Some Very Challenging Calculus Problems

Joseph Breen

Here are two difficult calculus problems, solved using only (sophisticated and clever applications of) elementary calculus. In particular, there is no complex analysis or use of the residue theorem, Fourier series, or anything like that. Both problems were the basis for talks given at the UCLA GSO Seminar.

The integral is the concatenation of two integrals from [3]. The infinite series was originally evaluated by other methods in [2], and the solution presented below is inspired by the solution from [4], together with other computations found on the internet and my own computational decisions.

Contents

1 A Really Hard Integral

The integral we will evaluate is

$$\boxed{I = \int_0^{\frac{\pi}{2}} \arccos\left(\frac{\cos x}{1 + 2\cos x}\right) dx.} \tag{1}$$

Step 1: Rewrite the integrand with trigonometry and then introduce a double integral.

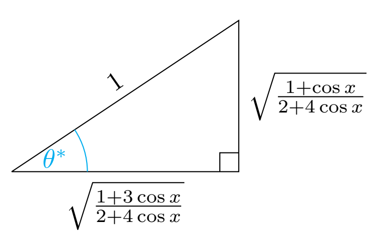
We begin with some trigonometry. Recall the double angle identity $\cos(2\theta) = 2\cos^2\theta - 1$. This implies $2\theta = \arccos\left(2\cos^2\theta - 1\right)$. Letting $\alpha = 2\cos^2\theta - 1$ yields $\theta = \arccos\left(\sqrt{\frac{1+\alpha}{2}}\right)$, and thus

$$\arccos(\alpha) = 2\arccos\left(\sqrt{\frac{1+\alpha}{2}}\right).$$

Using this, the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \underbrace{2\arccos\left(\sqrt{\frac{1+3\cos x}{2+4\cos x}}\right)}_{\theta^*} dx.$$

Next, consider a right triangle with angle θ^*.



This picture implies

$$\arccos\left(\sqrt{\frac{1+3\cos x}{2+4\cos x}}\right) = \arctan\left(\sqrt{\frac{1+\cos x}{1+3\cos x}}\right)$$

and so

$$I = 2\int_0^{\frac{\pi}{2}} \arctan\left(\sqrt{\frac{1+\cos x}{1+3\cos x}}\right)\,dx.$$

With the goal of using the aforementioned double angle identity again, we make the substitution $x = 2y$. Then $dx = 2\,dy$ and we have

$$\begin{aligned} I &= 4\int_0^{\frac{\pi}{4}} \arctan\left(\sqrt{\frac{1+\cos 2y}{1+3\cos 2y}}\right)\,dy \\ &= 4\int_0^{\frac{\pi}{4}} \arctan\left(\sqrt{\frac{2\cos^2 y}{-2+6\cos^2 y}}\right)\,dy \\ &= 4\int_0^{\frac{\pi}{4}} \arctan\underbrace{\left(\frac{\cos y}{\sqrt{2-3\sin^2 y}}\right)}_{b}\,dy. \end{aligned}$$

Recall that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C$. Thus,

$$\begin{aligned} \int_0^1 \frac{1}{1+b^2t^2}\,dt = \frac{1}{b^2}\int_0^1 \frac{1}{b^{-2}+t^2}\,dt &= \frac{1}{b}\arctan(bt)\Big|_0^1 \\ &= \frac{1}{b}\arctan(b). \end{aligned}$$

This implies

$$\begin{aligned} I &= 4\int_0^{\frac{\pi}{4}} \frac{\cos y}{\sqrt{2-3\sin^2 y}}\int_0^1 \frac{1}{1+\frac{\cos^2 y}{2-3\sin^2 y}t^2}\,dt\,dy \\ &= 4\int_0^{\frac{\pi}{4}}\int_0^1 \frac{\cos y\sqrt{2-3\sin^2 y}}{2-3\sin^2 y+\cos^2 y t^2}\,dt\,dy \\ &= 4\int_0^{\frac{\pi}{4}}\int_0^1 \frac{\cos y\sqrt{2-3\sin^2 y}}{(t^2+2)-(t^2+3)\sin^2 y}\,dt\,dy. \end{aligned}$$

