Topics in Integration and Infinite Series

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Last updated: January 5, 2020

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Preface

In short, what follows is an expression of my love for integrals and infinite series. Despite having studied and taught this level of calculus for years, I am continuously fascinated and surprised by the wonder of these topics.

At length, the reason for this book's existence is that a typical second course in calculus — one that covers the standard techniques of integration and the fundamentals of infinite series and power series — barely scratches the surface of a deep and beautiful realm of math. There are integration techniques and tricks that are never taught, even though a dedicated calculus student is more than capable of putting them to good use; there are delicate and interesting tests for convergence of infinite series that are not well known; finally, there are fascinating (albeit difficult) problems, examples, and phenomena that an undergraduate calculus course simply doesn't have time to discuss. There is a world of exciting and accessible material for an eager calculus student to explore, and the standard calculus curriculum (understandably) does not make this known.

Thus, I have collected a number of captivating topics from the world of integration and infinite series and given them a home in this book. All of the content in these pages is known and documented in various books, articles, and Math Stack Exchange threads, but having them all under the same roof is evidently rare. Furthermore, many of these topics are located in sources that are too advanced for an undergraduate calculus student and explained using real or complex analysis. I have done my best present all of the topics in this book using only elementary calculus so that *anyone* can appreciate them.

Who this book is written for

I had two kinds people in mind while writing this book.

- A motivated calculus student who has taken a standard course in integration and infinite series (or is taking one for the first time) who is keen to explore the beautiful and challenging math beyond the conventional curriculum.
- A graduate student or lecturer who wants a deeper appreciation of this level of calculus, either for the purposes of teaching (i.e., finding interesting supplemental material to discuss) or simply for personal satisfaction.[1] Even as a graduate student with a decent analysis background, many of the topics and techniques in this book were unknown and interesting to me.

How this book is organized

Chapter 1 is an overview of the calculus that I expect you (as a reader) to know[2] in order to understand what the rest of the book has to offer. More than just a review, I have also

[1] If you are a grad student or mathematician reading this, keep in mind that this is a *calculus* book, so I skip over many analytic issues.

[2] The exceptions are two of the solutions in Chapter 5 and some of the problems in 6, which use a little bit of multivariable calculus.

injected some of my own insights and perspectives to help a calculus learn the material. *If you are a UCLA Math 31B student reading this, you can more or less treat Chapter 1 as a partial study guide for the class. It may or may not be comprehensive.*

Chapter 2 discusses a number of integration techniques and tricks that are not usually covered in calculus classes, and Chapter 3 analogously discusses tests for convergence of infinite series (the proofs of which I have deferred to Appendix A) that are not covered in calculus classes. Chapter 4 contains a handful of additional topics involving infinite series that are not necessarily tests for convergence.

After this group of chapters, things get more interesting and significantly more difficult. Chapter 5 is an entire chapter devoted to the Basel problem, i.e., the evaluation of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This problem is usually considered inaccessible to undergraduate calculus students; Chapter 5 contains four of my favorite solutions which only rely on calculus. Chapter 6 is a ridiculous chapter for the brave reader containing some extremely difficult integrals and infinite series. I only use calculus to evaluate them (so no fancy complex analysis or anything like that), but they are truly a handful!

Final comments

???

Chapter 1

Preliminaries

Chapter 2

Integration Techniques

The study of integration techniques is a pillar of any calculus class; the main culprits are variable substitutions (including trigonometric substitutions), integration by parts, and partial fractions. These are arguably the most important techniques to learn, and you can compute an impressive zoo of integrals and antiderivatives. However, there are some other tricks that usually are not discussed. I will describe a few of them in this section. For many more interesting integrals, [3], [4], and [2] are wonderful resources; much of the following chapter is inspired by those books.

2.1 Imaginary numbers

The first technique I'm going to talk about requires the introduction of a new number: the imaginary number i. You may have come across this suspicious quantity in other contexts and wondered why such a thing exists. Turns out, i can come in handy for evaluating integrals! Before we start doing examples, I need to explain a few things about imaginary numbers in general. If you feel comfortable with imaginary numbers already, feel free to skip reading the following subsection.

2.1.1 Generalities on imaginary numbers

The number i is defined to be the square root of -1, so that its defining property is

$$i^2 = -1.$$

That's it! Usually, we can more or less treat i like any other constant number and do calculus as normal.[1] For example, if $f(x) = 2ix^2 + i$, then $f'(x) = 4ix$. Similarly,[2]

$$\int_0^1 e^{ix}\, dx = \frac{e^{ix}}{i}\bigg|_0^1 = \frac{e^i - 1}{i}.$$

More generally, a *complex number* is any number of the form

$$a + bi$$

where a and b are real numbers. For example, $1+i$, $\pi - 3i$ and $-i$ are all complex numbers.

[1] There *are* some things to worry about, but since this is a calculus document and not an analysis or complex analysis document, I won't discuss them here. See ??

[2] You may be wondering *but what does e^i mean*, and that is a good question to ask. Like I mentioned in the previous footnote, we won't worry about that here.

One thing that is important to know about complex numbers is how to take their absolute value. We define[3]

$$|a + bi| = \sqrt{a^2 + b^2}.$$

So $|i| = 1$, $|1 + i| = \sqrt{2}$, etc.

The next important thing to learn about complex numbers is an amazing formula called *Euler's formula*, which relates the trig functions sine and cosine to the exponential function. This will form the basis for a number of examples we will do in this section. I will state the formula and give you two different proofs.

Theorem 2.1 (Euler's formula)**.** *For any real number x,*

$$e^{ix} = \cos x + i \sin x.$$

Proof 1. The first proof is not conventional, but quite clever. Let $f(x) = e^{-ix}(\cos x + i \sin x)$. Then

$$\begin{aligned} f'(x) &= -ie^{-ix}(\cos x + i \sin x) + e^{-ix}(-\sin x + i \cos x) \\ &= -ie^{-ix} \cos x + ie^{-ix} \cos x + e^{-ix} \sin x - e^{-ix} \sin x \\ &= 0. \end{aligned}$$

Since $f'(x) = 0$, it follows that $f(x)$ must be constant! Thus,

$$e^{-ix}(\cos x + i \sin x) = C$$

for some constant C. Plugging in $x = 0$ yields $C = \cos 0 = 1$. Therefore,

$$e^{-ix}(\cos x + i \sin x) = 1$$

and therefore

$$e^{ix} = \cos x + i \sin x.$$

□

Proof 2. This next proof is the one usually presented in calculus textbooks, as it nicely makes use of MacLaurin series. In particular, recall the following MacLaurin series:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots . \end{aligned}$$

Using powers of i, we can make the above power series fit together nicely. Note that

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

[3]The reason we define the absolute value this way is because we can pretend that complex numbers live on a plane. In particular, the x axis represents the real direction and the y axis represents the imaginary direction, so that the number $1 + 2i$ would live at the coordinate $(1, 2)$. The absolute value of a complex number $a + bi$ is then the distance from the origin to the coordinate (a, b).

and then the pattern repeats. Thus,

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ i\sin x &= ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots . \end{aligned}$$

and so

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots \\ i\sin x &= ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots . \end{aligned}$$

By inspection, adding the latter two power series together gives the former. Therefore,

$$e^{ix} = \cos x + i\sin x.$$

□

Before we move on to examples, we can use Euler's formula to solve for $\sin x$ and $\cos x$ in terms of the exponential function. Note that

$$e^{-ix} = \cos(x) + i\sin(-x) = \cos x - i\sin x.$$

Thus,

$$e^{ix} + e^{-ix} = (\cos x + i\sin x) + (\cos x - i\sin x) = 2\cos x$$

and

$$e^{ix} - e^{-ix} = (\cos x + i\sin x) - (\cos x - i\sin x) = 2i\sin x.$$

From both of these computations, we have:

Theorem 2.2. *For any real number x,*

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \textit{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

2.1.2 Some examples

If you have a little faith and believe everything I said above, let's look at a simple example to see i in action. The first example we'll see is an integral that we could do otherwise, but Euler's formula provides another nice solution.

Example 2.3. Compute the following antiderivative:

$$\int \sin^2 x\, dx.$$

Solution. There are two standard ways to evaluate this antiderivative: with a clever application of integration by parts, or with a double angle identity. Here is the double angle identity computation, just for reference:

$$\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{\cos 2x}{2} \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

Alternatively, we can use Euler's formula; or rather, the fact that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Applying this to the integrand gives

$$\begin{aligned} \int \sin^2 x \, dx = \int \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^2 dx &= -\frac{1}{4} \int e^{2ix} - 2e^{ix}e^{-ix} + e^{-2ix} \, dx \\ &= -\frac{1}{4} \int e^{2ix} - 2 + e^{-2ix} \, dx. \end{aligned}$$

Now we can integrate the exponential terms as usual.

$$-\frac{1}{4} \int e^{2ix} - 2 + e^{-2ix} \, dx = -\frac{1}{4} \left(\frac{e^{2ix}}{2i} - 2x + \frac{e^{-2ix}}{-2i} \right) + C.$$

Now we can finish the problem off by using the exponential formula for the sine function again. Note that

$$-\frac{1}{4} \left(\frac{e^{2ix}}{2i} - 2x + \frac{e^{-2ix}}{-2i} \right) + C = -\frac{1}{4} \left(\frac{e^{2ix} - e^{-2ix}}{2i} - 2x \right) + C = -\frac{1}{4} (\sin 2x - 2x) + C.$$

We have arrived at the same answer as above:

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

□

More generally, you could approach any integral of the form

$$\int \sin^m x \, \cos^n x \, dx$$

in this exact same way. Expand $\sin x$ and $\cos x$ into complex exponentials using Euler's formula, multiply everything out and then integrate each exponential term easily, then regroup in terms of sines and cosines.

A similar line of attack can handle the following integral.

Example 2.4. Compute the following antiderivative:

$$\int \sin 3x \, \cos 2x \, dx.$$

Solution. Like the previous example, this antiderivative could be evaluated by a clever (double) application of integration by parts, or repeated uses of summation identities for trig functions. Instead, we'll expand $\sin 3x$ and $\cos 2x$ into complex exponentials.

$$\begin{aligned} \sin 3x \, \cos 2x &= \left(\frac{e^{3ix} - e^{-3ix}}{2i} \right) \left(\frac{e^{2ix} + e^{-2ix}}{2} \right) \\ &= \frac{1}{4i} \left(e^{5ix} - e^{-ix} + e^{ix} - e^{-5ix} \right). \end{aligned}$$

Thus,

$$\int \sin 3x\ \cos 2x\, dx = \frac{1}{4i}\int e^{5ix} - e^{-ix} + e^{ix} - e^{-5ix}\, dx$$
$$= \frac{1}{4i}\left(\frac{e^{5ix}}{5i} - \frac{e^{-ix}}{-i} + \frac{e^{ix}}{i} - \frac{e^{-5ix}}{-5i}\right) + C.$$

As before, we can regroup terms to revert sines and cosines.

$$\frac{1}{4i}\left(\frac{e^{5ix}}{5i} - \frac{e^{-ix}}{-i} + \frac{e^{ix}}{i} - \frac{e^{-5ix}}{-5i}\right) + C = \frac{1}{4i}\left(\frac{e^{5ix}+e^{-5ix}}{5i}\right) + \frac{1}{4i}\left(\frac{e^{ix}+e^{-ix}}{i}\right) + C$$
$$= -\frac{1}{10}\left(\frac{e^{5ix}+e^{-5ix}}{2}\right) - \frac{1}{2}\left(\frac{e^{ix}+e^{-ix}}{2}\right) + C.$$

The last equality comes from the fact that $(4i)(5i) = -20$ and $(4i)i = -4$. Thus,

$$\int \sin 3x\ \cos 2x\, dx = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C.$$

□

ADD MORE EXAMPLES

2.2 Differentiating a parametric integral

2.3 Symmetry and unconventional substitutions

Chapter 3

Convergence Tests for Infinite Series

Here, I have collected a number of convergence tests that are not covered in a typical undergraduate calculus course. My approach is as follows: I present the statement of each test or technique, followed by demonstrative examples. If you're curious about the proofs of each test, I have collected them in Appendix A. Note that in most cases I have chosen to start indexing each sum at $n = 1$, but as usual any of the following tests have natural generalizations to other starting indices.

