
  |T_n(x) - f(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}
  \]

  where \( K \) is a number such that \( |f^{(n+1)}(u)| \leq K \) for all \( u \) between \( a \) and \( x \).

- For reference, we include a table of standard Maclaurin and Taylor polynomials.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( a )</th>
<th>Maclaurin or Taylor Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x )</td>
<td>0</td>
<td>( T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>0</td>
<td>( T_{2n+1}(x) = T_{2n+2}(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>0</td>
<td>( T_{2n}(x) = T_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} )</td>
</tr>
<tr>
<td>( \ln x )</td>
<td>1</td>
<td>( T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n )</td>
</tr>
<tr>
<td>( \frac{1}{1-x} )</td>
<td>0</td>
<td>( T_n(x) = 1 + x + x^2 + \cdots + x^n )</td>
</tr>
</tbody>
</table>

### 11.7 EXERCISES

**Preliminary Questions**

1. What is \( T_3 \) centered at \( a = 3 \) for a function \( f \) such that \( f(3) = 9 \), \( f'(3) = 8 \), \( f''(3) = 4 \), and \( f'''(3) = 12 \)?

2. The dashed graphs in Figure 7 are Taylor polynomials for a function \( f \). Which of the two is a Maclaurin polynomial?

![Figure 7](image)

**Exercise Questions**

3. For which value of \( x \) does the Maclaurin polynomial \( T_n \) satisfy \( T_n(x) = f(x) \), no matter what \( f \) is?

4. Let \( T_n \) be the Maclaurin polynomial of a function \( f \) satisfying \( |f^{(n+1)}(x)| \leq 1 \) for all \( x \). Which of the following statements follow from the Error Bound?
   (a) \( |T_2(x) - f(x)| \leq \frac{2}{3} \)
   (b) \( |T_3(x) - f(x)| \leq \frac{2}{3} \)
   (c) \( |T_2(x) - f(x)| \leq \frac{1}{3} \)

Scanned with CamScanner
Exercises

In Exercises 1–16, calculate the Taylor polynomials $T_2$ and $T_3$ centered at $x = a$ for the given function and value of $a$.

1. $f(x) = \sin x, \ a = 0$

2. $f(x) = \sin x, \ a = \pi / 2$

3. $f(x) = \frac{1}{1+x}, \ a = 2$

4. $f(x) = \frac{1}{1+x^2}, \ a = -1$

5. $f(x) = x^4 - 2x, \ a = 3$

6. $f(x) = \frac{x^3 + 1}{x + 1}, \ a = -2$

7. $f(x) = \sqrt{x}, \ a = 1$

8. $f(x) = \sqrt{x}, \ a = 9$

9. $f(x) = \tan x, \ a = 0$

10. $f(x) = \tan x, \ a = \pi / 4$

11. $f(x) = e^{-x} + e^{2x}, \ a = 0$

12. $f(x) = e^{2x}, \ a = \ln 2$

13. $f(x) = x^2 e^{-x}, \ a = 1$

14. $f(x) = \cosh 2x, \ a = 0$

15. $f(x) = \frac{\ln x}{x}, \ a = 1$

16. $f(x) = \ln(x + 1), \ a = 0$

17. Show that the second Taylor polynomial for $f(x) = px^2 + qx + r$, centered at $a = 1$, is $f(x)$.

18. Show that the third Maclaurin polynomial for $f(x) = (x - 3)^3$ is $f(x)$.

19. Show that the $n$th Maclaurin polynomial for $f(x) = e^x$ is

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

20. Show that the $n$th Taylor polynomial for $f(x) = \frac{1}{x + 1}$, $a = 1$ is

$$T_n(x) = \frac{1}{2} - \frac{(x - 1)^2}{4} + \frac{(x - 1)^3}{8} + \cdots + (-1)^n \frac{(x - 1)^n}{2^n + 1}$$

21. Show that the Maclaurin polynomials for $f(x) = \sin x$ are

$$T_{2n+1}(x) = T_{2n+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n + 1)!}$$

22. Show that the Maclaurin polynomials for $f(x) = \ln(1 + x)$ are

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n}$$

In Exercises 23–30, find $T_n$ centered at $x = a$ for all $n$.

23. $f(x) = \frac{1}{1+x}, \ a = 0$

24. $f(x) = \frac{1}{x - 1}, \ a = 4$

25. $f(x) = e^x, \ a = 1$

26. $f(x) = e^x, \ a = -2$

27. $f(x) = x^2, \ a = 1$

28. $f(x) = x^2, \ a = 2$

29. $f(x) = \cos x, \ a = \pi / 4$

30. $f(x) = \sin 3x, \ a = 0$

In Exercises 31–34, find $T_2$ and use a calculator to compute the error $|f(x) - T_2(x)|$ for the given values of $a$ and $x$.

31. $y = e^x, \ a = 0, \ x = -0.5$

32. $y = \cos x, \ a = 0, \ x = \pi / 12$

33. $y = x^{3/2}, \ a = 1, \ x = 1.2$

34. $y = e^{2x}, \ a = \pi / 2, \ x = 1.5$

35. Compute $T_3$ for $f(x) = \sqrt{x}$ centered at $a = 1$. Then use a plot of the error $|f(x) - T_3(x)|$ to find a value $c > 1$ such that the error on the interval $[1, c]$ is at most 0.25.

36. Plot $f(x) = 1/(1 + x)$ together with the Taylor polynomials $T_n$ at $a = 1$ for $1 \leq n \leq 4$ on the interval $[-2, 8]$ (be sure to limit the upper plot range).

(a) Over which interval does $T_2$ appear to approximate $f$ closely?

(b) What happens for $x < -1$?

(c) Use a computer algebra system to produce and plot $T_4$ together with $f$ on $[-2, 8]$. Over which interval does $T_4$ appear to give a close approximation?

37. Let $T_3$ be the Maclaurin polynomial of $f(x) = e^x$. Use the Error Bound to find the maximum possible value of $|f(1.1) - T_3(1.1)|$. Show that we can take $K = e^{1.1}$.

38. Let $T_3$ be the Taylor polynomial of $f(x) = \sqrt{x}$ centered at $a = 4$. Apply the Error Bound to find the maximum possible error of the value $f(3.9) - T_3(3.9)$.

In Exercises 39–42, compute the Taylor polynomial indicated and use the Error Bound to find the maximum possible size of the error. Verify your result with a calculator.

39. $y = \cos x, \ a = 0; \ |\cos(0.25) - T_2(0.25)|$

40. $f(x) = x^{1/2}, \ a = 1; \ |f(1.2) - T_4(1.2)|$

41. $f(x) = e^{-x^2}, \ a = 4; \ |f(-4.3) - T_3(-4.3)|$

42. $f(x) = \sqrt{1 + x}, \ a = 8; \ |f(9.02) - T_5(8.02)|$

43. Calculate the Maclaurin polynomial $T_3$ for $f(x) = \tan^{-1} x$. Compute $T_3(1)$ and use the Error Bound to find a bound for $|\tan^{-1} \frac{1}{2} - T_3(1/2)|$. Refer to the graph in Figure 8 to find an acceptable value of $K$. Verify your result by computing $|\tan^{-1} \frac{1}{2} - T_3(1/2)|$ using a calculator.

44. Let $f(x) = \ln(x^2 - x + 1)$. The third Taylor polynomial at $a = 1$ is $T_3(x) = 2(x - 1) + (x - 1)^2 + \frac{7}{3}(x - 1)^3$.

![Figure 8](image-url)
Find the maximum possible value of \( f(1.1) - T_3(1.1) \), using the graph in Figure 9 to find an acceptable value of \( K \). Verify your result by computing \( f(1.1) - T_3(1.1) \) using a calculator.

**Figure 9** Graph of \( f^{(4)} \), where \( f(x) = \ln(x^3 - x + 1) \).

45. **(GU)** Let \( T_2 \) be the Taylor polynomial at \( a = 0.5 \) for \( f(x) = \cos(x^2) \). Use the Error Bound to find the maximum possible value of \( f(0.5) - T_2(0.5) \). Plot \( f^{(5)} \) to find an acceptable value of \( K \).

46. **(GU)** Calculate the Maclaurin polynomial \( T_2 \) for \( f(x) = \sec x \) and use the Error Bound to estimate the error \( |f(\frac{1}{2}) - T_2(\frac{1}{2})| \). Plot \( f^{(7)} \) to find an acceptable value of \( K \).

In Exercises 47–50, use the Error Bound to find a value of \( n \) for which the given inequality is satisfied. Then verify your result using a calculator.

47. \[ |\cos 0.1 - T_2(0.1)| \leq 10^{-7}, \quad a = 0 \]

48. \[ |\ln 1.3 - T_n(1.3)| \leq 10^{-4}, \quad a = 1 \]

49. \[ |\sqrt{3} - T_n(\sqrt{3})| \leq 10^{-6}, \quad a = 1 \]

50. \[ |e^{-0.1} - T_n(-0.1)| \leq 10^{-9}, \quad a = 0 \]

51. Let \( f(x) = e^{-x} \) and \( T_3(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} \).

(a) Use the Error Bound to show that for all \( x \geq 0 \),

\[ |f(x) - T_3(x)| \leq \frac{x^4}{24} \]

(b) **(GU)** Illustrate this inequality by plotting \( y = f(x) - T_3(x) \) and \( y = x^4/24 \) together over \([0, 1]\).

52. Use the Error Bound with \( n = 4 \) to show that

\[ |\sin x - \left( x - \frac{x^3}{6} \right) | \leq \frac{|x|^5}{120} \quad \text{(for all } x\text{)} \]

53. Let \( T_n \) be the Taylor polynomial for \( f(x) = \ln x \) at \( a = 1 \), and let \( c > 1 \). Show that

\[ |\ln c - T_n(c)| \leq \frac{|c-1|^n+1}{n+1} \]

Then find a value of \( n \) such that \( |\ln 1.5 - T_n(1.5)| \leq 10^{-2} \).

