PLANE PRACTICE PROBLEMS

Joseph Breen

Here are some practice problems involving finding the equation of a plane. Recall that the equation of any plane can be written

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where $\langle a, b, c\rangle$ is a normal vector to the plane and (x_0, y_0, z_0) is a point on the plane. In this set of problems, you will be given this information implicitly in different guises; I tried to think of as many variations as I could. The starred problems may be slightly challenging, while the un-starred problems are standard. Solutions begin on the next page. If you spot mistakes or think of other solutions, let me know!

Problems

1. Find the equation of the plane with normal vector $\langle 1, 2, 5\rangle$ which passes through the point $(-1, 3, 4)$.

2. Find the equation of the plane containing the points $(1, 2, 3)$, $(3, 2, 1)$, and $(-1, 0, 2)$.

3. Find the equation of the plane containing the line $\mathbf{r}(t) = \langle t + 1, 3t - 1, t\rangle$ and the point $(10, 10, 10)$.

4. Find the equation of the plane which is parallel to the plane $x - y + 9z = 0$ and contains the point $(1, 1, 1)$.

5. Let $\mathbf{r}_1(t) = \langle t + 3, 1 - t, 3t + 3\rangle$ and $\mathbf{r}_2(t) = \langle t + 5, -t, 3t\rangle$ be two parallel lines (that do not intersect). Find the equation of the plane containing both lines.

6. Let $\mathbf{r}_1(t) = \langle 4-2t, 5t-2, 2t\rangle$ and $\mathbf{r}_2(t) = \langle t+1, 2t+1, t+1\rangle$ be two lines. Find their point of intersection and then find the equation of the plane containing both lines.

7. Find the equation of a plane perpendicular to the planes $x + y - 3z = 0$ and $-x + 2y + 2z = 1$.

8. (*) Find the equation of a plane which contains the line $\mathbf{r}(t) = \langle 2t, 2t + 1, t + 2\rangle$ and minimizes the (acute) angle with the plane $x + y + z = 0$.

9. (*) Let $\mathbf{r}_1(t) = \langle t, t + 2, 2t\rangle$ and $\mathbf{r}_2(t) = \langle t + 1, 2t + 1, t + 1\rangle$ be two skew lines. Find the equation of the plane which lies exactly halfway between the two lines and intersects neither.

10. (*) Let $x + 2z = 1$ and $x + y + 2z = 2$ be two planes. Find the equation of the plane which bisects the obtuse angle formed by the intersection of the two planes.

11. (*) Let $2x + y - z = 0$ and $-x + 2y + 3z = 0$ be two planes. Find the equation of a plane which, together with the given planes, encloses a region which is an (infinitely long) **isosceles triangular prism** such that the **area of any (isosceles) triangular cross section of the prism** is 1. (Note that there are four correct answers).

Solutions

1. **Find the equation of the plane with normal vector $\langle 1, 2, 5\rangle$ which passes through the point $(-1, 3, 4)$.**

   Solution. To find the equation of a plane, we need a normal vector and a point. We are given both of these directly. Thus, the equation of the plane we seek is

   $$1(x-(-1)) + 2(y-3) + 5(z-4) = 0 \qquad \Rightarrow \qquad x + 2y + 5z = 25.$$

   □

2. **Find the equation of the plane containing the points $(1, 2, 3)$, $(3, 2, 1)$, and $(-1, 0, 2)$.**

   Solution. We are given a point on the plane (in fact, we are given three) so all we need to do is find a normal vector. We can do this by forming two vectors from the three points and taking their cross product. This will result in a vector orthogonal to both vectors, and since both vectors connect all three points, the resulting vector will be orthogonal to the plane we seek. Figure 1 contains a helpful picture.

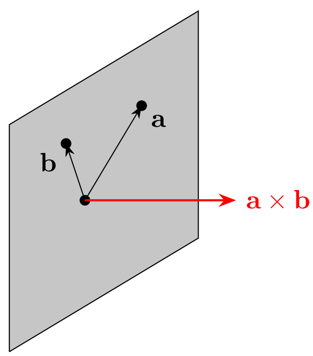


Figure 1: Finding the normal vector to a plane containing three points.