3.1 The condensation test

The condensation test, usually called the Cauchy condensation test, allows you replace all n's that appear in an infinite series with 2^n's, multiply the whole sequence by 2^n, then test for convergence of the resulting series. Oftentimes this can be helpful with series involving logarithms. Here is the formal statement.

Theorem 3.1 (Condensation test). *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series such that a_n is a positive, decreasing sequence. Then*

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} 2^n\, a_{2^n}$$

either both converge or diverge.

Let's look at a number of examples. We'll start off with a couple simple examples, just to get a feel for how the condensation test works.

Example 3.2. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

converge or diverge?

Solution. You probably already know that the above infinite sum diverges, so this is just an exercise in using the condensation test and seeing that it does in fact work. First, note that $\frac{1}{n}$ is a nonnegative and decreasing sequence, so we are allowd to invoke the condensation test. The condensation test says to replace all instances of n in the summand with 2^n. Doing this yields

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \rightsquigarrow \quad \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n}.$$

If we simplify the latter series we get

$$\sum_{n=1}^{\infty} 1.$$

This series diverges by the divergence test. By the condensation test, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges, which matches what we knew already. □

Example 3.3. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge or diverge?

Solution. Again, you probably already know that this series converges. Let's confirm that via the condensation test. First, the sequence $\frac{1}{n^2}$ is nonnegative and decreasing, so we can use the condensation test. We get

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \rightsquigarrow \quad \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^2} = \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

The resulting series is a geometric series with ratio $\frac{1}{2}$, thus, it converges. By the condensation test, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also converges. □

Those two examples were not that interesting, so let's try something more complicated. Like I mentioned above, the condensation test can be handy when dealing with logarithms.

Example 3.4. Does the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

converge or diverge?

Solution. This one is slightly more interesting, although you may observe that we could use the integral test. The condensation test gives another solution. First, since $\frac{1}{n(\ln n)^2}$ is nonnegative and decreasing, we are allowed to use the condensation test. This yields

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \quad \rightsquigarrow \quad \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{2^n(\ln(2^n))^2}.$$

Let's simplify the resulting series, using properties of logs.

$$\sum_{n=2}^{\infty} 2^n \cdot \frac{1}{2^n(\ln(2^n))^2} = \sum_{n=2}^{\infty} \frac{1}{(n\ \ln 2)^2} = \frac{1}{(\ln 2)^2} \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Wow! The scary looking series with a logarithm turned into a simple p-series. In particular, the resulting series converges, so by the condensation test, the original series converges as well. $\square$

Let's look at one more example, one with many logarithms. This is a good example demonstrating how you can use the condensation test multiple times to attack a series.

Example 3.5. Does the infinite series

$$\sum_{n=3}^{\infty} \frac{1}{n\ \ln n\ \sqrt{\ln(\ln n)}}$$

converges or diverge?

Solution. Again, we could use the integral test here. For fun, we'll try simplifying this with condensation. Note that $\frac{1}{n\ \ln n\ \sqrt{\ln(\ln n)}}$ is nonnegative and decreasing. Thus, we may use the condensation test. Consider

$$\sum_{n=3}^{\infty} \frac{1}{n\ \ln n\ \sqrt{\ln(\ln n)}} \quad \rightsquigarrow \quad \sum_{n=3}^{\infty} 2^n \frac{1}{2^n\ \ln 2^n\ \sqrt{\ln(\ln 2^n)}} = \frac{1}{\ln 2} \sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n \ln 2)}}.$$

Let's apply the condensation test one more time.

$$\frac{1}{\ln 2} \sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln(n \ln 2)}} \quad \rightsquigarrow \quad \frac{1}{\ln 2} \sum_{n=3}^{\infty} 2^n \frac{1}{2^n \sqrt{\ln(2^n \cdot \ln 2)}} = \frac{1}{\ln 2} \sum_{n=3}^{\infty} \frac{1}{\sqrt{\ln n + \ln 2 + \ln\ln 2}}.$$

Let's use condensation one more time!

$$\frac{1}{\ln 2} \sum_{n=3}^{\infty} \frac{1}{\sqrt{\ln n + \ln 2 + \ln\ln 2}}$$
$$\rightsquigarrow \quad \frac{1}{\ln 2} \sum_{n=3}^{\infty} 2^n \frac{1}{\sqrt{\ln 2^n + \ln 2 + \ln\ln 2}} = \frac{1}{\ln 2} \sum_{n=3}^{\infty} \frac{2^n}{\sqrt{n \ln 2 + \ln 2 + \ln\ln 2}}.$$

Having introduced a factor of 2^n in the numerator, the sequence in the above series should not go to 0 and thus the series would diverge by the divergence test. We can confirm this with L'Hopital's rule.

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{\sqrt{n \ln 2 + \ln 2 + \ln\ln 2}} &= \lim_{x\to\infty} \frac{2^x}{\sqrt{x \ln 2 + \ln 2 + \ln\ln 2}} \\
(L'H) &= \lim_{x\to\infty} \frac{\ln 2 \cdot 2^x}{\frac{\ln 2}{2\sqrt{x \ln 2 + \ln 2 + \ln\ln 2}}} \\
&= \lim_{x\to\infty} 2 \cdot 2^x \cdot \sqrt{x \ln 2 + \ln 2 + \ln\ln 2} = \infty > 1.
\end{aligned}$$

By the divergence test, the series $\sum_{n=3}^{\infty} \frac{2^n}{\sqrt{n \ln 2 + \ln 2 + \ln\ln 2}}$ diverges. Thus, by our three applications of the condensation test, the series $\sum_{n=3}^{\infty} \frac{1}{n\ \ln n\ \sqrt{\ln(\ln n)}}$ also diverges. $\square$

Example 3.6. Does the infinite series

$$\sum_{n=3}^{\infty} \frac{1}{\ln(\ln(\ln n))}$$

converges or diverge?

Solution. We could probably use the integral test here, but it would take some cleverness. Instead, let's begin with the condensation test, observing that $\frac{1}{\ln(\ln(\ln n))}$ is indeed a nonnegative and decreasing sequence.

$$\sum_{n=4}^{\infty} \frac{1}{\ln(\ln(\ln n))} \quad \rightsquigarrow \quad \sum_{n=3}^{\infty} 2^n \cdot \frac{1}{\ln(\ln(\ln 2^n))} = \sum_{n=3}^{\infty} \frac{2^n}{\ln(\ln n \ + \ln(\ln 2)))}.$$

Having introduced the factor of 2^n in the numerator, the resulting series should diverge by the divergence test. Let's confirm this. We could evaluate the following limit in a number of ways, but we may as well get some practice with L'Hopital's rule:

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{\ln(\ln n \ + \ln(\ln 2)))} &= \lim_{x\to\infty} \frac{2^x}{\ln(\ln x \ + \ln(\ln 2)))} \\
(L'H) &= \lim_{x\to\infty} \frac{\ln 2 \cdot 2^x}{\frac{1}{\ln x + \ln(\ln 2)} \cdot \frac{1}{x}} \\
&= \lim_{x\to\infty} \ln 2 \cdot 2^x \cdot (\ln x + \ln(\ln 2)) \cdot x = \infty > 0.
\end{aligned}$$

As suspected, the series $\sum_{n=3}^{\infty} \frac{2^n}{\ln(\ln n \ + \ln(\ln 2)))}$ diverges by the divergence test. By the condensation test, it follows that $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln(\ln n))}$ diverges. □

3.1.1 The condensation test in the context of the integral test

One interesting observation is to observe how the condensation test arises from the integral test via a substitution! Abstractly, consider the series

$$\sum_{n=1}^{\infty} f(n)$$

for some continuous, positive, decreasing function $f(x)$. By the integral test, to determine convergence or divegrence of the above series, we can consider

$$\int_1^{\infty} f(x)\,dx.$$

Make the substitution $x = 2^t$. Then $dx = 2^t\,dt$, $t = 0$ when $x = 1$, and as $x \to \infty$, $t \to \infty$ as well. Thus,

$$\int_1^{\infty} f(x)\,dx = \int_0^{\infty} 2^t f(2^t)\,dt.$$

Using the converse of the integral test, we can further determine convergence or divergence of this new integral by considering the series

$$\sum_{n=0}^{\infty} 2^n f(2^n).$$

We have more or less arrived at the conclusion of the condensation test. Neat!

3.2 The second ratio test

Next up is a refinement of the usual ratio test, which can sometimes be conclusive when the usual ratio test is inconclusive. Here is the statement.

Theorem 3.7 (Second ratio test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Let*

$$L_1 = \lim_{n\to\infty} \left|\frac{a_{2n}}{a_n}\right| \quad \textit{and} \quad L_2 = \lim_{n\to\infty} \left|\frac{a_{2n+1}}{a_n}\right|.$$

- *If both L_1 and L_2 are $< \frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n$ converges.*
- *If both L_1 and L_2 are $> \frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
- *Otherwise (i.e. if $L_1 = \frac{1}{2}$, or $L_2 = \frac{1}{2}$, or if $L_1 > \frac{1}{2}$ and $L_2 < \frac{1}{2}$ or vice versa) the test is inconclusive.*

In words, we compute *two* different limits, instead of just one as in the usual ratio test, and the conclusion relies on both of these limits.

The point of the second ratio test is to use it when the normal ratio test fails. Here is a simple example.

Example 3.8. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1}.$$

converge or diverge?

Solution. This is an infinite series we could understand with e.g. the limit comparison test, but we'll try the regular ratio test to see what happens. Note that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{(n+1)^3+1} \cdot \frac{n^3+1}{n} = 1.$$

Thus, the ratio test is inconclusive.

Let's try the second ratio test. To do so, we need to compute the two limits L_1 and L_2.

$$\begin{aligned} L_1 = \lim_{n\to\infty} \left|\frac{a_{2n}}{a_n}\right| &= \lim_{n\to\infty} \frac{2n}{(2n)^3+1} \cdot \frac{n^3+1}{n} \\ &= \lim_{n\to\infty} \frac{2n(n^3+1)}{n(8n^3+1)} \\ &= \lim_{n\to\infty} \frac{2n^3+2}{8n^3+1} = \frac{2}{8} = \frac{1}{4}. \end{aligned}$$

Similarly,

$$L_1 = \lim_{n\to\infty} \left|\frac{a_{2n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n+1}{(2n+1)^3+1} \cdot \frac{n^3+1}{n} = \frac{2}{8} = \frac{1}{4}.$$

Since both $L_1 = \frac{1}{4} < \frac{1}{2}$ and $L_2 = \frac{1}{4} < \frac{1}{2}$, this infinite series converges by the second ratio test. □

Example 3.9. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

converge or diverge?

Solution. As usual, this is an infinite series that you could otherwise attack with direct comparison, limit comparison, or even the integral test. Also, note that the usual ratio test fails:

$$\lim_{n\to\infty} \left| \frac{\ln(n+1)}{(n+1)^2} \cdot \frac{n^2}{\ln n} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \cdot \frac{\ln(n+1)}{\ln n} = 1.$$

Fortunately, the second ratio test will save the day. Note that

$$\begin{aligned} L_1 = \lim_{n\to\infty} \left| \frac{a_{2n}}{a_n} \right| &= \lim_{n\to\infty} \frac{\ln(2n)}{(2n)^2} \cdot \frac{n^2}{\ln n} \\ &= \lim_{n\to\infty} \frac{\ln 2 + \ln n}{\ln n} \cdot \frac{n^2}{(2n)^2} \\ &= \lim_{n\to\infty} \left(1 + \frac{\ln 2}{\ln n} \right) \cdot \frac{1}{4} \\ &= \frac{1}{4}. \end{aligned}$$

The computation for L_2 is almost identical and yields $L_2 = \frac{1}{4}$. Since L_1 and L_2 are both less than $\frac{1}{2}$, the series converges by the second ratio test. □

Here is one more similar example, just for fun.

Example 3.10. Does the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \ln n}$$

converge or diverge?