54. Let \( n \geq 1 \). Show that if \(|x|\) is small, then

\[ (x + 1)^{1/n} \approx 1 + \frac{x}{n} + \frac{1 - n^{-1} x^2}{2n^2} \]

Use this approximation with \( n = 6 \) to estimate \( 1.5^{1/6} \).

55. Verify that the third Maclaurin polynomial for \( f(x) = e^x \sin x \) is equal to the product of the third Maclaurin polynomials of \( f(x) = e^x \) and \( f(x) = \sin x \) (after discarding terms of degree greater than 3 in the product).

56. Find the fourth Maclaurin polynomial for \( f(x) = \sin x \cos x \) by multiplying the fourth Maclaurin polynomials for \( f(x) = \sin x \) and \( f(x) = \cos x \).

57. Find the Maclaurin polynomials \( T_n \) for \( f(x) = \cos(x^2) \). You may use the fact that \( T_n(x) \) is equal to the sum of the terms up to degree \( n \) obtained by substituting \( x^2 \) for \( x \) in the \( n \)th Maclaurin polynomial of \( \cos x \).

58. Find the Maclaurin polynomials of \( 1/(1 + x^2) \) by substituting \( -x^2 \) for \( x \) in the Maclaurin polynomials of \( 1/(1 - x) \).

59. Let \( f(x) = 3x^3 + 2x^2 - x - 4 \). Calculate \( T_j \) for \( j = 1, 2, 3, 4, 5 \) when both \( a = 0 \) and \( a = 1 \). Show that \( T_n(x) = f(x) \) in both cases.

60. Let \( T_n \) be the \( n \)th Taylor polynomial at \( x = a \) for a polynomial \( f \) of degree \( n \). Based on the result of Exercise 59, prove the value of \( |f(x) - T_n(x)| \). Prove that your guess is correct using the Error Bound.

61. Let \( s(t) \) be the distance of a truck to an intersection. At time \( t = 0 \), the truck is 60 m from the intersection, travels away from it at a velocity of 24 m/s, and begins to slow down with an acceleration of \( a = -4 \text{ } m/s^2 \). Determine the second Taylor polynomial of \( s \), and use it to estimate the truck's distance from the intersection after 4 s.

62. A bank owns a portfolio of bonds whose value \( P(t) \) depends on the interest rate \( r \) (measured in percent; e.g., \( r = 5 \) means a 5% interest rate). The bank's quantitative analyst determines that

\[ P(5) = 100,000, \quad \frac{d^2 P}{dr^2} \bigg|_{r=5} = 50,000 \]

In finance, this second derivative is called bond convexity. Find the second Taylor polynomial of \( P(t) \) centered at \( r = 5 \) and use it to estimate the value of the portfolio if the interest rate moves to \( r = 5.5 \% \).

63. A narrow, negatively charged ring of radius \( R \) exerts a force on a positively charged particle \( P \) located at distance \( x \) above the center of the ring of magnitude

\[ F(x) = \frac{kx}{(x^2 + R^2)^{3/2}} \]

where \( k > 0 \) is a constant (Figure 10).

(a) Compute the third-degree Maclaurin polynomial for \( F \).
(b) Show that \( F \approx -k/R^2x \) to second order. This shows that when \( x \) is small, \( F(x) \) behaves like a restoring force similar to the force exerted by a spring.
(c) Show that \( F(x) \approx -k/R^2x \) when \( x \) is large by showing that

\[ \lim_{x \to \infty} F(x) = 0 \]

Thus, \( F(x) \) behaves like an inverse square law, and the charged ring looks like a point charge from far away.

**Figure 10**
64. A light wave of wavelength \( \lambda \) travels from \( A \) to \( B \) passing through an aperture (circular region) located in a plane that is perpendicular to \( \overline{AB} \) (see Figure 11 for the notation). Let \( f(r) = r + h \); that is, \( f(r) \) is the distance \( AC + CB \) as a function of \( r \).
(a) Show that \( f(r) = \sqrt{d^2 + r^2} + \sqrt{h^2 + r^2} \), and use the Maclaurin polynomial of order 2 to show that

\[
f(r) = d + h + \frac{1}{2} \left( \frac{1}{d} + \frac{1}{h} \right) r^2
\]

(b) The Fresnel zones, used to determine the optical disturbance at \( B \), are the concentric bands bounded by the circles of radius \( R_n \) such that \( f(R_n) = d + h + n\lambda/2 \). Show that \( R_n \) can be approximated by

\[
R_n \approx \sqrt{nL}, \quad \text{where} \quad L = (d^{-1} + h^{-1})^{-1}.
\]

(c) Estimate the radii \( R_1 \) and \( R_{100} \) for blue light (\( \lambda = 475 \times 10^{-7} \) cm) if \( d = 100 \) cm.

**FIGURE 11** The Fresnel zones are the regions between the circles of radius \( R_n \).

65. Referring to Figure 12, let \( a \) be the length of the chord \( \overline{AC} \) of angle \( \theta \) of the unit circle. Derive the following approximation for the excess of the arc over the chord:

\[
\theta - a \approx \frac{\theta^3}{24}
\]

Hint: Show that \( \theta - a = \theta - 2 \sin(\theta/2) \) and use the third Maclaurin polynomial as an approximation.

**FIGURE 12** Unit circle.

66. To estimate the length \( \theta \) of a circular arc of the unit circle, the seventeenth-century Dutch scientist Christian Huygens used the approximation \( \theta \approx (8b - a) / 3 \), where \( a \) is the length of the chord \( \overline{AC} \) of angle \( \theta \) and \( b \) is the length of the chord \( \overline{AB} \) of angle \( \theta/2 \) (Figure 12).
(a) Prove that \( a = 2 \sin(\theta/2) \) and \( b = 2 \sin(\theta/4) \), and show that the Huygens' approximation amounts to the approximation

\[
\theta \approx \frac{16}{3} \sin \frac{\theta}{4} + \frac{2}{5} \sin \frac{\theta}{2}
\]

(b) Compute the fifth Maclaurin polynomial of the function on the right.

(c) Use the Error Bound to show that the error in the Huygens' approximation is less than 0.00022|\( \theta \)|.

**Further Insights and Challenges**

67. Show that the \( n \)th Maclaurin polynomial of \( f(x) = \arcsin x \) for \( n \) odd is

\[
T_n(x) = x + \frac{1}{2} x^3 + \frac{1 \cdot 3}{2 \cdot 4} x^5 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (n-2)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (n-1)} x^n
\]

68. Let \( x \geq 0 \) and assume that \( f^{(n+1)}(x) \geq 0 \) for all \( 0 \leq x \leq 1 \). Use Taylor's Theorem to show that the \( n \)th Maclaurin polynomial \( T_n(x) \) satisfies

\[
T_n(x) \leq f(x), \quad \text{for all} \quad x \geq 0
\]

69. Use Exercise 68 to show that for \( x \geq 0 \) and all \( n \),

\[
e^n \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}
\]

Sketch the graphs of \( y = e^x \), \( y = T_n(x) \), and \( y = T_2(x) \) on the same coordinate axes. Does this inequality remain true for \( x < 0 \)?

70. This exercise is intended to reinforce the proof of Taylor's Theorem.
(a) Show that \( f(x) = T_0(x) + \int_a^x f'(u) \, du \).
(b) Use Integration by Parts to prove the formula

\[
\int_a^x (x-u)f''(u) \, du = -f'(x)(x-a) + \int_a^x f'(u) \, du
\]

(c) Prove the case \( n = 2 \) of Taylor's Theorem:

\[
f(x) = T_2(x) + \int_a^x (x-u)f''(u) \, du
\]

71. Find the fourth Maclaurin polynomial \( T_4 \) for \( f(x) = e^{-x^2} \), and calculate \( I = \int_0^{1/2} T_4(x) \, dx \) as an estimate for \( \int_0^{1/2} e^{-x^2} \, dx \). A CAS yields the value \( I \approx 0.461281 \). How large is the error in your approximation? Hint: \( T_4 \) is obtained by substituting \(-x^2\) in the second Maclaurin polynomial for \( e^x \).

72. Approximating Integrals Let \( L > 0 \). Show that if two functions \( f \) and \( g \) satisfy \(|f(x) - g(x)| < L\) for all \( x \in [a, b] \), then

\[
\left| \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \right| < L(b-a)
\]

73. Let \( T_4 \) be the fourth Maclaurin polynomial for \( f(x) = \cos x \).
(a) Show that

\[
|\cos x - T_4(x)| \leq \frac{1}{6} \frac{x^6}{6!} \quad \text{for all} \quad x \in \left[0, \frac{1}{2}\right]
\]

**Hint:** \( T_4(x) = T_2(x) \).
(b) Evaluate \( \int_0^{1/2} T_4(x) \, dx \) as an approximation to \( \int_0^{1/2} \cos x \, dx \). Use Exercise 72 to find a bound for the size of the error.
74. Let \( Q(x) = 1 - x^2/6 \). Use the Error Bound for \( f(x) = \sin x \) to show that
\[
\left| \frac{\sin x}{x} - Q(x) \right| \leq \frac{|x|^4}{5!}
\]
Then calculate \( \int_0^1 Q(x) \, dx \) as an approximation to \( \int_0^1 (\sin x/x) \, dx \) and find a bound for the error.

75. (a) Compute the sixth Maclaurin polynomial \( T_6 \) for \( f(x) = \sin(x^2) \) by substituting \( x^2 \) in \( P(x) = x - x^3/6 \), the third Maclaurin polynomial for \( f(x) = \sin x \).
(b) Show that \( |\sin(x^2) - T_6(x)| \leq \frac{|x|^6}{5!} \).
\[ \text{Hint: Substitute } x^2 \text{ for } x \text{ in the Error Bound for } |\sin x - P(x)|, \text{ noting that } P \text{ is also the fourth Maclaurin polynomial for } f(x) = \sin x. \]
(c) Use \( T_6 \) to approximate \( \int_0^{1/2} \sin(x^2) \, dx \) and find a bound for the error.

