The Theorems of Vector Calculus

Joseph Breen

Introduction

One of the more intimidating parts of vector calculus is the wealth of so-called *fundamental theorems*:

i. The Gradient Theorem[1]

ii. Green's Theorem

iii. Stokes' Theorem

iv. The Divergence Theorem

Understanding when and how to use each of these can be confusing and overwhelming. The following discussion is meant to give some insight as to how each of these theorems are related. Our guiding principle will be that **the four theorems above arise as generalizations of the Fundamental Theorem of Calculus**.

Review: The Fundamental Theorem of Calculus

As with just about everything in multivariable and vector calculus, the theorems above are generalizations of ideas that we are familiar with from one dimension. Therefore, the first step in understanding the fundamental theorems of vector calculus is understanding the single variable case. Here is a brief review, with a perspective tailored to suit the language of the vector calculus theorems.

Say I give you a differentiable function f, defined on an interval $[a, b]$. The **Fundamental Theorem of Calculus** says that:

$$\int_a^b f'(x)\,dx = f(b) - f(a) \tag{1}$$

Consider the right hand side of the equation for a moment. Suppose I gave you a two-point set $\{a, b\}$ and I wanted to integrate the function f over $\{a, b\}$. Well, an integral is just a weighted sum, right? So the integral of f over the set $\{a, b\}$ would look something like $f(a) + f(b)$, where we weight each element of the set by the function at that point. The tricky part is that since the points a and b are disjoint, they have different *orientations*. So we'll give the weighted value at a a negative sign. Thus, the "integral of f over $\{a, b\}$ is $f(b) - f(a)$.[2] Rewriting (1), we get:

$$\int_a^b f'(x)\,dx = \int_{\{a,b\}} f(x)\,dx \tag{2}$$

[1]This is often referred to as *The Fundamental Theorem of Line Integrals*, or something of that sort.

[2]Don't think about this too much, because we're not being very formal. Just building intuition!

The next thing to notice is that the two point set $\{a, b\}$ is the **boundary** of the interval $[a, b]$. So if we denote the interval $[a, b]$ by I, we can use the symbol ∂ to denote the boundary of the interval I: $\partial I = \{a, b\}$. Using this new notation, (2) becomes:

$$\int_I f'(x)\,dx = \int_{\partial I} f(x)\,dx \tag{3}$$

Think about what this is saying: we have a function f, and a region (an interval) I, and we are equating **the integral over the interior of the derivative of the function** to **the integral over the boundary of the function**. For our purpose, this is the best way to think about the Fundamental Theorem of Calculus, and it is the underlying principle for all of the vector calculus theorems:

$$\left\{ \begin{array}{c} \text{integral over interior} \\ \text{of derivative of function} \end{array} \right\} = \left\{ \begin{array}{c} \text{integral over boundary} \\ \text{of function} \end{array} \right\} \tag{4}$$

Before we move on, here's one more way to think about the Fundamental Theorem of Calculus. You probably noticed how the symbol we used for the *boundary* is the same as the symbol we use for partial derivatives: ∂. This is not a coincidence! Here's why: we can think of an *integral* as a function that takes in two parameters (a region, and a function) and spits out the integral. Let's write this two-parameter function as $\langle \cdot, \cdot \rangle$. Explicitly, the integral of a function g over the interval I would be:

$$\langle I, g \rangle = \int_I g(x)\,dx$$

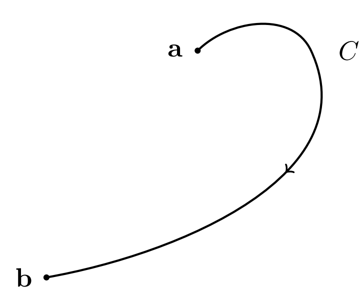
Using this notation in (3) gives us:

$$\langle I, f' \rangle = \langle \partial I, f \rangle \tag{5}$$

So the Fundamental Theorem of Calculus gives us a way to "move" the derivative symbol from one parameter to the other (informally, $\partial f = f'$). Pretty cool! We will continue to see that the ideas of the **derivative** and the **boundary** are closely related.