Next, we adjust constants in order to simplify the expression in the numerator. In particular, let $\sin y = \sqrt{\frac{2}{3}}\sin w$. Then $\cos y\,dy = \sqrt{\frac{2}{3}}\cos w\,dw$, and the integral becomes

$$\begin{aligned} I &= 4\int_0^{\frac{\pi}{3}}\int_0^1 \frac{\sqrt{2}\cos w}{(t^2+2)-(t^2+3)\frac{2}{3}\sin^2 w}\,dt\,\sqrt{\frac{2}{3}}\cos w\,dw \\ &= 8\sqrt{3}\int_0^{\frac{\pi}{3}}\int_0^1 \frac{\cos^2 w}{(3t^2+6)-(2t^2+6)\sin^2 w}\,dt\,dw \\ &= 8\sqrt{3}\int_0^{\frac{\pi}{3}}\int_0^1 \frac{\cos^2 w}{t^2+(2t^2+6)\cos^2 w}\,dt\,dw \\ &= 8\sqrt{3}\int_0^{\frac{\pi}{3}}\int_0^1 \frac{1}{t^2\sec^2 w+(2t^2+6)}\,dt\,dw. \end{aligned}$$

Step 2: Use a trig substitution, partial fractions, then integration by parts.

Let $s = \tan w$. Then $ds = \sec^2 w\,dw$. Since $1+\tan^2 w = \sec^2 w$, we have $\sec^2 w = 1+s^2$ and $dw = \frac{1}{1+s^2}\,ds$. Thus,

$$\begin{aligned} I &= 8\sqrt{3}\int_0^{\sqrt{3}}\int_0^1 \frac{1}{(1+s^2)t^2+(2t^2+6)}\,dt\,\frac{1}{1+s^2}\,ds \\ &= 8\sqrt{3}\int_0^{\sqrt{3}}\int_0^1 \frac{1}{(1+s^2)(3t^2+t^2s^2+6)}\,dt\,ds. \end{aligned}$$

Next, we decompose the integrand with partial fractions. We have

$$I = 8\sqrt{3}\int_0^{\sqrt{3}}\int_0^1 \frac{1}{2t^2+6}\left(\frac{1}{1+s^2} - \frac{t^2}{3t^2+t^2s^2+6}\right) dt\, ds.$$

The terms in the parentheses can be integrated with respect to s using the inverse tangent. Thus, we switch the order of integration.

$$\begin{aligned} I &= 8\sqrt{3}\int_0^1\int_0^{\sqrt{3}} \frac{1}{2t^2+6}\left(\frac{1}{1+s^2} - \frac{1}{3+\frac{6}{t^2}+s^2}\right) ds\, dt \\ &= 4\sqrt{3}\int_0^1 \frac{1}{t^2+3}\left(\frac{\pi}{3} - \frac{1}{\sqrt{3+\frac{6}{t^2}}}\arctan\left(\frac{\sqrt{3}}{\sqrt{3+\frac{6}{t^2}}}\right)\right) dt. \end{aligned}$$

Here we have again used the fact that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan(\frac{x}{a}) + C$. Next, the $\frac{1}{t^2+3}$ term can be integrated similarly with respect to t. This gives

$$\begin{aligned} I &= \frac{4\pi\sqrt{3}}{3}\int_0^1 \frac{1}{t^2+3}\,dt - 4\sqrt{3}\int_0^1 \frac{1}{(t^2+3)}\frac{1}{\sqrt{3+\frac{6}{t^2}}}\arctan\left(\frac{\sqrt{3}}{\sqrt{3+\frac{6}{t^2}}}\right) dt \\ &= \frac{4\pi\sqrt{3}}{3}\frac{1}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right) - 4\int_0^1 \frac{1}{(t^2+3)}\frac{t}{\sqrt{t^2+2}}\arctan\left(\frac{t}{\sqrt{t^2+2}}\right) dt \\ &= \frac{2\pi^2}{9} - 4\int_0^1 \frac{t}{(t^2+3)\sqrt{t^2+2}}\arctan\left(\frac{t}{\sqrt{t^2+2}}\right) dt. \end{aligned}$$