76. Prove by induction that for all \( k \),
\[
\begin{align*}
\frac{d^k}{dx^k} \left( \frac{x-a)^k}{k!} \right) &= \frac{(x-j)\cdots(x-j+k-1)(x-a)^{k-j}}{k!} \\
\left. \frac{d^k}{dx^k} \left( \frac{x-a)^k}{k!} \right) \right|_{x=a} &= \begin{cases} 1 & \text{for } k = j \\ 0 & \text{for } k \neq j \end{cases}
\end{align*}
\]
Use this to prove that \( T_n \) agrees with \( f \) at \( x = a \) to order \( n \).

77. Let \( a \) be any number and let
\[
P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0
\]
be a polynomial of degree \( n \) or less.
(a) Show that if \( P^{(j)}(a) = 0 \) for \( j = 0, 1, \ldots, n \), then \( P(x) = 0 \), that is, \( a_j = 0 \) for all \( j \). \[ \text{Hint: Use induction, noting that if the statement is true for degree } n - 1, \text{ then } P(0) = 0. \]
(b) Prove that \( T_n \) is the only polynomial of degree \( n \) or less that agrees with \( f \) at \( x = a \) to order \( n \). \[ \text{Hint: If } Q \text{ is another such polynomial, apply } (a) \text{ to } P(x) = T_n(x) - Q(x). \]

### 11.8 Taylor Series

In this section, we extend the Taylor polynomial to the Taylor series of a given function \( f \), obtained by including terms of all orders in the Taylor polynomial.

**Definition** **Taylor Series** If \( f \) is infinitely differentiable at \( x = c \), then the Taylor series for \( f(x) \) centered at \( c \) is the power series
\[
T(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n
\]

While this definition enables us to construct a power series using information from the function \( f \), we do not yet know whether this series defines a function that equals \( f \). The next two theorems settle the matter.

**Theorem 1** **Taylor Series Expansion** If \( f(x) \) is represented by a power series centered at \( c \) in an interval \( |x - c| < R \) with \( R > 0 \), then that power series is the Taylor series
\[
T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n
\]

**Proof** Suppose that \( f(x) \) is represented by a power series centered at \( x = c \) on an interval \( (c-R, c+R) \) with \( R > 0 \):
\[
f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots
\]
According to Theorem 2 in Section 11.6, we can compute the derivatives of \( f \) by differentiating the series term by term:
\[
f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots
\]
\[
f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots
\]
\[
f''(x) = 2a_2 + 2 \cdot 3a_3(x-c) + 3 \cdot 4a_4(x-c)^2 + 4 \cdot 5a_5(x-c)^3 + \cdots
\]
In general,
\[ f^{(k)}(x) = k!a_k + (2 \cdot 3 \cdots (k+1))a_{k+1}(x-c) + \cdots \]

Setting \( x = c \) in each of these series, we find that
\[ f'(c) = a_0, \quad f''(c) = a_1, \quad f'''(c) = 2a_2, \quad \ldots, \quad f^{(k)}(c) = k!a_k, \quad \ldots \]

It follows that \( a_k = \frac{f^{(k)}(c)}{k!} \). Therefore, \( f(x) = T(x) \), where \( T(x) \) is the Taylor series of \( f(x) \) centered at \( x = c \).

In the special case \( c = 0 \), \( T(x) \) is also called the Maclaurin series:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots
\]

**Example 1** Find the Taylor series for \( f(x) = x^{-3} \) centered at \( c = 1 \).

**Solution** It often helps to create a table, as in Table 1, to see the pattern. The derivatives of \( f(x) \) are
\[ f'(x) = -3x^{-4}, \quad f''(x) = (-3)(-4)x^{-5}, \text{ and in general,} \]
\[ f^{(n)}(x) = (-1)^n(3)(4)\cdots(n+2)x^{-n-3} \]

Note that \((3)(4)\cdots(n+2) = \frac{1}{2}(n+2)!\). Therefore,
\[ f^{(n)}(1) = (-1)^n\frac{1}{2}(n+2)! \]

Noting that \((n+2)! = (n+2)(n+1)n!\), we write the coefficients of the Taylor series as
\[ a_n = \frac{f^{(n)}(1)}{n!} = (-1)^n\frac{1}{2}(n+2)! \frac{1}{n!} = (-1)^n\frac{n+2}{2} \]

The Taylor series for \( f(x) = x^{-3} \) centered at \( c = 1 \) is
\[
T(x) = 1 - 3(x-1) + 6(x-1)^2 - 10(x-1)^3 + \cdots
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \frac{n+2}{2} (x-1)^n
\]

Theorem 1 tells us that if we want to represent a function \( f \) by a power series centered at \( c \), then the only candidate for the job is the Taylor series:

\[
T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n
\]

However, there is no guarantee that \( T(x) \) converges to \( f(x) \), even if \( T(x) \) converges. To study convergence, we consider the \( k \)th partial sum, which is the Taylor polynomial of degree \( k \):
\[
T_k(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(k)}(c)}{k!}(x-c)^k
\]

In Section 11.7, we defined the remainder
\[
R_k(x) = f(x) - T_k(x)
\]

Since \( T(x) \) is the limit of the partial sums \( T_k(x) \), we see that

The Taylor series converges to \( f(x) \) if and only if \( \lim_{k \to \infty} R_k(x) = 0 \).

There is no general method for determining whether \( R_k(x) \) tends to zero, but the following theorem can be applied in some important cases.
**Theorem 2** Let \( I = (c - R, c + R) \), where \( R > 0 \), and assume that \( f \) is infinitely differentiable on \( I \). Suppose there exists \( K > 0 \) such that all derivatives of \( f \) are bounded by \( K \) on \( I \):

\[
|f^{(k)}(x)| \leq K \quad \text{for all} \quad k \geq 0 \quad \text{and} \quad x \in I
\]

Then \( f \) is represented by its Taylor series in \( I \):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n \quad \text{for all} \quad x \in I
\]

**Proof** According to the Error Bound for Taylor polynomials (Theorem 2 in Section 11.7),

\[
|R_k(x)| = |f(x) - T_k(x)| \leq K \frac{|x - c|^{k+1}}{(k+1)!}
\]

If \( x \in I \), then \( |x - c| < R \) and

\[
|R_k(x)| \leq K \frac{R^{k+1}}{(k+1)!}
\]

We showed in Example 10 of Section 11.1 that \( R^k/k! \) tends to zero as \( k \to \infty \). Therefore,

\[
\lim_{k \to \infty} R_k(x) = 0 \quad \text{for all} \quad x \in (c - R, c + R), \quad \text{as required.}
\]

**Example 2** Expansions of Sine and Cosine

Show that the following Maclaurin expansions are valid for all \( x \):

\[
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

\[
\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

**Solution** Recall that the derivatives of \( f(x) = \sin x \) and their values at \( x = 0 \) form a repeating pattern of period 4:

\[
\begin{array}{c|c|c|c|c|c|c}
 f(x) & f'(x) & f''(x) & f'''(x) & f^{(4)}(x) & \cdots \\
 \sin x & \cos x & -\sin x & -\cos x & \sin x & \cdots \\
 0 & 1 & 0 & -1 & 0 & \cdots
\end{array}
\]

In other words, the even derivatives are zero and the odd derivatives alternate in sign: \( f^{(2n+1)}(0) = (-1)^n \). Therefore, the nonzero Taylor coefficients for \( \sin x \) are

\[
 a_{2n+1} = \frac{(-1)^n}{(2n+1)!}
\]

For \( f(x) = \cos x \), the situation is reversed. The odd derivatives are zero and the even derivatives alternate in sign: \( f^{(2n)}(0) = (-1)^n \cos 0 = (-1)^n \). Therefore, the nonzero Taylor coefficients for \( \cos x \) are \( a_{2n} = \frac{(-1)^n}{(2n)!} \).

We can apply Theorem 2 with \( K = 1 \) and any value of \( R \) because both sine and cosine satisfy \( |f^{(n)}(x)| \leq 1 \) for all \( x \) and \( n \). The conclusion is that the Taylor series converges to \( f(x) \) for \( |x| < R \). Since \( R \) is arbitrary, the Taylor expansions hold for all \( x \).
EXAMPLE 3  Taylor Expansion of \( f(x) = e^x \) at \( x = c \)

Find the Taylor series \( T(x) \) of \( f(x) = e^x \) at \( x = c \).

Solution  We have \( f^{(n)}(c) = e^c \) for all \( x \). Thus,

\[
T(x) = \sum_{n=0}^{\infty} \frac{e^c}{n!} (x - c)^n
\]

Because \( f(x) = e^x \) is increasing for all \( R > 0 \), \( |f^{(k)}(x)| \leq e^{c+R} \) for \( x \in (c - R, c + R) \).

Applying Theorem 2 with \( K = e^{c+R} \), we conclude that \( T(x) \) converges to \( f(x) \) for all \( x \in (c - R, c + R) \). Since \( R \) is arbitrary, the Taylor expansion holds for all \( x \). For \( c = 0 \), we obtain the standard Maclaurin series

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

Shortcuts to Finding Taylor Series

There are several methods for generating new Taylor series from known ones. First of all, we can differentiate and integrate Taylor series term by term within its interval of convergence, by Theorem 2 of Section 11.6. We can also multiply two Taylor series or substitute one Taylor series into another (we omit the proofs of these facts).

EXAMPLE 4  Find the Maclaurin series for \( f(x) = x^2 e^x \).

Solution  Multiply the known Maclaurin series for \( e^x \) by \( x^2 \):

\[
x^2 e^x = x^2 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \right)
\]

\[
= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \frac{x^7}{5!} + \cdots = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}
\]

EXAMPLE 5  Substitution  Find the Maclaurin series for \( f(x) = e^{-x^2} \).

Solution  Substitute \(-x^2\) for \( x \) in the Maclaurin series for \( e^x \):

\[
e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots
\]

The Taylor expansion of \( e^x \) is valid for all \( x \), so this expansion is also valid for all \( x \).

EXAMPLE 6  Integration  Find the Maclaurin series for \( f(x) = \ln(1 + x) \).