The Gradient Theorem

Having reviewed the Fundamental Theorem of Calculus in the one-dimensional case, we can try to abstract the idea in (4) to more complicated domains. So let's say I give you a curve C in n dimensions and a differentiable scalar field $f : \mathbb{R}^n \to \mathbb{R}$. Here, the curve C takes the role of the interval I, and the scalar field f acts like the function f. Suppose that the curve starts at the point $\mathbf{a} \in \mathbb{R}^n$ and ends at the point $\mathbf{b} \in \mathbb{R}^n$:



The left-hand side of (4) says we need an integral over the interior of our region. Since our region C is a curve, integrating over the length of C gives us a line integral! The left-hand side specifies that we are integrating the *derivative* of the function, so we need to take a kind of "derivative" of the scalar field f. In this case, our derivative will be the *gradient*, given by ∇f. Putting all this together, the left-hand side of (4) will be the line integral of ∇f over C:

$$\int_C \nabla f \cdot d\mathbf{r}$$

where $\mathbf{r} : [a, b] \to \mathbf{R}^n$ is a parametrization of the curve C.

On to the right-hand side of (4). As in the single variable case, the boundary of C is the two point set $\partial C = \{\mathbf{a}, \mathbf{b}\}$. So the *integral over the boundary of the function* is just $f(\mathbf{b}) - f(\mathbf{a})$. Thus, equating the left and right sides of (4), we get:

$$\int_C \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}) \tag{6}$$

This is the **Gradient Theorem**! It really is just a reinterpretation of the ideas from the Fundamental Theorem of Calculus in the context of a curve in more than one dimension. It relates an integral over the interior of our region to an integral over the boundary.

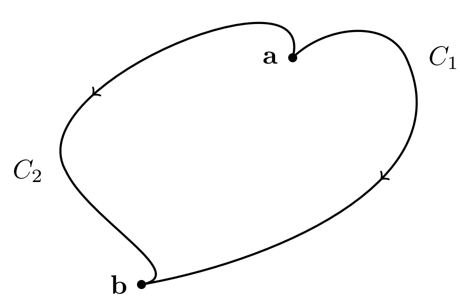
Just for fun, let's rewrite (6) using the $\langle \cdot, \cdot \rangle$ notation established in the previous section:

$$\langle C, \nabla f \rangle = \langle \partial C, f \rangle \tag{7}$$

We can see very explicitly the *duality* of the **gradient** operator ∇ and the **boundary** operator ∂.

Implications of the Gradient Theorem: Path Independence

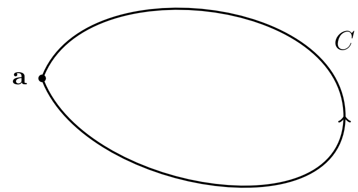
Recall that a vector field $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is called **conservative** if there is a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$ (remember, the gradient of a function is a vector field). Next, notice that the value of the line integral in (6) only depends on the endpoints of the curve, $\mathbf{a}$ and $\mathbf{b}$. So the Gradient Theorem implies that if we integrate something that looks like ∇f, we only have to worry about the starting and ending points. So, for example, suppose we have two curves C_1 and C_2:



The Gradient Theorem then says that:

$$\int_{C_1} \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}) = \int_{C_2} \nabla f(\mathbf{r}) \cdot d\mathbf{r}$$

In other words, in travelling from $\mathbf{a}$ to $\mathbf{b}$, the path we take doesn't matter! This phenomenon is called **path independence**. This holds for any conservative vector field (since by definition every conservative vector field $\mathbf{F}$ looks like ∇f). In particular, suppose that we integrate a conservative vector field $\mathbf{F} = \nabla f$ over a curve C such that $\mathbf{a} = \mathbf{b}$, i.e., a closed curve C:



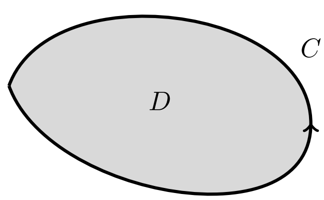
By the Gradient Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{a}) - f(\mathbf{a}) = 0$$

So the line integral of a conservative vector field around any closed loop is 0! This gives an alternate way to characterize conservative vector fields.

