Some 32A Practice Problems: Solutions

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Particle Motion

1. *A particle moves along a path given by the curve:*

$$\mathbf{r}(t) = \left\langle 2t^2 + 1,\, 2e^t + 2,\, t^3 \right\rangle$$

What are the normal and tangential components of acceleration when the particle is moving in the direction of the y-axis?

Solution. This problem asks us to find the tangential and normal components of acceleration at a certain time. All we need to do is compute $\mathbf{r}'$ and $\mathbf{r}''$, then use the formulas we know. Before that, though, there is something else we have to do first. Because the problem asks us to find the acceleration components *when the particle is moving in the direction of the y-axis,* we need to figure out at what time that happens.

The particle moves in the direction of the y-axis when its velocity vector is parallel to the vector $\langle 0,\, 1,\, 0\rangle$. So let's calculate $\mathbf{v}(t)$:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 4t,\, 2e^t,\, 3t^2 \right\rangle$$

We need the x and z components of this vector to be 0, so we know that $t = 0$ (otherwise, the vector wouldn't be parallel to $\langle 0,\, 1,\, 0\rangle$). This tells us that:

$$\mathbf{r}'(0) = \langle 0,\, 2,\, 0\rangle$$

Next, we need to calculate $\mathbf{r}''(t)$:

$$\mathbf{r}''(t) = \left\langle 4,\, 2e^t,\, 6t \right\rangle$$

At time $t = 0$, this vector is:

$$\mathbf{r}''(0) = \langle 4,\, 2,\, 0\rangle$$

Now it's a matter of plugging and chugging. The tangential component of acceleration at time $t = 0$ is:

$$a_T = \frac{\mathbf{r}'(0)\cdot\mathbf{r}''(0)}{\|\mathbf{r}'(0)\|} = \frac{\langle 0,\, 2,\, 0\rangle\cdot\langle 4,\, 2,\, 0\rangle}{\|\langle 0,\, 2,\, 0\rangle\|} = \frac{4}{2} = 2$$

The normal component of acceleration at $t = 0$ is:

$$a_N = \frac{\|\mathbf{r}'(0)\times\mathbf{r}''(0)\|}{\|\mathbf{r}'(0)\|} = \frac{\|\langle 0,\, 2,\, 0\rangle\times\langle 4,\, 2,\, 0\rangle\|}{\|\langle 0,\, 2,\, 0\rangle\|} = \frac{\|\langle 0,\, 0,\, -8\rangle\|}{2} = \frac{8}{2} = 4$$

$\square$

Projectile Motion

1. *One night, Joe sneaks onto the roof of Swift Hall — which is* 10 *meters tall — and throws his shoe at an angle of* 30 *degrees to the horizontal with an initial speed of* $2\,m/s$*. How fast is his shoe going when it hits the ground? Take gravity to be* $10\,m/s^2$*.*

   Solution. Because this situation is implicitly two dimensional, and because there is no wind or anything that would give my shoe any extra acceleration, we can use the standard equations for projectile motion:

   $$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle = \left\langle x_0 + v_0\cos(\theta)t,\, y_0 + v_0\sin(\theta)t - \frac{1}{2}gt^2\right\rangle$$

   In the context of this problem, $g = 10$, $v_0 = 2$, and $\theta = \frac{\pi}{6}$. Since I'm at the top of Swift Hall, we can take my initial position to be $(x_0, y_0) = (0, 10)$. Plugging these values into our position equation gives us:

   $$\mathbf{r}(t) = \left\langle \sqrt{3}t,\, 10 + t - 5t^2\right\rangle$$

   In order to calculate how fast the shoe is travelling when it hits the ground, we need to know *when* this happens. Well, the shoe hits the ground when the y component of our position vector is 0, so:

   $$10 + t - 5t^2 = 0$$

   Solving this for t gives us $t \approx 1.5$. To find the speed of the ball at this time, we first need to find the velocity vector. This is just the derivative of the position:

   $$\mathbf{v}(t) = \langle x'(t),\, y'(t)\rangle = \left\langle \sqrt{3},\, 1 - 10t\right\rangle$$

   So the velocity vector at time $t = 1.5$ is:

   $$\mathbf{v}(1.5) = \left\langle \sqrt{3},\, -14\right\rangle$$

   Thus, the speed of the shoe when it hits the ground is:

   $$\|\mathbf{v}(1.5)\| = \sqrt{(\sqrt{3})^2 + (-14)^2} \approx 14.1$$

   □

2. *Joe is having a little too much fun on Dillo Day, makes a poor judgement call, and shoots a firework off in front of Lunt Hall. The firework is launched in the western direction (in the direction of the negative* x *axis) at an angle of* 60 *degrees, with an initial speed of* $20\,m/s$*. The wind gives the firework a northerly acceleration of* $3\,m/s^2$*. Take gravity to be* $10\,m/s^2$*. How far away from Lunt does the firework land?*

Solution. As always, we need to find the position function $\mathbf{r}(t)$. First, let's figure out what the acceleration function $\mathbf{a}(t)$ looks like. Since gravity is pulling the firework downwards, the z component of $\mathbf{a}(t)$ will be -10. The wind gives a "northerly" acceleration of 3, so the y component of $\mathbf{a}(t)$ will be 3. There is no acceleration in the x direction. Hence,

$$\mathbf{a}(t) = \langle 0,\, 3,\, -10\rangle$$

We also want the initial velocity and position vectors. Since I launched the firework in the negative x direction, there will be no y-velocity. So we only need to find the x and z components of the initial acceleration. This is basically the same as the two dimensional case: the x component will be $-v_0\cos(\theta)$ (its negative because I launched in the negative direction) and the z component will be $v_0\sin(\theta)$. Hence,

$$\mathbf{v}(0) = \langle -20\cos(\pi/3),\, 0,\, 20\sin(\pi/3)\rangle = \left\langle -10,\, 0,\, 10\sqrt{3}\right\rangle$$

We implicitly assume that Lunt Hall is located at the point $(0, 0, 0)$, so

$$\mathbf{r}(0) = \langle 0,\, 0,\, 0\rangle$$

Now, we're ready to find the position function. Let's start by integrating the acceleration function to get the velocity function:

$$\mathbf{v}(t) = \langle C_1,\, 3t + C_2,\, -10t + C_3\rangle$$

To find the constants, plug in $t = 0$. We know this has to equal the initial velocity vector:

$$\mathbf{v}(0) = \langle C_1,\, C_2,\, C_3\rangle = \left\langle -10,\, 0,\, 10\sqrt{3}\right\rangle$$

Hence, the velocity function is:

$$\mathbf{v}(t) = \left\langle -10,\, 3t,\, -10t + 10\sqrt{3}\right\rangle$$

Next, we integrate again to get position:

$$\mathbf{r}(t) = \left\langle -10t + C_1,\, \frac{3}{2}t^2 + C_2,\, -5t^2 + 10\sqrt{3}t + C_3\right\rangle$$