Solution  We integrate the geometric series with common ratio \(-x\) (valid for \(|x| < 1\)):

\[
\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots
\]

\[
\ln(1 + x) = \int \frac{dx}{1 + x} = A + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = A + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}
\]

The constant of integration \( A \) on the right is zero because \( \ln(1 + x) = 0 \) for \( x = 0 \), so

\[
\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}
\]

This expansion is valid for \(|x| < 1\). It also holds for \( x = 1 \) (see Exercise 92).

In many cases, there is no convenient formula for the coefficients of a Taylor series for a given function, but we can still compute as many coefficients as desired, as the next example demonstrates.
EXAMPLE 7 Multiplying Taylor Series  Write out the terms up to degree 5 in the Maclaurin series for \( f(x) = e^x \cos x \).

Solution  We multiply the fifth-order Maclaurin polynomials of \( e^x \) and \( \cos x \) together, dropping the terms of degree greater than 5:

\[
\left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \right) \left( 1 - \frac{x^2}{2} + \frac{x^4}{24} \right)
\]

Distributing the term on the left (and ignoring products that result in terms of degree greater than 5), we obtain

\[
\left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} \right) - \left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \right) \left( \frac{x^2}{2} \right) + (1 + x) \left( \frac{x^4}{24} \right)
\]

\[
= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30}
\]

We conclude that the Maclaurin series for \( f(x) = e^x \cos x \) (with the terms up to degree 5) appears as

\[
e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \cdots
\]

In the next example, we express a definite integral of \( \sin(x^2) \) as an infinite series. This is useful because the definite integral cannot be evaluated directly by finding an antiderivative of \( \sin(x^2) \).

EXAMPLE 8 Let \( J = \int_0^1 \sin(x^2) \, dx \).

(a) Express \( J \) as an infinite series.

(b) Determine \( J \) to within an error less than \( 10^{-4} \).

Solution

(a) The Maclaurin expansion for \( f(x) = \sin x \) is valid for all \( x \), so we have

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \implies \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}
\]

We obtain an infinite series for \( J \) by integration:

\[
J = \int_0^1 \sin(x^2) \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^1 x^{4n+2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{1}{4n+3} \right)
\]

\[
= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75,600} + \cdots
\]

(b) The infinite series for \( J \) is an alternating series with decreasing terms, so the sum of the first \( N \) terms is accurate to within an error that is less than the \( (N+1) \)st term. The absolute value of the fourth term \( 1/75,600 \) is smaller than \( 10^{-4} \), so we obtain the desired accuracy using the first three terms of the series for \( J \):

\[
J \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \approx 0.31028
\]

The error satisfies

\[
|J - \left( \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \right)| < \frac{1}{75,600} \approx 1.3 \times 10^{-5}
\]

The percentage error is less than 0.005% with just three terms.

The next example demonstrates how power series can be used to assist in the evaluation of limits.
EXAMPLE 9 Determine \( \lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x} \).

Solution This limit is of indeterminate form \( \frac{0}{0} \), so we could use L'Hôpital's Rule repeatedly. However, instead, we will work with the Maclaurin series. We have

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
\]

Hence, the limit becomes

\[
\lim_{x \to 0} \frac{x - \sin x}{x^3 \cos x} = \lim_{x \to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^3 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}
\]

\[
= \lim_{x \to 0} \frac{\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}{x^3 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}
\]

\[
= \lim_{x \to 0} \frac{\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}{x^3 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}
\]

\[
= \lim_{x \to 0} \frac{1}{3!} = \frac{1}{6}
\]

Binomial Series

Isaac Newton discovered an important generalization of the Binomial Theorem around 1665. For any number \( a \) (integer or not) and integer \( n \geq 0 \), we define the binomial coefficient:

\[
\binom{a}{n} = \frac{a(a-1)(a-2) \cdots (a-n+1)}{n!}, \quad \binom{a}{0} = 1
\]

For example,

\[
\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20,
\]

\[
\binom{4}{3} = \frac{4 \cdot \frac{1}{3} \cdot \left(-\frac{2}{3}\right)}{3 \cdot 2 \cdot 1} = -\frac{4}{81}
\]

Let

\[
f(x) = (1 + x)^a
\]

The Binomial Theorem of algebra (see Appendix C) states that for any whole number \( a \),

\[
(r + s)^a = r^a + \binom{a}{1} r^{a-1} s + \binom{a}{2} r^{a-2} s^2 + \cdots + \binom{a}{a-1} r s^{a-1} + s^a
\]

Setting \( r = 1 \) and \( s = x \), we obtain the expansion of \( f(x) \):

\[
(1 + x)^a = 1 + \binom{a}{1} x + \binom{a}{2} x^2 + \cdots + \binom{a}{a-1} x^{a-1} + x^a
\]

We derive Newton's generalization by computing the Maclaurin series of \( f(x) \) without assuming that \( a \) is a whole number. Observe that the derivatives follow a pattern:

\[
f(x) = (1 + x)^a \quad \quad f(0) = 1
\]

\[
f'(x) = a(1 + x)^{a-1} \quad \quad f'(0) = a
\]

\[
f''(x) = a(a - 1)(1 + x)^{a-2} \quad \quad f''(0) = a(a - 1)
\]

\[
f'''(x) = a(a - 1)(a - 2)(1 + x)^{a-3} \quad \quad f'''(0) = a(a - 1)(a - 2)
\]
In general, \( f^{(n)}(0) = a(a-1)(a-2) \cdots (a-n+1) \) and
\[
\frac{f^{(n)}(0)}{n!} = \frac{a(a-1)(a-2) \cdots (a-n+1)}{n!} = \binom{a}{n}
\]

Hence, the Maclaurin series for \( f(x) = (1 + x)^a \) is the binomial series
\[
\sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + ax + \binom{a}{2} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \cdots + \binom{a}{n} x^n + \cdots
\]

The Ratio Test shows that this series has radius of convergence \( R = 1 \) (Exercise 94), and an additional argument (developed in Exercise 95) shows that it converges to \((1 + x)^a\) for \(|x| < 1\).

**Theorem 3** The Binomial Series  For any exponent \( a \) and for \(|x| < 1\),
\[
(1 + x)^a = 1 + \binom{a}{1} x + \binom{a}{2} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \cdots + \binom{a}{n} x^n + \cdots
\]

**Example 10** Find the terms through degree 4 in the Maclaurin expansion of
\[ f(x) = (1 + x)^{4/3} \]

**Solution** The binomial coefficients \( \binom{4/3}{n} \) for \( 0 < n < 4 \) are
\[
\begin{align*}
1, & \quad \frac{4}{3} \left( \frac{4}{3} \right) = \frac{4}{3} \cdot \frac{4}{3} = \frac{4}{9},
\frac{4}{3} \left( \frac{4}{3} \right) \left( -\frac{2}{3} \right) = \frac{4}{9} \cdot \frac{4}{3} \cdot \left( -\frac{2}{3} \right) = \frac{4}{27},
\frac{4}{3} \left( \frac{4}{3} \right) \left( -\frac{2}{3} \right) \left( -\frac{5}{3} \right) = \frac{4}{27} \cdot \frac{4}{3} \cdot \left( -\frac{2}{3} \right) \cdot \left( -\frac{5}{3} \right) = \frac{5}{243}
\end{align*}
\]

Therefore, \((1 + x)^{4/3} \approx 1 + \frac{4}{3} x + \frac{4}{9} x^2 - \frac{4}{27} x^3 + \frac{5}{243} x^4 + \cdots\).

**Example 11** Find the Maclaurin series for
\[ f(x) = \frac{1}{\sqrt{1 - x^2}} \]

**Solution** First, let's find the coefficients in the binomial series for \((1 + x)^{-1/2}\):
\[
1, \quad -\frac{1}{2} = -\frac{1}{2}, \quad -\frac{1}{2} - \frac{3}{4} = -\frac{5}{4}, \quad -\frac{1}{2} - \frac{5}{4} - \frac{7}{6} = -\frac{35}{12}, \quad -\frac{1}{2} - \frac{5}{4} - \frac{7}{6} - \frac{9}{8} = -\frac{217}{48}
\]

The general pattern is
\[
\binom{-\frac{1}{2}}{n} = -\frac{1}{2} \cdot \frac{-\frac{5}{4}}{\frac{5}{4}} \cdot \frac{-\frac{7}{6}}{\frac{5}{4}} \cdot \cdots \cdot \frac{-\frac{3n-1}{2n-2}}{\frac{3n-1}{2n-2}} = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 2n}
\]

Thus, the following binomial expansion is valid for \(|x| < 1\):
\[
\frac{1}{\sqrt{1 + x}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} x^n = 1 - \frac{1}{2} x + \frac{1 \cdot 3}{2 \cdot 4} x^2 - \cdots
\]

If \(|x| < 1\), then \(|x|^2 < 1\), and we can substitute \(-x^2\) for \(x\) to obtain
\[
\frac{1}{\sqrt{1 - x^2}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} x^{2n} = 1 + \frac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \cdots
\]

Taylor series are particularly useful for studying the so-called special functions (such as Bessel and hypergeometric functions) that appear in a wide range of physics and engineering applications. One example is the following elliptic integral of the first kind, defined for \(|k| < 1\):
\[
E(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}
\]
This function is used in physics to compute the period $T$ of pendulum of length $L$ (meters) released from an angle $\theta$ (Figure 1). When $\theta$ is small, we can use the small-angle approximation $T \approx 2\pi\sqrt{L/g}$ where $g$ is the acceleration due to gravity, $9.8 \text{ m/s}^2$. This approximation breaks down for large angles (Figure 2). The exact value of the period is $T = 4\sqrt{L/g} E(k)$, where $k = \sin \frac{1}{2}\theta$.