Green's Theorem

Next, we will discuss *Green's Theorem*, which is the generalization of the Fundamental Theorem of Calculus to regions in the plane. We'll start by reconsidering the integral of a (not necessarily conservative) vector field $\mathbf{F}$ around a closed curve — in particular, a curve in two dimensions. The loop C defines a two-dimensional region, which we will call D:



Note that C is the boundary of D, i.e., $C = \partial D$. So parametrizing C with $\mathbf{r}$ and calculating the line integral of $\mathbf{F}$ around C is the same as calculating:

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

Look back to the fundamental idea in (4) — the right-hand side is about integrating a function over the boundary of some region. This suggests (by looking at the left-hand side) that we can relate this line integral to an integral over the interior D of some "derivative" of $\mathbf{F}$.

Since the interior of our region D is a two-dimensional region of the plane, the integral on the left-hand side of (4) will be a double integral. Therefore, a rough draft of Green's Theorem might look like:

$$\iint_D (\text{“derivative” of } \mathbf{F})\, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} \tag{8}$$

All we have to do now is decide what the “derivative” of $\mathbf{F}$ will be. In the case of the scalar field f, the “derivative” was the gradient ∇f. In the case of a vector field $\mathbf{F}$ defined on a two-dimensional region, the “derivative” of $\mathbf{F}$ will be the *curl* of $\mathbf{F}$: $\nabla \times \mathbf{F}$. Well, *almost* — we'll have to make some slight modifications.

Why?

Well, for one, the curl is only defined for vector fields in three dimensions (because of the cross product nature of $\nabla \times \mathbf{F}$), and our vector field is only in two dimensions. We can account for this by adding a third component of 0 to $\mathbf{F}$, effectively just pretending that we are three dimensions!

The other issue is that $\nabla \times \mathbf{F}$ is another vector field, and we don't know how to take the double integral of vector valued functions — we only know how to take double integrals of scalar valued functions. So how about we just consider the *length* of the curl instead? We could do this by computing $\|\nabla \times \mathbf{F}\|$, but we're actually going to be a bit more clever. Since $\mathbf{F}$ only has x and y components (we artificially gave it a z component of 0) the vector $\nabla \times \mathbf{F}$ will stick straight up in the z-direction. So the length we're looking for is exactly the z component of $\nabla \times \mathbf{F}$! We can strip off this component by taking the dot product with the following unit vector $\mathbf{n}$:

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The “derivative” of $\mathbf{F}$ is then $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$. Plugging this into (8) gives us **Green's Theorem**:

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} \tag{9}$$

Most of the time, you'll see Green's Theorem stated in slightly different way. Suppose that the vector field $\mathbf{F}$ has component functions P and Q; explicitly,

$$\mathbf{F}(x, y) = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$$

Also, we can think of $d\mathbf{r}$ as the vector $\begin{bmatrix} dx \\ dy \end{bmatrix}$ and ∇ as the vector $\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$. Plugging all of this into (9) gives us:

$$\iint_D \left(\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P(x, y) \\ Q(x, y) \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dx\, dy = \oint_{\partial D} \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Simplifying this (and then switching the left and right sides of the equation) gives us the typical formulation of **Green's Theorem**:

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \tag{10}$$

Although they look different, both (9) and (10) say the same thing as (4): that the integral of the vector field over the boundary curve equals the integral of the curl of the vector field over the interior.

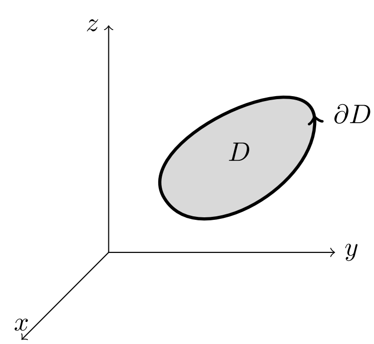
As we did before, we can rewrite (9) using our $\langle \cdot, \cdot \rangle$ notation to see the duality between the boundary operator ∂ and the curl operator $\nabla\times$:

$$\langle D, \nabla \times \mathbf{F} \rangle = \langle \partial D, \mathbf{F} \rangle \tag{11}$$

There is a lot to be said about the intuition, implications, and subtleties of Green's Theorem, but we'll save that for another day. The discussion here is meant to highlight the fact that Green's Theorem encodes the same idea as the Fundamental Theorem of Calculus and the Gradient Theorem — it is just a careful reinterpretation in a different context.