   Thus, let

   $$\mathbf{a} := \langle 3, 2, 1\rangle - \langle 1, 2, 3\rangle = \langle 2, 0, -2\rangle$$
   $$\mathbf{b} := \langle -1, 0, 2\rangle - \langle 1, 2, 3\rangle = \langle -2, -2, -1\rangle.$$

   Then the vectors $\mathbf{a}$ and $\mathbf{b}$ lie "in" the plane. A normal vector to the plane will be

   $$\mathbf{a} \times \mathbf{b} = \langle 2, 0, -2\rangle \times \langle -2, -2, -1\rangle = \langle -4, 6, -4\rangle.$$

   Thus, the equation of the plane we seek is

   $$-4(x-1) + 6(y-2) - 4(z-3) = 0.$$

   Simplifying gives

   $$2(x-1) - 3(y-2) + 2(z-3) = 0 \qquad \Rightarrow \qquad 2x - 3y + 2z = 2.$$

   □

3. **Find the equation of the plane containing the line $\mathbf{r}(t) = \langle t+1, 3t-1, t\rangle$ and the point $(10, 10, 10)$.**

Solution. We are given a point, so we only need to find a normal vector. To do this, we will take a cross product of some pair of vectors. As the plane we seek contains the given lines, a direction vector for the line will lie "in" the plane. To get another vector which lies "in" the plane, we could connect the given point $(10, 10, 10)$ to *any* point on the line (since our plane contains both).

Thus, let $\mathbf{a} = \langle 1, 3, 1 \rangle$. This is a direction vector for the line $\mathbf{r}(t)$. Note that $\mathbf{r}(0) = \langle 1, -1, 0 \rangle$. This some random point on the line. So let

$$\mathbf{b} = \langle 10, 10, 10 \rangle - \langle 1, -1, 0 \rangle = \langle 9, 11, 10 \rangle.$$

Then $\mathbf{a}$ and $\mathbf{b}$ are two vectors that lie "in" the plane. Thus, a normal vector for the plane we seek is

$$\mathbf{a} \times \mathbf{b} = \langle 1, 3, 1 \rangle \times \langle 9, 11, 10 \rangle = \langle 19, -1, -16 \rangle.$$

Thus, the equation of the plane we seek is

$$19(x - 10) - 1(y - 10) - 16(z - 10) = 0 \qquad \Rightarrow \qquad 19x - y - 16z = 20.$$

□

4. **Find the equation of the plane which is parallel to the plane $x - y + 9z = 0$ and contains the point $(1, 1, 1)$.**

Solution. We are again given a point, so we only need to find a normal vector. The key here is that *parallel planes have the same*[1] *normal vectors*. Since a normal vector for the given plane is $\langle 1, -1, 9 \rangle$, we can take this to be the normal vector for the plane that we seek. Thus, the equation for the plane we seek is

$$1(x - 1) - 1(y - 1) + 9(z - 1) = 0 \qquad \Rightarrow \qquad x - y + 9z = 9.$$

□

5. **Let $\mathbf{r}_1(t) = \langle t + 3, 1 - t, 3t + 3 \rangle$ and $\mathbf{r}_2(t) = \langle t + 5, -t, 3t \rangle$ be two parallel lines (that do not intersect). Find the equation of the plane containing both lines.**

Solution. Observe that the two lines are in fact parallel because they have parallel (the same) direction vectors. A direction vector for both lines is $\langle 1, -1, 3 \rangle$. This vector lies "in" the plane we seek.

To find the desired equation, we need a point and a normal vector. Because the plane contains both lines, to get a point we can use any point on either line. For example, $\mathbf{r}_2(0) = (5, 0, 0)$ is a point on the line. To get a normal vector, we will take the cross product of two vectors that lie "in" the plane as usual. One of those vectors is the direction vector $\mathbf{a} := \langle 1, -1, 3 \rangle$ from above. To get another vector, we connect any two points on the lines; see Figure 2 for a rough picture. For example, since $\mathbf{r}_1(0) = (3, 1, 3)$ and $\mathbf{r}_2(0) = (5, 0, 0)$, we may take

$$\mathbf{b} = \langle 3, 1, 3 \rangle - \langle 5, 0, 0 \rangle = \langle -2, 1, 3 \rangle.$$

Then a normal vector for our plane is

$$\mathbf{a} \times \mathbf{b} = \langle 1, -1, 3 \rangle \times \langle -2, 1, 3 \rangle = \langle -6, -9, -1 \rangle.$$