Next, we will integrate by parts. Let

$$u = \arctan\left(\frac{t}{\sqrt{t^2+2}}\right) \quad \text{and} \quad dv = \frac{t}{(t^2+3)\sqrt{t^2+2}}\,dt.$$

Then

$$du = \frac{1}{1+\frac{t^2}{t^2+2}}\cdot\frac{\sqrt{t^2+2}-\frac{t^2}{\sqrt{t^2+2}}}{t^2+2}\,dt = \frac{1}{(t^2+1)\sqrt{t^2+2}}\,dt. \tag{2}$$

Next, observe that

$$\frac{d}{dx}\arctan(\sqrt{t^2+2}) = \frac{1}{(1+t^2+2}\cdot\frac{t}{\sqrt{t^2+2}} = \frac{t}{(t^2+3)\sqrt{t^2+2}}.$$

Thus, $v = \arctan(\sqrt{t^2+2})$. Integrating by parts with this set up yields

$$\begin{aligned} I &= \frac{2\pi^2}{9} - 4\left(\arctan\left(\frac{t}{\sqrt{t^2+2}}\right)\arctan(\sqrt{t^2+2})\Bigg|_0^1 - \int_0^1 \frac{\arctan(\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\,dt\right) \\ &= \frac{2\pi^2}{9} - 4\left(\frac{\pi}{6}\cdot\frac{\pi}{3} - \int_0^1 \frac{\arctan(\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\,dt\right) \\ &= 4\int_0^1 \frac{\arctan(\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\,dt. \end{aligned}$$

Step 3: Differentiate under the integral.

We introduce an additional parameter in the integrand:

$$I(z) = 4\int_0^1 \frac{\arctan(z\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\,dt.$$

We seek $I = I(1)$. By the fundamental theorem of calculus,

$$\int_1^\infty I'(z)\,dz = \lim_{z\to\infty} I(z) - I(1) \qquad \Rightarrow \qquad I = \lim_{z\to\infty} I(z) - \int_1^\infty I'(z)\,dz.$$

We compute each term separately.

$$\begin{aligned}
\lim_{z\to\infty} I(z) &= 4\int_0^1 \frac{\lim_{z\to\infty}\arctan(z\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\,dt \\
&= 4\int_0^1 \frac{\frac{\pi}{2}}{(t^2+1)\sqrt{t^2+2}}\,dt \\
&= 2\pi\arctan\left(\frac{t}{\sqrt{t^2+2}}\right)\Bigg|_0^1 \\
&= \frac{\pi^2}{3}.
\end{aligned}$$

The second to last equality comes from our previous computation in (2).

Next,

$$\begin{aligned}
\int_1^\infty I'(z)\,dz &= 4\int_1^\infty\int_0^1 \frac{d}{dz}\left(\frac{\arctan(z\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\right)dt\,dz \\
&= 4\int_1^\infty\int_0^1 \frac{1}{(t^2+1)\sqrt{t^2+2}}\cdot\frac{1}{1+z^2(t^2+2)}\cdot\sqrt{t^2+2}\,dt\,dz \\
&= 4\int_1^\infty\int_0^1 \frac{1}{(t^2+1)(1+z^2t^2+2z^2)}\,dt\,dz.
\end{aligned}$$

We decompose the integrand with partial fractions.

$$\begin{aligned}
\int_1^\infty I'(z)\,dz &= 4\int_1^\infty\int_0^1 \frac{1}{1+z^2}\left(\frac{1}{1+t^2} - \frac{z^2}{1+z^2t^2+2z^2}\right)dt\,dz \\
&= 4\int_1^\infty \frac{1}{1+z^2}\left(\int_0^1\frac{1}{1+t^2}\,dt - \int_0^1\frac{z^2}{1+z^2t^2+2z^2}\,dt\right)dz \\
&= 4\int_1^\infty \frac{1}{1+z^2}\left(\frac{\pi}{4} - \int_0^1 \frac{1}{2+\frac{1}{z^2}+t^2}\,dt\right)dz \\
&= \pi\arctan(z)\Big|_1^\infty - 4\int_1^\infty \frac{1}{1+z^2}\cdot\frac{1}{\sqrt{2+\frac{1}{z^2}}}\arctan\left(\frac{1}{\sqrt{2+\frac{1}{z^2}}}\right)dz \\
&= \frac{\pi^2}{4} - 4\int_1^\infty \frac{1}{1+z^2}\cdot\frac{1}{\sqrt{2+\frac{1}{z^2}}}\arctan\left(\frac{1}{\sqrt{2+\frac{1}{z^2}}}\right)dz.
\end{aligned}$$

