Some 32A Practice Problems

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Particle Motion

1. A particle moves along a path given by the curve:

\[ r(t) = (2t^2 + 1, 2e^t + 2, t^3) \]

What are the normal and tangential components of acceleration when the particle is moving in the direction of the y-axis?

Projectile Motion

1. One night, Joe sneaks onto the roof of Swift Hall — which is 10 meters tall — and throws his shoe at an angle of 30 degrees to the horizontal with an initial speed of 2 m/s. How fast is his shoe going when it hits the ground? Take gravity to be 10 m/s².

2. Joe is having a little too much fun on Dillo Day, makes a poor judgement call, and shoots a firework off in front of Lunt Hall. The firework is launched in the western direction (in the direction of the negative x axis) at an angle of 60 degrees, with an initial speed of 20 m/s. The wind gives the firework a northerly acceleration of 3 m/s². Take gravity to be 10 m/s². How far away from Lunt does the firework land?

3. (Challenging Problem) Joe is about to take the halfcourt shot at the NU-Wisconsin basketball game. The basketball hoop is 3 meters high, and is 28 meters from half court. Unfortunately, Joe hasn’t played basketball since 8th grade and has no idea how hard to shoot the ball, so he shoots at an angle of 45 degrees with an initial speed of 20 m/s. Also, due to the faltering structural integrity of Welsh-Ryan Arena, there is a draft that gives the ball a left-ward acceleration of 1 m/s². Take gravity to be 10 m/s², and suppose for the sake of simplicity that Joe shoots from the point (0, 0, 0). How close does the ball come to going in?

Multivariable Functions

1. Let \( f(x, y) = x + y^2 \). Determine the domain of \( f \) and draw a contour plot with three level curves corresponding to the values \(-1, 0, \) and 1.

2. Let \( f(x, y) = \frac{x^2 - y^2}{x + y} \). Describe the domain of the function, and draw a contour plot.

3. Describe the level surfaces of the function \( f(x, y, z) = x^2 + y^2 + z^2 \).
Multivariable Limits and Continuity

1. (Technique: Continuity) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{x^2 - y + 3}{\cos(xy) + 1}
\]

2. (Technique: Different Path Approaches) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{2x^3 - xy - 3y^3}{3x^3 - 2y^3}
\]

3. (Technique: Different Path Approaches) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{x^2 + xy + y^2}{xy + x^2 + 2y}
\]

4. (Technique: Different Path Approaches) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2 + 3y^2}
\]

5. (Technique: Different Path Approaches) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{x^4}{x^8 + y^2}
\]

6. (Technique: Factoring) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^2 + y}
\]

7. (Technique: Conjugate) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{x^3 + xy + y^2}{2 - \sqrt{x^3 + xy + y^2} + 4}
\]

8. (Technique: Squeeze Theorem / Comparison) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{x^4 y^2}{x^4 + 3y^6}
\]

9. (Technique: Polar Coordinates) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{2xy \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}
\]

10. (Technique: Polar Coordinates) Evaluate the following limit:
\[
\lim_{(x,y)\to(0,0)} \frac{2xy \sqrt{x^2 + y^2}}{x^2 + y^2}
\]

11. Is the following function continuous at the point \((0,0,0)\)?
\[
f(x, y, z) = \begin{cases} 
\frac{x^2 + y^2 + z}{x + y + z} & (x, y, z) \neq (0,0,0) \\
1 & (x, y, z) = (0,0,0)
\end{cases}
\]
Partial Derivatives

1. Let \( f(x, y) = \frac{e^{xy}}{x^2 + y^2} \). Compute \( f_x(x, y) \) and \( f_y(x, y) \).

2. Let \( g(r, \theta, z) = r^2 \cos 2z \theta + zr \theta - 2r - z \). Compute \( g_z(1, \frac{\pi}{4}, 2) \).

3. Joe just bought an Old Fashioned Buttermilk donut and a Valrhona Triple Chocolate donut from Firecakes, and is trying to decide which one to eat first. After extensive testing and data collection, Joe has modeled his happiness as a function of consumption of donuts. Let \( \textit{H} \) denote happiness, \( \textit{B} \) denote the number of Old Fashioned Buttermilk donuts consumed, and \( \textit{C} \) denote the number of Valrhona Triple Chocolate donuts consumed. Then

\[
\textit{H}(\textit{B}, \textit{C}) = e^{3\textit{B}} + (\textit{B} + 1)(\textit{C} + 1)^3 + e^\text{C}
\]

If Joe only has time to take a single bite of one donut, which one should he eat?

4. Let \( f(x, y) = \ln(x^2 + y^2 + 1) \). Prove that \( f_{xy} = f_{yx} \).

5. Consider the contour plot of the function \( f(x, y) \) below.

Determine whether each of the following quantities is positive, negative, or zero:

(a) \( f_x(P) \)  
(b) \( f_y(P) \)  
(c) \( f_{yy}(P) \)  
(d) \( f_y(Q) \)  
(e) \( f_x(Q) \)  
(f) \( f_{xx}(Q) \)

Tangent Planes and Approximation

1. Let \( f(x, y) = 2xy + 2^y \). Find the equation of the tangent plane to \( f \) at the point \((1, 2)\).
2. Let \( f(x, y, z) = xy + xz + yz \). Find the linear approximation to \( f \) at the point \((1, 2, 3)\).

3. Let \( f(x, y) = x^3y^3 + x + y \). Find the linear approximation and quadratic approximation to \( f \) at the point \((1, 1)\). Find \( \Delta z \) in moving from the point \((1, 1)\) to \((2, 2)\). Estimate \( \Delta z \) with the linear and quadratic approximation.

The Chain Rule

1. Let \( f(x, y) = x^2y + xy^2 \) and \( x(u, v) = 2uv \) and \( y(u, v) = u^2 + v^2 \). Consider \( f \) as a function of \( u \) and \( v \) by \( f(x(u, v), y(u, v)) \). Use the chain rule to compute \( \frac{\partial f}{\partial u} \).

2. Let \( g(t) = f(x(t), y(t)) \), where \( x(t) = t^2 \) and \( y(t) = 2t + 1 \). Use the following table of values to compute \( g'(1) \).

\[
\begin{align*}
  f_x(1, 1) &= 2 & f_y(1, 1) &= 3 \\
  f_x(2, 1) &= 4 & f_y(2, 1) &= -2 \\
  f_x(1, 3) &= -1 & f_y(1, 3) &= 5
\end{align*}
\]

Directional Derivatives and Gradients

1. Let \( f(x, y, z) = x^2y - 2xz \). Compute the gradient of \( f \), and find the directional derivative of \( f \) in the direction of the line \( y = x = z \).

2. Consider the contour plot of the function \( f(x, y) \) below.

Draw the gradient vector at point \( R \), and determine whether the directional derivative in the southeast direction is positive, negative, or 0.
3. Consider the surface \( y = x^2 + z^2 - 2 \). Determine at what points on the surface, if any, the tangent plane is parallel to the \( x - z \) plane.

4. After getting home from the Northwestern - Purdue game, Joe had a choice - he could either stay home all day and watch football, or go to the library and do work, or do some combination of both. Joe modeled his happiness \( H \) as a function of time spent in the library \( (L) \) and time spent watching football \( (F) \):

\[
H = F - L
\]

What fraction of his time should he spend doing each to be as happy as possible?