Thus, the equation of the plane we seek is

$$-6(x - 5) - 9(y - 0) - 1(z - 0) = 0 \qquad \Rightarrow \qquad 6x + 9y + z = 30.$$

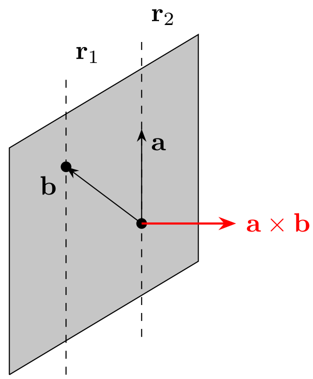


Figure 2: Finding the normal vector to a plane containing two parallel lines.

□

6. **Let $\mathbf{r}_1(t) = \langle 4-2t, 5t-2, 2t\rangle$ and $\mathbf{r}_2(t) = \langle t+1, 2t+1, t+1\rangle$ be two lines. Find their point of intersection and then find the equation of the plane containing both lines.**

Solution. First, we find the point of intersection between the two lines. Because the lines may intersect at different "times", we seek a solution to the system of equations defined by

$$\mathbf{r}_1(t) = \mathbf{r}_2(s)$$

where t and s are different parameters. This system of equation is given by

$$\langle 4-2t, 5t-2, 2t\rangle = \langle s+1, 2s+1, s+1\rangle \qquad \Rightarrow \qquad \begin{cases} 4-2t = s+1 \\ 5t-2 = 2s+1 \\ 2t = s+1 \end{cases}.$$

The first and last equation together (which both equal $s+1$ on the right hand side) imply that $4-2t = 2t$ and thus $4t = 4$ and so $t = 1$. Since $2t = s+1$, when $t = 1$ we must have $2 = s+1$ and so $s = 1$. Plugging this into the middle equation gives $5-3 = 2+1$, which is indeed true. So the intersection of the two lines is given by parameters $s = 1$ and $t = 1$.[2] Plugging either of these parameters into their respective line equations gives the point

$$\mathbf{r}_1(1) = \mathbf{r}_2(1) = (2, 3, 2).$$

This is the point of intersection.

To find the equation of the plane containing both lines, we need a point on the plane and a normal vector. We can use the point we just found, so we only need to find a normal vector. Since the plane contains both lines, the direction vectors for both lines lie "in" the plane and thus their cross product will be orthogonal to the plane. So a normal vector for the plane is

$$\langle -2, 5, 2\rangle \times \langle 1, 2, 1\rangle = \langle 1, 4, -9\rangle.$$

[1] To say this more precisely: any two normal vectors for parallel planes are parallel.
[2] So the lines actually do intersect at the same time! This is interesting but not relevant for the problem.

Thus, the equation of the plane we seek is

$$1(x-2)+4(y-3)-9(z-2)=0 \qquad \Rightarrow \qquad x+4y-9z=-4.$$

□

7. **Find the equation of a plane perpendicular to the planes $x+y-3z=0$ and $-x+2y+2z=1$.**

Solution. One difference about this problem is that there are infinitely many correct solutions! If there was an additional condition specifying a point or something like that, there would be a single correct answer, but we are free to find *any* plane which is perpendicular to both of the given planes. That means we only need to find an appropriate normal vector (and then we can take the point $(0,0,0)$, or any other point we like.

The key concept for this problem is the following: if two planes are orthogonal, their normal vectors are orthogonal. So we seek a plane with a normal vector orthogonal to both $\langle 1,1,-3\rangle$ and $\langle -1,2,2\rangle$ (the normal vectors for the given planes). We can get such a vector by taking their cross product:

$$\langle 1,1,-3\rangle \times \langle -1,2,2\rangle = \langle 8,1,3\rangle.$$

Thus, the equation of a plane satisfying the desired condition is

$$8x+y+3z=0.$$

In fact, the plane $8x+y+3z=\lambda$ for *any* choice of scalar λ would be a correct answer! □

8. **(*) Find the equation of a plane which contains the line $\mathbf{r}(t)=\langle 2t, 2t+1, t+2\rangle$ and minimizes the (acute) angle with the plane $x+y+z=0$.**

Solution. As usual, we need a point and a normal vector. Because the plane contains the line $\mathbf{r}(t) = \langle 2t, 2t+1, t+2\rangle$, we may take our point to be any point on the line. For example, let's use $\mathbf{r}(0) = (0,1,2)$. The normal vector requires some more work. Figure 3 is my poor attempt at giving a visual aid for the following argument.