Stokes' Theorem

Next in line is *Stokes' Theorem.* Fortunately, Stokes' Theorem isn't all that different from Green's Theorem. The main difference is that we're *actually* in three dimensions, instead of just pretending to be. Given a pancake-like surface D sitting in three dimensions (instead of two dimensions), its boundary ∂S is a closed curve:



As before, we have a region (the surface D) and its boundary (the curve ∂D). To apply the ideas in (4), we need a function to integrate. As with Green's Theorem, the function will be a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$. The right-hand side of (4) concerns the integral of the function over the boundary, which in this context, is the line integral of $\mathbf{F}$ over ∂D (parametrized by $\mathbf{r}$):

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

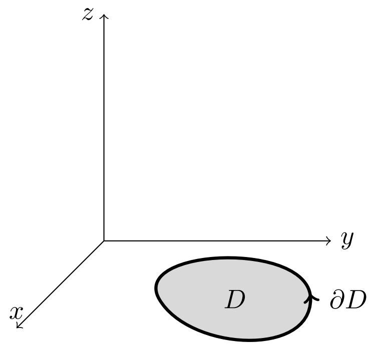
The left-hand side of (4) talks about taking the "derivative" of $\mathbf{F}$ and integrating it over the interior of our region. As in Green's Theorem, the "derviative" of $\mathbf{F}$ will be the curl of $\mathbf{F}$, given by $\nabla \times \mathbf{F}$. This time, since we are already in three dimensions, we don't have to make any adjustments! So we integrate $\nabla \times \mathbf{F}$ over the interior of our region, which is a *surface*. Therefore, the left-hand side of (4) will be a surface integral over D (parametrized by $\mathbf{S} : [a, b] \times [c, d] \to \mathbb{R}^3$):

$$\iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

Equating these two integrals as in (4) gives us **Stokes' Theorem**:

$$\iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} \tag{12}$$

Not surprisingly, this looks very similar to the statement of Green's Theorem in (9). In fact, suppose that the region D is a region lying exactly in the xy plane:



In order to calculate the surface integral of $\nabla \times \mathbf{F}$ over D, we need to find the normal vector $\mathbf{n}$ to the surface. But since D lies in the xy plane, its (unit) normal vector sticks straight up in the z direction!

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So the surface integral turns into a double integral of the form:

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA$$

In this particular case, Stokes' Theorem gives us the following:

$$\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$$

This is exactly the formulation of Green's Theorem in (9)! Because Green's Theorem and Stokes' Theorem are both generalizations of the ideas in (4), this makes a lot of sense.

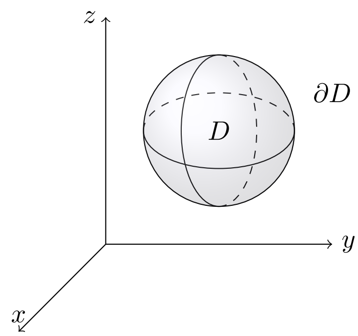
As before, writing (12) in the $\langle \cdot, \cdot \rangle$ notation yields the exact same equation as in (11):

$$\langle D, \nabla \times \mathbf{F} \rangle = \langle \partial D, \mathbf{F} \rangle \tag{13}$$

Like Green's Theorem, discussing the subtleties (for example, the issue of orientation) and implications of Stokes' Theorem is a discussion in and of itself. The important point here is to note, yet again, that Stokes' Theorem does not say anything crazy or new. It is simply the idea of relating the derivative and the boundary in the context of surfaces in three dimensions.