**EXAMPLE 12 Elliptic Function** Find the Maclaurin series for $E(k)$ and estimate $E(k)$ for $k = \sin \frac{\pi}{3}$.

**Solution** Substitute $x = k \sin t$ in the Taylor expansion (3):

$$
\frac{1}{\sqrt{1 - k^2 \sin^2 t}} = 1 + \frac{1}{2} k^2 \sin^2 t + \frac{1}{2} \cdot \frac{3}{4} k^4 \sin^4 t + \frac{1}{2} \cdot \frac{3 \cdot 5}{2 \cdot 4} k^6 \sin^6 t + \cdots
$$

This expansion is valid because $|k| < 1$ and hence, $|k| = |k \sin t| < 1$. Thus, $E(k)$ is equal to

$$
\int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} = \int_0^{\pi/2} dt + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdot (2n)} \left( \int_0^{\pi/2} \sin^{2n} t \, dt \right) k^{2n}
$$

According to Exercise 76 in Section 8.2,

$$
\int_0^{\pi/2} \sin^{2n} t \, dt = \left( \frac{1 \cdots (2n-1)}{2 \cdots 2n} \right) \frac{\pi}{2^n}
$$

This yields

$$
E(k) = \frac{\pi}{2} + \frac{\pi}{2} \sum_{n=1}^{\infty} \left( \frac{1 \cdots (2n-1)^2}{2 \cdots 2n} \right) k^{2n}
$$

We approximate $E(k)$ for $k = \sin \left( \frac{\pi}{3} \right) = \frac{1}{2}$ using the first five terms:

$$
E \left( \frac{1}{2} \right) \approx \frac{\pi}{2} \left( 1 + \left( \frac{1}{2} \right)^2 \left( \frac{1}{2} \right)^2 + \left( \frac{1}{2} \cdot \frac{3}{2} \right)^2 \left( \frac{1}{2} \right)^4 + \left( \frac{1}{2} \cdot \frac{3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 \left( \frac{1}{2} \right)^6 \right)
$$

$$
\approx 1.68517
$$

**Euler's Formula**

Euler's formula expresses a surprising relationship between the exponential function, $f(x) = e^x$, and the basic trigonometric functions, $g(x) = \sin x$ and $h(x) = \cos x$. This formula holds for all complex numbers. The complex numbers are numbers in the form $a + bi$, where $a$ and $b$ are real numbers and $i$ is defined to be the square root of $-1$, that is, $i^2 = -1$. We can add and multiply complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)i
$$

$$(a + bi)(c + di) = ac + adi + bci + dbi^2 = (ac - bd) + (ad + bc)i
$$

As a result, we can compute polynomial functions of a complex variable $z = a + bi$:

$$
f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n
$$

We can measure the distance $d$ between complex numbers:

$$
d(a + bi, c + di) = \sqrt{(c - a)^2 + (d - b)^2}
$$

With such a measure of distance, we can address the convergence of a sequence or a series of complex numbers to a limit that is a complex number, just as we do with real numbers. In particular, the power series associated with $e^x$, $\sin x$, and $\cos x$ each can be shown to converge for all complex numbers (the proofs are essentially the same as the proofs...
for the real-number case). In the field of complex variables, these series are often used to define the corresponding functions for all complex numbers \( z \):

\[
e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots, \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots, \quad \cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots
\]

We can combine these series to obtain Euler's Formula:

**THEOREM 4 Euler's Formula**  For all complex numbers \( z \),

\[
e^{iz} = \cos z + i \sin z
\]

**Proof**  The key to the proof lies in the pattern of the powers of \( i \). First, note that

\[
i^0 = 1, \quad i^1 = i, \quad i^2 = -1, \quad i^3 = i^2 i = (-1) i = -i
\]

Furthermore, \( i^4 = i^2 i^2 = (-1)(-1) = 1 \) and \( i^5 = i^4 i = (1)i = i \), and the cycle of values \( 1, i, -1, -i \) is now repeating. Therefore, starting with \( n = 0 \), the values of \( i^n \) repeatedly cycle through \( 1, i, -1, -i \). We have

\[
e^{iz} = 1 + iz + \frac{(iz)^2}{2} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots
\]

\[
= 1 + iz - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots
\]

\[
= 1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots + i \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \right)
\]

\[
= \cos z + i \sin z
\]

Euler's Formula is particularly useful in electrical engineering. Periodic signals are often expressed in terms of sine and cosine functions. Mathematical operations and computations involving combinations of signals are often more conveniently approached with the signals expressed in terms of complex exponential functions such as \( f(z) = e^{iz} \).

If we substitute \( z = \pi \) into Euler's Formula, we obtain

\[
e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1
\]

Rearranging, this relation is expressed as

\[
e^{i\pi} + 1 = 0
\]

This equation is known as Euler's Identity and is particularly pleasing because it relates five of the more important numbers used in mathematics and its applications.

In Table 2, we provide a list of useful Maclaurin series and the values of \( x \) for which they converge.

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>Maclaurin series</th>
<th>Converges to ( f(x) ) for</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^x )</td>
<td>( \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots )</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots )</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots )</td>
<td>All ( x )</td>
</tr>
<tr>
<td>( \frac{1}{1-x} )</td>
<td>( \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots )</td>
<td>(</td>
</tr>
</tbody>
</table>
### 11.8 SUMMARY

- **Taylor series** of \( f(x) \) centered at \( x = c \):

  \[
  T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n
  \]

  The partial sum \( T_k(x) \) is the \( k \)th Taylor polynomial.

- **Maclaurin series** (\( c = 0 \)):

  \[
  T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
  \]

  - If \( f(x) \) is represented by a power series \( \sum_{n=0}^{\infty} a_n(x - c)^n \) for \( |x - c| < R \) with \( R > 0 \), then this power series is necessarily the Taylor series centered at \( x = c \).
  - A function \( f \) is represented by its Taylor series \( T(x) \) if and only if the remainder \( R_k(x) = f(x) - T_k(x) \) tends to zero as \( k \to \infty \).
  - Let \( I = (c - R, c + R) \) with \( R > 0 \). Suppose that there exists \( K > 0 \) such that \( |f^{(k)}(x)| < K \) for all \( x \in I \) and all \( k \). Then \( f \) is represented by its Taylor series on \( I \); that is, \( f(x) = T(x) \) for \( x \in I \).
  - A good way to find the Taylor series of a function is to start with known Taylor series and apply one of the following operations: differentiation, integration, multiplication, or substitution.
  - For any exponent \( a \), the binomial expansion is valid for \( |x| < 1 \):

    \[
    (1 + x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + \cdots
    \]

### 11.8 EXERCISES

**Preliminary Questions**

1. Determine \( f(0) \) and \( f''(0) \) for a function \( f \) with Maclaurin series

   \[
   T(x) = 3 + 2x + 12x^2 + 5x^3 + \cdots
   \]

2. Determine \( f(-2) \) and \( f^{(4)}(-2) \) for a function with Taylor series

   \[
   T(x) = 3(x + 2) + (x + 2)^2 - 4(x + 2)^3 + 2(x + 2)^4 + \cdots
   \]

3. What is the easiest way to find the Maclaurin series for the function \( f(x) = \sin(x^2) \)?

4. Find the Taylor series for \( f \) centered at \( c = 3 \) if \( f(3) = 4 \) and \( f'(x) \) has a Taylor expansion

   \[
   f'(x) = \sum_{n=1}^{\infty} \frac{(x - 3)^n}{n}
   \]
5. Let $T(x)$ be the Maclaurin series of $f(x)$. Which of the following guarantees that $f(2) = T(2)$?
(a) $T(x)$ converges for $x = 2$.
(b) The remainder $R_n(2)$ approaches a limit as $k \to \infty$.
(c) The remainder $R_n(2)$ approaches zero as $k \to \infty$.

Exercises

1. Write out the first four terms of the Maclaurin series of $f(x)$ if
   
   \[ f(0) = 2, \quad f'(0) = 3, \quad f''(0) = 4, \quad f'''(0) = 12 \]

2. Write out the first four terms of the Taylor series of $f(x)$ centered at $c = 3$ if
   
   \[ f(3) = 1, \quad f'(3) = 2, \quad f''(3) = 12, \quad f'''(3) = 3 \]

In Exercises 3–18, find the Maclaurin series and find the interval on which the expansion is valid.

3. \[ f(x) = \frac{1}{1 + 10x} \]
4. \[ f(x) = \frac{x^2}{1 - x^3} \]
5. \[ f(x) = \cos 3x \]
6. \[ f(x) = \sin(2x) \]
7. \[ f(x) = \sin(x^2) \]
8. \[ f(x) = e^{2x} \]
9. \[ f(x) = \ln(1 - x^2) \]
10. \[ f(x) = (1 - x)^{-1/2} \]
11. \[ f(x) = \tan^{-1}(x^2) \]
12. \[ f(x) = x^2 e^{x^2} \]
13. \[ f(x) = e^{x^2} \]
14. \[ f(x) = \frac{1 - \cos x}{x} \]
15. \[ f(x) = \ln(1 - 5x) \]
16. \[ f(x) = (x^2 + 2x)e^x \]
17. \[ f(x) = \sin x \]
18. \[ f(x) = \cosh x \]

In Exercises 19–30, find the terms through degree 4 of the Maclaurin series of $f(x)$. Use multiplication and substitution as necessary.

19. \[ f(x) = e^x \sin x \]
20. \[ f(x) = e^x \ln(1 - x) \]
21. \[ f(x) = \frac{\sin x}{1 - x} \]
22. \[ f(x) = \frac{x}{1 + \sin x} \]
23. \[ f(x) = (1 + x)^{1/4} \]
24. \[ f(x) = (1 + x)^{-3/2} \]
25. \[ f(x) = e^{\tan^{-1}x} \]
26. \[ f(x) = \sin(x^3 - x) \]
27. \[ f(x) = e^{\sin x} \]
28. \[ f(x) = e^{e^x} \]
29. \[ f(x) = \cosh(x^2) \]
30. \[ f(x) = \sin(x) \cosh(x) \]

In Exercises 31–40, find the Taylor series centered at $c$ and the interval on which the expansion is valid.