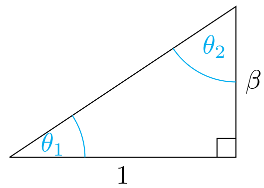
Next, let $r = \frac{1}{z}$. Then $dz = -\frac{1}{r^2}\,dr$, and the above integral becomes

$$\begin{aligned}
\int_1^\infty I'(z)\,dz &= \frac{\pi^2}{4} - 4\int_0^1 \frac{1}{1+\frac{1}{r^2}}\cdot\frac{1}{\sqrt{2+r^2}}\arctan\left(\frac{1}{\sqrt{2+r^2}}\right)\frac{1}{r^2}\,dr \\
&= \frac{\pi^2}{4} - 4\int_0^1 \frac{1}{(r^2+1)\sqrt{2+r^2}}\arctan\left(\frac{1}{\sqrt{2+r^2}}\right)dr.
\end{aligned}$$

Recall that

$$I = 4\int_0^1 \frac{\arctan(\sqrt{t^2+2})}{(t^2+1)\sqrt{t^2+2}}\,dt.$$

The integral in the above expression is very similar to I, with an inverted argument in the inverse tangent. Motivated by this, we invoke a handy trig identity. Consider the following right triangle:



This implies that

$$\theta_1 + \theta_2 = \frac{\pi}{2} \qquad \Rightarrow \qquad \arctan(\beta) + \arctan\left(\frac{1}{\beta}\right) = \frac{\pi}{2}.$$

Using this identity,

$$\begin{aligned}
\int_1^\infty I'(z)\,dz &= \frac{\pi^2}{4} - 4\int_0^1 \frac{1}{(r^2+1)\sqrt{2+r^2}}\left(\frac{\pi}{2} - \arctan(\sqrt{t^2+2})\right)\,dr \\
&= \frac{\pi^2}{4} - 2\pi\int_0^1 \frac{1}{(r^2+1)\sqrt{r^2+2}}\,dr + 4\int_0^1 \frac{\arctan(\sqrt{r^2+2})}{(r^2+1)\sqrt{r^2+2}}\,dr \\
&= \frac{\pi^2}{4} - \frac{\pi^2}{3} + I
\end{aligned}$$

where in the last equality we again made use of (2). Thus,

$$\begin{aligned}
I &= \lim_{z\to\infty} I(z) - \int_1^\infty I'(z)\,dz \\
&= \frac{\pi^2}{3} - \left(\frac{\pi^2}{4} - \frac{\pi^2}{3} + I\right).
\end{aligned}$$

Solving for I in this last equation gives a final answer of

$$\boxed{\int_0^{\frac{\pi}{2}} \arccos\left(\frac{\cos x}{1+2\cos x}\right)\,dx = \frac{5\pi^2}{24}}.$$

2 A Really Hard Infinite Series

The infinite series we will evaluate is

$$\boxed{S = \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)^2.} \tag{3}$$

We adopt the shorthand notation $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ to denote the *nth harmonic number*. Recall that $H_n \sim \log n$, so S clearly converges.

Step 1: Decompose S into two sums.

We begin by rewriting the summand. Observe that $H_{n+1} = H_n + \frac{1}{n+1}$. Thus,

$$\begin{aligned} H_n^2 = H_n \left(H_{n+1} - \frac{1}{n+1}\right) &= H_n H_{n+1} - \frac{H_n}{n+1} \\ &= H_n H_{n+1} - \frac{1}{n+1} \left(H_{n+1} - \frac{1}{n+1}\right) \\ &= H_n H_{n+1} - \frac{H_{n+1}}{n+1} + \frac{1}{(n+1)^2}. \end{aligned}$$

