SOME CHALLENGE PROBLEMS

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1. **Evaluate the following infinite series:**

   (a) $\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n-1}}{4^n(2n+1)!}$

   (b) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

   (c) $\sum_{n=0}^{\infty} \frac{n+1}{n!}$.

   Solution.

   (a) Recall that $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x. So

$$\sin(x/2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x/2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2^{2n+1}(2n+1)!}.$$

   Plugging in π for x in this series gives

$$1 = \sin(\pi/2) = \sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}.$$

   So

$$1 = \sum_{n=0}^{\infty}(-1)^n \frac{\pi^2 \cdot \pi^{2n-1}}{4^n \cdot 2 \cdot (2n+1)!} = \frac{\pi^2}{2}\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n-1}}{4^n(2n+1)!}.$$

   Thus,

$$\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n-1}}{4^n(2n+1)!} = \frac{2}{\pi^2}.$$

   (b) Recall that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$. Taking the derivative of both sides gives

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{nx^{n-1}}.$$

   Thus,

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{nx^n}.$$

   Taking another derivative, we have

$$\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \sum_{n=1} n^2 x^{n-1}.$$

   Simplifying the left hand side gives

$$\frac{1-2x+x^2+2x-2x^2}{(1-x)^4} = \frac{1-x^2}{(1-x)^4} = \frac{(1-x)(1+x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}.$$

   Thus,

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}$$

and so

$$\sum_{n=1} n^2 x^n = \frac{x(1+x)}{(1-x)^3}.$$

So

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1} n^2 (1/2)^n = \frac{(1/2)(1+1/2)}{(1-1/2)^3} = \frac{3/4}{(1/2)^3} = \frac{8 \cdot 3}{4} = 6.$$

(c) Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x, so

$$xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}.$$

Taking the derivative of both sides gives

$$e^x + xe^x = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}.$$

Plugging in $x = 1$ yields

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} = e^1 + 1 \cdot e^1 = 2e.$$

□