Plugging in $t = 0$ gives us:

$$\mathbf{r}(0) = \langle C_1,\, C_2,\, C_3\rangle = \langle 0,\, 0,\, 0\rangle$$

So the position function is:

$$\mathbf{r}(t) = \left\langle -10t,\, \frac{3}{2}t^2,\, -5t^2 + 10\sqrt{3}t\right\rangle$$

Awesome - the hard work is done. Now we can actually answer the question. We want to know how far away the firework lands, so we need to know *when* this happens. The firework hits the ground precisely when the z component is 0 (note that in three dimensions, "hitting the ground" means $z = 0$, while in two dimensions, it means $y = 0$!):

$$-5t^2 + 10\sqrt{3}t = 0$$

Solving this gives us $t = 2\sqrt{3}$. If we want to know *where* the ball lands, we just plug this time into the position function:

$$\begin{aligned}\mathbf{r}(2\sqrt{3}) &= \left\langle -10(2\sqrt{3}),\, \frac{3}{2}(2\sqrt{3})^2,\, -5(2\sqrt{3})^2 + 10\sqrt{3}(2\sqrt{3}) \right\rangle \\ &= \left\langle -20\sqrt{3},\, 18,\, 0 \right\rangle\end{aligned}$$

To figure out how far away this is from Lunt, we just take the magnitude of the vector:

$$\left\| \mathbf{r}(-2\sqrt{3}) \right\| = \sqrt{(20\sqrt{3})^2 + (18)^2} \approx 39$$

Therefore, the firework lands about 39 meters away from Lunt. □

3. *(Challenging Problem) Joe is about to take the halfcourt shot at the NU-Wisconsin basketball game. The basketball hoop is* 3 *meters high, and is* 28 *meters from half court. Unfortunately, Joe hasn't played basketball since* 8*th grade and has no idea how hard to shoot the ball, so he shoots at an angle of* 45 *degrees with an initial speed of* $20\, m/s$*. Also, due to the faltering structural integrity of Welsh-Ryan Arena, there is a draft that gives the ball a left-ward acceleration of* $1\, m/s$*. Take gravity to be* $10\, m/s^2$*, and suppose for the sake of simplicity that Joe shoots from the point* $(0, 0, 0)$*. How close does the ball come to going in?*

Solution. Suppose that half-court is at $(0, 0, 0)$. Then if the basketball hoop is in the positive y direction (so positive x points to my right into the crowd), the hoop would be located at the point $(0, 28, 3)$.

Our first goal is to find the position function of the ball. Gravity pulls downwards, and the wind blows the ball to the left (in the negative x direction), so the acceleration function is:

$$\mathbf{a}(t) = \langle -1,\, 0,\, -10 \rangle$$

The initial velocity will only have y and z components, since I'm shooting towards the hoop. Similarly to before, the y component will be given by $v_0 \cos(\theta)$ and the z component will be given by $v_0 \sin(\theta)$:

$$\mathbf{v}(0) = \langle 0,\, 20\cos(\pi/4),\, 20\sin(\pi/4) \rangle = \left\langle 0,\, \frac{20}{2}\sqrt{2},\, \frac{20}{2}\sqrt{2} \right\rangle$$

The initial position vector is:

$$\mathbf{r}(0) = \langle 0,\, 0,\, 0 \rangle$$

Integrating the acceleration vector and plugging in $t = 0$ to solve for constants, we get:

$$\mathbf{v}(t) = \left\langle -t,\, \frac{20}{2}\sqrt{2},\, -10t + \frac{20}{2}\sqrt{2} \right\rangle$$

Integrating again to get position (and plugging in $t = 0$ to solve for constants), we get:

$$\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2,\, \frac{20}{2}\sqrt{2}t,\, -5t^2 + \frac{20}{2}\sqrt{2}t \right\rangle$$

We now have a complete description of the position of the ball. If we want to know how close the ball gets to the hoop, we should calculate the distance between $\mathbf{r}(t)$ (the position of the ball) and $(0, 28, 3)$ (the position of the hoop). I'll denote this distance — which depends on t — by $d(t)$:

$$\begin{aligned} d(t) &= \|\langle 0,\, 28,\, 3\rangle - \mathbf{r}(t)\| \\ &= \left\| \left\langle \frac{1}{2}t^2,\, 28 - \frac{20}{2}\sqrt{2}t,\, 5t^2 - \frac{20}{2}\sqrt{2}t \right\rangle + 3 \right\| \\ &= \sqrt{\left(\frac{1}{2}t^2\right)^2 + \left(28 - \frac{20}{2}\sqrt{2}t\right)^2 + \left(5t^2 - \frac{20}{2}\sqrt{2}t + 3\right)^2} \end{aligned}$$

We want to find the *minimum* distance, so we need to find when $d'(t) = 0$. Taking the derivative of d would be very annoying, so instead, we can minimize the *square* of the distance, $d^2(t)$:

$$d^2(t) = \left(\frac{1}{2}t^2\right)^2 + \left(28 - \frac{20}{2}\sqrt{2}t\right)^2 + \left(5t^2 - \frac{20}{2}\sqrt{2}t + 3\right)^2$$

Now, taking the derivative (using the chain rule a couple of times) isn't so bad. Doing a lot of simplification gives us:

$$(d^2)'(t) = 101t^3 - 300\sqrt{2}t^2 + 860t - 620\sqrt{2}$$

Setting this equal to 0 and solving for t gives us $t \approx 2.1155$. In other words, the ball comes closest to the hoop at time 2.1155. Plugging this into $d(t)$ gives:

$$d(2.1155) = \sqrt{\left(\frac{1}{2}(2.1155)^2\right)^2 + \left(28 - \frac{20}{2}\sqrt{2}(2.1155)\right)^2 + \left(5(2.1155)^2 - \frac{20}{2}\sqrt{2})(2.1155) + 3\right)^2} \approx 5.9$$

So the closed the ball got was 5.9 meters from the rim. (It was actually a lot closer, I swear.)

□

Multivariable Functions

1. *Let $f(x, y) = x + y^2$. Determine the domain of f and draw a contour plot with three level curves corresponding to the values -1, 0, and 1.*

Solution. First, let's consider the domain of f. We want to ask ourselves the following question: *what can go wrong if I stick certain values in for x and y?* Well, nothing! The quantity $x + y^2$ is always defined, so the domain of f is all of $\mathbb{R}^2$.