The next step is seemingly out of nowhere. We introduce two infinite series with the eventual goal of using summation by parts. We have

$$\begin{aligned} H_n^2 &= H_n H_{n+1} - \frac{H_{n+1}}{n+1} + \frac{1}{(n+1)^2} + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{H_k}{k}\right) + \left(\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= H_n H_{n+1} - \sum_{k=1}^{n+1} \frac{H_k}{k} + \sum_{k=1}^{n+1} \frac{1}{k^2} + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= H_n H_{n+1} - \sum_{k=2}^{n+1} \frac{H_k}{k} + \sum_{k=2}^{n+1} \frac{1}{k^2} + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right). \end{aligned}$$

The last equality comes from the observation that the terms corresponding to $k = 1$ in the first and second summations are both 1. Continuing,

$$\begin{aligned} H_n^2 &= H_n H_{n+1} - \sum_{k=2}^{n+1} \frac{1}{k} \left(H_k - \frac{1}{k}\right) + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= H_n H_{n+1} - \sum_{k=2}^{n+1} \frac{1}{k} \cdot H_{k-1} + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= H_n H_{n+1} - \sum_{k=2}^{n+1} \left(H_k - H_{k-1}\right) H_{k-1} + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= H_n H_{n+1} - \sum_{k=2}^{n+1} H_k H_{k-1} + \sum_{k=2}^{n+1} H_{k-1}^2 + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= H_n H_{n+1} - \sum_{k=3}^{n+2} H_{k-1} H_{k-2} + \sum_{k=2}^{n+1} H_{k-1}^2 + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= H_n H_{n+1} - \sum_{k=3}^{n+2} H_{k-1} H_{k-2} + \sum_{k=2}^{n+1} H_{k-1}^2 + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= 1 + \sum_{k=3}^{n+1} \left(H_{k-1}^2 - H_{k-1} H_{k-2}\right) + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right). \end{aligned}$$

The last equality comes from picking off the last term of the first sum and the first term of the second sum. Simplifying,

$$\begin{aligned} H_n^2 &= 1 + \sum_{k=3}^{n+1} H_{k-1}\left(H_{k-1} - H_{k-2}\right) + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= 1 + \sum_{k=3}^{n+1} \frac{H_{k-1}}{k-1} + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= \sum_{k=2}^{n+1} \frac{H_{k-1}}{k-1} + \left(\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}\right) \\ &= 2\sum_{k=1}^{n} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{1}{k^2}. \end{aligned}$$

All of this shows that

$$S = \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = 2\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{H_k}{k}}_{S_1} - \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k^2}}_{S_2}. \tag{4}$$

Step 2: Simplify S_1.

Next we consider S_1. Note that

$$\begin{aligned} \frac{H_k}{k} = \frac{1}{k}\sum_{j=1}^{k} \frac{1}{j} = \frac{1}{k}\sum_{j=1}^{k} \int_0^1 x^{j-1}\,dx \\ = \frac{1}{k}\int_0^1 \sum_{j=1}^{k} x^{j-1}\,dx \\ = \frac{1}{k}\int_0^1 \frac{1-x^k}{1-x}\,dx \end{aligned}$$

where we have used the partial sum formula for a geometric series in the last equality. Continuing,

$$\frac{H_k}{k} = \frac{1}{k}\int_0^1 \frac{1-x^k}{1-x}\,dx = \int_0^1 \frac{1}{1-x}\int_x^1 t^{k-1}\,dt\,dx.$$

Next, we change the order of integration, taking care to change the bounds appropriately.

$$\frac{H_k}{k} = \int_0^1 t^{k-1}\int_0^t \frac{1}{1-x}\,dx\,dt = -\int_0^1 t^{k-1}\log(1-t)\,dt.$$

We continue simplifying S_1 in the same manner. In particular,

$$\begin{aligned} \sum_{k=1}^{n} \frac{H_k}{k} &= -\sum_{k=1}^{n} \int_0^1 t^{k-1}\log(1-t)\,dt \\ &= -\int_0^1 \frac{1-t^n}{1-t}\log(1-t)\,dt \\ &= -n\int_0^1 \frac{\log(1-t)}{1-t}\int_t^1 y^{n-1}\,dy\,dt \\ &= -n\int_0^1 y^{n-1}\int_0^y \frac{\log(1-t)}{1-t}\,dt\,dy. \end{aligned}$$