31. \[ f(x) = \frac{1}{x}, \quad c = 1 \]
32. \[ f(x) = e^{3x}, \quad c = -1 \]
33. \[ f(x) = \frac{1}{1 - x}, \quad c = 5 \]
34. \[ f(x) = \sin x, \quad c = \frac{\pi}{2} \]
35. \[ f(x) = x^4 + 3x - 1, \quad c = 2 \]
36. \[ f(x) = x^4 + 3x - 1, \quad c = 0 \]
37. \[ f(x) = \frac{1}{x^2}, \quad c = 4 \]
38. \[ f(x) = \frac{\sqrt{x}}{2}, \quad c = 4 \]
39. \[ f(x) = \frac{1}{x^2 - 1}, \quad c = 3 \]
40. \[ f(x) = \frac{1}{3x^2 - 2}, \quad c = -1 \]

41. Use the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to find the Maclaurin series for $f(x) = \cos^2 x$. 
42. Show that for $|x| < 1$,
   \[ \tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \]
   Hint: Recall that $\frac{d}{dx} \tanh^{-1}x = \frac{1}{1 - x^2}.$
43. Use the Maclaurin series for $\ln(1 + x)$ and $\ln(1 - x)$ to show that
   \[ \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \]
   for $|x| < 1$. What can you conclude by comparing this result with that of Exercise 42?
44. Differentiate the Maclaurin series for $\frac{1}{1 - x}$ twice to find the Maclaurin series of $\frac{1}{(1 - x)^2}$.
45. Show, by integrating the Maclaurin series for $f(x) = \frac{1}{\sqrt{1 - x^2}}$, that for $|x| < 1$,
   \[ \sin^{-1}x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n + 1} \]
46. Use the first five terms of the Maclaurin series in Exercise 45 to approximate $\sin^{-1} \frac{1}{2}$. Compare the result with the calculator value.
47. How many terms of the Maclaurin series of $f(x) = \ln(1 + x)$ are needed to compute $\ln 1.2$ to within an error of at most 0.0001? Make the computation and compare the result with the calculator value.
48. Show that
   \[ \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots \]
   converges to zero. How many terms must be computed to get within 0.01 of zero?
49. Use the Maclaurin expansion for $e^{-x^2}$ to express the function $F(x) = \int_0^x e^{-t^2} \, dt$ as an alternating power series in $x$ (Figure 3).
(a) How many terms of the Maclaurin series are needed to approximate the integral for $x = 1$ to within an error of at most 0.001?
(b) (CAS) Carry out the computation and check your answer using a computer algebra system.

**FIGURE 3** The Maclaurin polynomial $T_{15}(x)$ for $F(x) = \int_0^x e^{-t^2} \, dt$.
50. Let $F(x) = \int_0^x \sin t \, dt$, Show that
   \[ F(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]
   Evaluate $F(1)$ to three decimal places.
In Exercises 51–54, express the definite integral as an infinite series and find its value to within an error of at most $10^{-4}$.

51. $\int_0^1 \cos(x^2) \, dx$ 
52. $\int_0^1 \tan^{-1}(x^2) \, dx$

53. $\int_0^1 e^{-x^2} \, dx$
54. $\int_0^1 \frac{dx}{\sqrt{x + 1}}$

In Exercises 55–58, express the integral as an infinite series.

55. $\int_0^1 \frac{1 - \cos t}{t} \, dt$, for all $t$
56. $\int_0^1 \frac{1 - \sin t}{t} \, dt$, for all $t$

57. $\int_0^x \ln(1 + t^2) \, dt$, for $|x| < 1$
58. $\int_0^x \frac{dt}{\sqrt{1 - t^4}}$, for $|x| < 1$

59. Which function has a Maclaurin series $\sum_{n=0}^{\infty} (-1)^n 2^n x^n$?

60. Which function has the following Maclaurin series?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} (x - 3)^n$$

For which values of $x$ is the expansion valid?

61. Using Maclaurin series, determine to exactly what value the following series converges:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\pi/2)^n$$

62. Using Maclaurin series, determine to exactly what value the following series converges:

$$\sum_{n=0}^{\infty} \frac{(\ln 5)^n}{n!}$$

In Exercises 51–64, use Theorem 2 to prove that the $f(x)$ is represented by its Maclaurin series for all $x$.

63. $f(x) = \sin(x/2) + \cos(x/3)$
64. $f(x) = e^{-x}$
65. $f(x) = \sin x$
66. $f(x) = (1 + x)^{100}$

In Exercises 67–70, find the functions with the following Maclaurin series (refer to Table 2 prior to the section summary).

67. $1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \ldots$
68. $1 - 4x + 4x^2 - 4x^3 + 4x^4 - 4x^5 + \ldots$
69. $1 - \frac{5x^2}{3!} + \frac{5x^4}{5!} - \frac{5x^6}{7!} + \ldots$
70. $x^4 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{12}}{7} + \ldots$

In Exercises 71 and 72, let

$$f(x) = \frac{1}{1 - x(1 - 2x)}$$

71. Find the Maclaurin series of $f(x)$ using the identity

$$f(x) = \frac{2}{1 - 2x} - \frac{1}{1 + x}$$

72. Find the Taylor series for $f(x)$ at $c = 2$. Hint: Rewrite the identity of Exercise 71 as

$$f(x) = \frac{2}{-3 - 2(x - 2)} - \frac{1}{-1 - (x - 2)}$$

73. When a voltage $V$ is applied to a series circuit consisting of a resistor $R$ and an inductor $L$, the current at time $t$ is

$$i(t) = \frac{V}{R} (1 - e^{-Rt/L})$$

Expand $i(t)$ in a Maclaurin series. Show that $i(t) \approx \frac{Vt}{L}$ for small $t$.

74. Use the result of Exercise 73 and your knowledge of alternating series to show that

$$\frac{Vt}{L} \left(1 - \frac{R}{2L}\right) \leq i(t) \leq \frac{Vt}{L} \quad \text{(for all $t$)}$$

75. Find the Maclaurin series for $f(x) = \cos(x^3)$ and use it to determine $f^{(6)}(0)$.

76. Find $f^{(7)}(0)$ and $f^{(8)}(0)$ for $f(x) = \tan^{-1}x$ using the Maclaurin series.

77. Use substitution to find the first three terms of the Maclaurin series for $f(x) = e^{x^2}$. How does the result show that $f^{(n)}(0) = 0$ for $1 \leq n \leq 19$?

78. Use the binomial series to find $f^{(8)}(0)$ for $f(x) = \sqrt{1 - x^2}$.

79. Does the Maclaurin series for $f(x) = (1 + x)^{2/3}$ converge to $f(x)$ at $x = 2$? Give numerical evidence to support your answer.

80. Explain the steps required to verify that the Maclaurin series for $f(x) = e^x$ converges to $f(x)$ for all $x$.

81. Let $f(x) = \sqrt{1 + x}$.

(a) Use a graphing calculator to compare the graph of $f$ with the graphs of the first five Taylor polynomials for $f$. What do they suggest about the interval of convergence of the Taylor series?

(b) Investigate numerically whether or not the Taylor expansion for $f$ is valid for $x = 1$ and $x = -1$.

82. Use the first five terms of the Maclaurin series for the elliptic integral $E(x)$ to estimate the period $T$ of a 1-m pendulum released at an angle $\theta = \frac{\pi}{3}$ (see Example 12).

83. Use Example 12 and the approximation $\sin x \approx x$ to show that the period $T$ of a pendulum released at an angle $\theta$ has the following second-order approximation:

$$T \approx 2\pi \sqrt{\frac{L}{g} \left(1 + \frac{\theta^2}{16}\right)}$$

In Exercises 84–87, the limits can be done using multiple L'Hôpital's Rule steps. Power series provide an alternative approach. In each case substitute in the Maclaurin series for the trig function or the inverse trig function involved, simplify, and compute the limit.

84. $\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$
85. $\lim_{x \to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$
86. $\lim_{x \to 0} \frac{\tan^{-1} x - x \cos x - \frac{1}{2} x^3}{x^5}$
87. $\lim_{x \to 0} \frac{(\sin x)^2 - \cos x}{x^2}$

88. Use Euler's Formula to express each of the following in $a + bi$ form.

(a) $\cos 2i$
(b) $4e^{\frac{3i}{2}}$
(c) $i e^{\frac{5i}{2}}$

89. Use Euler's Formula to express each of the following in $a + bi$ form.

(a) $-e^{\frac{i}{2}}$
(b) $2e^{i\pi}$
(c) $3i e^{\frac{3i}{2}}$

In Exercises 90–91, use Euler's Formula to prove that the identity holds. Note the similarity between these relationships and the definitions of the hyperbolic sine and cosine functions.

90. $\cos z = \frac{e^{iz} + e^{-iz}}{2}$
91. $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$
Further Insights and Challenges

92. In this exercise, we show that the Maclaurin expansion of \( f(x) = \ln(1 + x) \) is valid for \( x = 1 \).

(a) Show that for all \( x \neq -1, \)
\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n + \frac{(-1)^{N+1} x^{N+1}}{1 + x}
\]

(b) Integrate from 0 to 1 to obtain
\[
\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} + \frac{(-1)^{N+1} \int_0^1 x^{N+1} dx}{1 + x}
\]

(c) Verify that the integral on the right tends to zero as \( N \to \infty \) by showing that it is smaller than \( \int_0^1 x^{N+1} dx \).

(d) Prove the formula
\[
\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots
\]

93. Let \( g(t) = \frac{1}{1 + t^2} \).

(a) Show that \( \int_0^1 g(t) dt = \frac{\pi}{4} = \frac{1}{2} \ln 2 \).

(b) Show that \( g(t) = 1 - t^2 + t^4 - t^6 + \cdots \).

(c) Evaluate \( S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \).

In Exercises 94 and 95, we investigate the convergence of the binomial series
\[
T_n(x) = \sum_{n=0}^{\infty} \binom{n}{k} x^n
\]

94. Prove that \( T_n(x) \) has radius of convergence \( R = 1 \) if \( a \) is not a whole number. What is the radius of convergence if \( a \) is a whole number?

95. By Exercise 94, \( T_n(x) \) converges for \( |x| < 1 \), but we do not yet know whether \( T_n(x) = (1 + x)^a \).

(a) Verify the identity
\[
a \binom{n}{k} = n \binom{n-1}{k} + (n+1) \binom{n}{k+1}
\]

(b) Use (a) to show that \( y = T_n(x) \) satisfies the differential equation \((1 + x)y' = ay\) with initial condition \( y(0) = 1 \).