To get level curves, we set f equal to -1, 0, and 1. First,

$$f(x, y) = -1 \quad \Rightarrow \quad x + y^2 = -1 \quad \Rightarrow \quad x = -y^2 - 1$$

This is a leftward opening parabola with a base at the point $(-1, 0)$. Next,

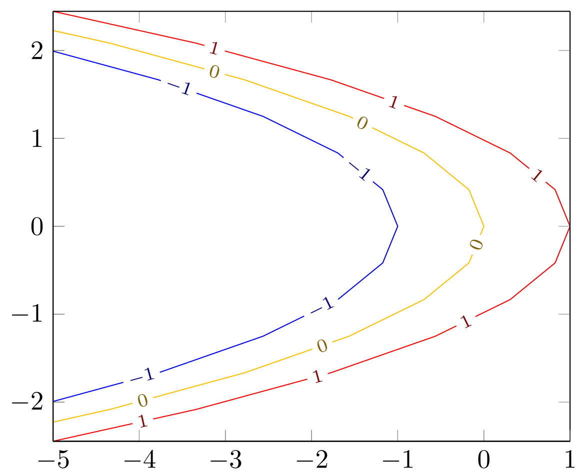
$$f(x, y) = 0 \quad \Rightarrow \quad x + y^2 = 0 \quad \Rightarrow \quad x = -y^2$$

This is a leftward opening parabola with a base at the point $(0,0)$. Finally,

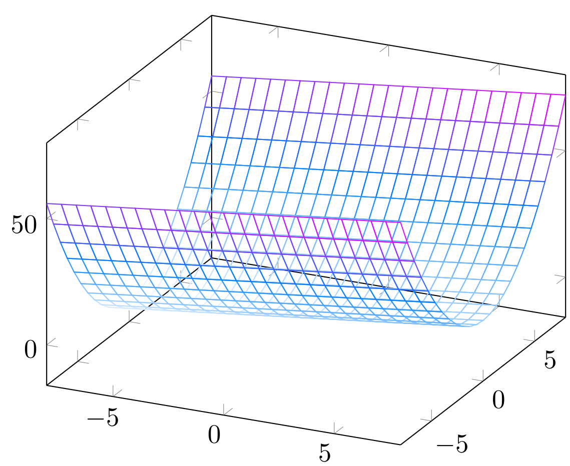
$$f(x,y)=1 \quad \Rightarrow \quad x+y^2=-1 \quad \Rightarrow \quad x=-y^2+1$$

This is a leftward opening parabola with a base at the point $(1,0)$.

Plotting these three curves gives us our contour plot:



Here's a graph of the function, for the sake of completeness:



$\square$

2. *Let* $f(x,y)=\frac{x^2-y^2}{x+y}$. *Describe the domain of the function, and draw a contour plot.*

Solution. Anytime we have a fraction, we have to watch out for the possibility that we divide by 0. This happens when $x+y=0$, and hence $y=-x$. Nothing else can go wrong, so our domain is all of $\mathbb{R}^2$, minus the off-diagonal line determined by the equation $y=-x$.

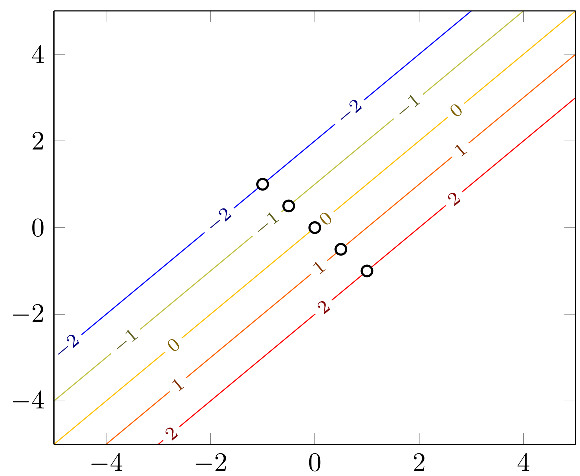
To draw the contour plots, pick some values for k (say $-1, 0, 1$) and consider:

$$f(x, y) = \frac{x^2 - y^2}{x + y} = k$$

Note that we can factor the top:

$$\frac{x^2 - y^2}{x + y} = \frac{(x + y)(x - y)}{x + y} = x - y$$

So our level curves look like $x - y = k$. That means that they're lines! But because y cannot equal $-x$ (by our domain restriction), there will be empty holes where the lines intersect $y = -x$:



$\square$

3. *Describe the level surfaces of the function* $f(x, y, z) = x^2 + y^2 + z^2$.

Solution. To find the level surfaces of f, we fix the value of f at some k:

$$f(x, y, z) = x^2 + y^2 + z^2 = k$$

This gives us equations that look like:

$$x^2 + y^2 + z^2 = k$$

If $k > 0$, then this is a sphere with radius $\sqrt{k}$. For example, if $k = 4$, then

$$x^2 + y^2 + z^2 = 4$$

is a sphere of radius 2. If $k = 0$, then the only point that satisfies the equation

$$x^2 + y^2 + z^2 = 0$$

is the point $(0, 0, 0)$, so the level surface is a single point. If $k < 0$, then we get nothing. $\square$

Multivariable Limits and Continuity

1. *(Technique: Continuity) Evaluate the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{x^{2-y}+3}{\cos(xy)+1}$$

Solution. The first thing you should always try in evaluating a limit is plugging the target point in. So let's give it a shot:

$$\lim_{(x,y)\to(0,0)} \frac{x^{2-y}+3}{\cos(xy)+1} = \frac{(0)^{2-0}+3}{\cos(0)+1} = \frac{3}{2}$$

Nothing bad happened (we didn't divide by 0), so we're done! The function $\frac{x^{2-y}+3}{\cos(xy)+1}$ is continuous at the point $(0,0)$, so plugging it in works. $\square$

2. *(Technique: Different Path Approaches) Evaluate the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{2x^3-xy-3y^3}{3x^3-2y^3}$$

Solution. As always, let's try plugging in the point $(0,0)$ before we do anything else:

$$\lim_{(x,y)\to(0,0)} \frac{2x^3-xy-3y^3}{3x^3-2y^3} = \frac{0}{0}$$

That's bad news — we can't divide by 0, so plugging in the target point tells us nothing about the limit. The natural thing to try next is to approach the point $(0,0)$ from different paths. If we get different answers, then this means the limit doesn't exist! Approaching along the x-axis and the y-axis is pretty easy to do, because these are defined by the equations $y=0$ and $x=0$, respectively.

Let's travel along the x-axis first. This means that $y=0$, so our limit becomes:

$$\lim_{(x,y)\to(0,0)} \frac{2x^3-x(0)-3(0)^3}{3x^3-2(0)^3} = \lim_{x\to 0}\frac{2x^3}{3x^3} = \lim_{x\to 0}\frac{2}{3} = \frac{2}{3}$$

So if we approach along the x-axis, our function approaches $\frac{2}{3}$.

Next, let's try the y-axis. This means that $x=0$:

$$\lim_{(x,y)\to(0,0)} \frac{2(0)^3-(0)y-3y^3}{3(0)^3-2y^3} = \lim_{y\to 0}\frac{-3y^3}{-2y^3} = \lim_{y\to 0}\frac{3}{2} = \frac{3}{2}$$

So if we approach along the y-axis, our function approaches $\frac{3}{2}$.