The inner integral can be evaluated by a simple substitution to yield

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{n}{2} \int_0^1 y^{n-1} \log^2(1-y)\, dy.$$

So

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{H_k}{k} &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 y^{n-1} \log^2(1-y)\, dy \\ &= \frac{1}{2} \int_0^1 \frac{1}{y} \left(\sum_{n=1}^{\infty} \frac{y^n}{n} \right) \log^2(1-y)\, dy.\end{aligned}$$

Recall that $\log(1-y) = -\sum_{n=1}^{\infty} \frac{y^n}{n}$ for $-1 \leq y < 1$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{H_k}{k} = -\frac{1}{2} \int_0^1 \frac{\log^3(1-y)}{y}\, dy.$$

Now we make the change of variables $z = 1-y$. This yields

$$-\frac{1}{2} \int_0^1 \frac{\log^3(1-y)}{y}\, dy = -\frac{1}{2} \int_0^1 \frac{\log^3 z}{1-z}\, dz = -\frac{1}{2} \int_0^1 \log^3 z \left(\sum_{n=0}^{\infty} z^n \right) dz = -\frac{1}{2} \sum_{n=0}^{\infty} \int_0^1 z^n \log^3 z\, dz.$$

Next, we integrate by parts three times. First, an observation. By L'Hopital's rule,

$$\lim_{z \to 0} \frac{\log^m z}{\frac{1}{z^j}} = 0$$

for any $m, j > 0$. Thus, if we let $u = \log^3 z$ and $dv = z^n\, dz$ and integrate by parts, the boundary terms disappear and we have

$$\int_0^1 z^n \log^3 z\, dz = -\frac{3}{n+1} \int_0^1 z^n \log^2 z\, dz.$$

Integrating by parts two more times yields

$$\int_0^1 z^n \log^3 z\, dz = \frac{6}{(n+1)^2} \int_0^1 z^n \log z\, dz = -\frac{6}{(n+1)^3} \int_0^1 z^n\, dz = -\frac{6}{(n+1)^4}.$$

Putting all of this together yields

$$S_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{H_k}{k} = -\frac{1}{2} \sum_{n=0}^{\infty} \int_0^1 z^n \log^3 z\, dz = 3 \sum_{n=0}^{\infty} \frac{1}{(n+1)^4} = 3 \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Step 3: Simplify S_2.

Next, we consider S_2. We adopt the shorthand notation $H_n^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$ to denote the *nth harmonic number of order* 2. Then

$$S_2 = \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2}.$$

We proceed by summation by parts. We have

$$\sum_{n=1}^{N} \frac{1}{n^2} H_n^{(2)} = 1 + \sum_{n=2}^{N} \left(H_n^{(2)} - H_{n-1}^{(2)} \right) H_n^{(2)} = 1 + \sum_{n=2}^{N} H_n^{(2)} H_n^{(2)} - \sum_{n=2}^{N} H_{n-1}^{(2)} H_n^{(2)}.$$

Next, we reindex the first sum and pull off terms from both in order to simplify.

$$\begin{aligned}1+\sum_{n=2}^{N} H_n^{(2)} H_n^{(2)}-\sum_{n=2}^{N} H_{n-1}^{(2)} H_n^{(2)} &= 1+H_N^{(2)} H_N^{(2)}-\left(1+\frac{1}{4}\right)+\sum_{n=3}^{N} H_{n-1}^{(2)} H_{n-1}^{(2)}-\sum_{n=3}^{N} H_{n-1}^{(2)} H_n^{(2)} \\ &= -\frac{1}{4}+\left(H_N^{(2)}\right)^2+\sum_{n=3}^{N} H_{n-1}^{(2)}\left(H_{n-1}^{(2)}-H_n^{(2)}\right) \\ &= -\frac{1}{4}+\left(H_N^{(2)}\right)^2-\sum_{n=3}^{N} H_{n-1}^{(2)} \cdot \frac{1}{n^2} \\ &= \left(H_N^{(2)}\right)^2-\sum_{n=2}^{N} H_{n-1}^{(2)} \cdot \frac{1}{n^2}.\end{aligned}$$