Solution. You can verify that the usual ratio test is inconclusive, which shouldn't be surprising due to the lack of exponentials or factorials. If we try the second ratio test, we have

$$L_1 = \lim_{n\to\infty} \left| \frac{a_{2n}}{a_n} \right| = \lim_{n\to\infty} \frac{\sqrt{n} \ln n}{\sqrt{2n} \ln(2n)} = \lim_{n\to\infty} \frac{1}{\sqrt{2}} \left(1 + \frac{\ln 2}{\ln n} \right) = \frac{1}{\sqrt{2}}.$$

Note that $\frac{1}{\sqrt{2}} > \frac{1}{2}$. A nearly identical computation shows that $L_2 = \frac{1}{\sqrt{2}} > \frac{1}{2}$. Since $L_1, L_2 > \frac{1}{2}$, the second ratio test implies that this series diverges. □

Here is one last remark about the second ratio test, to contextualize some of the examples we've considered. The usual ratio test fails for any p series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, since $\left(\frac{n}{n+1} \right)^p \to 1$ as $n \to \infty$. This implies that the ratio test will fail in general for series with ratios of polynomials, such as

$$\sum_{n=1}^{\infty} \frac{3n^2 - 2n + 1}{n^7 + 6}.$$

Since $\left(\frac{\ln n}{\ln(n+1)}\right)^p \to 1$ as $n \to \infty$, we can further introduce logarithms and the ratio test will still fail. For example, we could introduce a logarithm into the previous series:

$$\sum_{n=1}^{\infty} \frac{(3n^2 - 2n + 1) \cdot \ln n}{n^7 + 6}.$$

In contrast, the second ratio test will be conclusive for any p-series with $p \neq 1$, since

$$\lim_{n\to\infty} \left(\frac{n}{2n}\right)^p = \left(\frac{1}{2}\right)^p.$$

In particular, if $p \neq 1$, this limit is either greater than or less than $\frac{1}{2}$. As before, introducing a logarithm doesn't change the asymptotic behavior in terms of the second ratio test computations. So it makes sense that we were able to analyze series such as

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$$

with the second ratio test. On the other hand, you could check that both the ratio test and the second ratio test would be inconclusive for the following infinite series:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

3.3 Raabe's ratio test

Raabe's ratio test is another refinement of the usual ratio test, in the sense that it can sometimes make a conclusion about an infinite series $\sum_{n=1}^{\infty} a_n$ when $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ (and so the usual ratio test fails). We will state Raabe's test for series with *positive* terms.

Theorem 3.11 (Raabe's ratio test). *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series such that $a_n > 0$. Assuming the following limit exists, let*

$$L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right).$$

- *If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*
- *If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
- *If $L = 1$, then the test is inconclusive.*

It is important to note that the conclusions are somewhat "flipped" from the usual ratio test conclusions. Roughly, this is because the above limit involves $\frac{a_n}{a_{n+1}}$ rather than a term like $\frac{a_{n+1}}{a_n}$.

We'll start with a few straightforward examples to get a feel for the type of computation that Raabe's test involves, and then we'll see a more complicated example where the usual ratio test fails.

Example 3.12. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converge or diverge?

Solution. To apply Raabe's test, we need to compute $L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right)$. We have

$$\begin{aligned} L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(n+1)^3}{n^3} - 1\right) &= \lim_{n\to\infty} n\left(\frac{(n+1)^3 - n^3}{n^3}\right) \\ &= \lim_{n\to\infty} \frac{n^3 + 3n^2 + 3n + 1 - n^3}{n^2} \\ &= \lim_{n\to\infty} \frac{3n^2 + 3n + 1}{n^2} = 3. \end{aligned}$$

Since $3 > 1$, by Raabe's ratio test, this series converges. □

Example 3.13. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

converge or diverge?

Solution. Note that

$$\begin{aligned} L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) &= \lim_{n\to\infty} n\left(\frac{\sqrt{n+1}}{\sqrt{n}} - 1\right) \\ &= \lim_{n\to\infty} n\left(\frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}}\right) \\ &= \lim_{n\to\infty} n\left(\frac{n+1-n}{\sqrt{n}\left(\sqrt{n+1} + \sqrt{n}\right)}\right) \\ &= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \\ &= \frac{1}{2}. \end{aligned}$$

Since $\frac{1}{2} < 1$, by Raabe's ratio test, this series diverges.

□

Example 3.14. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{n!}{2^n}$$

converge or diverge?

Solution. Note that

$$L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n!}{2^n}\cdot\frac{2^{n+1}}{(n+1)!} - 1\right)$$
$$= \lim_{n\to\infty} n\left(\frac{2}{n+1} - 1\right).$$

Because $\frac{2}{n+1} - 1 \to -1$ as $n \to \infty$, it follows that $L = -\infty$. Since $L < 1$, by Raabe's ratio test, this series diverges. □

Now, let's look at a couple examples where most of our usual tests will fail. Before we do so, I'm going to introduce some new notation: the *double factorial*. The name and notation may be deceiving; a double factorial is *not* a factorial operation applied twice! In words, a double factorial acts like a normal factorial but with jumps of size 2 rather than size 1. For example,

$$6!! = 6\cdot 4\cdot 2 \quad \text{and} \quad 7!! = 7\cdot 5\cdot 3\cdot 1.$$

A double factorial of an even number will always finish on 2, and a double factorial of an odd number will always finish on 1 (or 3, equivalently). Thus, to define the double factorial in general we may write

$$(2n)!! = (2n)(2n-2)\cdots 2 \quad \text{and} \quad (2n+1)!! = (2n+1)(2n-1)\cdots 3\cdot 1.$$

With this, we can consider infinite series that don't typically arise in a calculus class.

Example 3.15. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$$

converge or diverge?

Solution. First, let's try the usual ratio test. We'll have to be a little careful with the double factorials, which may be unfamiliar. Note that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(2(n+1)-1)!!}{(2(n+1))!!}\cdot\frac{(2n)!!}{(2n-1)!!} = \lim_{n\to\infty}\frac{(2n+1)!!}{(2n+2)!!}\cdot\frac{(2n)!!}{(2n-1)!!}$$
$$= \lim_{n\to\infty}\frac{(2n)!!}{(2n+2)!!}\cdot\frac{(2n+1)!!}{(2n-1)!!}$$
$$= \lim_{n\to\infty}\frac{1}{2n+2}\cdot\frac{2n+1}{1}$$
$$= 1.$$

Thus, the ratio test is inconclusive.

Let's try Raabe's ratio test. In the previous computation we shows that

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2}.$$

Thus,

$$\begin{aligned} L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) &= \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - 1\right) \\ &= \lim_{n\to\infty} n\left(\frac{2n+2-(2n+1)}{2n+1}\right) \\ &= \lim_{n\to\infty} n\left(\frac{1}{2n+1}\right) \\ &= \frac{1}{2}. \end{aligned}$$

Since $L = \frac{1}{2} < 1$, by Raabe's test, this series diverges. Nice!

□

Solution 2. Here is another solution, just for fun. The series

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$$

may look a little weird, but we can actually use an easy direct comparison argument by observing that

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3 \geq (2n-2)\cdot(2n-4)\cdots 2 = (2n-2)!!.$$

It follows that

$$\frac{(2n-1)!!}{(2n)!!} \geq \frac{(2n-2)!!}{(2n)!!} = \frac{1}{2n} > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, by direct comparison, $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$ also diverges. □

Example 3.16. Does the infinite series

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!\cdot(2n+1)}$$

converge or diverge?

Solution. First, a couple remarks about this example before we jump into Raabe's test. As with the previous example, the usual ratio test will fail. This should make sense intuitively, since the ratio test failed for the previous example and this example differs only by the presence of a linear factor. Let's check, anyway:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(2(n+1)-1)!!}{(2(n+1))!!}\cdot\frac{(2n)!!}{(2n-1)!!}\cdot\frac{2n+1}{2(n+1)+1} \\ &= \lim_{n\to\infty} \frac{(2n+1)!!}{(2n+2)!!}\cdot\frac{(2n)!!}{(2n-1)!!}\cdot\frac{2n+1}{2n+3} \\ &= \lim_{n\to\infty} \frac{2n+1}{2n+2}\cdot\frac{2n+1}{2n+3} \\ &= 1. \end{aligned}$$

Unlike the previous example, direct comparison arguments are more elusive. The natural comparisons to try are

$$\frac{(2n-1)!!}{(2n)!!\cdot(2n+1)} \geq \frac{(2n-2)!!}{(2n)!!\cdot(2n+1)} = \frac{1}{(2n)\cdot(2n+1)}$$

and

$$\frac{(2n-1)!!}{(2n)!!\cdot(2n+1)} \leq \frac{(2n)!!}{(2n)!!\cdot(2n+1)} = \frac{1}{2n+1}$$

but neither of these lead to conclusions via the direct comparison test. The extra linear factor turns out to be quite muddlesome.

Fortunately, Raabe's test will work! The ratio test computation that we just did shows that

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2}\cdot\frac{2n+1}{2n+3}$$

so that

$$\begin{aligned} L = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) &= \lim_{n\to\infty} n\left(\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1\right) \\ &= \lim_{n\to\infty} n\left(\frac{(2n+2)(2n+3)-(2n+1)^2}{(2n+1)^2}\right) \\ &= \lim_{n\to\infty} n\left(\frac{4n^2+10n+6-(4n^2+4n+1)}{(2n+1)^2}\right) \\ &= \lim_{n\to\infty} n\left(\frac{6n+5}{(2n+1)^2}\right) \\ &= \frac{6}{4}. \end{aligned}$$

Since $L > 1$, by Raabe's test, this series converges. □

3.3.1 Generalizing Raabe's ratio test even further: Kummer's ratio test

Kummer's test is a vast generalization of Raabe's ratio test — and in fact, that usual ratio test as well! This full level of generality is difficult to use in practice, so using a specific case such as the ratio test or Raabe's test is a better way to approach a series. Nonetheless, it is interesting to see where these tests come from and how one could generalize even further. As usual, I will refer to the appendix for a proof.

Theorem 3.17 (Kummer's ratio test). *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series such that $a_n > 0$. Let $\sum_{n=1}^{\infty} b_n$ be a divergent infinite series such that $b_n > 0$. Assuming the following limit exists, let*

$$\alpha = \lim_{n\to\infty} \frac{1}{b_n}\frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}}.$$

- *If $\alpha > 0$, then $\sum_{n=1}^{\infty} a_n$ converges.*
- *If $\alpha < 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
- *If $\alpha = 0$, then the test is inconclusive.*

I claim that both the usual ratio test and Raabe's ratio test are specific cases of Kummer's test, i.e., come from specific choices of the sequence b_n. Suppose that we choose $b_n = 1$;

note that $b_n > 0$ and $\sum_{n=1}^{\infty} 1$ diverges, so this choice of sequence is compatible with the statement of Kummer's test. In this case,

$$\alpha = \lim_{n\to\infty} \frac{1}{1}\frac{a_n}{a_{n+1}} - \frac{1}{1} = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} - 1.$$

To see this this is equivalent to the usual ratio test, note that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{1+\alpha}.$$

This limit is < 1 and > 1 if $\alpha > 0$ and $\alpha < 0$, respectively. Next, suppose that $b_n = \frac{1}{n}$. Again, $b_n > 0$ and $\sum_{n=1}^{\infty} 1$ diverges, so this choice of sequence is compatible with the statement of Kummer's test. Here we have

$$\alpha = \lim_{n\to\infty} n\frac{a_n}{a_{n+1}} - (n+1).$$

Thus,

$$\alpha + 1 = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = L.$$

Because $L > 1$ and $L < 1$ according to whether $\alpha > 0$ and $\alpha < 0$, we have indeed produced Raabe's ratio test.

3.4 A generalization of the alternating series test: Dirichlet's test

The motivation for the next test is the following series:

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}.$$

This series eludes all of the usual convergence tests. It is tempting to want to use the alternating series test, because $\sin n$ vaguely oscillates through numbers between -1 and 1 and thus the series might behaves similarly to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. Unfortunately, the alternating series test only applies to series that are *literally alternating*, i.e., the sign of every single term has to change from positive to negative and back. The function $\sin n$ does not literally alternate: $\sin 1 > 0$, $\sin 2 > 0$, $\sin 3 > 0$, then finally $\sin 4 < 0$, and so on.

With that being said, it feels like the alternating series test should morally apply and imply convergence. Fortunately, there is a generalization of the alternating series test, usually called *Dirichlet's test*, which will save the day.