Since $\frac{2}{3}\neq\frac{3}{2}$, this means that the general limit as we approach the origin *cannot exist.*

$\square$

3. *(Technique: Different Path Approaches) Evaluate the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{xy+x^2+2y}$$

Solution. First things first, plug in $(0,0)$. We get $\frac{0}{0}$, so we need to do something else. Let's try approaching along the axes like in the previous problem. If $x=0$, we get:

$$\lim_{(x,y)\to(0,0)} \frac{(0)^2+(0)y+y^2}{(0)y+(0)^2+2y} = \lim_{y\to 0} \frac{y^2}{2y} = \lim_{y\to 0} \frac{y}{2} = 0$$

and if $y=0$, we get:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+x(0)+(0)^2}{x(0)+x^2+2(0)} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} = 1$$

Thus, since $0 \neq 1$, the limit doesn't exist. □

4. *(Technique: Different Path Approaches) Evaluate the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+3y^2}$$

Solution. Step one, plug in $(0,0)$. This gives us $\frac{0}{0}$, no good. Next, let's try the axes: if $x=0$,

$$\lim_{(x,y)\to(0,0)} \frac{(0)y}{2(0)^2+3y^2} = \lim_{y\to 0} \frac{0}{3y^2} = \lim_{y\to 0} 0 = 0$$

If $y=0$,

$$\lim_{(x,y)\to(0,0)} \frac{x(0)}{2x^2+3(0)^2} = \lim_{x\to 0} \frac{0}{2x^2} = \lim_{x\to 0} 0 = 0$$

Since approaching along the axes gives the same answer, this doesn't tell us anything (note: it does *not* imply that the limit equals 0). Let's try another path of approach. Because of the almost-but-not-quite-symmetric nature of the equation $\frac{xy}{2x^2+3y^2}$, it might be a good idea to try the diagonal line $y=x$. If we make all of the y's in the equation equal to x, the limit becomes:

$$\lim_{x\to 0} \frac{x(x)}{2x^2+3(x)^2} = \lim_{x\to 0} \frac{x^2}{5x^2} = \lim_{x\to 0} \frac{1}{5} = \frac{1}{5}$$

This approach gave us a different answer than 0, so therefore the limit does not exist. □

5. *(Technique: Different Path Approaches) Evaluate the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^8+y^2}$$

Solution. Plugging in $(0,0)$ gives us $\frac{0}{0}$. It also isn't hard to see that evaluating the limit along the axes (when $x=0$ and $y=0$) yields a limit of 0 for both. So we have to try other things.

We could try $y=x$ like before, but this gives us:

$$\lim_{x\to 0} \frac{x^4x}{x^8+x^2} = \lim_{x\to 0} \frac{x^5}{x^8+x^2}$$

Factoring out an x^2 from everything, we get:

$$\lim_{x\to 0} \frac{x^3}{x^6+1} = 0$$

So we get 0 again — darn.

Here's the thing to notice. We can make the powers of x and y on the bottom equal if we make $y = x^4$, because $4 \cdot 2 = 8$. So let's try approaching the origin along the curve $y = x^4$:

$$\lim_{x \to 0} \frac{x^4(x^4)}{x^8 + (x^4)^2} = \lim_{x \to 0} \frac{x^8}{x^8 + x^8} = \lim_{x \to 0} \frac{x^8}{2x^8} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}$$

This is different than 0, so the limit does not exist. □

6. *(Technique: Factoring) Evaluate the following limit:*

$$\lim_{(x,y) \to (0,0)} \frac{x^4 - y^2}{x^2 + y}$$

Solution. Plugging in $(0, 0)$ gets us $\frac{0}{0}$. Trying the usual methods of approach, like $x = 0$, $y = 0$, etc. will all get you a limit of 0. You might suspect that the limit equals 0, so we need to do something other than just plugging things in. Here's what you should notice: the numerator is a *difference of squares.*

$$\lim_{(x,y) \to (0,0)} \frac{(x^2)^2 - y^2}{x^2 + y}$$

Hence, we can factor it, and things cancel out nicely:

$$\lim_{(x,y) \to (0,0)} \frac{(x^2 + y)(x^2 - y)}{x^2 + y} = \lim_{(x,y) \to (0,0)} x^2 - y = 0$$

□

7. *(Technique: Conjugate) Evaluate the following limit:*

$$\lim_{(x,y) \to (0,0)} \frac{x^3 + xy + y^2}{2 - \sqrt{x^3 + xy + y^2 + 4}}$$

Solution. Plugging in 0 gives us $\frac{0}{0}$, and other paths don't get us very far. Focus in on the denominator:

$$2 - \sqrt{x^3 + xy + y^2 + 4}$$

Because this looks like $\sqrt{blah} - \sqrt{blah}$, it might be worthwhile to multiply the top and bottom by the conjugate:

$$2 + \sqrt{x^3 + xy + y^2 + 4}$$

If we do that, the difference of squares on the bottom will work out nicely:

$$\begin{aligned}
\lim_{(x,y) \to (0,0)} \frac{x^3 + xy + y^2}{2 - \sqrt{x^3 + xy + y^2 + 4}} &= \lim_{(x,y) \to (0,0)} \frac{x^3 + xy + y^2}{2 - \sqrt{x^3 + xy + y^2 + 4}} \cdot \frac{2 + \sqrt{x^3 + xy + y^2 + 4}}{2 + \sqrt{x^3 + xy + y^2 + 4}} \\
&= \lim_{(x,y) \to (0,0)} \frac{(x^3 + xy + y^2)(2 + \sqrt{x^3 + xy + y^2 + 4})}{2^2 - (\sqrt{x^3 + xy + y^2 + 4})^2} \\
&= \lim_{(x,y) \to (0,0)} \frac{(x^3 + xy + y^2)(2 + \sqrt{x^3 + xy + y^2 + 4})}{-(x^3 + xy + y^2)}
\end{aligned}$$

Now, we can cross out the denominator with the term on top:

$$= \lim_{(x,y)\to(0,0)} \frac{(x^3+xy+y^2)(2+\sqrt{x^3+xy+y^2+4})}{-(x^3+xy+y^2)}$$
$$= \lim_{(x,y)\to(0,0)} -(2+\sqrt{x^3+xy+y^2+4})$$
$$= -4$$

□

8. *(Technique: Squeeze Theorem / Comparison) Evaluate the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4+3y^6}$$

Solution. As usual, plugging in $(0,0)$ gives us $\frac{0}{0}$, and trying different paths won't get us very far. This gets tricky.