The Divergence Theorem

The final theorem we will discuss is the *Divergence Theorem*. So far, we've developed fundamental theorems for two kinds of regions: curves (in which case the boundary is a two-point set), and two-dimensional surfaces (in which case the boundary is a closed curve). In three dimensions, there is one more kind of region we can consider: a three-dimensional region (which we denote by D), bounded by a closed surface (∂D):



As before, our function will be a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$. Given this set up, we can do our thing and interpret (4) in this context:

The right-hand side of (4) suggests that we should integrate the function $\mathbf{F}$ over the boundary of the region D, which is a surface. This will be the surface integral of $\mathbf{F}$ over the closed surface ∂S (parametrized by $\mathbf{S}$):

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S}$$

The left-hand side of (4) is an integral of the "derivative" of $\mathbf{F}$ over the interior of the closed surface, which is the three-dimensional volume D. Since this region is just a subset of three-space, the integral itself will be a regular triple integral:

$$\iiint_D (\text{“derivative” of } \mathbf{F})\, dV$$

The tricky part is deciding what our "derviative" will be. When the region was a curve (e.g. the Gradient Theorem), the derivative was the *gradient*. In the case of surfaces (e.g. Green's and Stokes' theorems), the derivative was the *curl*. As the name of the present section might suggest, the derivative in the case of three-dimensional regions will be the *divergence*. Explicitly, given the vector field $\mathbf{F}$, its "derivative" will be $\nabla \cdot \mathbf{F}$.[3] Therefore, the left-hand side of (4) becomes:

$$\iiint_D \nabla \cdot \mathbf{F}\, dV$$

Equating the two integrals as in (4) gives us the **Divergence Theorem**:

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} \tag{14}$$

[3]Note that $\nabla \cdot \mathbf{F}$ is *not the same* as the gradient, ∇f.

Writing (14) in the bracket notation yields:

$$\langle D, \nabla \cdot \mathbf{F} \rangle = \langle \partial D, \mathbf{F} \rangle \tag{15}$$

Here, we can see the duality between the boundary operator ∂ and the divergence operator $\nabla\cdot$.

As before, we won't discuss the details and consequences of the Divergence Theorem here. I only want to emphasize the relationship between this theorem, the ones before it, and the underlying idea in (4).

Looking Ahead: The Generalized Stokes' Theorem[4]

To recap what we just did, we took the Fundamental Theorem of Calculus in one-dimension and generalized it to two and three dimensions, considering all the possible kinds of regions that we can integrate over. This gave us four new theorems: the Gradient Theorem, Green's Theorem, Stokes' Theorem, and the Divergence Theorem. Why should we stop at three dimensions? If we bump up to four dimensions, we have new kinds of regions to integrate over! For example, we could take a subset of three-dimensional Euclidean space, twist it and bend it in the fourth dimension, and integrate over that. Or we could integrate over plain-old four-dimensional space. This would give us *two more* fundamental theorems. Then we could go to five dimensions, six dimensions, etc., at each stage considering the new kinds of regions and constructing more and more fundamental theorems. While this may seem cool, there is something unsettling about the fact that generalizing the Fundamental Theorem of Calculus gives tons and tons of fundamental theorems. As we've seen, the idea is the same in all of them (namely the idea in (4)), but their technical statement varies wildly with the context.

If I were you, I would be asking the following question:

Why are there so many fundamental theorems? Why can't there just be one?

It turns out that there is! It's called the **Generalized Stokes' Theorem**, and it is an extremely general and all-encompassing interpretation of the idea in (4). Here's what it looks like:

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega \tag{16}$$

Inside this equation is the Fundamental Theorem of Calculus, the Gradient Theorem, Green's Theorem, Stokes' Theorem, the Divergence Theorem, and so much more.

It takes a lot of difficult math to formally understand what (16) means, but here's the idea: we have this function-like thing ω, and a region-like thing Ω. On the left-hand side, we are integrating $d\omega$, which is a kind of "derivative" of ω, over all of Ω, and on the right-hand side we are integrating ω over $\partial\Omega$, the boundary of Ω. Basically, it's a fancy way to rephrase (4)! We can rewrite (16) in bracket notation:

$$\langle \Omega, d\omega \rangle = \langle \partial\Omega, \omega \rangle \tag{17}$$

This ultimately expresses the duality of the idea of the *boundary* and the idea of the *derivative*. Pretty cool!

[4] This is beyond the scope of a multivariable calculus class, so you don't have to know this!