The last inequality comes from the observation that $\frac{H_1^{(2)}}{2^2}=\frac{1}{4}$. Continuing,

$$\left(H_N^{(2)}\right)^2-\sum_{n=2}^{N} \frac{H_{n-1}^{(2)}}{n^2}=\left(H_N^{(2)}\right)^2-\sum_{n=2}^{N} \frac{H_n^{(2)}-\frac{1}{n^2}}{n^2}=\left(H_N^{(2)}\right)^2-\sum_{n=1}^{N} \frac{H_n^{(2)}}{n^2}+\sum_{n=1}^{N} \frac{1}{n^4}.$$

Putting all of this together and sending $N \to \infty$ yields

$$S_2=\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^2-S_2+\sum_{n=1}^{\infty} \frac{1}{n^4}$$

and so

$$S_2=\frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^2+\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Step 4: Algebraically relate $\sum_{n=1}^{\infty} \frac{1}{n^4}$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

There are many ways to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Typical arguments use Fourier series, complex analysis, or infinite product expansions. In the spirit of presenting a complete solution which only uses integrals, infinite sums, and ordinary power series, we present a pure algebraic manipulation of $\sum_{n=1}^{\infty} \frac{1}{n^4}$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This technique comes from https://math.stackexchange.com/q/1006510 and the reference therein. Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^4}=\frac{1}{5} \sum_{n=1}^{\infty}\left(\frac{2}{n \cdot n^3}+\frac{1}{n^2 \cdot n^2}+\frac{2}{n^3 \cdot n}\right).$$

Let

$$a_{m,n}=\frac{2}{m n^3}+\frac{1}{m^2 n^2}+\frac{2}{m^3 n}.$$

Observe[1] that

$$\begin{aligned}\sum_{n=1}^{\infty} a_{n,n} &= \sum_{m,n \geq 1} a_{m,n}-\sum_{n>m \geq 1} a_{m,n}-\sum_{m>n \geq 1} a_{m,n} \\ &= \sum_{m,n \geq 1} a_{m,n}-\sum_{m,n \geq 1} a_{m,m+n}-\sum_{m,n \geq 1} a_{m+n,n} \\ &= \sum_{m,n \geq 1}\left(a_{m,n}-a_{m,m+n}-a_{m+n,n}\right).\end{aligned}$$

[1] Note that there are some divergent double sums here. It's probably okay though!

Now we simplify the above double sequence.

$$a_{m,n} - a_{m,m+n} - a_{m+n,n} = \left(\frac{2}{mn^3} + \frac{1}{m^2n^2} + \frac{2}{m^3n}\right) \underbrace{- \left(\frac{2}{m(m+n)^3} + \frac{1}{m^2(m+n)^2} + \frac{2}{m^3(m+n)}\right) - \left(\frac{2}{(m+n)n^3} + \frac{1}{(m+n)^2n^2} + \frac{2}{(m+n)^3n}\right)}_{A}.$$

Isolating and simplifying A,

$$\begin{aligned}
A &= -\left(\frac{2m^2n^3 + mn^3(m+n) + 2n^3(m+n)^2 + 2m^3n^2 + m^3n(m+n) + 2m^3(m+n)^2}{m^3n^3(m+n)^3}\right) \\
&= -\left(\frac{2m^2n^2(m+n) + mn^3(m+n) + 2n^3(m+n)^2 + m^3n(m+n) + 2m^3(m+n)^2}{m^3n^3(m+n)^3}\right) \\
&= -\left(\frac{2m^2n^2 + mn^3 + 2n^3(m+n) + m^3n + 2m^3(m+n)}{m^3n^3(m+n)^2}\right) \\
&= -\left(\frac{mn(2mn + n^2 + m^2) + 2n^3(m+n) + 2m^3(m+n)}{m^3n^3(m+n)^2}\right) \\
&= -\left(\frac{mn(n+m)^2 + 2n^3(m+n) + 2m^3(m+n)}{m^3n^3(m+n)^2}\right) \\
&= -\left(\frac{mn(n+m) + 2n^3 + 2m^3}{m^3n^3(m+n)}\right)
\end{aligned}$$