One thing to notice is that on the bottom, we have x^4+3y^6. Since $3y^6$ is always positive, $x^4+3y^6 \geq x^4$, and therefore $\frac{1}{x^4+3y^6} \leq \frac{1}{x^4}$. This tells us that:

$$\frac{x^4y^2}{x^4+3y^6} \leq \frac{x^4y^2}{x^4} = y^2$$

Also, since every term in $\frac{x^4y^2}{x^4+3y^6}$ is always positive (because of the even powers), it follows that

$$0 \leq \frac{x^4y^2}{x^4+3y^6}$$

Putting this together, we have

$$0 \leq \frac{x^4y^2}{x^4+3y^6} \leq y^2$$

Then, taking limits on all sides,

$$0 \leq \frac{x^4y^2}{x^4+3y^6} \leq y^2$$
$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4+3y^6} \leq \lim_{(x,y)\to(0,0)} y^2$$
$$0 \leq \lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4+3y^6} \leq \lim_{y\to 0,0} y^2$$
$$0 \leq \lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4+3y^6} \leq 0$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{x^4+3y^6} = 0$$

□

9. *(Technique: Polar Coordinates) Evaluate the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{2xye^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

Solution. Plugging in $(0,0)$ gives us $\frac{0}{0}$, different path approaches tell us nothing, etc. But notice that the term $\sqrt{x^2+y^2}$ appears in the function. Since $r^2 = x^2+y^2$, and hence $r = \sqrt{x^2+y^2}$, this suggest that we should switch the limit into polar coordinates. So,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{2xye^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{2(r\cos\theta)(r\sin\theta)e^r}{r} \\ &= \lim_{r\to 0} 2r\cos\theta\sin\theta e^r \\ &= 0\end{aligned}$$

□

10. *(Technique: Polar Coordinates) Evaluate the following limit:*

$$\lim_{(x,y)\to(0,0)} \frac{2xye^{\sqrt{x^2+y^2}}}{x^2+y^2}$$

Solution. This is almost the same problem as the previous one, but instead of $\sqrt{x^2+y^2}$ on the bottom, we have x^2+y^2. Let's see how this changes things. Converting to polar coordinates and simplifying gives us:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{2xye^{\sqrt{x^2+y^2}}}{x^2+y^2} &= \lim_{r\to 0} \frac{2(r\cos\theta)(r\sin\theta)e^r}{r^2} \\ &= \lim_{r\to 0} 2\cos\theta\sin\theta e^r \\ &= 2\cos\theta\sin\theta\end{aligned}$$

So our limit depends on what θ is! If $\theta = 0$, then since $\sin 0 = 0$, the limit is 0. But if $\theta = \frac{\pi}{4}$, then

$$2\cos\theta\sin\theta = 2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = 1$$

Since $1 \neq 0$, the limit does not exist.

□

11. *Is the following function continuous at the point* $(0,0,0)$*?*

$$f(x,y,z) = \begin{cases} \frac{xy+yz+z^2}{x+y+z^2} & (x,y,z) \neq (0,0,0) \\ 1 & (x,y,z) = (0,0,0) \end{cases}$$

Solution. We need to determine if

$$\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(0,0,0)$$

If this is true, then yes, f is continuous at the origin. If this isn't true, then f is not continuous. By the definition of f, this means that we need

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+z^2}{x+y+z^2} = 1$$

Let's look at the limit on the left. If we approach along the x-axis (which means that $y = 0$ and $z = 0$), then we get:

$$\begin{aligned}\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + z^2}{x + y + z^2} &= \lim_{x\to 0} \frac{0}{x} \\ &= \lim_{x\to 0} 0 \\ &= 0\end{aligned}$$

Think about what this means: we approach the origin along one path, and we got 0. Therefore, the limit *cannot* equal 1. The limit may or may not exist, but it doesn't matter! We can say for sure that f is not continuous at $(0, 0, 0)$.

□

Partial Derivatives

1. *Let* $f(x, y) = \frac{e^{xy}}{x^2+y^2}$. *Compute* $f_x(x, y)$ *and* $f_y(x, y)$.

   Solution. Let's do $f_x(x, y)$ first. We want to take the partial derivative of f with respect to x, so we'll pretend that y is a constant. Here, we have to use the quotient rule, because our function is a fraction:

$$\begin{aligned}f_x(x, y) = \frac{\partial}{\partial x}\left(\frac{e^{xy}}{x^2 + y^2}\right) &= \frac{(x^2 + y^2)\frac{\partial}{\partial x}(e^{xy}) - (e^{xy})\frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2} \\ &= \frac{(x^2 + y^2)(ye^{xy}) - (e^{xy})(2x)}{(x^2 + y^2)^2}\end{aligned}$$

   Similarly for y,

$$\begin{aligned}f_y(x, y) = \frac{\partial}{\partial y}\left(\frac{e^{xy}}{x^2 + y^2}\right) &= \frac{(x^2 + y^2)\frac{\partial}{\partial y}(e^{xy}) - (e^{xy})\frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2} \\ &= \frac{(x^2 + y^2)(xe^{xy}) - (e^{xy})(2y)}{(x^2 + y^2)^2}\end{aligned}$$

   Nice.

   □

2. *Let* $g(r, \theta, z) = r^2 \cos 2z\theta + zr\theta - 2^{r-z}$. *Compute* $g_z(1, \frac{\pi}{4}, 2)$.

   Solution. Then first step is to calculate the function $g_z(r, \theta, z)$. Then we can plug in the point $(1, \frac{\pi}{4}, 2)$. So in taking the partial of g with respect to z, we pretend r and θ are constants, and we get:

$$\begin{aligned}g_z(r, \theta, z) &= \frac{\partial}{\partial z}(r^2 \cos 2z\theta + zr\theta - 2^{r-z}) \\ &= r^2(-\sin 2z\theta)(2\theta) + r\theta - (\ln 2)2^{r-z}(-1)\end{aligned}$$

Therefore,

$$\begin{aligned} g_z\left(1, \frac{\pi}{4}, 2\right) &= 1^2\left(-\sin 2(2)\frac{\pi}{4}\right)\left(2\frac{\pi}{4}\right) + (1)\frac{\pi}{4} - (\ln 2)2^{1-2}(-1) \\ &= \frac{\pi}{4} + \frac{\ln 2}{2} \end{aligned}$$

$\square$

3. *Joe just bought an Old Fashioned Buttermilk donut and a Valrhona Triple Chocolate donut from Firecakes, and is trying to decide which one to eat first. After extensive testing and data collection, Joe has modeled his happiness as a function of consumption of donuts. Let H denote happiness, B denote the number of Old Fashioned Buttermilk donuts consumed, and C denote the number of Valrhona Triple Chocolate donuts consumed. Then*

$$H(B, C) = e^{3B} + (B+1)(C+1)^3 + e^{\frac{C}{2}}$$

If Joe only has time to take a single bite of one donut, which one should he eat?