(c) Prove \( T_n(x) = (1 + x)^a \) for \( |x| < 1 \) by showing that the derivative of the ratio \( \frac{T_n(x)}{1 + x} \) is zero.

96. The function \( G(k) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1 - k^2 \sin^2 t}{t} dt \) is called an elliptic integral of the second kind. Prove that for \( |k| < 1 \),
\[
G(k) = \frac{\pi}{2} = \sum_{n=1}^{\infty} \left( \begin{array}{c} 1 \cdots (2n-1) \\ 2 \cdots 4 \end{array} \right) \frac{k^{2n}}{2n+1}
\]

97. Assume that \( a < b \) and let \( L \) be the arc length (circumference) of the ellipse \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \) shown in Figure 4. There is no explicit formula for \( L \), but it is known that \( L = 4bG(k) \), where \( G(k) \) as in Exercise 96 and \( k = \sqrt{1 - a^2/b^2} \). Use the first three terms of the expansion of Exercise 96 to estimate \( L \) when \( a = 4 \) and \( b = 5 \).

FIGURE 4 The ellipse \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \).

98. Use Exercise 96 to prove that if \( a < b \) and \( a/b \) is near 1 (a nearly circular ellipse), then
\[
L \approx \frac{\pi}{2} \left( 3a^2 + \frac{a^2}{b^2} \right)
\]

Hint: Use the first two terms of the series for \( G(k) \).

99. Irrationality of \( e \) Prove that \( e \) is an irrational number using the following argument by contradiction. Suppose that \( e = M/N \), where \( M, N \) are nonzero integers.

(a) Show that \( M! e^{-1} \) is a whole number.

(b) Use the power series for \( f(x) = e^x \) at \( x = -1 \) to show that there is an integer \( B \) such that \( M! e^{-1} \) equals
\[
B + (-1)^{M+1} \left( \frac{1}{M+1} - \frac{1}{(M+1)(M+2)} + \cdots \right)
\]

(c) Use your knowledge of alternating series with decreasing terms to conclude that \( 0 < |M! e^{-1} - B| < 1 \) and observe that this contradicts (a). Hence, \( e \) is not equal to \( M/N \).

100. Use the result of Exercise 71 in Section 7.5 to show that the Maclaurin series of the function
\[
f(x) = \begin{cases} \frac{e^{-1/x}}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}
\]
is \( T(x) = 0 \). This provides an example of a function \( f \) whose Maclaurin series converges but does not converge to \( f(x) \) (except at \( x = 0 \)).

CHAPTER REVIEW EXERCISES

1. Let \( a_n = \frac{n - 3}{n!} \) and \( b_n = a_{n+3} \). Calculate the first three terms in each sequence.
   (a) \( a_n^2 \)  
   (b) \( b_n \)  
   (c) \( a_n b_n \)  
   (d) \( 2a_n + 3a_n \)

2. Prove that \( \lim_{n \to \infty} \frac{2n-1}{3n+2} = \frac{2}{3} \) using the limit definition.

In Exercises 3–8, compute the limit (or state that it does not exist) assuming that \( \lim_{n \to \infty} a_n = 2 \).

3. \( \lim_{n \to \infty} (5a_n - 2a_n^2) \)  
4. \( \lim_{n \to \infty} \frac{1}{a_n} \)  
5. \( \lim_{n \to \infty} e^{a_n} \)  
6. \( \lim_{n \to \infty} \cos(\pi a_n) \)  
7. \( \lim_{n \to \infty} (-1)^n a_n \)  
8. \( \lim_{n \to \infty} \frac{a_n + n}{a_n + n^2} \)
In Exercises 9–22, determine the limit of the sequence or show that the sequence diverges.

9. \( a_n = \sqrt{n + 5} - \sqrt{n + 2} \)
10. \( a_n = \frac{3n^3 - n}{1 - 2n^3} \)
11. \( a_n = 2^{1/n^2} \)
12. \( a_n = \frac{10^n}{n!} \)
13. \( b_m = 1 + (-1)^m \)
14. \( b_m = \frac{1 + (-1)^m}{m} \)
15. \( b_n = \tan^{-1} \left( \frac{n + 2}{n + 3} \right) \)
16. \( a_n = \frac{100^n}{n!} - \frac{3 + \pi^n}{5^n} \)
17. \( b_n = \sqrt{n^2 + n} - \sqrt{n^2 + 1} \)
18. \( c_n = \sqrt{n^2 + n} - \sqrt{n^2 - n} \)
19. \( b_m = \left( 1 + \frac{1}{m} \right)^{3m} \)
20. \( c_n = \left( 1 + \frac{2}{n} \right)^n \)
21. \( b_n = n \left( \ln(n + 1) - \ln n \right) \)
22. \( c_n = \frac{\ln(n^2 + 1)}{\ln(n^3 + 1)} \)

23. Use the Squeeze Theorem to show that \( \lim_{n \to \infty} \frac{\arctan(n^2)}{\sqrt{n}} = 0. \)

24. Give an example of a divergent sequence \( \{a_n\} \) such that \( \{\sin a_n\} \) is convergent.

25. Calculate \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \), where \( a_n = \frac{1}{2} n^2 - \frac{1}{3} n^2 \).

26. Define \( a_{n+1} = \sqrt{a_n} + 6 \) with \( a_1 = 2. \)
   (a) Compute \( a_n \) for \( n = 2, 3, 4, 5. \)
   (b) Show that \( \{a_n\} \) is increasing and is bounded by 3.
   (c) Prove that \( \lim_{n \to \infty} a_n \) exists and find its value.

27. Calculate the partial sums \( S_5 \) and \( S_7 \) of the series \( \sum_{n=1}^{\infty} \frac{n - 2}{n^2 + 2n}. \)

28. Find the sum \( 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots. \)
29. Find the sum \( \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots. \)
30. Use series to determine a reduced fraction that has decimal expansion 0.12121212···.
31. Use series to determine a reduced fraction that has decimal expansion 0.108108108···.

32. Find the sum \( \sum_{n=2}^{\infty} \left( \frac{2}{e} \right)^n. \)
33. Find the sum \( \sum_{n=1}^{\infty} \frac{2^n + 3}{3^n}. \)
34. Show that \( \sum_{n=1}^{\infty} \left( b - \tan^{-1} n^2 \right) \) diverges if \( b \neq \frac{\pi}{2}. \)

35. Give an example of divergent series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) such that \( \sum_{n=1}^{\infty} (a_n + b_n) = 1. \)

36. Let \( S = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right) \). Compute \( S_N \) for \( N = 1, 2, 3, 4. \) Find \( S \) by showing that \( S_N = \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}. \)

37. Evaluate \( S = \sum_{n=1}^{\infty} \frac{1}{n(n+3)}. \)

38. Find the total area of the infinitely many circles on the interval \([0, 1]\) in Figure 1.

![Figure 1](image)

In Exercises 39–42, use the Integral Test to determine whether the infinite series converges.

39. \( \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1} \)
40. \( \sum_{n=1}^{\infty} \frac{n^2}{(n+1)^{1.01}} \)
41. \( \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 + 2n)} \)
42. \( \sum_{n=1}^{\infty} \frac{n^3}{e^n} \)

In Exercises 43–50, use the Direct Comparison or Limit Comparison Test to determine whether the infinite series converges.

43. \( \sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{1}{2}}} \)
44. \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + n} \)
45. \( \sum_{n=2}^{\infty} \frac{n^2 + 1}{n^3.5 - 2} \)
46. \( \sum_{n=1}^{\infty} \frac{1}{n - \ln n} \)
47. \( \sum_{n=2}^{\infty} \frac{n}{n^2 + 5} \)
48. \( \sum_{n=1}^{\infty} \frac{1}{3^n - 2^n} \)
49. \( \sum_{n=1}^{\infty} \frac{n^{10} + 10^n}{n^{11} + 11^n} \)
50. \( \sum_{n=1}^{\infty} \frac{n^{20} + 21^n}{n^{21} + 20^n} \)

51. Determine the convergence of \( \sum_{n=1}^{\infty} \frac{3^n + n}{3^n - 2} \) using the Limit Comparison Test with \( b_n = \left( \frac{3}{2} \right)^n. \)

52. Determine the convergence of \( \sum_{n=1}^{\infty} \frac{\ln n}{n^{1.5n}} \) using the Limit Comparison Test with \( b_n = \frac{1}{1.4^n}. \)
53. Let \( a_n = 1 - \sqrt{1 - \frac{1}{n^2}} \). Show that \( \lim_{n \to \infty} a_n = 0 \) and that \( \sum_{n=1}^{\infty} a_n \) diverges. \( \text{Hint:} \) Show that \( a_n \geq \frac{1}{2n} \).

54. Determine whether \( \sum_{n=2}^{\infty} \left( 1 - \sqrt{1 - \frac{1}{n^2}} \right) \) converges.

55. Consider \( \sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)^{3/2}} \):
   
   (a) Show that the series converges.
   
   (b) \( \text{CAS} \) Use the inequality in (4) from Exercise 83 of Section 11.3 with \( M = 99 \) to approximate the sum of the series. What is the maximum size of the error?

In Exercises 56–59, determine whether the series converges absolutely, if it does not determine whether it converges conditionally.

56. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + 2n} \)

57. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n! \ln(n+1)} \)

58. \( \sum_{n=1}^{\infty} \frac{\cos \left( \frac{\pi}{n} + \pi n \right)}{\sqrt{n}} \)

59. \( \sum_{n=1}^{\infty} \frac{\cos \left( \frac{\pi}{n} + 2\pi n \right)}{\sqrt{n}} \)

60. \( \text{CAS} \) Use a computer algebra system to approximate \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \sqrt{n}} \) to within an error of at most \( 10^{-3} \).

61. Catalan's constant is defined by \( K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \).
   
   (a) How many terms of the series are needed to calculate \( K \) with an error of less than \( 10^{-6} \)?
   