So

$$\begin{aligned}
a_{m,n} - a_{m,m+n} - a_{m+n,n} &= \left(\frac{2m^2 + mn + 2n^2}{m^3n^3}\right) - \left(\frac{mn(n+m) + 2n^3 + 2m^3}{m^3n^3(m+n)}\right) \\
&= \frac{2m^2(n+m) + 2n^2(n+m) - 2n^3 - 2m^3}{m^3n^3(m+n)} \\
&= \frac{2m^2n + 2n^2m}{m^3n^3(m+n)} \\
&= \frac{2m + 2n}{m^2n^2(m+n)} \\
&= \frac{2}{m^2n^2}.
\end{aligned}$$

Putting everything together, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{5}\sum_{n=1}^{\infty} a_{n,n} &= \frac{1}{5}\sum_{m,n\geq 1} (a_{m,n} - a_{m,m+n} - a_{m+n,n}) \\
&= \frac{1}{5}\sum_{m,n\geq 1} \frac{2}{m^2n^2} \\
&= \frac{2}{5}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^2} \cdot \frac{1}{n^2} \\
&= \frac{2}{5}\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^2.
\end{aligned}$$

Step 5: Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Again, there are numerous ways to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We present a solution due to [1] which is thematically consistent with everything else done thus far. Note that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Solving for the desired sum in the above equation gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

We evaluate the latter sum by converting it to a double integral. Note that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} &= \sum_{n=0}^{\infty} \left(\int_0^1 x^{2n}\,dx\right)\left(\int_0^1 y^{2n}\,dy\right)\\ &= \int_0^1\int_0^1 \sum_{n=0}^{\infty} (x^2y^2)^n\,dx\,dy\\ &= \int_0^1\int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy.\end{aligned}$$

Next, we make the change of variables

$$x = \frac{\sin u}{\cos v} \quad \text{and} \quad y = \frac{\sin v}{\cos u}.$$

Note that

$$\left|\begin{bmatrix}\partial_u x & \partial_v x\\ \partial_u y & \partial_v y\end{bmatrix}\right| = \left|\begin{bmatrix}\frac{\cos u}{\cos v} & \frac{\sin u \sin v}{\cos^2 v}\\ \frac{\sin u \sin v}{\cos^2 u} & \frac{\cos v}{\cos u}\end{bmatrix}\right| = 1 - \tan^2 u \tan^2 v.$$

It follows that $dx\,dy = (1-\tan^2 u\tan^2 v)\,du\,dv$. Let E be the region in the the square $0 \le u, v \le \frac{\pi}{2}$ which is the image of $0 \le x, y \le 1$ under this transformation. Note that this region is defined by $\sin u \le \cos v$ and $\sin v \le \cos u$. Note that equality occurs in both inequalities if $v = \frac{\pi}{2} - u$. This line divides the square $0 \le u, v \le \frac{\pi}{2}$ into two triangles. The inequalities dictate that E is the triangle with vertices $(0,0)$, $(\pi/2, 0)$, and $(0, \pi/2)$. Therefore,

$$\int_0^1\int_0^1 \frac{1}{1-x^2y^2}\,dx\,dy = \iint_E \frac{1-\tan^2 u\tan^2 v}{1-\tan^2 u\tan^2 v}\,du\,dv = \iint_E du\,dv = \frac{1}{2}\cdot\frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

Step 6: Put it all together.

Summarizing steps 1 through 5 gives the final answer:

$$\begin{aligned}S = 2S_1 - S_2 = 6\sum_{n=1}^{\infty}\frac{1}{n^4} - \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^2 - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} &= \frac{11}{2}\sum_{n=1}^{\infty}\frac{1}{n^4} - \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^2\\ &= \frac{11}{2}\cdot\frac{2}{5}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^2 - \frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^2\\ &= \frac{17}{10}\left(\frac{\pi^2}{6}\right)^2.\end{aligned}$$

Thus,

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)^2 = \frac{17\pi^4}{360}.}$$

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[1] Beukers, F., Calabi, E. Kolk, J. *Sums of generalized harmonic series and volumes.* Nieuw Arch. Wisk., 1993.

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