Theorem 3.18 (Dirichlet's test). *Let $\sum_{n=1}^{\infty} b_n\, a_n$ be an infinite series. Suppose that*

- *a_n is positive, decreasing, and $a_n \to 0$*
- *the partial sums of $\sum_{n=1}^{\infty} b_n$ are bounded, i.e.,*

$$\left|\sum_{n=1}^{N} b_n\right| \le K$$

for some constant K (which does not depend on N).

Then $\sum_{n=1}^{\infty} b_n\, a_n$ converges.

The first condition of this test looks like the conditions of the alternating series test, but the second condition is a little intimidating. Notice that if $b_n = (-1)^n$, the first condition alone gives the alternating series test. The point here is that if $b_n = (-1)^n$, we can verify that the partial sums are bounded:

$$\left|\sum_{n=1}^{2}(-1)^n\right| = |-1+1| = 0$$
$$\left|\sum_{n=1}^{3}(-1)^n\right| = |-1+1-1| = |-1| = 1$$
$$\left|\sum_{n=1}^{4}(-1)^n\right| = |-1+1-1+1| = 0$$
$$\vdots$$

We can see that

$$\left|\sum_{n=1}^{N}(-1)^n\right| \le 1$$

for all N. Thus, the second condition of Dirichlet's test is satisfied and we have verified the alternating series test.

Now let's see if we can tackle the motivating series. Even with Dirichlet's test at hand, this series is difficult and will take a lot of work.[1] Buckle up!

Example 3.19. Does the infinite series

$$\sum_{n=1}^{\infty}\frac{\sin n}{n}.$$

converge or diverge?[a]

[a] A bonus, challenging question: does the series converge absolutely or conditionally?

Solution. In order to use Dirichlet's test, we will decompose the summand as follows:

$$\sum_{n=1}^{\infty}\overbrace{\sin n}^{b_n}\underbrace{\frac{1}{n}}_{a_n}.$$

First, note that $\frac{1}{n}$ is a positive, decreasing sequence, and $\frac{1}{n} \to 0$. Thus, in order to invoke Dirichlet's test, the only thing we have to verify is that

$$\left|\sum_{n=1}^{N}\sin n\right| \le K$$

for some constant K. This is the tricky part. I will present two arguments:

i. The first argument is simpler (and nicer), but uses complex numbers. In particular, it uses Euler's formula, which I discussed in 2.1.1. Recall that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

[1] To be honest, this example is beyond the scope of a calculus course.

From this it follows that

$$\sum_{n=1}^{N} \sin n = \frac{1}{2i}\left(\sum_{n=1}^{N} e^{in} - \sum_{n=1}^{N} e^{-in}\right).$$

Both sums are geometric sums, the first with ratio e^i and the second with ratio e^{-i}. Recall the formula for the partial sums of a geometric series:

$$\sum_{n=1}^{N} r^n = \frac{r - r^{N+1}}{1-r}.$$

Using this, we conclude

$$\sum_{n=1}^{N} \sin n = \frac{1}{2i}\left(\frac{e^i - e^{i(N+1)}}{1-e^i} - \frac{e^{-i} - e^{-i(N+1)}}{1-e^{-i}}\right).$$

The only terms that depend on N on the right hand side are $e^{i(N+1)}$ and $e^{-i(N+1)}$. Also recall from 2.1.1 that $|e^{ix}| = 1$ for any real number x. Thus,

$$|e^{i(N+1)}| = |e^{-i(N+1)}| = 1.$$

This implies[2] that

$$\left|\sum_{n=1}^{N} \sin n\right| = \left|\frac{1}{2i}\left(\frac{e^i - e^{i(N+1)}}{1-e^i} - \frac{e^{-i} - e^{-i(N+1)}}{1-e^{-i}}\right)\right|$$

is bounded above by some constant independent of N, as desired.

ii. The next solution does not use complex numbers, but requires an extreme amount of cleverness involving trig identities. Recall the cosine sum identity:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

It follows that

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Here I have used the fact that cosine is an even function, and sine is an odd function. Subtracting these two identities gives

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta.$$

The quantity we care about is $\sum_{n=1}^{N}$, so the next thing to do is plug in $\alpha = n$. This yields

$$\cos(n-\beta) - \cos(n+\beta) = 2\sin n\sin\beta.$$

Next we will choose an appropriate choice of β. The motivation here is to produce a telescoping sum on the left hand side. We can accomplish this by picking $\beta = \frac{1}{2}$. This yields the trig identity

$$\cos\left(n - \frac{1}{2}\right) - \cos\left(n + \frac{1}{2}\right) = 2\sin n \sin\left(\frac{1}{2}\right). \tag{3.1}$$

[2] To be honest, there is more work to be done here. In particular, we should use the triangle inequality: $|a+b| \le |a| + |b|$.

Let's sum both sides of (3.1) from 1 to N:

$$\sum_{n=1}^{N}\left[\cos\left(n-\frac{1}{2}\right)-\cos\left(n+\frac{1}{2}\right)\right]=2\sin\left(\frac{1}{2}\right)\sum_{n=1}^{N}\sin n.$$

The sum we care about is on the right. As mentioned above, the sum on the left telescopes! In particular,

$$\begin{aligned}\sum_{n=1}^{N}&\left[\cos\left(n-\frac{1}{2}\right)-\cos\left(n+\frac{1}{2}\right)\right]\\ &=\left[\cos\left(\frac{1}{2}\right)-\cos\left(\frac{3}{2}\right)\right]+\left[\cos\left(\frac{3}{2}\right)-\cos\left(\frac{5}{2}\right)\right]+\cdots\\ &\qquad\cdots+\left[\cos\left(N-\frac{1}{2}\right)-\cos\left(N+\frac{1}{2}\right)\right].\end{aligned}$$

Cancelling all of the corresponding terms yields

$$\sum_{n=1}^{N}\left[\cos\left(n-\frac{1}{2}\right)-\cos\left(n+\frac{1}{2}\right)\right]=\cos\left(\frac{1}{2}\right)-\cos\left(N+\frac{1}{2}\right).$$

All of this together implies

$$\sum_{n=1}^{N}\sin n=\frac{\cos\left(\frac{1}{2}\right)-\cos\left(N+\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}.$$

Since $|\cos x|\leq 1$ for all x, we have

$$\left|\sum_{n=1}^{N}\sin n\right|\leq\frac{2}{2\sin\left(\frac{1}{2}\right)}.$$

Either of these arguments shows that

$$\left|\sum_{n=1}^{N}\sin n\right|\leq K$$

for some constant K. Thus, by Dirichlet's test, the series

$$\sum_{n=1}^{\infty}\frac{\sin n}{n}$$

converges. Whew!

□

3.5 A one-two punch: limit comparison and Taylor polynomials

The technique I'll discuss next is not necessarily a new test, but rather a powerful concoction of two techniques you already know. In particular, I will demonstrate how useful the limit comparison test can be together with the knowledge of Taylor polynomials. This is an idea that is sometimes explored in calculus classes, but only briefly and implicitly.

For convenience I will remind you of the statement of the limit comparison test here.

Theorem 3.20 (Limit comparison test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series such that $a_n > 0$. Let $b_n > 0$, and let*
$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$
If $0 < L < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or diverge.

The point of the limit comparison test is the following: given an infinite series $\sum_{n=1}^{\infty} a_n$, you identify a simpler series $\sum_{n=1}^{\infty} b_n$ for which you already know the behavior of (typically a p-series), then compute L as above to make your desired conclusion. The tricky part about limit comparison is that you have to pick an appropriate sequence b_n which exhibits similar asymptotic decay as a_n. This is straightforward in many of the cases from a calculus class; for example, given
$$\sum_{n=1}^{\infty} \frac{n^3+1}{n^5+n+2}$$
you would identify the dominate growth behavior in the numerator and denominator as n^3 and n^5, respectively, and thus would choose $b_n = \frac{n^3}{n^5} = \frac{1}{n^2}$. The point of this section is that we can push this to the extreme by using Taylor polynomials.

Let's start of with a simple example to see this principle (that I have yet to describe) in action.

Example 3.21. Does the infinite series
$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$
converge or diverge?

Solution. I claim that we can run limit comparison with $b_n = \frac{1}{n}$ to conclude that this series diverges. The actual solution will be very simple, but how did I decide on $b_n = \frac{1}{n}$? Unlike the rational series above, we cannot easily identify the dominant behavior of the sequence $\sin\left(\frac{1}{n}\right)$. This is where Taylor polynomials come into play. Recall that the MacLaurin expansion of $\sin x$ is
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$
This implies that for x values close to 0, $\sin x \approx x$. Said differently, the 1st MacLaurin polynomial of $\sin x$ at 0 is $T_1(x) = x$. Since $\frac{1}{n}$ is a number close to 0 for large values of n, it follows that
$$\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}.$$
Thus, the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ should behave the same as $\sum_{n=1}^{\infty} \frac{1}{n}$. This is what motivated my choice of b_n.

With all of that preambling out of the way, let's actually finish the solution. Note that
$$\begin{aligned}\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} &= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \\ (L'H) &= \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1.\end{aligned}$$
Since $0 < 1 < \infty$, limit comparison implies that $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ behaves the same as $\sum_{n=1}^{\infty} \frac{1}{n}$. Since the latter is a divergent p-series, the former diverges as well. □

That example was relatively tame, so here are a number of more complicated examples that exhibit the same technique.

Example 3.22. Does the infinite series

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right)$$

converge or diverge?

Solution. Recall the MacLaurin series expansion[3] for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots .$$

Consequently, for large values of n we expect $\arctan\left(\frac{1}{n^2}\right) \approx \frac{1}{n^2}$. Thus, we run limit comparison with $b_n = \frac{1}{n^2}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} \\
(L'H) &= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x^4}} \cdot -\frac{2}{x^3}}{-\frac{2}{x^3}} \\
&= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^4}} = 1.
\end{aligned}$$

Since $0 < 1 < \infty$, by limit comparison, $\sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^2}\right)$ behaves the same as $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since the latter is a convergent p-series, the former converges as well. □

Example 3.23. Does the infinite series

$$\sum_{n=1}^{\infty} \ln\left(\frac{1}{1-\sin^2\left(\frac{1}{n}\right)}\right)$$

converge or diverge?

Solution. Recall the MacLaurin series expansions for $\ln(1-x)$ and $\sin x$:

$$\begin{aligned}
\ln(1-x) &= -x - \frac{x^2}{2} - \cdots \\
\sin x &= x - \frac{x^3}{3!} + \cdots .
\end{aligned}$$

Thus, we expect $\sin^2\left(\frac{1}{n}\right) \approx \frac{1}{n^2}$ for large n and therefore

$$\ln\left(\frac{1}{1-\sin^2\left(\frac{1}{n}\right)}\right) = -\ln\left(1-\sin^2\left(\frac{1}{n}\right)\right) \approx -\ln\left(1-\frac{1}{n^2}\right) \approx \frac{1}{n^2}$$

[3]If this is something you forgot, you can quickly re-derive this by integrating the geometric series

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

term by term.

for large n. Thus, we run limit comparison with $b_n = \frac{1}{n^2}$. We compute:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln\left(\frac{1}{1-\sin^2\left(\frac{1}{n}\right)}\right)}{\frac{1}{n^2}} &= \lim_{x\to\infty} \frac{\ln\left(\frac{1}{1-\sin^2\left(\frac{1}{x}\right)}\right)}{\frac{1}{x^2}} \\
(L'H) &= \lim_{x\to\infty} \frac{\left(1-\sin^2\left(\frac{1}{x}\right)\right)\cdot 2\sin\left(\frac{1}{x}\right)\cdot\cos\left(\frac{1}{x}\right)\cdot -\frac{1}{x^2}}{-\frac{2}{x^3}} \\
&= \lim_{x\to\infty}\left(1-\sin^2\left(\frac{1}{x}\right)\right)\cos\left(\frac{1}{x}\right)\cdot\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}.
\end{aligned}$$

Note that

$$\lim_{x\to\infty}\left(1-\sin^2\left(\frac{1}{x}\right)\right)\cos\left(\frac{1}{x}\right) = 1$$

by continuity and that

$$\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$$

by another application of L'Hopital (see the first example in this section). Thus,

$$\lim_{n\to\infty} \frac{\ln\left(\frac{1}{1-\sin^2\left(\frac{1}{n}\right)}\right)}{\frac{1}{n^2}} = 1.$$

By limit comparison, the series $\sum_{n=1}^{\infty} \ln\left(\frac{1}{1-\sin^2\left(\frac{1}{n}\right)}\right)$ behaves the same as $\sum_{n=1}^{\infty}\frac{1}{n^2}$. As the latter sum is a convergent p-series, the former converges as well. □

Example 3.24. Does the infinite series

$$\sum_{n=1}^{\infty}\left(1-3^{-\frac{1}{n}}\right)$$

converge or diverge?