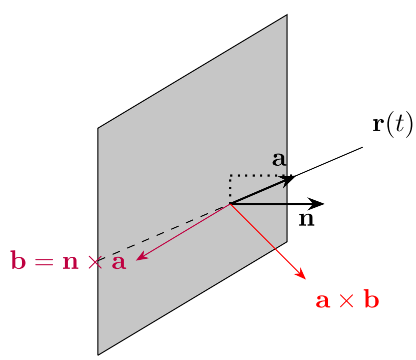


Figure 3: Finding the normal vector for a plane which contains a line and minimizes the angle with another plane. Here the gray plane is the plane $x+y+z=0$ and $\mathbf{n}$ is its normal vector. The purple vector $\mathbf{b}$ is the cross product of a direction vector $\mathbf{a}$ for the given line and $\mathbf{n}$ and lies "in" the given plane. The red vector $\mathbf{a}\times\mathbf{b}$ is the normal vector we ultimately seek.

We will find the appropriate normal vector as usual by taking the cross product of two vectors that lie "in" the plane. Since the desired plane contains the line $\mathbf{r}(t)$, we may take one of these vectors to be any direction vector for the line. In particular, let $\mathbf{a} = \langle 2, 2, 1 \rangle$.

In order to minimize the angle between our desired plane and the given plane, we need to minimize the angle between $\mathbf{n} = \langle 1, 1, 1 \rangle$ (the normal vector for the given plane) and our desired normal vector. This part is something you just have to *see*: given a vector $\mathbf{a}$ and another vector $\mathbf{n}$, the vector orthogonal to $\mathbf{a}$ which minimizes the angle with $\mathbf{n}$ must lie in the plane generated by $\mathbf{a}$ and $\mathbf{n}$; see Figure 4.

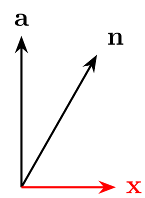


Figure 4: View the page as the plane spanned by $\mathbf{a}$ and $\mathbf{n}$. Picture the red vector $\mathbf{x}$ rotating around $\mathbf{a}$ (so that it rotates out of the paper towards you, then behind the paper, etc.) Among all such vectors $\mathbf{x}$, the one which minimizes the angle with $\mathbf{n}$ lies on the page; that is, the red vector pictured above is the desired vector.

If you buy all of that, here's what we need to do: we need to find a vector which is orthogonal to $\mathbf{a}$ and lies in the plane spanned by $\mathbf{a}$ and $\mathbf{n}$. Because the plane spanned by $\mathbf{a}$ and $\mathbf{n}$ has normal vector $\mathbf{b} := \mathbf{a} \times \mathbf{n}$, the normal vector we then seek is

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\mathbf{a} \times \mathbf{n}) = \langle 2, 2, 1 \rangle \times (\langle 2, 2, 1 \rangle \times \langle 1, 1, 1 \rangle) = \langle 1, 1, -4 \rangle.$$

Thus, we have a point $(0, 1, 2)$ and a normal vector $\langle 1, 1, -4 \rangle$. The plane we seek is thus

$$1(x - 0) + 1(y - 1) - 4(z - 2) = 0 \qquad \Rightarrow \qquad x + y - 4z = -7.$$

□

9. **(*) Let $\mathbf{r}_1(t) = \langle t, t+2, 2t \rangle$ and $\mathbf{r}_2(t) = \langle t+1, 2t+1, t+1 \rangle$ be two skew lines. Find the equation of the plane which lies exactly halfway between the two lines and intersects neither.**

Solution. Because we want our plane to not intersect either of the given lines, we need the direction vectors of both $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ to lie "in" the plane. That means that we can get a normal vector for the plane that we seek by taking the cross product of direction vectors for the given lines:

$$\langle 1, 1, 2 \rangle \times \langle 1, 2, 1 \rangle = \langle -3, 1, 1 \rangle.$$