Solution. As with any word problem, your immediate goal is to strip away all the words and nonsense and figure out what the questions is asking you to find. This question deals with the interpretation of partial derivatives. Here's why: we have a function H in terms of two variables, B and C. Presumably, I can only eat one bite of one of the donuts, which means that (starting at the point $(0,0)$, since I haven't eaten any donuts), I can either move a tiny bit in the positive B direction (by taking one bite of the Old Fashioned) or I can move a tiny bit in the positive C direction (by taking a bite of the Triple Chocolate). We want to measure how my happiness H responds when I do each of these, so we can calculate $H_B(0,0)$ and $H_C(0,0)$. First, we have:

$$H_B(B, C) = 3e^{3B} + (C+1)^3 \qquad H_C(B_C) = 3(B+1)(C+1)^2 + \frac{1}{2}e^{\frac{C}{2}}$$

Which gives us:

$$H_B(0,0) = 3e^0 + (0+1)^3 = 4 \qquad H_C(B_C) = 3(0+1)(0+1)^2 + \frac{1}{2}e^{\frac{0}{2}} = 3.5$$

Therefore, if I take one bite of the Old Fashioned, my happiness will increase by 4 units. If I take one bite of the Triple Chocolate, my happiness will increase by 3.5 units. Thus, the Buttermilk is the way to go. $\square$

4. *Let $f(x, y) = \ln(x^2 + y^2 + 1)$. Prove that $f_{xy} = f_{yx}$.*

Solution. This is routine; we just need to take the partial derivative of the partial derivatives. Let's find the first partials:

$$\begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}\ln(x^2 + y^2 + 1) \\ &= \frac{2x}{x^2 + y^2 + 1} \end{aligned}$$

and

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}\ln(x^2+y^2+1) \\ &= \frac{2y}{x^2+y^2+1} \end{aligned}$$

Next, let's take the partial with respect to y of f_x:

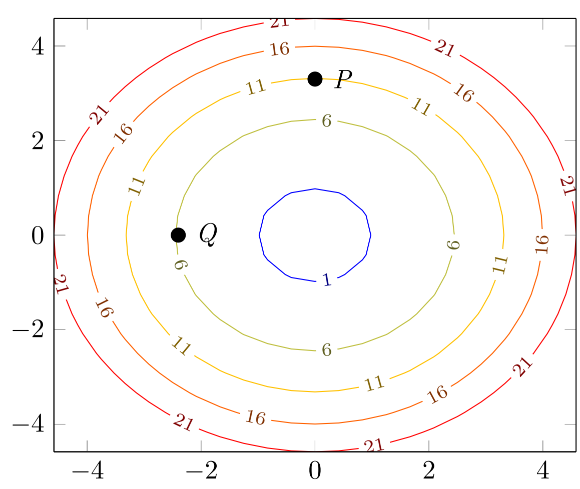
$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y}\frac{2x}{x^2+y^2+1} \\ &= \frac{(x^2+y^2+1)(0)-(2x)(2y)}{(x^2+y^2+1)^2} \\ &= \frac{4xy}{(x^2+y^2+1)^2} \end{aligned}$$

Similarly, the partial with respect to x of f_y is:

$$\begin{aligned} f_{yx}(x,y) &= \frac{\partial}{\partial x}\frac{2y}{x^2+y^2+1} \\ &= \frac{(x^2+y^2+1)(0)-(2y)(2x)}{(x^2+y^2+1)^2} \\ &= \frac{4xy}{(x^2+y^2+1)^2} \end{aligned}$$

Hence, $f_{xy} = f_{yx}$. □

5. *Consider the contour plot of the function $f(x,y)$ below.*



Determine whether each of the following quantities is positive, negative, or zero:

(a) $f_x(P)$

(b) $f_y(P)$

(c) $f_{yy}(P)$

(d) $f_y(Q)$

(e) $f_x(Q)$

(f) $f_{xx}(Q)$

Solution. (a) $f_x(P) = 0$. This is because the x-direction is tangent to the level curve at P. Explicitly, if we zoom in super closely to P, and we move a teeny bit to the right (in the positive x direction), we are basically moving right on top of the level curve. Hence, since level curves represent where the function doesn't change, the the function isn't changing if we move in this direction. Therefore, the partial derivative of f with respect to x at P must be 0.

(b) $f_y(P) > 0$. As we move vertically from P (in the positive y direction), the level curves increase in value. This means that the function is getting bigger and bigger. Hence, the the change in the function as we move north is positive, and so the partial derivative of f with respect to Y at P is positive.

(c) $f_{yy}(P) > 0$. Note that as we move in the positive y direction from P, the level curves are getting closer together. This means that the change is getting more extreme, because it takes a shorter and shorter distance to travel to the next level of height. Furthermore, since $f_y(P) > 0$, this means that the slope in the y direction is becoming more and more positive. Hence, f is concave-up in the y direction, and so $f_{yy}(P) > 0$.

(d) $f_y(Q) = 0$. Similar to part (a), since the y direction is tangent to the level curve at Q, the function is not changing when we move a tiny bit in the positive y direction. Hence, the partial derivative with respect to y at Q is 0.

(e) $f_x(Q) < 0$. As we move to the right (in the positive x direction), the level curves are decreasing in value. This means that the function is decreasing, and hence the partial derivative with respect to x at Q is negative.

(f) $f_{xx}(Q) > 0$. As we move to the right from Q, the level curves are getting farther and farther apart. This means that the change is getting less and less extreme, since we have to travel a farther distance to move to the next level. Since $f_x(Q) < 0$, this means that the slopes are getting less and less negative. In other words, the slopes are increasing in value, and so the change in the slopes is positive. Hence, $f_{xx}(Q)$ is positive.

□

Tangent Planes and Approximation

1. *Let* $f(x, y) = 2xy + 2^{xy}$. *Find the equation of the tangent plane to* f *at the point* $(1, 2)$.

Solution. The equation for the tangent plane is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Here, $(a, b) = (1, 2)$. We have:

$$f(1, 2) = 2(1)(2) + 2^{(1)(2)} = 8$$

The partial derivatives of f are:

$$f_x(x, y) = 2y + 2^{xy} y \ln 2$$

and

$$f_y(x, y) = 2x + 2^{xy} x \ln 2$$

Hence,

$$f_x(1, 2) = 2(2) + 2^{(1)(2)}(2) \ln 2 = 4 + 8 \ln 2$$

and

$$f_y(1, 2) = 2(1) + 2^{(1)(2)}(1) \ln 2 = 2 + 4 \ln 2$$

Therefore, the equation of the tangent plane is:

$$z = 8 + (4 + 8 \ln 2)(x - 1) + (2 + 4 \ln 2)(y - 2)$$

□

2. *Let* $f(x, y, z) = xy + xz + yz$*. Find the linear approximation to* f *at the point* $(1, 2, 3)$*.*

Solution. The equation for the linear approximation is:

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

Evaluating the function at $(1, 2, 3)$ gives us:

$$f(1, 2, 3) = (1)(2) + (1)(3) + (2)(3) = 11$$

The partial derivatives are:

$$f_x(x, y, z) = y + z$$
$$f_y(x, y, z) = x + z$$
$$f_z(x, y, z) = x + y$$

Hence,

$$f_x(1, 2, 3) = 2 + 3 = 5$$
$$f_y(1, 2, 3) = 1 + 3 = 4$$
$$f_z(1, 2, 3) = 1 + 2 = 3$$

Therefore, the equation for the approximation is:

$$L(x, y, z) = 11 + 5(x - 1) + 4(y - 2) + 3(z - 3)$$

□

3. *Let* $f(x,y) = x^3y^3+x+y$. *Find the linear approximation and quadratic approximation to* f *at the point* $(1,1)$. *Find* Δz *in moving from the point* $(1,1)$ *to* $(2,2)$. *Estimate* Δz *with the linear and quadratic approximation.*