Solution. For this problem, we can cleverly use the MacLaurin expansion of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots.$$

It follows that

$$3^x = e^{(\ln 3)\cdot x} = 1 + (\ln 3)x + \frac{((\ln 3)x)^2}{2!} + \cdots$$

and so

$$3^{-\frac{1}{n}} = 1 - \frac{\ln 3}{n} + \frac{\frac{(\ln 3)^2}{n^2}}{2!} + \cdots.$$

So for large values of n,

$$1 - 3^{-\frac{1}{n}} \approx \frac{\ln 3}{n}.$$

Thus, we run limit comparison with $b_n = \frac{1}{n}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{1-3^{-\frac{1}{n}}}{\frac{1}{n}} &= \lim_{x\to\infty} \frac{1-3^{-\frac{1}{x}}}{\frac{1}{x}} \\
(L'H) &= \lim_{x\to\infty} \frac{-3^{-\frac{1}{x}} \cdot \ln 3 \cdot \frac{1}{x^2}}{\frac{-1}{x^2}} \\
&= \ln 3.
\end{aligned}$$

Since $0 < \ln 3 < \infty$, by limit comparison, $\sum_{n=1}^{\infty}\left(1-3^{-\frac{1}{n}}\right)$ behaves the same as $\sum_{n=1}^{\infty}\frac{1}{n}$. Since the latter is a divergent p-series, the former diverges as well. □

One more example. This one looks wild, and is a testament to the power of this method of attack.

Example 3.25. Does the infinite series

$$\sum_{n=1}^{\infty}\left(\int_0^{\frac{1}{n}} \frac{\sin(x^2)}{x}\,dx\right)$$

converge or diverge?

Solution. This series appears intimidating; after all, there is an integral in the summand! As a remark, you could conceivably try to evaluate the integral with the goal of simplifying the infinite series, but you'll find that integrating $\frac{\sin(x^2)}{x}$ is not easy. Instead, we will trust the intuition we have from Taylor polynomials and fearlessly press on.

Because the MacLaurin series for $\sin x$ is

$$\sin x = x - \frac{1}{3!}x^3 + \cdots$$

we expect $\sin(x^2) \approx x^2$ and therefore $\frac{\sin(x^2)}{x} \approx \frac{x^2}{x} = x$ for small values of x. Therefore, we expect

$$\int_0^{\frac{1}{n}} \frac{\sin(x^2)}{x}\,dx \approx \int_0^{\frac{1}{n}} x\,dx = \frac{x^2}{2}\Big|_0^{\frac{1}{n}} = \frac{1}{2n^2}$$

for large values of n. So let's run limit comparison with $b_n = \frac{1}{n^2}$. Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\int_0^{\frac{1}{n}} \frac{\sin(x^2)}{x}\,dx}{\frac{1}{n^2}} &= \lim_{t\to\infty} \frac{\int_0^{\frac{1}{t}} \frac{\sin(x^2)}{x}\,dx}{\frac{1}{t^2}} \\
(L'H) &= \lim_{t\to\infty} \frac{\frac{d}{dt}\int_0^{\frac{1}{t}} \frac{\sin(x^2)}{x}\,dx}{\frac{-2}{t^3}}
\end{aligned}$$

by L'Hopital, as the initial limit is indeterminate of type $\frac{0}{0}$. By the fundamental theorem of calculus,

$$\frac{d}{dt}\int_0^{\frac{1}{t}} \frac{\sin(x^2)}{x}\,dx = \frac{\sin\left(\frac{1}{t^2}\right)}{\frac{1}{t}} \cdot \frac{-1}{t^2} = -\sin\left(\frac{1}{t^2}\right) \cdot \frac{1}{t}.$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\int_0^{\frac{1}{n}} \frac{\sin(x^2)}{x}\,dx}{\frac{1}{n^2}} &= \lim_{t\to\infty} \frac{-\sin\left(\frac{1}{t^2}\right)\frac{1}{t}}{\frac{-2}{t^3}} = \lim_{t\to\infty} \frac{\sin\left(\frac{1}{t^2}\right)}{\frac{2}{t^2}} \\
(L'H) &= \lim_{t\to\infty} \frac{\cos\left(\frac{1}{t^2}\right)\cdot\frac{-2}{t^3}}{\frac{-4}{t^3}} \\
&= \frac{1}{2}.
\end{aligned}$$

Since $0 < 1/2 < \infty$, by limit comparison, $\sum_{n=1}^{\infty} \int_0^{\frac{1}{n}} \frac{\sin(x^2)}{x}\,dx$ behaves the same as $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since the latter is a convergent p-series, the former converges as well. □

Chapter 4

Additional Infinite Series Topics

4.1 Rearrangement of conditionally convergent series

4.2 Summation by parts

One of the most well known integration techniques is integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Example 4.1. Evaluate the following infinite series:

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Solution. You may have seen this series evaluated by taking the derivative of the power series expansion

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Summation by parts gives another method of evaluation.

The first key observation is that

$$\frac{1}{2^n} = \frac{1}{2^{n-1}} - \frac{1}{2^n} \qquad \left(= \frac{2-1}{2^n}\right).$$

Thus,

$$\begin{aligned}\sum_{n=1}^{N} \frac{n}{2^n} &= \sum_{n=1}^{N} n\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) \\ &= \sum_{n=1}^{N} \frac{n}{2^{n-1}} - \sum_{n=1}^{N} \frac{n}{2^n}.\end{aligned}$$

Next, I will reindex the first series so that the denominator is also 2^n. This gives

$$\sum_{n=1}^{N} \frac{n}{2^{n-1}} - \sum_{n=1}^{N} \frac{n}{2^n} = \sum_{n=0}^{N-1} \frac{n+1}{2^n} - \sum_{n=1}^{N} \frac{n}{2^n}.$$

I did this because I wanted to combine the two sums and pull out a factor of $\frac{1}{2^n}$. Now, the beginning and ending index of the two sums do not match. To fix this, we can pull off the $n = 0$ term from the first series and the $n = N$ term from the second sum.

$$\sum_{n=0}^{N-1} \frac{n+1}{2^n} - \sum_{n=1}^{N} \frac{n}{2^n} = 1 - \frac{N}{2^N} + \sum_{n=1}^{N-1} \frac{n+1}{2^n} - \sum_{n=1}^{N-1} \frac{n}{2^n}.$$

Now we can combine the two sums:

$$1 - \frac{N}{2^N} + \sum_{n=1}^{N-1} \frac{n+1}{2^n} - \sum_{n=1}^{N-1} \frac{n}{2^n} = 1 - \frac{N}{2^N} + \sum_{n=1}^{N-1} \left(\frac{n+1}{2^n} - \frac{n}{2^n} \right) = 1 - \frac{N}{2^N} + \sum_{n=1}^{N-1} \frac{1}{2^n}.$$

In summary, we have shown

$$\sum_{n=1}^{N} \frac{n}{2^n} = 1 - \frac{N}{2^N} + \sum_{n=1}^{N-1} \frac{1}{2^n}.$$

Sending $N \to \infty$ and using the fact that $\lim_{N\to\infty} \frac{N}{2^N} = 0$ gives

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

We're left with a familiar geometric series! So

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 1 + \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 2.$$

□

4.3 Acceleration methods

4.4 Thinning out the harmonic series

4.5 Open infinite series

4.6 Fascinating formulas

Chapter 5

The Basel Problem

One of the most famous problems in the history of math is the *Basel problem*. The problem, first posed in 1650, is to evaluate the following infinite sum of numbers:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots . \tag{5.1}$$

You should recognize (5.1) as the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{5.2}$$

One of the first things you learn in a calculus course is that the series (5.2) converges to some finite number. In fact, (5.2) is a p*-series*, and the convergence behavior of p-series is well understood. A typical calculus class does not usually consider how to *evaluate* such sums. The main culprits for evaluation are things like geometric series, telescoping series, or obvious evaluations of MacLaurin series; the subject of the Basel problem is none of these.

It is no wonder then that the Basel problem remained unsolved for a long time. In fact, it was unsolved for almost 100 years! A young Leonhard Euler was the first to present a solution in 1735. He discovered the (perhaps surprising) result:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}. \tag{5.3}$$

The presence of a transcendental constant like π may come as a surprise. Even if you don't find it surprising, you have to admire how lovely the answer is.

So how *do* you evaluate an infinite sum like (5.2) and why don't calculus classes teach this? It turns out that evaluating a p-series is in general much more difficult than evaluating a geometric series or a p-series. There are a ton of different proofs out in the world, many of which use fancy math that most calculus students likely wouldn't understand. However, there are purely calculus based solutions out there, and in this text I've prepared four of my favorites. The first solution only assumes L'Hopital's rule and some basic knowledge about infinite series — all you need beyond that is a little faith. The next two solutions use double integrals, so multivariable calculus is a prerequisite. The fourth solution does not use multivariable calculus, but it does use complex numbers and Euler's formula from 2.1.1.

5.1 The first solution: A polynomial with infinitely many roots

The first solution begins with a seemingly unrelated discussion about polynomials. For organizational sake, I'll split this off into a subsection.

A digression on polynomials.

Let's play a hypothetical game. I give you a list of numbers, for example, $1, 2, 3$, and you give me a polynomial having exactly those numbers as roots. You would have no trouble in giving me an answer. For example,

$$p(x) = (x-1)(x-2)(x-3)$$

is such a polynomial. One issue is that there is no *unique* answer; for example,

$$q(x) = 100(x-1)(x-2)(x-3)$$

is another such polynomial. The next thing I'll do is write down the polynomials a little differently: instead of factors of the form $(x-c)$ like we're used to, I'm going to write $(1-\frac{x}{c})$. This quantity is 0 when $x = c$, so therefore

$$k(x) = \left(1-\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\left(1-\frac{x}{3}\right)$$

is yet another polynomial with $1, 2, 3$ as roots. Note if 0 is one of our desired roots, we can't write the factor as $(1-\frac{x}{0})$, simply because division by 0 is not allowed. Thus, we can only write x. So for example, if I asked for a polynomial with roots $-3, 0, \pi, 77$, you might give me the following answers:

$$\ell(x) = x\left(1-\frac{x}{-3}\right)\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{77}\right)$$

or

$$m(x) = 12x\left(1-\frac{x}{-3}\right)\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{77}\right).$$

The last observation I'll make is one about the leading constant, specifically for the polynomials $\ell(x)$ and $m(x)$ which have 0 as a root. Note that

$$\lim_{x\to 0}\frac{m(x)}{x} = \lim_{x\to 0} 12\left(1-\frac{x}{-3}\right)\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{77}\right) = 12,$$
$$\lim_{x\to 0}\frac{\ell(x)}{x} = \lim_{x\to 0}\left(1-\frac{x}{-3}\right)\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{77}\right) = 1.$$

The point is that if $p(x)$ is a polynomial with x as a root written with factors in the form $(1-\frac{x}{c})$, we can recover the leading constant by computing $\lim_{x\to 0}\frac{p(x)}{x}$.

Back to the solution.

The point of the above discussion about writing down polynomials is that we're going to the same thing, but with the function $\sin x$. Obviously, $\sin x$ is not a polynomial, but it turns out that we can treat $\sin x$ as a polynomial with infinitely many roots! If you've studied power series and specifically MacLaurin series, this might not be surprising: the MacLaurin series expansion of $\sin x$ is more or less an infinite-degree polynomial written as an infinite sum. This time, we will expand $\sin x$ into an infinite polynomial as an *infinite product*.