The slightly more difficult part is finding a point on the plane. The relevant condition is the fact that the plane lies exactly halfway between the skew lines. We can locate any point on this plane by connecting a point on one line to a point on the other (any such points will work — prove this as an extra exercise!) and travelling halfway along this vector; see Figure 5. To make this precise, let

$$\mathbf{p} := \mathbf{r}_1(0) = \langle 0, 2, 0 \rangle$$
$$\mathbf{q} := \mathbf{r}_2(0) = \langle 1, 1, 1 \rangle$$

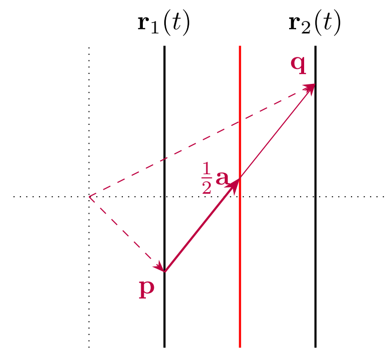


Figure 5: A heuristic two dimensional picture to capture the relevant situation in three dimensions. There are two skew lines (in black) and the red line is the plane that we seek. The vector $\mathbf{a} = \mathbf{q} - \mathbf{p}$ *is the difference between two points on the line. The vector* $\mathbf{p} + \frac{1}{2}\mathbf{a}$ *lands on the plane we seek.*

be points on the first and second line, respectively. Let

$$\mathbf{a} := \mathbf{q} - \mathbf{p} = \langle 1, 1, 1\rangle - \langle 0, 2, 0\rangle = \langle 1, -1, 1\rangle.$$

Then a point on the plane we seek is

$$\mathbf{p} + \frac{1}{2}\mathbf{a} = \langle 0, 2, 0\rangle + \frac{1}{2}\langle 1, -1, 1\rangle = \langle \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\rangle.$$

Using the normal vector we computed above, the plane we seek is

$$-3\left(x - \frac{1}{2}\right) + 1\left(y - \frac{3}{2}\right) + 1\left(z - \frac{1}{2}\right) = 0.$$

Simplifying gives

$$-3x + y + z = \frac{1}{2}.$$

□

10. **(*) Let $x + 2z = 1$ and $x + y + 2z = 2$ be two planes. Find the equation of the plane which bisects the obtuse angle formed by the intersection of the two planes.**

Solution. As usual, to find the desired plane we need a point and a normal vector. Because the plane we seek bisects some angle formed by the other planes, in particular the plane passes through the intersection of the two planes. So to get a point on the desired plane, we seek a point on the intersection of the planes $x + 2z = 1$ and $x + y + 2z = 2$. Any point will work. In particular, by inspection or divine intervention or luck or algebra, the point $(1, 1, 0)$ lies on both planes since $1 + 2(0) = 1$ and $1 + 1 + 2(0) = 2$. Thus, our point is $(1, 1, 0)$.

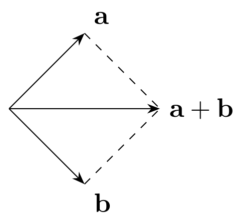
The trickier part is finding an appropriate normal vector. Because we want our plane to bisect the two given planes, we want the normal vector of our plane to bisect the obtuse angle formed by the two normal vectors. Here is a nice fact: if $\mathbf{a}$ and $\mathbf{b}$ are vectors of the same length, then $\mathbf{a} + \mathbf{b}$ bisects the angle formed by $\mathbf{a}$ and $\mathbf{b}$. To see this, observe that

$$\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 + \mathbf{a} \cdot \mathbf{b}$$

and

$$\mathbf{b}\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{b}\cdot\mathbf{a} + \|\mathbf{b}\|^2 .$$

Since $\|\mathbf{a}\|^2 = \|\mathbf{b}\|^2$, it follows that $\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{b}\cdot(\mathbf{a}+\mathbf{b})$ and therefore the angle between $\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$ is the same as the angle between $\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$. The geometric intuition should be clear from the picture:



Thus, a step in the right direction is to find normal vectors for the given planes that have the same length. By picking off the coefficients of the equations $x + 2z = 1$ and $x + y + 2z = 2$ we get normal vectors $\langle 1, 0, 2\rangle$ and $\langle 1, 1, 2\rangle$. These vectors do not have the same length, so we can normalize each of them. That is, let

$$\begin{aligned}
\mathbf{a} &:= \frac{1}{\sqrt{1^2+2^2}}\langle 1,0,2\rangle = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\rangle \\
\mathbf{b} &:= \frac{1}{\sqrt{1^2+1^2+2^2}}\langle 1,1,2\rangle = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle .
\end{aligned}$$

Then $\mathbf{a}$ and $\mathbf{b}$ are both *unit* normal vectors of their respective planes.