Solution. At the point $(1,1)$, we have $f(1,1) = 3$. The partial derivatives and their evaluations are:

$$\begin{aligned} f_x(x,y) &= 3x^2y^3 + 1 & \qquad f_x(1,1) &= 4 \\ f_y(x,y) &= 3x^3y^2 + 1 & \qquad f_y(1,1) &= 4 \end{aligned}$$

Hence, the linear approximation to f at $(1,1)$ is:

$$T(x,y) = 3 + 4(x-1) + 4(y-1)$$

To find the quadratic approximation, we need the second partial derivatives:

$$\begin{aligned} f_{xx}(x,y) &= 6xy^3 & \qquad f_{xx}(1,1) &= 6 \\ f_{yy}(x,y) &= 6x^3y & \qquad f_{yy}(1,1) &= 6 \\ f_{xy}(x,y) &= 9x^2y^2 & \qquad f_{xy}(1,1) &= 9 \end{aligned}$$

Then the quadratic approximation to f at $(1,1)$ is:

$$\begin{aligned} L(x,y) &= T(x,y) + \frac{1}{2}\left(f_{xx}(1,1)(x-1)^2 + 2f_{xy}(1,1)(x-1)(y-1) + f_{yy}(1,1)(y-1)^2\right) \\ &= 3 + 4(x-1) + 4(y-1) + \frac{1}{2}\left(6(x-1)^2 + 18(x-1)(y-1) + 6(y-1)^2\right) \\ &= 3 + 4(x-1) + 4(y-1) + 3(x-1)^2 + 9(x-1)(y-1) + 3(y-1)^2 \end{aligned}$$

To calculate Δz, we just plug in the two points into $f(x,y)$ and subtract:

$$\begin{aligned} \Delta z &= f(2,2) - f(1,1) \\ &= 68 - 3 \\ &= 65 \end{aligned}$$

To calculate dz, which is the approximation to Δz using the linear approximation, we have:

$$\Delta z \approx dz = 4(2-1) + 4(2-1) = 8$$

Using the quadratic approximation, we have:

$$\Delta z \approx 8 + 3(2-1)^2 + 9(2-1)(2-1) + 3(2-1)^2 = 23$$

$\square$

The Chain Rule

1. *Let $f(x,y) = x^2y + xy^2$ and $x(u,v) = 2uv$ and $y(u,v) = u^2 + v^2$. Consider f as a function of u and v by $f(x(u,v), y(u,v))$. Use the chain rule to compute $\frac{\partial f}{\partial u}$.*

   Solution. The first thing to do in a chain rule problem is to draw out the diagram of functions that depicts how which things depend on which variables. Because f is a function that depends on x and y, we'll draw two arrows descending from f to x and y. Each of x and y depend on u and v, so they will both have two arrows as well. Here's the diagram:

$$\begin{tikzcd} & & f \arrow[dl] \arrow[dr] & & \\ & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & \\ u & v & & u & v \end{tikzcd}$$

   If we want to calculate $\frac{\partial f}{\partial u}$, we need to add up all the ways in which f changes when u changes. This happens through x, and it also happens through y. These two paths are highlighted in red:

$$\begin{tikzcd} & & f \arrow[dl, red] \arrow[dr, red] & & \\ & x \arrow[dl, red] \arrow[d] & & y \arrow[d, red] \arrow[dr] & \\ u & v & & u & v \end{tikzcd}$$

   Let's consider the left "leg." There are two levels of change: one in which f changes as x changes, and one in which x changes as u changes. These are represented by the values $\frac{\partial f}{\partial x}$ and $\frac{\partial x}{\partial u}$. To get the composite change, we just multiply the terms: $\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u}$. This gives us the amount that f changes with respect to u *through* x. We can repeat the same thing with the "leg" on the right to get $\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$. This the amount that f changes with respect to u through y. To get the total change, we add the two quantities together:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

   Calculating all of these quantities gives the following:

$$\begin{aligned} f_x(x,y) &= 2xy + y^2 \\ f_y(x,y) &= x^2 + 2xy \\ x_u(u,v) &= 2v \\ y_u(u,v) &= 2u \end{aligned}$$

   Therefore,

$$\frac{\partial f}{\partial u} = (2xy + y^2)(2v) + (x^2 + 2xy)(2u)$$

But we probably want to express $\frac{\partial f}{\partial u}$ completely in terms of u and v; all we have to do is plug in the expressions of x and y in terms of u and v:

$$\frac{\partial f}{\partial u} = (2(2uv)(u^2+v^2) + (u^2+v^2)^2)(2v) + ((2uv)^2 + 2(2uv)(u^2+v^2))(2u)$$

$\square$

2. *Let $g(t) = f(x(t), y(t))$, where $x(t) = t^2$ and $y(t) = 2t+1$. Use the following table of values to compute $g'(1)$.*

$$f_x(1,1) = 2 \qquad f_y(1,1) = 3$$
$$f_x(2,1) = 4 \qquad f_y(2,1) = -2$$
$$f_x(1,3) = -1 \qquad f_y(1,3) = 5$$

Solution. As before, we'll draw a diagram to start off. This is slightly more confusing than in the previous problem, because our composite function is g, and only depends on t. The intermediate functions are $f(x,y)$ and $x(t)$ and $y(t)$. Here's what the diagram might look like (the dotted arrow is just to demonstrate that g is the composite function; it's not too important):

$$\begin{tikzcd} & f \arrow[dl] \arrow[dr] \arrow[dd, dashed, "g"] & \\ x \arrow[dr] & & y \arrow[dl] \\ & t & \end{tikzcd}$$

To get $g'(t)$, which is just $\frac{dg}{dt}$, we need to find how f changes with respect to t through x, and also through y. Thus,

$$\frac{dg}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Note that, since x and y only depend on one variable, t, they are just regular derivatives and not partial derivatives. It does make a difference! We can can calculate the derivatives of x and y to get:

$$x'(t) = 2t \qquad x'(1) = 2$$
$$y'(t) = 2 \qquad y'(1) = 2$$

Now, we just need f_x and f_y. But we don't what the function looks like, so we can't actually calculate these partial derivatives. Fortunately, we are evaluating $g'(t)$ at the point $t = 1$, so we can use the table of values that we were given. We just have to be careful which values we use. Since

$(x(t), y(t)) = (t^2, 2t+1)$, we know that $(x(1), y(1)) = (1, 3)$. Therefore, the (x, y) point that we care about is $(1, 3)$. So:

$$f_x(1,3) = -1 \qquad f_y(1,3) = 5$$

Therefore,

$$\begin{aligned} g'(1) &= f_x(1,3)x'(1) + f_y(1,3)y'(1) \\ &= (-1)(2) + (5)(2) \\ &= 8 \end{aligned}$$

□

Directional Derivatives and Gradients

1. *Let $f(x, y, z) = x^2y - 2xz$. Compute the gradient of f, and find the directional derivative of f in the direction of the line $y = x = z$.*