What are the roots of $\sin x$? We know that $\sin(n\pi) = 0$ for every integer n, so the roots of $\sin x$ are

$$0, \pi, -\pi, 2\pi, -2\pi, 3\pi, -3\pi, \ldots.$$

By the previous discussion, it follows that[1]

$$\sin x = Cx\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{-\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{-2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1-\frac{x}{-3\pi}\right)\cdots \quad (5.4)$$

[1] This is not a formal derivation. Fully justifying this *would* require some fancy math!

for some constant C. Also by the above discussion, we can revover the constant C with the following L'Hopital computation:

$$C = \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = 1.$$

Thus, $C = 1$ and so

$$\sin x = x \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 - \frac{x}{-3\pi}\right)\cdots . \tag{5.5}$$

Now we're in business. The rest of the solution uses some clever manipulations and observations about the above infinite product. First, plugging in $x = \pi y$ into (5.5) gives

$$\begin{aligned}\sin(\pi y) &= \pi y \left(1 - \frac{y}{1}\right)\left(1 - \frac{y}{-1}\right)\left(1 - \frac{y}{2}\right)\left(1 - \frac{y}{-2}\right)\left(1 - \frac{y}{3}\right)\left(1 - \frac{y}{-3}\right)\cdots \\ &= \pi y (1-y)(1+y)\left(1 - \frac{y}{2}\right)\left(1 + \frac{y}{2}\right)\left(1 - \frac{y}{3}\right)\left(1 + \frac{y}{3}\right)\cdots .\end{aligned}$$

Notice that every term after the initial factor of πy comes paired with a conjugate term:

$$\pi y \underbrace{(1-y)(1+y)}\underbrace{\left(1 - \frac{y}{2}\right)\left(1 + \frac{y}{2}\right)}\underbrace{\left(1 - \frac{y}{3}\right)\left(1 + \frac{y}{3}\right)}\cdots .$$

Using difference of squares, we can multiply each of these paired terms together to get

$$\sin(\pi y) = \pi y \left(1 - y^2\right)\left(1 - \frac{y^2}{4}\right)\left(1 - \frac{y^2}{9}\right)\cdots . \tag{5.6}$$

You can already begin to see the desired $1, 4, 9, 16, \ldots$ pattern emerge! The pattern is locked away in a *product*, and we want the pattern to be exhibited by a *sum*. One useful tool for converting multiplication to addition is the logarithm. Let's take the log of both sides of (5.6):

$$\begin{aligned}\ln(\sin(\pi y)) &= \ln\left[\pi y \left(1 - y^2\right)\left(1 - \frac{y^2}{4}\right)\left(1 - \frac{y^2}{9}\right)\cdots\right] \\ &= \ln \pi + \ln y + \ln\left(1 - y^2\right) + \ln\left(1 - \frac{y^2}{4}\right) + \ln\left(1 - \frac{y^2}{9}\right) + \cdots .\end{aligned}$$

Next, we differentiate both sides with respect to y. This yields

$$\frac{\pi \cos(\pi y)}{\sin(\pi y)} = \frac{1}{y} + \frac{-2y}{1 - y^2} + \frac{-\frac{2y}{4}}{1 - \frac{y^2}{4}} + \frac{-\frac{2y}{9}}{1 - \frac{y^2}{9}} + \cdots .$$

Moving the $\frac{1}{y}$ to the left side and then dividing both sides by $-2y$ then gives

$$\frac{1}{2y^2} - \frac{\pi \cos(\pi y)}{2y \sin(\pi y)} = \frac{1}{1 - y^2} + \frac{1}{4 - y^2} + \frac{1}{9 - y^2} + \cdots .$$

It should be apparent that to get the desired sum, all that remains to do is "plug in $y = 0$" on the right hand side. Unfortunately, the left side is not defined at 0, so we need to take a limit. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \lim_{y \to 0}\left(\frac{1}{2y^2} - \frac{\pi \cos(\pi y)}{2y \sin(\pi y)}\right).$$

Buckle up, because we will need to use L'Hopital's rule a couple of times to evaluate this limit. The first thing I'll do is combine the fractions; then, we will have an indeterminate form of the type $\frac{0}{0}$ which allows us to use L'Hopital's rule.

$$\begin{aligned}\lim_{y\to 0}\left(\frac{1}{2y^2}-\frac{\pi\cos(\pi y)}{2y\sin(\pi y)}\right) &= \lim_{y\to 0}\left(\frac{\sin(\pi y)-\pi y\cos(\pi y)}{2y^2\sin(\pi y)}\right)\\ &= \lim_{y\to 0}\frac{\pi\cos(\pi y)-\pi\cos(\pi y)+\pi^2 y\sin(\pi y)}{4y\sin(\pi y)+2\pi y^2\cos(\pi y)}.\end{aligned}$$

The last equality comes from one application of L'Hopital's rule. Let's simplify a little bit.

$$\begin{aligned}\lim_{y\to 0}\frac{\pi\cos(\pi y)-\pi\cos(\pi y)+\pi^2 y\sin(\pi y)}{4y\sin(\pi y)+2\pi y^2\cos(\pi y)} &= \lim_{y\to 0}\frac{\pi^2 y\sin(\pi y)}{4y\sin(\pi y)+2\pi y^2\cos(\pi y)}\\ &= \lim_{y\to 0}\frac{\pi^2\sin(\pi y)}{4\sin(\pi y)+2\pi y\cos(\pi y)}.\end{aligned}$$

This limit is still a $\frac{0}{0}$ indeterminate form, so we can use L'Hopital's rule one more time:

$$\begin{aligned}\lim_{y\to 0}\frac{\pi^2\sin(\pi y)}{4\sin(\pi y)+2\pi y\cos(\pi y)} &= \lim_{y\to 0}\frac{\pi^3\cos(\pi y)}{4\pi\cos(\pi y)+2\pi\cos(\pi y)-2\pi^2 y\sin(\pi y)}\\ &= \frac{\pi^3\cdot 1}{4\pi\cdot 1+2\pi\cdot 1-2\pi^2\cdot 0}\\ &= \frac{\pi^2}{6}.\end{aligned}$$

There we go! Putting everything together, we have shown that

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots=\frac{\pi^2}{6}.$$

5.2 The second solution: A double integral

For the second solution, we cleverly turn the infinite sum (5.2) into a double integral. The first observation is

$$\int_0^1 x^{n-1}\,dx = \left.\frac{x^n}{n}\right|_0^1 = \frac{1}{n}.$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{n} &= \sum_{n=1}^{\infty}\left(\int_0^1 x^{n-1}\,dx\right)\left(\int_0^1 y^{n-1}\,dy\right)\\ &= \int_0^1\int_0^1\sum_{n=1}^{\infty}(xy)^{n-1}\,dx\,dy.\end{aligned}$$

Note that the inner sum is a geomtric series with ratio xy. Using the geometric series summation formula, we then have

$$\int_0^1\int_0^1\sum_{n=1}^{\infty}(xy)^{n-1}\,dx\,dy = \int_0^1\int_0^1\frac{1}{1-xy}\,dx\,dy.$$

Summarizing our progress so far, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \int_0^1\int_0^1\frac{1}{1-xy}\,dx\,dy. \tag{5.7}$$

It remains to evaluate the latter double integral. We make the change of variables $u = x+y$ and$v = y - x$. Then $x = \frac{u-v}{2}$, $y = \frac{u+v}{2}$, and

$$\left|\frac{\partial x\, \partial y}{\partial u\, \partial v}\right| = \left|\begin{bmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{bmatrix}\right| = \left|\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right| = \frac{1}{2}.$$

This shows that $dx\, dy = \frac{1}{2}\, du\, dv$. Thus,

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dy\, dx = \frac{1}{2} \iint_E \frac{1}{1 - \frac{u^2-v^2}{4}}\, du\, dv = 2 \iint_E \frac{1}{4-u^2+v^2}\, du\, dv$$

where E is the square with vertices $(0,0)$, $(1,-1)$, $(2,0)$, and $(1,1)$. Next, we can make a further reduction by noticing that the integrand is even with respect to v. That is,

$$\frac{1}{4-u^2+(-v)^2} = \frac{1}{4-u^2+v^2}.$$

Because the square E is symmetric across the u-axis, we have

$$2 \iint_E \frac{1}{4-u^2+v^2}\, du\, dv = 2 \iint_{E^+} \frac{1}{4-u^2+v^2}\, du\, dv$$

where E^+ is the triangle with vertices $(0,0)$, $(2,0)$, and $(1,1)$, which is the top half of the (tilted) square E.

FINISH

5.3 The third solution: Another double integral

The third solution is rooted in another double integral computation, due to [1].[2] First, we will cleverly reduce the sum of squared reciprocals to the sum of squared *odd* reciprocals. Then, we proceed as in the previous solution and use our knowledge of geometric series to write the sum as an integral.

Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

All I've done here is decompose the sum over all n to two sums, one over even numbers and one over odd numbers (using the dummy variable n for both sums). Pulling out the factor of $\frac{1}{2^2}$ in the first sum on the right yields the following equation:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

We can algebraically solve for desired sum:

$$\frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

[2] In fact, [1] contains a more general computation which evaluates

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

for all $k > 0$.

So all we have to do now is evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. The initial set up will be very similar to that in the second solution, this time hinging on the observation that

$$\int_0^1 x^{2n}\,dx = \frac{x^{2n+1}}{2n+1}\bigg|_0^1 = \frac{1}{2n+1}.$$

In particular, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \cdot \frac{1}{(2n+1)} &= \sum_{n=0}^{\infty} \left(\int_0^1 x^{2n}\,dx\right)\left(\int_0^1 y^{2n}\,dy\right) \\ &= \int_0^1 \int_0^1 \sum_{n=0}^{\infty} (x^2y^2)^n \,dx\,dy \\ &= \int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy.\end{aligned}$$

We will evaluate this double integral in a much different way than the double integral from the second solution. The trick is an absurdly clever variable substitution:

$$x = \frac{\sin u}{\cos v} \quad \text{and} \quad y = \frac{\sin v}{\cos u}.$$

Note that

$$\left|\frac{\partial x\,\partial y}{\partial u\,\partial v}\right| = \left|\begin{bmatrix}\partial_u x & \partial_v x \\ \partial_u y & \partial_v y\end{bmatrix}\right| = \left|\begin{bmatrix}\frac{\cos u}{\cos v} & \frac{\sin u \sin v}{\cos^2 v} \\ \frac{\sin u \sin v}{\cos^2 u} & \frac{\cos v}{\cos u}\end{bmatrix}\right| = 1 - \tan^2 u \tan^2 v.$$

It follows that $dx\,dy = (1 - \tan^2 u \tan^2 v)\,du\,dv$.

Now we have to figure out how the region of integration transforms under this change of variables. Let E be the region in the the square $0 \leq u, v \leq \frac{\pi}{2}$ which is the image of $0 \leq x, y \leq 1$ under this transformation. Note that this region is defined by $\sin u \leq \cos v$ and $\sin v \leq \cos u$, since $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Note that equality occurs in both inequalities if $v = \frac{\pi}{2} - u$, since

$$\cos\left(\frac{\pi}{2} - v\right) = \sin v \quad \text{and} \quad \sin\left(\frac{\pi}{2} - u\right) = \cos u.$$

This line $v = \frac{\pi}{2} - u$ divides the square $0 \leq u, v \leq \frac{\pi}{2}$ into two triangles. The inequalities dictate that E is the triangle with vertices $(0,0)$, $(\pi/2, 0)$, and $(0, \pi/2)$. If this last comment is suspicious, it is instructive to plug in a test point like $(u = \frac{\pi}{4}, v = \frac{\pi}{6})$, which lies below the diagonal $v = \frac{\pi}{2} - u$. You can verify that this point satisfies both of the inequalities $\sin u \leq \cos v$ and $\sin v \leq \cos u$. Likewise, any point *above* the diagonal will satisfy neither inequality.