Based on the previous discussion it might seem reasonable to suspect that the normal vector we seek is $\mathbf{a}+\mathbf{b}$. Using such a normal vector would definitely give us a plane that bisects some angle formed by the given planes, but it might not be the angle we actually want. For example, Figure 6 indicates the two different planes that bisect an angle formed by two planes.

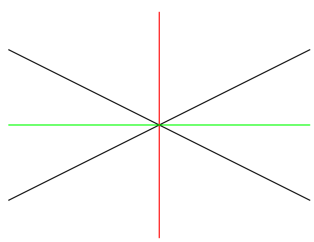


Figure 6: An aerial view of two given planes in black and two planes, one red and one green, which bisect the angles formed by the black planes. We specifically seek the red plane (the one that bisects the obtuse angle.

We can produce the correct plane by either letting $\mathbf{a}+\mathbf{b}$ or $\mathbf{a}-\mathbf{b}$ be the normal vector; these two choices of normal vectors will produce the two different planes, for example, see Figure 7.

It is clear that the choice of $\mathbf{a}+\mathbf{b}$ or $\mathbf{a}-\mathbf{b}$ depends on whether the angle between $\mathbf{a}$ and $\mathbf{b}$ is acute or obtuse. We can determine this via the dot product. Note that

$$\mathbf{a}\cdot\mathbf{b} = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle = \frac{1}{\sqrt{30}} + \frac{4}{\sqrt{30}} = \frac{5}{\sqrt{30}}.$$

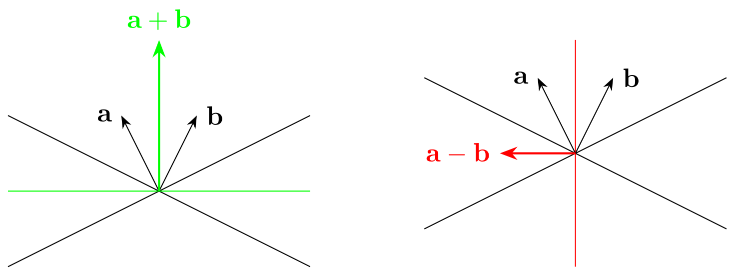


Figure 7: Computing a bisecting normal vector in two methods. Note that if the normal vectors have an acute angle between them as above, the normal vector $\mathbf{a}+\mathbf{b}$ *produces a plane which bisects the acute angle and* $\mathbf{a}-\mathbf{b}$ *produces a plane which bisects the obtuse angle.*

Since $\mathbf{a}\cdot\mathbf{b} > 0$, it follows that the angle between $\mathbf{a}$ and $\mathbf{b}$ is acute. Thus, the situation depicted in Figure 7 is accurate: to get the plane which bisects the obtuse angle, we may take the normal vector to be $\mathbf{a}-\mathbf{b}$:

$$\mathbf{a}-\mathbf{b} = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\rangle - \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle = \left\langle \frac{\sqrt{6}-\sqrt{5}}{\sqrt{30}}, -\frac{1}{\sqrt{6}}, 2\frac{\sqrt{6}-\sqrt{5}}{\sqrt{30}} \right\rangle.$$

We have finally found a normal vector to pair with our point $(1,1,0)$. Thus, the equation of the plane we seek is

$$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{30}}(x-1) - \frac{1}{\sqrt{6}}(y-1) + 2\frac{\sqrt{6}-\sqrt{5}}{\sqrt{30}}(z-0) = 0.$$

Simplifying gives

$$\left(\sqrt{6}-\sqrt{5}\right)x - \sqrt{5}y + 2(\sqrt{6}-\sqrt{5})z = \sqrt{6}-2\sqrt{5}.$$

Whew!

□

11. **(*) Let $2x + y - z = 0$ and $-x + 2y + 3z = 0$ be two planes. Find the equation of a plane which, together with the given planes, encloses a region which is an (infinitely long) isosceles triangular prism such that the area of any (isosceles) triangular cross section of the prism is 1. (Note that there are four correct answers).**

Solution. To be updated. This is relatively similar to the previous problem. □