   Solution. To calculate the gradient of f, we simply have to compute the partial derivatives and throw everything in a vector:

   $$\nabla f(x, y, z) = \langle f_x,\, f_y,\, f_z \rangle$$

   Therefore, the gradient is:

   $$\nabla f(x, y, z) = \left\langle 2xy - 2z,\, x^2,\, 2x \right\rangle$$

   Next, recall that the directional derivative in the direction of the unit vector $\mathbf{u}$ is calculated as follows:

   $$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

   So if we can find a unit vector in the direction of the line $y = x = z$, we can compute the dot product above. There are a couple ways to determine a unit vector in the direction of the line $y = x = z$. The first thing I thought of is to parametrize the line. To do this, I'm just going to set $x(t) = t$ (why can I do this? Why not!). Then we can figure out what $y(t)$ and $z(t)$ are using the equation. We know that $y = x$, so therefore, $y(t) = t$. Same for z: $z(t) = t$. Hence, we can paramterize the line as follows:

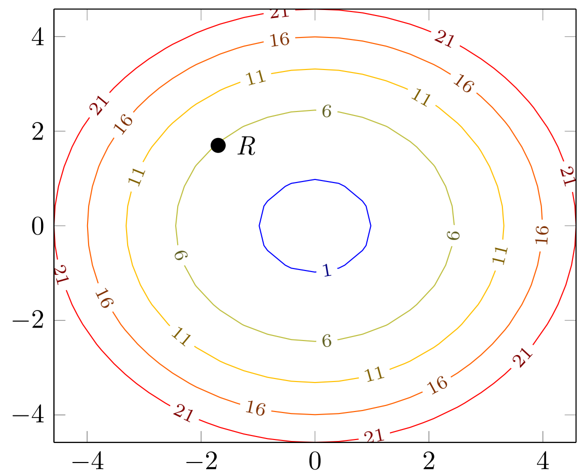
   $$\mathbf{r}(t) = \langle t,\, t,\, t \rangle = t\, \langle 1,\, 1,\, 1 \rangle$$

   Therefore, a direction vector for the line is $\langle 1, 1, 1 \rangle$. But that's not a unit vector, so we just have to divide by the length: $\mathbf{u} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$. Thus, the directional derivative in that direction is:

   $$\begin{aligned} D_{\mathbf{u}} f(x, y, z) &= \nabla f(x, y, z) \cdot \mathbf{u} \\ &= \left\langle 2xy - 2z,\, x^2,\, 2x \right\rangle \cdot \frac{1}{\sqrt{3}} \langle 1,\, 1,\, 1 \rangle \\ &= \frac{1}{\sqrt{3}} \left(2xy - 2z + x^2 + 2x \right) \end{aligned}$$

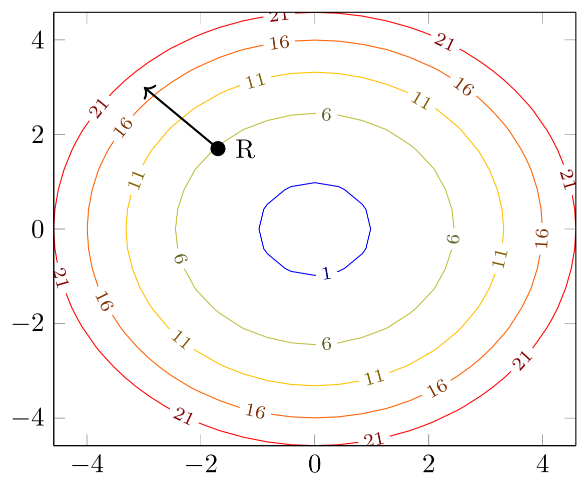
   □

2. *Consider the contour plot of the function $f(x, y)$ below.*



Draw the gradient vector at point R, and determine whether the directional derivative in the southeast direction is positive, negative, or 0.

Solution. The gradient vector is always perpendicular to contour curves. Therefore, the gradient vector at R looks something like this:



Next, if we head in the southeast direction from R, the value of the contour lines decrease from 6 to 1. Therefore, the directional derivative in the southest direction is negative. $\square$

3. *Consider the surface $y = x^2 + z^2 - 2$. Determine at what points on the surface, if any, the tangent plane is parallel to the $x - z$ plane.*

Solution. In order to answer this question, we'll use the fact that the gradient vector of a function $f(x, y, z)$ is always orthogonal to the level surfaces of f. The reason that this is useful is because we can rewrite the equation of the surface as a level curve of a function $f(x, y, z)$. First, move everything over to one side:

$$x^2 + z^2 - 2 - y = 0$$

Next, define $f(x, y, z) = x^2 + z^2 - 2 - y$. Then the surface we're considering is exactly the level surface of f corresponding to the value 0:

$$f(x, y, z) = 0$$

So calculating the gradient of f will give a vector normal to the surface at every point:

$$\nabla f(x, y, z) = \langle 2x,\ -1,\ 2z \rangle$$

If we want a tangent plane to the surface to be parallel to the $x - z$ plane, we want the *normal* vector of the plane to be parallel to a normal vector of the xz plane. A normal vector of the tangent plane to the surface is the same as a vector normal to the surface, so therefore the gradient vector that we calculated above gives us the normal vector to our tangent plane. Hence, we want

$$\langle 2x,\ -1,\ 2z \rangle$$

to be parallel to the vector

$$\langle 0,\ 1,\ 0 \rangle$$

which is a normal vector to the xz plane. This means that the x and z components of the gradient have to be 0, which means that $x = 0$ and $z = 0$. Plugging this into our original equation gives us

$$y = 0^2 + 0^2 - 2 = -2$$

So the only point at which the tangent plane is parallel to the xz plane is at $(0, -2, 0$. $\square$

4. *After getting home from the Northwestern - Purdue game, Joe had a choice - he could either stay home all day and watch football, or go to the library and do work, or do some combination of both. Joe modeled his happiness H as a function of time spent in the library (L) and time spent watching football (F):*

$$H = F - L$$

What fraction of his time should he spend doing each to be as happy as possible?

Solution. This is a really vague word problem designed to test how well you understand the gradient conceptually. The important part here is that the gradient gives the *direction of fastest increase.* So somehow, we could use the gradient to calculate the "direction I should go in" to increase my happiness as much as possible. The word problems you'll be faced with will be more straightforward; I'm just seeing how far we can push the intuition of the gradient.

We can think of my model as a two variable function:

$$H(F, L) = F - L$$

Let's calculate the gradient:

$$\nabla H(F, L) = \langle H_F,\, H_L \rangle = \langle 1,\, -1 \rangle$$

This gives the direction of greatest increase of H. Here's (vaguely) how to interpret that: in order for me to be as happy as possible, for every 1 minute I spend watching football, I should spend -1 minutes in the library. Here's where this problem gets stupid, so just bear with me: because of the laws of physics, I can't spend negative time in the library. Therefore, I should spend at most 0 minutes in the library.

Thus, in order for me to be as happy as possible, I should spend all my time watching football and none of my time in the library. Surprise, surprise. $\square$