Anyway, we can now transform the double integral. We have

$$\begin{aligned}\int_0^1 \int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy &= \iint_E \frac{1}{1 - \left(\frac{\sin u}{\cos v}\right)^2 \left(\frac{\sin v}{\cos u}\right)^2} (1 - \tan^2 u \tan^2 v)\,du\,dv \\ &= \iint_E \frac{1 - \tan^2 u \tan^2 v}{1 - \tan^2 u \tan^2 v}\,du\,dv \\ &= \iint_E du\,dv.\end{aligned}$$

This final integral just gives the area of the region E. Recall that E was the triangle with with vertices $(0,0)$, $(\pi/2, 0)$, and $(0, \pi/2)$. This triangle has a base width of $\frac{\pi}{2}$ and a height of $\frac{\pi}{2}$. Thus,

$$\iint_E du\,dv = \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{8}.$$

Putting all of our computations together, we have:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2} &= \frac{4}{3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\\ &= \frac{4}{3}\int_0^1\int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy\\ &= \frac{4}{3}\iint_E du\,dv\\ &= \frac{4}{3}\cdot\frac{\pi^2}{8}\\ &= \frac{\pi^2}{6}.\end{aligned}$$

5.4 The fourth solution: A complex integral

The next solution begins with the computation of a difficult integral, one which seems wholly unrelated to the problem at hand. Consider

$$\int_0^{\frac{\pi}{2}} \ln(2\cos x)\,dx.$$

Note that for $0 < x < \frac{\pi}{2}$, $\cos x > 0$ and so the function is well-defined. To deal with this integral we will use Euler's formula as in 2.1.1. Recall that

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}.$$

Thus, $2\cos x = e^{ix} + e^{-ix}$. The integral in question is then

$$\int_0^{\frac{\pi}{2}} \ln(2\cos x)\,dx = \int_0^{\frac{\pi}{2}} \ln(e^{ix}+e^{-ix})\,dx.$$

In order to make sense of this integral, we wish to use the MacLaurin series expansion for $\ln(1+x)$:

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}. \tag{5.8}$$

Towards this goal, we can factor out a factor of e^{ix} in the argument of the natural logarithm.

$$\int_0^{\frac{\pi}{2}} \ln(e^{ix}+e^{-ix})\,dx = \int_0^{\frac{\pi}{2}} \ln\left(e^{ix}\left(1+e^{-2ix}\right)\right)\,dx.$$

Now we can use the multiplicative property of the logarithm to split the integrand.[3]

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \ln\left(e^{ix}\left(1+e^{-2ix}\right)\right)\,dx &= \int_0^{\frac{\pi}{2}} \ln(e^{ix}) + \ln(1+e^{-2ix})\,dx\\ &= \int_0^{\frac{\pi}{2}} ix + \ln(1+e^{-2ix})\,dx.\end{aligned}$$

[3] I'm being a little dishonest by blatantly using properties of the logarithm with complex numbers. It turns out that extending the function $\ln x$ to complex numbers requires some care, but that's the subject of a complex analysis class. For our sake, rooted in the world of calculus, we carelessly apply our knowledge of the real-valued function $\ln x$, cross our fingers, and hope that everything works out. If you're curious, since $0 < x < \frac{\pi}{2}$, we can take $\ln z$ to be something called the *principal branch of the logarithm*.

The first term is easy to integrate and yields $\int_0^{\frac{\pi}{2}} ix\,dx = \frac{ix^2}{2}\Big|_0^{\frac{\pi}{2}} = i\frac{\pi^2}{8}$. For the second term, we use[4] (5.8).

$$\int_0^{\frac{\pi}{2}} ix + \ln(1+e^{-2ix})\,dx = i\frac{\pi^2}{8} + \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-2inx}}{n}\,dx. \tag{5.9}$$

Integrating the resulting infinite series[5] term by term gives

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-2inx}}{n}\,dx &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-2inx}}{(-2in)n}\Bigg|_0^{\frac{\pi}{2}} \\ &= -\frac{1}{2i} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-in\pi}-1}{n^2}. \end{aligned}$$

In order to make sense of this infinite series, we need to carefully analyse the term $e^{-in\pi}-1$. Recall that Euler's formula gives

$$e^{-in\pi} = \cos(n\pi) + i\sin(n\pi).$$

Since $\sin(n\pi) = 0$ for all n and $\cos(n\pi) = (-1)^n$, we have

$$e^{-in\pi} - 1 = (-1)^n - 1 = \begin{cases} 0 & \text{if } n = 2k \\ -2 & \text{if } n = 2k+1 \end{cases}.$$

Since terms with even index are 0, we can reindex the sum and only consider those with odd index. Symbolically,

$$-\frac{1}{2i} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-in\pi}-1}{n^2} = -\frac{1}{2i} \sum_{k=0}^{\infty} (-1)^{2k+2} \frac{-2}{(2k+1)^2} = \frac{1}{i} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

Next, I will multiply the top and bottom of the coefficient out from by i in order to combine this sum with the $i\frac{\pi^2}{8}$ term. Since $i^2 = -1$, this yields

$$\frac{1}{i} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = -i \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

Plugging this into (5.9) and then relating back to the original integral, we have concluded

$$\int_0^{\frac{\pi}{2}} \ln(2\cos x)\,dx = \left(\frac{\pi^2}{8} - \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \right) i. \tag{5.10}$$

Let's think about this for a little bit. The integral we began with on the left of (5.10) is an integral of a real-valued function, so the final answer should be a *real* number. But on the right hand side, we have some (completely real) quantity multiplied by i. In other words, the right hand side is purely imaginary number. The only way this can possibly make sense is if both sides are equal to 0. Said differently, if $x = iy$ for real numbers x and y, then $x = y = 0$.

[4]Here we have another issue with the complex logarithm. We know that the series in (5.8) converges for real values of x in the interval $-1 < x \leq 1$. It turns out that the *complex* series converges for complex numbers satisfying $|z| \leq 1$, except for $z = -1$. Since $0 < x < \frac{\pi}{2}$, $e^{-2ix} \neq -1$ and $|e^{-2ix}| = 1$. Thus, we are more or less allowed to use the power series for $\ln x$. In the spirit of calculus, we also won't deal with this issue carefully.

[5]This is no longer a power series, so in an analysis class we would need some justification for being able to integrate term by term like we do with power series.

This gives two amazing facts, one of which is

$$\int_0^{\frac{\pi}{2}} \ln(2\cos x)\, dx = 0.$$

Our final goal is the Basel problem, but this is a lovely result in itself! Equating the right side of (5.10) to 0 gives

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

If you read the third solution to the Basel problem in 5.3, this result should look familiar! Another important fact from the third solution is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

I will defer you to the beginning of 5.3 for the proof of this fact. Since $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

Chapter 6

Some Really Hard Problems

In this chapter, we turn the calculus dial up to 11. I have chosen a number of exceedingly difficult (but fascinating and fun) integrals and infinite series to evaluate. These problems demonstrate just how crazy a normal looking calculus problem can be. If you decide to take the plunge into this chapter, buckle up — it will be an adventure.

As a warning, I do use a bit of multivariable calculus at times!

6.1 A really hard integral

The following integral is known as *Coxeter's integral*, and originated in CITE. Along the way (in particular, in Step 3) this integral gets reduced to one known as Ahmed's integral, which was introduced in CITE. The solution I've presented below is a concatenation of two solutions from [3].

Example 6.1. Evaluate the following integral:

$$I = \int_0^{\frac{\pi}{2}} \arccos\left(\frac{\cos x}{1+2\cos x}\right)\, dx.$$

Solution. I have broken up the solution into three major steps.

Step 1: Rewrite the integrand with trigonometry and then introduce a double integral.

We begin with some trigonometry. In particular the first thing I will use is the double angle identity

$$\cos(2\theta) = 2\cos^2\theta - 1.$$

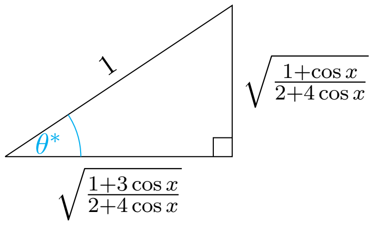
This implies $2\theta = \arccos\left(2\cos^2\theta - 1\right)$. Letting $\alpha = 2\cos^2\theta - 1$ and then solving for θ yields $\theta = \arccos\left(\sqrt{\frac{1+\alpha}{2}}\right)$, and thus

$$\arccos(\alpha) = 2\arccos\left(\sqrt{\frac{1+\alpha}{2}}\right).$$

Using this, the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \underbrace{2\arccos\left(\sqrt{\frac{1+3\cos x}{2+4\cos x}}\right)}_{\theta^*}\, dx.$$

Next, consider a right triangle with angle θ^*.



Since

$$\cos\theta^* = \sqrt{\frac{1+3\cos x}{2+4\cos x}} \quad \text{and} \quad \tan\theta^* = \frac{\sqrt{\frac{1+\cos x}{2+4\cos x}}}{\sqrt{\frac{1+3\cos x}{2+4\cos x}}} = \sqrt{\frac{1+\cos x}{1+3\cos x}}$$

it follows that

$$\arccos\left(\sqrt{\frac{1+3\cos x}{2+4\cos x}}\right) = \arctan\left(\sqrt{\frac{1+\cos x}{1+3\cos x}}\right)$$

and so

$$I = 2\int_0^{\frac{\pi}{2}} \arctan\left(\sqrt{\frac{1+\cos x}{1+3\cos x}}\right) dx.$$

With the goal of using the aforementioned double angle identity again, we make the substitution $x = 2y$. Then $dx = 2\,dy$ and we have

$$\begin{aligned} I &= 4\int_0^{\frac{\pi}{4}} \arctan\left(\sqrt{\frac{1+\cos 2y}{1+3\cos 2y}}\right) dy \\ &= 4\int_0^{\frac{\pi}{4}} \arctan\left(\sqrt{\frac{2\cos^2 y}{-2+6\cos^2 y}}\right) dy \\ &= 4\int_0^{\frac{\pi}{4}} \arctan\underbrace{\left(\frac{\cos y}{\sqrt{2-3\sin^2 y}}\right)}_{b} dy. \end{aligned}$$

In the last equality I used the identity $\sin^2 y + \cos^2 y = 1$. Recall that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C$. Thus,

$$\begin{aligned} \int_0^1 \frac{1}{1+b^2t^2}\,dt = \frac{1}{b^2}\int_0^1 \frac{1}{b^{-2}+t^2}\,dt &= \frac{1}{b}\arctan(bt)\Big|_0^1 \\ &= \frac{1}{b}\arctan(b). \end{aligned}$$

This implies

$$\begin{aligned} I &= 4\int_0^{\frac{\pi}{4}} \frac{\cos y}{\sqrt{2-3\sin^2 y}} \int_0^1 \frac{1}{1+\frac{\cos^2 y}{2-3\sin^2 y}t^2}\,dt\,dy \\ &= 4\int_0^{\frac{\pi}{4}}\int_0^1 \frac{\cos y\sqrt{2-3\sin^2 y}}{2-3\sin^2 y+\cos^2 y t^2}\,dt\,dy \\ &= 4\int_0^{\frac{\pi}{4}}\int_0^1 \frac{\cos y\sqrt{2-3\sin^2 y}}{(t^2+2)-(t^2+3)\sin^2 y}\,dt\,dy. \end{aligned}$$

Next, we will adjust constants in order to simplify the expression in the numerator. In particular, let $\sin y = \sqrt{\frac{2}{3}} \sin w$. Then $\cos y\, dy = \sqrt{\frac{2}{3}} \cos w\, dw$, and the integral becomes

$$\begin{aligned} I &= 4 \int_0^{\frac{\pi}{3}} \int_0^1 \frac{\sqrt{2}\cos w}{(t^2+2) - (t^2+3)\frac{2}{3}\sin^2 w}\, dt \sqrt{\frac{2}{3}} \cos w\, dw \\ &= 8\sqrt{3} \int_0^{\frac{\pi}{3}} \int_0^1 \frac{\cos^2 w}{(3t^2+6) - (2t^2+6)\sin^2 w}\, dt\, dw \\ &= 8\sqrt{3} \int_0^{\frac{\pi}{3}} \int_0^1 \frac{\cos^2 w}{t^2 + (2t^2+6)\cos^2 w}\, dt\, dw \\ &= 8\sqrt{3} \int_0^{\frac{\pi}{3}} \int_0^1 \frac{1}{t^2 \sec^2 w + (2t^2+6)}\, dt\, dw. \end{aligned}$$

Step 2: Use a trig substitution, partial fractions, then integration by parts.