   (b) \( \text{CAS} \) Carry out the calculation.

62. Give an example of a conditionally convergent series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) such that \( \sum_{n=1}^{\infty} (a_n + b_n) \) converges absolutely.

63. Let \( \sum_{n=1}^{\infty} a_n \) be an absolutely convergent series. Determine whether the following series are convergent or divergent:
   
   (a) \( \sum_{n=1}^{\infty} \left( a_n + \frac{1}{n^2} \right) \)
   
   (b) \( \sum_{n=1}^{\infty} (-1)^n a_n \)
   
   (c) \( \sum_{n=1}^{\infty} \frac{1}{n + 2} \)
   
   (d) \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} \)

64. Let \( \{a_n\} \) be a positive sequence such that \( \lim_{n \to \infty} \sqrt[n]{a_n} = \frac{1}{2} \). Determine whether the following series converge or diverge:
   
   (a) \( \sum_{n=1}^{\infty} 2a_n \)
   
   (b) \( \sum_{n=1}^{\infty} 3^na_n \)
   
   (c) \( \sum_{n=1}^{\infty} \sqrt{n}a_n \)

In Exercises 65–72, apply the Ratio Test to determine convergence or divergence, or state that the Ratio Test is inconclusive.

65. \( \sum_{n=1}^{\infty} \frac{n^5}{5^n} \)

66. \( \sum_{n=1}^{\infty} \frac{\sqrt{n} + 1}{n^8} \)

67. \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n^3} \)

68. \( \sum_{n=1}^{\infty} \frac{n^4}{n^2!} \)

69. \( \sum_{n=1}^{\infty} \frac{2n^2}{n!} \)

70. \( \sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}} \)

71. \( \sum_{n=1}^{\infty} \left( \frac{n}{2} \right)^n \frac{1}{n!} \)

72. \( \sum_{n=1}^{\infty} \left( \frac{n}{4} \right)^n \frac{1}{n!} \)

In Exercises 73–76, apply the Root Test to determine convergence or divergence, or state that the Root Test is inconclusive.

73. \( \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n \)

74. \( \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \)

75. \( \sum_{n=1}^{\infty} \left( \frac{3}{4} \right)^n \)

76. \( \sum_{n=1}^{\infty} \left( \frac{\cos \frac{1}{n}}{n} \right)^3 \)

In Exercises 77–100, determine convergence or divergence using any method covered in the text.

77. \( \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \)

78. \( \sum_{n=1}^{\infty} \left( \frac{7n}{5n} \right)^n \)

79. \( \sum_{n=1}^{\infty} e^{-0.02n} \)

80. \( \sum_{n=1}^{\infty} ne^{-0.02n} \)

81. \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} + \sqrt{n+1}} \)

82. \( \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^{3/2}} \)

83. \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n} \)

84. \( \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \)

85. \( \sum_{n=1}^{\infty} \frac{n}{1 + 100n} \)

86. \( \sum_{n=1}^{\infty} \frac{n^3 - 2n^2 + n - 4}{2n^3 + 3n^3 - 4n^2 - 1} \)

87. \( \sum_{n=1}^{\infty} \frac{\cos n}{n^{3/2}} \)

88. \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3 + 1}} \)

89. \( \sum_{n=1}^{\infty} \left( \frac{n}{5n+2} \right)^n \)

90. \( \sum_{n=1}^{\infty} \left( \frac{n}{n} \right)^n \)

91. \( \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n} + \ln n} \)

92. \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(1 + \sqrt{n})} \)

93. \( \sum_{n=1}^{\infty} \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right) \)

94. \( \sum_{n=1}^{\infty} \left( \ln n - \ln(n+1) \right) \)

95. \( \sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}} \)

96. \( \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}} \)

97. \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \)

98. \( \sum_{n=1}^{\infty} \frac{1}{\ln^2 n} \)

99. \( \sum_{n=1}^{\infty} \frac{\sin^2 \frac{\pi}{n}}{n} \)

100. \( \sum_{n=0}^{\infty} \frac{2^n}{n!} \)
In Exercises 101–106, find the interval of convergence of the power series.

101. \[ \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \]
102. \[ \sum_{n=0}^{\infty} \frac{x^n}{n+1} \]
103. \[ \sum_{n=0}^{\infty} \frac{n^6}{n!} (x-3)^n \]
104. \[ \sum_{n=0}^{\infty} nx^n \]
105. \[ \sum_{n=0}^{\infty} (nx)^n \]
106. \[ \sum_{n=0}^{\infty} \frac{(2x-3)^n}{n \ln n} \]

107. Expand \( f(x) = \frac{2}{3x - 4} \) as a power series centered at \( c = 0 \). Determine the values of \( x \) for which the series converges.

108. Prove that
\[ \sum_{n=0}^{\infty} \frac{ne^{-nx}}{n+1} = \frac{e^{-x}}{(1-e^{-x})^2} \]

HINT: Express the left-hand side as the derivative of a geometric series.

109. Let \( F(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k \cdot k!} \).
(a) Show that \( F(x) \) has infinite radius of convergence.
(b) Show that \( y = F(x) \) is a solution of
\[ y'' = xy' + y, \quad y(0) = 1, \quad y'(0) = 0 \]
(c) Plot the partial sums \( S_N \) for \( N = 1, 3, 5, 7 \) on the same set of axes.

110. Find a power series \( P(x) = \sum_{n=0}^{\infty} a_n x^n \) that satisfies the Laguerre differential equation
\[ xy'' + (1 - x)y' - y = 0 \]
with initial condition satisfying \( P(0) = 1 \).

111. Use power series to evaluate \( \lim_{x \to 0} \frac{x^2 e^x}{\cos x - 1} \).

112. Use power series to evaluate \( \lim_{x \to 0} \frac{x^2 (1 - \ln (1 + x))}{\sin x - x} \).

In Exercises 113–118, find the Taylor polynomial at \( x = a \) for the given function.

113. \( f(x) = x^3 \), \( T_3, \quad a = 1 \)
114. \( f(x) = 3(x^3 + 2) - 5(x + 2) \), \( T_3, \quad a = -2 \)
115. \( f(x) = x \ln(x) \), \( T_4, \quad a = 1 \)
116. \( f(x) = 3(x + 2)^{1/3} \), \( T_3, \quad a = 2 \)
117. \( f(x) = xe^{-x^2} \), \( T_4, \quad a = 0 \)
118. \( f(x) = \ln(\cos x) \), \( T_3, \quad a = 0 \)
119. Find the \( n \)-th Maclaurin polynomial for \( f(x) = e^{2x} \).
120. Use the fifth Maclaurin polynomial of \( f(x) = e^{x^2} \) to approximate \( \sqrt{x} \). Use a calculator to determine the error.

121. Use the third Taylor polynomial of \( f(x) = \tan^{-1} x \) at \( a = 1 \) to approximate \( f(1.1) \). Use a calculator to determine the error.

122. Let \( T_4 \) be the Taylor polynomial for \( f(x) = \sqrt{x} \) at \( a = 16 \). Use the Error Bound to find the maximum possible size of \( |f(17) - T_4(17)| \).

123. Find \( n \) such that \( |e - T_n(1)| < 10^{-8} \), where \( T_n \) is the \( n \)-th Maclaurin polynomial for \( f(x) = e^x \).

124. Let \( T_4 \) be the Taylor polynomial for \( f(x) = x \ln x \) at \( a = 1 \) computed in Exercise 115. Use the Error Bound to find a bound for \( |f(1.2) - T_4(1.2)| \).

125. Verify that \( T_n(x) = 1 + x + x^2 + \cdots + x^n \) is the \( n \)-th Maclaurin polynomial of \( f(x) = 1/(1-x) \). Show using substitution that the \( n \)-th Maclaurin polynomial for \( f(x) = 1/(1-x) \) is
\[ T_n(x) = 1 + \frac{1}{4} x + \frac{1}{4^2} x^2 + \cdots + \frac{1}{4^n} x^n \]

What is the \( n \)-th Maclaurin polynomial for \( g(x) = \frac{1}{1+x} \)?

126. Let \( f(x) = \frac{5}{4 + 3x - x^2} \) and let \( a_k \) be the coefficient of \( x^k \) in the Maclaurin polynomial \( T_n \) for \( k \leq n \).
(a) Show that \( f(x) = \left( \frac{1/4}{1-x/4} + \frac{1}{1+x} \right) \).
(b) Use Exercise 125 to show that \( a_k = \frac{1}{4^k} + (-1)^k \).
(c) Compute \( T_5 \).

In Exercises 127–136, find the Taylor series centered at \( c \).

127. \( f(x) = e^{4x} \), \( c = 0 \)
128. \( f(x) = e^{2x} \), \( c = -1 \)
129. \( f(x) = x^4 \), \( c = 2 \)
130. \( f(x) = x^3 - x \), \( c = -2 \)
131. \( f(x) = \sin x \), \( c = \pi \)
132. \( f(x) = e^{x-1} \), \( c = -1 \)
133. \( f(x) = \frac{1}{1-2x} \), \( c = -2 \)
134. \( f(x) = \frac{1}{(1-2x)^2} \), \( c = -2 \)
135. \( f(x) = \ln \left( \frac{x}{2} \right) \), \( c = 2 \)
136. \( f(x) = x \ln \left( 1 + \frac{x}{2} \right) \), \( c = 0 \)

In Exercises 137–140, find the first three terms of the Maclaurin series of \( f(x) \) and use it to calculate \( f^{(3)}(0) \).

137. \( f(x) = (x^2 - x)e^x \)
138. \( f(x) = \tan^{-1}(x^2 - x) \)
139. \( f(x) = \frac{1}{1 + \tan x} \)
140. \( f(x) = (\sin x)\sqrt{1+x} \)

141. Calculate \( \frac{\pi}{2} - \frac{\pi^3}{2^33!} + \frac{\pi^5}{2^55!} - \frac{\pi^7}{2^77!} + \cdots \).

142. Find the Maclaurin series of the function \( F(x) = \int_0^x e^t - 1 \ dt \).