Let $s = \tan w$. Then $ds = \sec^2 w\, dw$. Since $1 + \tan^2 w = \sec^2 w$, we have $\sec^2 w = 1 + s^2$ and $dw = \frac{1}{1+s^2}\, ds$. Thus,

$$\begin{aligned} I &= 8\sqrt{3} \int_0^{\sqrt{3}} \int_0^1 \frac{1}{(1+s^2)t^2 + (2t^2+6)}\, dt \frac{1}{1+s^2}\, ds \\ &= 8\sqrt{3} \int_0^{\sqrt{3}} \int_0^1 \frac{1}{(1+s^2)(3t^2+t^2s^2+6)}\, dt\, ds. \end{aligned}$$

Next, we decompose the integrand with partial fractions. We have

$$I = 8\sqrt{3} \int_0^{\sqrt{3}} \int_0^1 \frac{1}{2t^2+6} \left(\frac{1}{1+s^2} - \frac{t^2}{3t^2+t^2s^2+6} \right) dt\, ds.$$

The terms in the parentheses can be integrated with respect to s using the inverse tangent. Thus, we switch the order of integration.

$$\begin{aligned} I &= 8\sqrt{3} \int_0^1 \int_0^{\sqrt{3}} \frac{1}{2t^2+6} \left(\frac{1}{1+s^2} - \frac{1}{3+\frac{6}{t^2}+s^2} \right) ds\, dt \\ &= 4\sqrt{3} \int_0^1 \frac{1}{t^2+3} \left(\frac{\pi}{3} - \frac{1}{\sqrt{3+\frac{6}{t^2}}} \arctan\left(\frac{\sqrt{3}}{\sqrt{3+\frac{6}{t^2}}} \right) \right) dt. \end{aligned}$$

Here we have again used the fact that $\int \frac{1}{a^2+x^2}\, dx = \frac{1}{a}\arctan(\frac{x}{a}) + C$. Next, the $\frac{1}{t^2+3}$ term can be integrated similarly with respect to t. This gives

$$\begin{aligned} I &= \frac{4\pi\sqrt{3}}{3} \int_0^1 \frac{1}{t^2+3}\, dt - 4\sqrt{3} \int_0^1 \frac{1}{(t^2+3)} \frac{1}{\sqrt{3+\frac{6}{t^2}}} \arctan\left(\frac{\sqrt{3}}{\sqrt{3+\frac{6}{t^2}}} \right) dt \\ &= \frac{4\pi\sqrt{3}}{3} \frac{1}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}}\right) - 4 \int_0^1 \frac{1}{(t^2+3)} \frac{t}{\sqrt{t^2+2}} \arctan\left(\frac{t}{\sqrt{t^2+2}} \right) dt \\ &= \frac{2\pi^2}{9} - 4 \int_0^1 \frac{t}{(t^2+3)\sqrt{t^2+2}} \arctan\left(\frac{t}{\sqrt{t^2+2}} \right) dt. \end{aligned}$$

Next, we will integrate by parts. Let

$$u = \arctan\left(\frac{t}{\sqrt{t^2+2}} \right) \quad \text{and} \quad dv = \frac{t}{(t^2+3)\sqrt{t^2+2}}\, dt.$$

Then

$$du = \frac{1}{1+\frac{t^2}{t^2+2}} \cdot \frac{\sqrt{t^2+2} - \frac{t^2}{\sqrt{t^2+2}}}{t^2+2}\, dt = \frac{1}{(t^2+1)\sqrt{t^2+2}}\, dt. \tag{6.1}$$

Next, observe that

$$\frac{d}{dx} \arctan(\sqrt{t^2+2}) = \frac{1}{(1+t^2+2} \cdot \frac{t}{\sqrt{t^2+2}} = \frac{t}{(t^2+3)\sqrt{t^2+2}}.$$

Thus, $v = \arctan(\sqrt{t^2+2})$. Integrating by parts with this set up yields

$$\begin{aligned} I &= \frac{2\pi^2}{9} - 4\left(\arctan\left(\frac{t}{\sqrt{t^2+2}}\right) \arctan(\sqrt{t^2+2}) \Big|_0^1 - \int_0^1 \frac{\arctan(\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\, dt \right) \\ &= \frac{2\pi^2}{9} - 4\left(\frac{\pi}{6} \cdot \frac{\pi}{3} - \int_0^1 \frac{\arctan(\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\, dt \right) \\ &= 4\int_0^1 \frac{\arctan(\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\, dt. \end{aligned}$$

Step 3: Differentiate under the integral.

In the final step, we will use the technique of differentiating a parametric integral discussed in Section 2.2. Introduce an additional parameter in the integrand as follows:

$$I(z) = 4\int_0^1 \frac{\arctan(z\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\, dt.$$

We seek $I = I(1)$. By the fundamental theorem of calculus,

$$\int_1^\infty I'(z)\, dz = \lim_{z\to\infty} I(z) - I(1) \qquad \Rightarrow \qquad I = \lim_{z\to\infty} I(z) - \int_1^\infty I'(z)\, dz.$$

We compute each term separately.

$$\begin{aligned} \lim_{z\to\infty} I(z) &= 4\int_0^1 \frac{\lim_{z\to\infty} \arctan(z\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\, dt \\ &= 4\int_0^1 \frac{\frac{\pi}{2}}{(t^2+1)\sqrt{t^2+2}}\, dt \\ &= 2\pi \arctan\left(\frac{t}{\sqrt{t^2+2}}\right) \Big|_0^1 \\ &= \frac{\pi^2}{3}. \end{aligned}$$

The second to last equality comes from our previous computation in (6.1).

Next,

$$\begin{aligned} \int_1^\infty I'(z)\, dz &= 4\int_1^\infty \int_0^1 \frac{d}{dz}\left(\frac{\arctan(z\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\right) dt\, dz \\ &= 4\int_1^\infty \int_0^1 \frac{1}{(t^2+1)\sqrt{t^2+2}} \cdot \frac{1}{1+z^2(t^2+2)} \cdot \sqrt{t^2+2}\, dt\, dz \\ &= 4\int_1^\infty \int_0^1 \frac{1}{(t^2+1)(1+z^2t^2+2z^2)}\, dt\, dz. \end{aligned}$$

We decompose the integrand with partial fractions.

$$
\begin{aligned}
\int_1^\infty I'(z)\,dz &= 4\int_1^\infty \int_0^1 \frac{1}{1+z^2}\left(\frac{1}{1+t^2} - \frac{z^2}{1+z^2t^2+2z^2}\right) dt\,dz \\
&= 4\int_1^\infty \frac{1}{1+z^2}\left(\int_0^1 \frac{1}{1+t^2}\,dt - \int_0^1 \frac{z^2}{1+z^2t^2+2z^2}\,dt\right) dz \\
&= 4\int_1^\infty \frac{1}{1+z^2}\left(\frac{\pi}{4} - \int_0^1 \frac{1}{2+\frac{1}{z^2}+t^2}\,dt\right) dz \\
&= \pi\arctan(z)\Big|_1^\infty - 4\int_1^\infty \frac{1}{1+z^2}\cdot\frac{1}{\sqrt{2+\frac{1}{z^2}}}\arctan\left(\frac{1}{\sqrt{2+\frac{1}{z^2}}}\right) dz \\
&= \frac{\pi^2}{4} - 4\int_1^\infty \frac{1}{1+z^2}\cdot\frac{1}{\sqrt{2+\frac{1}{z^2}}}\arctan\left(\frac{1}{\sqrt{2+\frac{1}{z^2}}}\right) dz.
\end{aligned}
$$

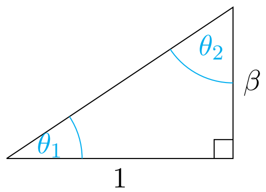
Next, let $r = \frac{1}{z}$. Then $dz = -\frac{1}{r^2}\,dr$, and the above integral becomes

$$
\begin{aligned}
\int_1^\infty I'(z)\,dz &= \frac{\pi^2}{4} - 4\int_0^1 \frac{1}{1+\frac{1}{r^2}}\cdot\frac{1}{\sqrt{2+r^2}}\arctan\left(\frac{1}{\sqrt{2+r^2}}\right)\frac{1}{r^2}\,dr \\
&= \frac{\pi^2}{4} - 4\int_0^1 \frac{1}{(r^2+1)\sqrt{2+r^2}}\arctan\left(\frac{1}{\sqrt{2+r^2}}\right) dr.
\end{aligned}
$$

Recall that

$$
I = 4\int_0^1 \frac{\arctan(\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\,dt.
$$

The integral in the above expression is very similar to I, with an inverted argument in the inverse tangent. Motivated by this, we invoke a handy trig identity. Consider the following right triangle:



Since the angles of a triangle add up to π radians, the picture implies that

$$
\theta_1 + \theta_2 = \frac{\pi}{2} \qquad \Rightarrow \qquad \arctan(\beta) + \arctan\left(\frac{1}{\beta}\right) = \frac{\pi}{2}.
$$

Using this identity,

$$
\begin{aligned}
\int_1^\infty I'(z)\,dz &= \frac{\pi^2}{4} - 4\int_0^1 \frac{1}{(r^2+1)\sqrt{2+r^2}}\left(\frac{\pi}{2} - \arctan(\sqrt{t^2+2})\right) dr \\
&= \frac{\pi^2}{4} - 2\pi\int_0^1 \frac{1}{(r^2+1)\sqrt{r^2+2}}\,dr + 4\int_0^1 \frac{\arctan(\sqrt{r^2+2})}{(r^2+1)\sqrt{r^2+2}}\,dr \\
&= \frac{\pi^2}{4} - \frac{\pi^2}{3} + I
\end{aligned}
$$

where in the last equality we again made use of (6.1). Thus,

$$\begin{aligned} I &= \lim_{z\to\infty} I(z) - \int_1^{\infty} I'(z)\, dz \\ &= \frac{\pi^2}{3} - \left(\frac{\pi^2}{4} - \frac{\pi^2}{3} + I\right). \end{aligned}$$

Solving for I in this last equation gives the final answer:

$$\boxed{\int_0^{\frac{\pi}{2}} \arccos\left(\frac{\cos x}{1+2\cos x}\right) dx = \frac{5\pi^2}{24}}.$$

□

Appendix A

Proofs of Convergence Tests

A.1 The condensation test

Theorem A.1. *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series such that* a_n *is a positive, decreasing sequence. Then*

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \sum_{n=1}^{\infty} 2^n a_{2^n}$$

either both converge or diverge.

Proof. Note that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= a_1 + a_2 + a_3 + \cdots \\ &= (a_1) + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots .\end{aligned}$$

Here I have suggestedly grouped all the terms of the series into groups of size 1, 2, 4, etc. The next group would have size 8, then 16, and in general the nth group would have size 2^n. Within each group, we use the assumption that a_n is decreasing. For example, $a_3 \leq a_2$, hence $(a_2 + a_3) \leq (a_2 + a_2) = 2a_2$. Similarly, $a_7 \leq a_6 \leq a_5 \leq a_4$, thus

$$(a_4 + a_5 + a_6 + a_7) \leq (a_4 + a_4 + a_4 + a_4) = 4a_4.$$

More generally,

$$(a_{2^n} + \cdots a_{2^{n+1}-1}) \leq (\overbrace{a_{2^n} + \cdots + a_{2^n}}^{2^n \text{ times}}) = 2^n a_{2^n}.$$

All of this shows that

$$0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=0}^{\infty} 2^n a_{2^n}.$$

Thus, if the latter series converges, then the former series converges as well.

Next, we have to show that if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well. The argument will be very similar, modified slightly to get a reversed inequality. Again using the fact that a_n is decreasing, we have

$$(a_4 + a_5 + a_6 + a_7) \geq (a_8 + a_8 + a_8 + a_8) = 4a_8$$

and in general,

$$(a_{2^n} + \cdots a_{2^{n+1}-1}) \geq (\overbrace{a_{2^{n+1}} + \cdots + a_{2^{n+1}}}^{2^n \text{ times}}) = 2^n a_{2^{n+1}}.$$

This shows that

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=0}^{\infty} 2^n a_{2^{n+1}} \geq 0.$$

If $\sum_{n=0}^{\infty} 2^n a_{2^n}$ diverges, then

$$\sum_{n=0}^{\infty} 2^n a_{2^{n+1}} \quad \left(= \frac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2^n} \right)$$

diverges, and so the above comparison implies that $\sum_{n=1}^{\infty} a_n$ diverges as well.

Finally, observe that changing the starting index from 0 to 1 in any of the above series does not affect convergence. □

A.2 The second ratio test

A.3 Raabe's ratio test

A.4 Dirichlet's test

References

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